Memorial conference on
Spectral Theory of Partial Differential operators

Abstracts

Boris Altshuler
Many Body Localization and Anderson Model on Random Regular Graphs
Abstract:
The talk is about statistics of the eigenstates and local spectra of the tight-binding model with onsite disorder (Anderson model) on the ensemble of large random regular graphs (RGG). The interest to these problems is determined by the similarity of these statistics with the statistics of Many-Body local spectra and wave-functions of a wide class of Many-Body models.
Recent progress in understanding dynamics of quantum Many-Body systems is based on the concept of Many-Body Localization: the Many-Body can have a discrete spectrum with eigenstates localized in the Hilbert space in a way similar to the conventional real space one-particle Anderson Localization. Depending on e.g. temperature the system finds itself either in MBL or in the Many-Body extended (MBE) phase. In the former case conventional Statistical Mechanics is invalid. Moreover, validity of the conventional Statistical Mechanics is not restored immediately after the transition to the MBE phase: there appears a new phase, which we call Non-ergodic extended (NEE). The eigenstates in this phase turn out to be multifractal.
I plan to discuss manifestations of the NEE states in the statistics of the eigenfunctions and explain why Anderson model on RGG is a good laboratory of understanding properties on Many Body eigenstates. I will also present the numerical and analytical justifications of the existence of the NEE states in the RGG Anderson model.

Nikolai Filonov
Maxwell operator in a cylinder with coefficients that do not depend on the longitudinal variable
Abstract:
We consider the Maxwell operator in a cylinder with arbitrary Lipschitz cross-section. We assume that the coefficients (dielectric and magnetic permittivities) depend on the transversal variables only. They are positive matrix-valued functions which are not assumed to be smooth. We show that
1) the spectrum of the Maxwell operator is absolutely continuous;
2) the geometry of the spectrum depends on the topology of the cross-section of the cylinder.

Rupert Frank
Trace ideal versions of the Stein-Tomas theorem
Abstract:
Birman and Solomyak made fundamental contributions to our understanding of trace ideal properties of operators of the form W_1(X) T(-i\nabla) W_2(X). In this talk we present some recent results about such operators when T is only a measure. This has applications to quantum many-body systems and to the study of eigenvalues with complex potentials. The talk is based on joint work with Julien Sabin.

Thomas Hoffmann-Ostenhof
Remarks on Pleijel's theorem.
Abstract:
I discuss some recent results concerning Pleijel's theorem. In particular some joint work with Helffer and some work with Charron and Helffer are presented.

Simon Gindikin
Curved version of the Radon inversion formula
Abstract:
This year we celebrate 100 years of the Radon formula reconstructing functions through their integrals along lines. This formula played an outstanding role as at theoretical aspects of Mathematics, so at applied ones. I'll give an generalization of this formula for curves.

Efim Gluskin
On the linearized version of the symplectic capacity

Alexei A. Ilyin
Sharp constants and correction terms in a class of interpolation inequalities
Abstract:
We develop a general method for finding sharp constants and possible correction terms for the imbeddings of Hilbert Sobolev spaces into the space of bounded functions. Both the continuous and discrete cases fit into this general scheme. Applications to spectral theory and Carlson-Landau inequalities will be given.

Victor Ivrii
Asymptotics of the ground state energy for relativistic heavy atoms and molecule.
Abstract:
We consider a multiparticle Hamiltonian

$$\mathsf{H} = \sum_k H_k + \sum_{k<l} |x_k - x_l|^{-1}$$

on $\bigwedge_{1\le n\le N} L^2(\mathbb{R}^3)$, with $H_k = (c^2D_k^2 + m^2c^4)^{1/2} - mc^2 - \sum_m Z_m|x_k - y_m|^{-1}$, where y_m, Z_m is a position and the charge of m-th nuclei and consider the asymptotics of the ground state as $Z = Z_1 + \ldots + Z_m \to \infty$, $c \ge \pi Z_m/2$ for all m.
It is work in the progress: currently remainder estimate is $O(Z^{5/3})$, which is improvement over the paper of J. P. Solovej, T. Ø. Sørensen and W. L. Spitzer. The relativistic Scott correction for atoms and molecules Comm. Pure Appl. Math., 63: 39–118 (2010), and the remainder estimate $o(Z^{5/3})$ requires next correction terms.

Yakar Kannai
Is (quasi) analyticity relevant to financial markets?
Abstract:
It is well known that time analyticity of solutions of parabolic differential equations plays a crucial role in demonstrating dynamic completeness of financial markets at equilibrium, making derivatives pricing possible. It is known that global conditions are required in order to obtain uniqueness and time analyticity for Cauchy problem, even for the standard heat equation. Observe that what is really relevant is quasi-analyticity. We obtain time-quasianalyticity for parabolic equations (both homogeneous and inhomogeneous) with quasi-analytic coefficients, under sufficiently general growth conditions to cover standard financial applications (jointly with R. Raimondo).

Vladimir Kozlov
Hadamard formula for eigenvalues of the Dirichlet and Neumann problems for elliptic operators on C^1 and C{1,\alpha} –domains
Abstract:
The classical Hadamard formula for the □rst eigenvalue of the Dirichlet- Laplacian in a bounded domain gives its asymptotic representation when we perturb the reference domain by

using a normal shift function. This asymptotic formula was derived under quite restrictive smoothness assumptions on the boundary of the reference domain and the shift function. Here we show that for the validity of the Hadamard formula it is sufficient to assume only C^1 smoothness of both the reference and perturbed domains. We demonstrate also that this smoothness is optimal. Namely, the result is not true for Lipschits domains and Lipschits perturbations. For C^1; -domains we give an optimal estimate of the remainder term in the asymptotics. Similar results are obtained also for the Neumann problem.
This is a joint work with Johan Thim.

Mark Malamud
Trace formulas for pairs of m-dissipative operators and contractions
Abstract:
We will discuss Krein type trace formulas for pairs of two resolvent comparable m-dissipative operators as well as for pairs of contractions with trace class difference. In particular, it will be shown that Krein-type formula remains valid for pairs of contractions and any operator Lipschitz function analytic in the unit disk.
The talk is based on works [MN2015] and [MNP17] joint with H. Neidhardt and V. Peller.
[MN2015] M. Malamud, H. Neidhardt, Trace formulas for additive and non-additive perturbations, Advances in Math., v. 274 (2015), p. 736-832.
[MNP17] M. Malamud, H. Neidhardt, V. Peller, Analytic operator Lipschitz functions in the disk and a trace formula for functions of contractions (submitted)

Vladimir Maz'ya
Bounds for eigenfunctions of the Laplacian on non-compact Riemannian manifolds
Abstract:
We deal with eigenvalue problems for the Laplacian on non-compact Riemannian manifolds M of finite volume. Sharp conditions ensuring L^q(M) and L^∞(M) bounds for eigenfunctions are exhibited in terms of either the isoperimetric function or the isocapacitary function of M. This is a joint work with Andrea Cianchi (Florence)

Boris S. Mityagin
The Spectrum of a Harmonic Oscillator Operator Perturbed by δ - Interactions
Abstract:
The operator

$$Ly = -(d/dx)^2 y(x) + x^2 y + iw(x)y, \quad y \text{ in } L^2(\mathbb{R}),$$

with a complex-valued perturbation generates a variety of questions on the spectra of differential operators and geometry of basis systems in a Hilbert space. We will focus on two results and the techniques used to prove them.

Claim 1. *Let the potential*

(1) $$w(x) \in L^p, \; 1 \le p < \infty, \quad \nu = \|w\|_p, \; \textit{or}$$

(2) $$w(x) = \sum_{k=1}^{\infty} c_k \delta\left(x - b_k\right), \quad \nu = \sum |c_k| < \infty,$$

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[2] Boris Mityagin. The spectrum of a harmonic oscillator operator perturbed by point interactions. International Journal of Theoretical Physics, 54(11):4068–4085, 2015.

[3] Boris Mityagin. The spectrum of a harmonic oscillator operator perturbed by δ-interactions. Integral Equations and Operator Theory, 85(4):451 – 495, 2016.

Alexander Nazarov
Spectral asymptotics for 1D perturbations of positive compact operators and small ball probabilities for Gaussian random functions

Vladimir Peller
A solution of M.G. Krein's problem concerning the Lifshitz - Krein trace formula
Abstract:

I am going to give a solution to M.G. Krein's problem to describe the maximal class of functions, for which the Lifshits - Krein trace formula holds for arbitrary pairs of self-adjoint operators with trace class difference.

Alexander Pushnitski
Functional calculus for self-adjoint operators under the Kato smoothness condition
Abstract:

This talk is based on work in progress with Rupert Frank. We develop a notion of Schatten class valued smoothness, which is an extension of Kato smoothness for operators in the Schatten class. We then combine this notion with the famous Birman-Solomyak formula for f(A)-f(B) (here A,B are self-adjoint operators) in terms of double operator integrals. This results in new Schatten norm estimates for f(A)-f(B), where A, B satisfy some assumptions typical for scattering theory; for example, A and B can be the free and the perturbed Schrödinger operators.

Didier Robert
Some recent results on growth of Sobolev norms for time dependent Schrödinger evolutions
Abstract:

When a time independent Schrödinger Hamiltonian H_0 is perturbed by a time dependent potential V(t) the modes of H_0 are not preserved by the evolution generated by H_0 + V(t) and exchange of energies may occur. We shall present a general approach connecting spectral properties of H_0, the size and the oscillations in time of the perturbation V(t).

Uzy Smilansky
Implications of Anderson Localization in the time domain.
Abstract:
A wave-packet is a solution of the wave equation, generated as a superposition of free propagating waves, which describe a short pulse of localized wave-action that travels as a unit, at a constant speed. Recently, ultra – short wave packets of electromagnetic radiation became experimentally available, and they are used to explore the scattering of light by atomic systems with high temporal resolution. In the present talk I shall discuss the modification of the wave packet by the scattering process, which cause a broadening of the wave-packet. This modified shape can be interpreted as the delay-time distribution due to the interaction. I shall illustrate the phenomenon by studying the scattering of wave-packets from a random potential on the real half-line.

Boris Solomyak
On the spectral theory of substitution dynamical systems.
Abstract:
We consider substitutions on a finite alphabet, the classical examples being the Thue-Morse: $0 \mapsto 01$, $1 \mapsto 10$, and Fibonacci: $0 \mapsto 01$, $1 \mapsto 0$ substitutions. Iterating the substitution (by concatenation) produces in the limit an infinite symbolic sequence, which is usually aperiodic. From this sequence, one gets a measure-preserving system, and it is of interest to investigate its spectral properties. I will survey what is known about the nature of the spectrum (when is it discrete? singular continuous? etc.), including some recent results, obtained jointly with A. I. Bufetov and A. Berlinkov.

Tatiana Suslina
Spectral approach to homogenization of periodic differential operators
Abstract:

The talk is devoted to the operator-theoretic (spectral) approach to homogenization suggested by M. Birman and T. Suslina.
In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider matrix elliptic second order differential operators $A = A(\mathbf{x}, \mathbf{D})$ admitting a factorization of the form $A = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$. Here an $(m \times m)$-matrix-valued function $g(\mathbf{x})$ is bounded, positive definite, and periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$. Next, $b(\mathbf{D}) = \sum_{l=1}^d b_l D_l$, where b_l are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{l=1}^d b_l \xi_l$ has rank n for any $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$.
We study the operator $A_\varepsilon := A(\mathbf{x}/\varepsilon, \mathbf{D})$ for small $\varepsilon > 0$. It turns out that the resolvent $(A_\varepsilon + I)^{-1}$ converges in the L_2-operator norm to the resolvent of the *effective operator* $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$. Here g^0 is a constant positive *effective matrix*. We prove that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon. \qquad (1)$$

Also, we obtain more accurate approximation for the resolvent:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon^2, \qquad (2)$$

and approximation for the resolvent in the norm of operators acting from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the Sobolev space $H^1(\mathbb{R}^d;\mathbb{C}^n)$:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C\varepsilon. \qquad (3)$$

Here $K(\varepsilon)$ and $K_1(\varepsilon)$ are the so called *correctors*; they contain rapidly oscillating factors and so depend on ε. Estimates (1)–(3) are order-sharp.
The method is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory. By the scaling transformation, the problem is reduced to approximation of the operator $\varepsilon^2(A+\varepsilon^2 I)^{-1}$. The operator A admits expansion in the direct integral of the operators $A(\mathbf{k})$ acting in $L_2(\Omega;\mathbb{C}^n)$ (where Ω is the cell of the lattice Γ). The operator $A(\mathbf{k})$ is given by the expression $b(\mathbf{D}+\mathbf{k})^* g(\mathbf{x}) b(\mathbf{D}+\mathbf{k})$ with periodic boundary conditions. This operator family is studied by means of the analytic perturbation theory. It is possible to approximate the resolvent $(A(\mathbf{k})+\varepsilon^2 I)^{-1}$ in terms of the spectral characteristics near the bottom of the spectrum. This shows that homogenization can be treated as a *spectral threshold effect*.
General results are applied to specific operators of mathematical physics.

Dmitri Vassiliev
Analytic definition of spin structure
Abstract:
The analysis of the massless Dirac equation on a manifold without boundary involves the topological concept of spin structure. We show that in lower dimensions spin structure can be defined in a purely analytic fashion.
Our analytic definition relies on the use of the concept of a non-degenerate two-by-two formally self-adjoint first order linear differential operator and gauge transformations of such operators. Our analytic construction works in dimensions four (Lorentzian signature) and three (Riemannian signature). We prove that our analytic definition of spin structure is equivalent to the traditional topological definition. A detailed exposition is provided in arXiv:1611.08297.
Spin structure has implications for spectral analysis. Say, a 3-torus has eight different spin structures and, hence, eight different massless Dirac operators. These eight different massless Dirac operators have different spectra.

Timo Weidl
Semiclassical spectral bounds with remainder terms
Abstract:
The Berezin and the Li-Yau inequalities state that the first term in Weyl's asymptotic formula serves also as uniform spectral bound on partial eigenvalue sums of the Dirichlet Laplacian. I will report on various attempts to improve these bounds by taking terms of lower order into account. Moreover, I will sketch some recent results on the magnetic Laplacian and on the Heisenberg Laplacian, respectively.
1. H. Kovarik, T. Weidl: ``Improved Berezin-Li-Yau inequalities with magnetic fields''. Proceedings of the Royal Society of Edinburgh, 145A, 145-160, 2015
2. H. Kovarik, B. Ruszkowski, T. Weidl: ``Melas-type bounds for the Heisenberg Laplacian on bounded domains''. to appear in Journal of Spectral Theory.
3. D. Barseghyan, P. Exner, H. Kovarik, T. Weidl: ``Semiclassical bounds in magnetic bottles''. to appear in Reviews in Mathematical Physics, 28 (1), 2016.

Abstracts

Normalization and factorization of linear ordinary differential operators

Sergei Yakovenko
(joint work with Leanne Mezuman, Shira Tanny)

The local theory of linear ordinary differential equations exists in two closely related but different flavors.

One may consider singular point of a system of first order differential equations which have the form

$$t^{1+r}\dot{x} = A(t)x, \quad t \in (\mathbb{C}, 0),\ A(t) = A_0 + tA_1 + t^2A_2 + \cdots \in \mathbb{C}[[t]] \otimes \mathrm{Mat}(n, \mathbb{C}),$$

where the nonnegative integer $r \in \mathbb{Z}_+$ is the Poincaré rank; if $r = 0$, the singularity is called Fuchsian. On the space of such systems there is a natural group action, called the gauge equivalence: two systems defined by two matrix series $A(t), B(t)$ are equivalent, if there exists a matrix series $H \in \mathrm{GL}(n, \mathbb{C}[[t]])$ such that

$$t^{1+r}\dot{H} = HA - BH, \qquad H = H_0 + tH_1 + t^2H_2 + \cdots, \ \det H_0 \neq 0.$$

The simplest form, to which a system can be transformed by a (always formal in our settings) gauge equivalence, depends on the Poincaré rank and the eigenvalues $\lambda_1, \dots, \lambda_n$ of the leading matrix A_0. For several reasons it is more convenient to write the systems using the Euler derivation $\boldsymbol{\epsilon} = t\,\frac{\mathrm{d}}{\mathrm{d}t}$ rather than the usual derivation in t, denoted by the dot above.

Theorem 1 (H. Poincaré, H. Dulac)**.** *If no two eigenvalues of a Fuchsian system differ by a positive integer, $\lambda_i - \lambda_j \notin \{1, 2, 3, \dots\}$, then the system is gauge equivalent to an Euler system $(\boldsymbol{\epsilon} - A_0)x = 0$.*

If d is the largest natural difference between the eigenvalues, then the Fuchsian system is equivalent to an integrable polynomial system $(\boldsymbol{\epsilon} - A_0 + \cdots + t^d A_d)x = 0$.

In the non-Fuchsian case with $r > 0$ we have the following result.

Theorem 2 (Diagonalization theorem, Hukuhara–Turritin–Levelt)**.** *If all eigenvalues of a non-Fuchsian system are pairwise different, $\lambda_i \neq \lambda_j$ for $i \neq j$, then the system can be transformed into a diagonal form $(t^r\boldsymbol{\epsilon} - B(t))x = 0$ with the diagonal matrix function $B(t) = \mathrm{diag}(\beta_1(t), \dots, \beta_n(t))$, $\beta_i \in \mathbb{C}[[t]]$, $\beta_i(0) = \lambda_i$.*

The normal form in the resonant case is more complicated.

Another flavor of the local linear theory is that of linear ordinary differential operators of higher order. Such equations can be written using *differential operators* under the form $Ly = 0$, where $L = a_0(t)\boldsymbol{\epsilon}^n + a_1(t)\boldsymbol{\epsilon}^{n-1} + \cdots + a_{n-1}(t)\boldsymbol{\epsilon} + a_n(t)$ is an operator with the coefficients $a_i \in \mathbb{C}[[t]]$, $a_0 \not\equiv 0$, and $\boldsymbol{\epsilon} = t\frac{\mathrm{d}}{\mathrm{d}t}$ is the Euler derivation operator. Any such operator can be re-expanded in the form $L = \sum_{k=0}^{\infty} t^k p_k(\boldsymbol{\epsilon})$, $k \in \mathbb{Z}_+$, $p_k \in \mathbb{C}[\boldsymbol{\epsilon}]$, in other words, $\mathscr{W} = \mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathbb{C}[\boldsymbol{\epsilon}]$ with the commutation identity $\boldsymbol{\epsilon}^j t^k = t^k(\boldsymbol{\epsilon} + k)^j$. An operator L is *Fuchsian*, if $\deg p_0 = n = \max_k \deg p_k$.

Fuchsian operators form a subset $\mathscr{F} \subseteq \mathscr{W}$ closed by composition, albeit not a subalgebra.

There is no natural group acting on differential operators, but they form a (non-commutative) $\mathbb{C}$-algebra $\mathscr{W}$ with respect to composition.

Definition 1 (cf. [O, TY]). *Two operators $L, M \in \mathscr{W}$ are $\mathscr{F}$-equivalent, if there exist two operators $K, H \in \mathscr{F}$ such that $KL - MH = 0$ and $\gcd(H, L) = 1$.*

This means that the operator H acting by $u = Hy$ sends solutions y of the operator $Ly = 0$ to those of $Mu = 0$, while not vanishing on any one of them.

Theorem 3 (cf. [TY]). *If $L = p_0(\boldsymbol{\epsilon}) + tp_1(\boldsymbol{\epsilon}) + \cdots$ is a Fuchsian operator and no two roots of the polynomial $p_0 \in \mathbb{C}[\boldsymbol{\epsilon}]$ differ by a positive integer, $\lambda_i - \lambda_j \notin \{1, 2, 3, \dots\}$, then the operator is $\mathscr{F}$-equivalent to an Euler equation $M = p_0(\boldsymbol{\epsilon}) \in \mathbb{C}[\boldsymbol{\epsilon}] \subseteq \mathscr{F}$.*

If d is the largest natural difference between the roots, then the Fuchsian operator is $\mathscr{F}$-equivalent to a Liouville integrable operator $(\boldsymbol{\epsilon} - \beta_1(t)) \cdots (\boldsymbol{\epsilon} - \beta_n(t))$ with polynomial coefficients $\beta_i \in \mathbb{C}[t]$ of degrees $\leq d$, $\beta_i(0) = \lambda_i$.

In the non-Fuchsian case we look for an analog of the Diagonalization theorem, which would describe factorization of an operator $L \in \mathscr{W} \setminus \mathscr{F}$ into a composition of operators of smaller orders. The answer depends on the growth pattern of the degrees $\deg p_k$, $k = 0, 1, 2, \dots$ expressed in terms of the Newton diagram. If $L = \sum_{j,k} c_{jk} t^k \boldsymbol{\epsilon}^j$ is the double series (all powers of t appear to the right from powers of $\boldsymbol{\epsilon}$), then the support $\operatorname{supp} L = \{(j, k) \colon c_{jk} \neq 0\}$ is a subset in $\mathbb{Z}^2_+$, and the Newton polygon Δ_L is the convex hull of the origin $(0, 0)$, the support $\operatorname{supp} L$ and its vertical translates by $(0, 1)$. The Newton polygon is an epigraph of a piecewise-affine convex monotone function $\chi \colon [0, d] \to \mathbb{R}_+$ with corners only at the lattice points $\mathbb{Z}^2_+ \subseteq \mathbb{R}^2_+$. The set of values of its derivative (slopes) is called the Poincaré spectrum of L, $S(L) \subseteq \mathbb{Q}_+$.

The main property of the Newton polygon is the identity $\Delta_{LM} = \Delta_{ML} = \Delta_L + \Delta_M$ with respect to the Minkowski sum, which holds exactly in the same form as for the *commutative algebra of pseudopolynomials* $\mathscr{P} = \mathbb{C}[[t]] \otimes \mathbb{C}[\xi] = \mathbb{C}[[t]][\xi]$. The latter case is well known since Newton's invention of the "rotating ruler method" [N]. It turns out that one can derive directly the results for the non-commutative algebra $\mathscr{W}$ from those for $\mathscr{P}$.

Definition 2. *A* single-slope *operator $L = \sum a_{j,k} t^k \boldsymbol{\epsilon}^j$ is the operator for which the function χ is linear, $\chi(j) = rj$, $r = \frac{p}{q} \in \mathbb{Q}_+$, so that $S(L) = \{r\}$.*

A symbol *of a single-slope operator is the polynomial*

$$\sigma_L(t, \xi) = \sum_{k - rj = 0} a_{jk} t^k \xi^j = \prod_{i=1}^{m} (\lambda_i - t^p \xi^q)$$

of degree $n = mq$, $n = \operatorname{ord} L$. The numbers $\lambda_i \in \mathbb{C}^*$ are characteristic roots.

A single-slope operator is totally resonant*, if all characteristic roots $\lambda_1, \dots, \lambda_m \in \mathbb{C}^*$ of its symbol coincide, $\lambda_1 = \cdots = \lambda_m$, in particular, if $m = 1$.*

Theorem 4 (cf. [M, R, vdPS]).

1. *A non-Fuchsian operator L with the Poincaré spectrum $S(L) = \{r_1, \dots, r_s\} \subseteq \mathbb{Q}_+$, $r_i \neq r_j$, admits factorization into single-slope terms $L = L_1 \cdots L_s$, $S(L_i) = r_i$.*
2. *A single-slope non-Fuchsian operator L with $S(L) = \{r\}$ and symbol σ admits factorization into totally resonant operators of the same slope.*
3. *In particular, if all characteristic roots $\lambda_1, \dots, \lambda_m \in \mathbb{C}^*$ of the single-slope operator L are pairwise different, $\lambda_i \neq \lambda_j$, then this operator admits factorization into m operators of the first order $L_i = t^p \boldsymbol{\epsilon}^q - \beta_i(t)$, $\beta_i \in \mathbb{C}[[t]]$, $\beta_i(0) = \lambda_i$.*

The known proofs of this theorem are based on involved considerations in the non-commutative algebra $\mathscr{W}$. In contrast with that, we developed a direct approach that allows direct transfer of factorization results in the commutative algebra $\mathscr{P}$ or in the local algebra $\mathbb{C}[[t, s]]$ to the non-commutative case.

More specifically, we consider weighed quasihomogeneous polynomials in two variables with the weight $w(t^k \xi^j) = k - rj$, $r \in \mathbb{Q}_+$ and the *homological equations*

$$P_\alpha u_{\gamma-\alpha} + Q_\beta v_{\gamma-\beta} = S_\gamma, \quad \operatorname{supp} P_\alpha \subseteq \Delta, \ \operatorname{supp} Q_\beta \subseteq \Delta'', \ \operatorname{supp} S_\gamma \subseteq \Delta' + \Delta'',$$

which have to be solved with respect to the unknown quasihomogeneous polynomials $u_{\gamma-\alpha}, v_{\gamma-\beta}$ subject to the constraints $\operatorname{supp} u_{\gamma-\alpha} \subseteq \Delta''$, $\operatorname{supp} v_{\gamma-\beta} \in \Delta'$. Solvability of these equations for any weight γ and any right hand side S_γ depends on the Newton polygons Δ', Δ'' and the polynomials P_α, Q_β in a very nontrivial way, but can be derived from the factorization results in $\mathscr{P}$.

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Reporter: Christopher Daw