# Stability, Bethe approximation
## and some annoying problems

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Joint work with various subsets of Ádám Schweitzer, Márton Borbényi, András Imolay, Nicholas Ruozzi and Shahab Shams.

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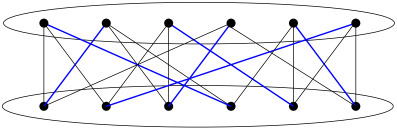
# Perfect matchings

**Theorem (A. Schrijver)**

*Let $G$ be a bipartite $d$–regular graph on $2n$ vertices, and let $\mathrm{pm}(G)$ denote the number of perfect matchings of the graph $G$. Then*

$$\mathrm{pm}(G) \geq \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right)^n .$$



$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

# Orientations

## Theorem (A. Schrijver)

*Let $\varepsilon(G)$ be the number of Eulerian orientations of graph $G$ withe degree sequence $d_1, \ldots, d_n$, where all $d_i$ are even. Then*

$$\varepsilon(G) \geq \prod_{j=1}^{n} \frac{\binom{d_j}{d_j/2}}{2^{d_j/2}}.$$

*In particular, if $G$ is a $d$–regular graph on $n$ vertices, where $d$ is even, then*

$$\varepsilon(G) \geq \left( \frac{\binom{d}{d/2}}{2^{d/2}} \right)^n .$$

## Example

*For $d = 4$ we have $\frac{\binom{d}{d/2}}{2^{d/2}} = \frac{3}{2}$, that is, a 4–regular graph has always at least $\left(\frac{3}{2}\right)^n$ Eulerian orientations.*

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## Definition

*A multivariate polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ is called real stable if it has real coefficients, and $P(z_1, \ldots, z_n) \neq 0$ if $\mathrm{Im}(z_i) > 0$ for all $i = 1, \ldots, n$.*

## Remark

*The univariate real stable polynomials are exactly the real rooted polynomials.*

## Example

*Let $\mathcal{T}(G)$ denote the set of spanning trees of a graph $G$. Then the polynomial*

$$\sum_{T \in \mathcal{T}(G)} \prod_{e \in E(T)} x_e$$

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# Capacity of a polynomial

## Definition

*Let* $P \in \mathbb{R}[x_1, \ldots, x_n]$ *and* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0}$. *Then*

$$\mathrm{cap}_{\underline{\alpha}}(P) := \inf_{x_1, \ldots, x_n > 0} \frac{P(x_1, \ldots, x_n)}{\prod_{i=1}^n x_i^{\alpha_i}}.$$

# General theme

**Theorem**

*Let $P$ be a real stable polynomial. Let $T$ be some operator acting on polynomials. Then*

$$\mathrm{cap}_{\underline{\beta}}(T(P)) \geq c(T, \underline{\alpha}, \underline{\beta})\mathrm{cap}_{\underline{\alpha}}(P).$$

# Orientations again (Ongoing conversation with Ádám Schweitzer)

Let

$$P_G((x_u)_{u\in V(G)}) = \prod_{(u,v)\in E(G)} (x_u + x_v).$$

This is clearly real stable.

Furthermore,

$$P_G((x_u)_{u\in V(G)}) = \prod_{(u,v)\in E(G)} (x_u + x_v) = \sum_{\mathcal{O}} \prod_{u\in V(G)} x_u^{d_{\mathcal{O}}(v)}.$$

In particular, if all degrees $d_1, \dots, d_n$ are even, then the coefficient of $x_1^{d_1/2} \dots x_n^{d_n/2}$ is the number of Eulerian orientations, that we denoted by $\varepsilon(G)$.

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# Capacity revisited (Ongoing conversation with Ádám Schweitzer)

Let

$$P_G((x_i)_{i\in V(G)}) = \prod_{(i,j)\in E(G)} (x_i + x_j).$$

and $\underline{\alpha} = (d_1/2, \ldots, d_n/2)$. Then

$$\mathrm{cap}_{\underline{\alpha}}(P_G) = \inf_{x_i>0} \frac{\prod_{(i,j)\in E(G)}(x_i + x_j)}{\prod_{i=1}^{n} x_i^{d_i/2}} = \inf_{x_i>0} \prod_{(i,j)\in E(G)} \frac{x_i + x_j}{\sqrt{x_i x_j}} \geq 2^{e(G)}.$$

Because of $\underline{x} = \underline{1}$ we have $\mathrm{cap}_{\underline{\alpha}}(P_G) = 2^{e(G)}$.

# Towards the coefficients

**Theorem**

*Let* $f(z) = \sum_{k=0}^{n} a_k z^k$ *be a real-rooted polynomial with* $a_k \geq 0$*. Then*

$$a_k \geq \binom{n}{k} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \inf_{t>0} \frac{f(t)}{t^k}.$$

**Theorem (Hoeffding)**

*Let* $f(z) = \sum_{k=0}^{n} p_k z^k$ *be a real-rooted polynomial with* $p_k \geq 0$*, and* $f(1) = 1$*, i. e.,* $\sum_{k=0}^{n} p_k = 1$*. Let* $p$ *be defined by the equation* $\sum_{k=0}^{n} k p_k = np$*. Suppose that for non-negative integers* $b$ *and* $c$ *we have* $b \leq np \leq c$*. Then*

$$\sum_{k=b}^{c} p_k \geq \sum_{k=b}^{c} \binom{n}{k} p^k (1-p)^{n-k}.$$

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# Hoeffding implies capacity bound

For a $t > 0$ let us consider the probability distribution $p_j = \frac{a_j t^j}{f(t)}$. Then $\sum_{j=0}^n p_j z^j$ is still real-rooted polynomial. Choose $t_k$ in such a way that $\sum_{j=0}^n j p_j = k = np$, i.e. $p = \frac{k}{n}$.
Next let us apply Hoeffding's theorem with $b = c = k$. Then

$$\frac{a_k t_k^k}{f(t_k)} = p_k \geq \binom{n}{k} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k}.$$

In other words,

$$\begin{aligned} a_k &\geq \binom{n}{k} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \frac{f(t_k)}{t_k^k} \\ &\geq \binom{n}{k} \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \inf_{t>0} \frac{f(t)}{t^k}. \end{aligned}$$

# Finishing the proof

- $\varepsilon(G)$ is the coefficient of $x_1^{d_1/2} \dots x_n^{d_n/2}$ in $P_G(\underline{x}) = \prod_{(i,j)\in E(G)}(x_i + x_j)$.
- $\mathrm{cap}_{\underline{\alpha}}(P_G) = 2^{e(G)}$ for $\underline{\alpha} = (d_1/2, \dots, d_n/2)$.
- If $T_k f$ is the coefficient of $x^k$ in $f$ of a real-rooted polynomial of degree at most $n$, then $T_k f \geq \binom{n}{k}\left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k} \inf_{t>0} \frac{f(t)}{t^k}$.
- Let us apply it with $n = d_j$ and $k = d_j/2$ subsequently for all $j$:

$$\varepsilon(G) \geq \prod_{j=1}^{n} \frac{\binom{d_j}{d_j/2}}{2^{d_j}} \cdot 2^{e(G)} = \prod_{j=1}^{n} \frac{\binom{d_j}{d_j/2}}{2^{d_j/2}}.$$

# Question

Where does the constants come from in the inequalities

$$\mathrm{pm}(G) \geq \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right)^n \quad \text{and} \quad \varepsilon(G) \geq \left( \frac{\binom{d}{d/2}}{2^{d/2}} \right)^n ?$$

# Graphical models

# Graphical models and factor graphs

A factor graph $\mathcal{G} = (F, V, E, (\phi_a)_{a \in F})$ is a bipartite graph equipped with a set of functions.

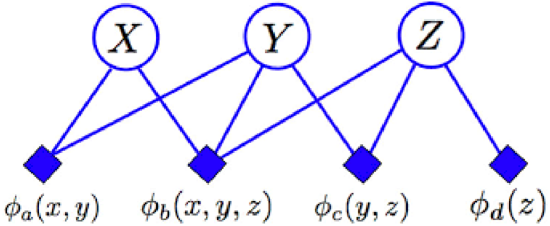
- The elements of $F$ are called *function nodes*.
- The elements of $V$ are called *variable nodes*.
- For each variable node $v$ we associate a variable $x_v$ taking its values from a set $\mathcal{X}$.
- For each $a$ there is an associated function $\phi_a : \mathcal{X}^{\partial a} \to \mathbb{R}_+$.

The partition function of $\mathcal{G}$ is

$$Z(\mathcal{G}) = \sum_{\underline{x} \in \mathcal{X}^V} \prod_{a \in F} \phi_a(\underline{x}_{\partial a}),$$

where $\underline{x}_{\partial a}$ is the restriction of $\underline{x}$ to the set $\partial a$.

# Example: a graphical model.




The partition function is

$$Z(\mathcal{G}) = \sum_{x,y,z} \phi_a(x, y)\phi_b(x, y, z)\phi_c(y, z)\phi_d(z).$$

# Examples

**Perfect matchings:**

- Variable nodes: edges of the original graph $G$
- Alphabet: $\mathcal{X} = \{0, 1\}$. $x_e = 1 \iff e \in M$.
- Function nodes: vertices of the the original graph $G$
- Functions $\phi_v(x_{e_1}, \ldots, x_{e_k}) = 1 \iff$ exactly one of them is $1$. Otherwise $\phi_v(x_{e_1}, \ldots, x_{e_k}) = 0$.

**Proper colorings with $q$ colors:**

- Variable nodes: vertices of the original graph $G$
- Alphabet: $\mathcal{X} = \{1, 2, \ldots, q\}$.
- Function nodes: edges of the the original graph $G$
- Functions $\phi_e(x_u, x_v) = 1 \iff x_u \neq x_v$. Otherwise $\phi_e(x_u, x_v) = 0$.

# Special cases

Further examples:

- Matchings
- Permanents
- Orientations
- Independent sets
- Homomorphisms
- Solutions of a conjunctive normal form
- Coding theory

Non-examples:

- Spanning trees or forests
- Acyclic orientations

# Bethe approximation

# Bethe approximation: pseudo marginal polytope

For each variable node $v$ we introduce a probability distribution $b_v$ on $\mathcal{X}$, and for each function node $a$ we also introduce a probability distribution $b_a$ on $\mathcal{X}^{\partial a}$:

$$\sum_{x \in \mathcal{X}} b_v(x) = 1 \ \ \forall v \in V, \ \ b_v(x) \geq 0 \ \ \ \forall x \in \mathcal{X}$$

$$\sum_{\underline{x} \in \mathcal{X}^{\partial a}} b_a(\underline{x}) = 1 \ \ \forall a \in F, \ \ b_a(\underline{x}) \geq 0 \ \ \ \forall \underline{x} \in \mathcal{X}^{\partial a}.$$

Furthermore, $b_v$ and $b_a$ have to be consistent in the following sense: for all $c \in \mathcal{X}, \ a \in F, v \in \partial a$ we have

$$\sum_{\underline{x} \in \mathcal{X}^{\partial a \setminus v}} b_a(\underline{x}, c) = b_v(c).$$

We will call a $\underline{b} = ((b_v)_{v \in V}, (b_a)_{a \in F})$ a locally consistent set of marginals. The set of such $\underline{b}$ will be denoted by $\mathrm{Mar}(\mathcal{G})$.

# Bethe-approximation

Let $\mathbb{F}$ be the following function evaluated on a $\underline{b} \in \mathrm{Mar}(\mathcal{G})$:

$$\mathbb{F}(\underline{b}) = \sum_{a \in F} \sum_{\underline{x} \in \mathcal{X}^{\partial a}} b_a(\underline{x}) \ln \frac{\phi_a(\underline{x})}{b_a(\underline{x})} - \sum_{v \in V} (1 - |\partial v|) \sum_{x \in \mathcal{X}} b_v(x) \ln b_v(x).$$

Finally, let

$$H_B(\mathcal{G}) = \sup_{\underline{b} \in \mathrm{Mar}(\mathcal{G})} \mathbb{F}(\underline{b}) \quad \text{and} \quad Z_B(\mathcal{G}) = \exp(H_B(\mathcal{G})).$$

Here $H_B(\mathcal{G})$ is the Bethe free entropy, and $Z_B(\mathcal{G})$ is the Bethe-approximation (or Bethe partition function).

# Gauge transformation

# Gauge transformations and loop calculus

**Theorem (M. Chertkov and V. Csernyak)**

*Let $G = (V, E)$ be a $d$–regular graph on $2n$ vertices. Then*

$$\mathrm{pm}(G) = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n \sum_{A \subseteq E} \frac{1}{(d-1)^{|A|}} \prod_{v \in V} (1 - d_A(v)),$$

*where $d_A(v)$ is the degree of the vertex $v$ in the graph $(V, A)$.*

# Eulerian orientation and gauge transformations

Let $\underline{c} = (c_0, c_1, \dots, c_d)$ be defined as follows.

$$c_k = \begin{cases} \frac{\binom{d}{d/2}\binom{d/2}{k/2}}{2^{d/2}\binom{d}{k}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

**Theorem (M. Borbényi and P. Csikvári)**

*Let $G$ be a $d$–regular graph $G$, where $d$ is even. Then the number of Eulerian orientations satisfy*

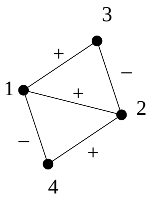
$$\varepsilon(G) = \sum_{A \subseteq E(G)} \prod_{v \in V(G)} c_{d_A(v)}.$$

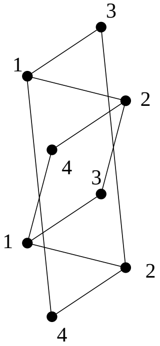
*In particular,*

$$\varepsilon(G) \geq \prod_{v \in V(G)} c_{d_\emptyset(v)} = \left( \frac{\binom{d}{d/2}}{2^{d/2}} \right)^n.$$

# Graph covers

# k-covers




## Definition

*A graph $H$ is a $k$-cover of a graph $G$ if $V(H) = V(G) \times \{0, 1, \ldots, k-1\}$ and if $(u, v) \in E(G)$, then there is a perfect matching between the vertices $(u, i)$ and $(v, j)$ for $i, j \in \{0, 1, \ldots, k-1\}$, and if $(u, v) \notin E(G)$ then there is no edge between these $2k$ vertices.*

# Bethe-approximation via covers

**Theorem (P. Vontobel)**

*Let $\mathcal{G}$ be a factor graph, and let $C_k(\mathcal{G})$ be the set of all $k$-covers of $\mathcal{G}$. Then*

$$Z_B(\mathcal{G}) = \lim_{k\to\infty} \left(\mathbb{E}Z(\mathcal{H})\right)^{1/k},$$

*where $\mathbb{E}Z(\mathcal{H})$ is the average of $Z(\mathcal{H})$ over all elements of $C_k(\mathcal{G})$.*

**Remark**

*We can take this theorem as a definition of the Bethe approximation for any graph parameter. For instance, we can speak about the Bethe approximation of the number of acyclic orientations.*

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# Cover method I

If the inequality $Z(G)^k \geq Z(H)$ holds true for all $k$-covers $H$ of $G$, then $Z(G) \geq Z_B(G)$ by Vontobel's theorem.

**Theorem (Ruozzi)**

*If for all local function $\phi_a : \{0,1\}^{\partial a} \to \mathbb{R}_+$ ($a \in F$) we have*

$$\phi_a(\underline{x})\phi_a(\underline{y}) \leq \phi_a(\underline{x} \vee \underline{y})\phi_a(\underline{x} \wedge \underline{y}),$$

*then $Z(G)^k \geq Z(H)$ holds true for all $k$-covers $H$ of $G$. Consequently, $Z(G) \geq Z_B(G)$.*

Applies to hard-core model on bipartite graphs, ferromagnetic Ising-model on all graphs.

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# Bethe-approximation via covers II

Alternative approach: take a sequence of graphs $G = G_0, G_1, G_2, \dots$ such that $G_k$ is a $2$-cover of $G_{k-1}$, and $g(G_k) \to \infty$, where $g(G_k)$ is the length of the shortest cycle.

Hope:

$$\lim_{k\to\infty} \frac{1}{|G_k|} \ln Z(G_k) = \frac{1}{|G|} \ln Z_B(G).$$

So practically we hope that the Bethe approximation only depends on the universal cover tree of the original graph.

This leads to the following more general question: given a sequence of graphs that are locally more and more alike (that is, Benjamini–Schramm convergent). Is it true that $\lim_{k\to\infty} \frac{1}{|G_k|} \ln Z(G_k)$ exists?

**Theorem (P. Csikvári)**

*If $G$ is a bipartite graph, and $H$ is a $2$-cover of it, then*

$$\mathrm{pm}(G)^2 \geq \mathrm{pm}(H).$$

**Theorem (P. Csikvári and A. Imolay)**

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# Homomorphisms

# Sidorenko's conjecture

Let $G$ and $H$ be two simple graphs.
Let $\hom(G, H)$ denote the number of homomorphism of $G$ to $H$.
Furthermore, let

$$\hom_S(G, H) := v(H)^{v(G)} \left( \frac{2e(H)}{v(H)^2} \right)^{e(G)} .$$

**Conjecture**

*Let $G$ be a bipartite graph and $H$ be any simple graph. Then*

$$\hom(G, H) \geq \hom_S(G, H).$$

**Proposition (P. Csikvári, N. Ruozzi, S. Shams (good news))**

*Let* $\hom_B(G,H)$ *be the Bethe approximation of* $\hom(G,H)$*. Then*

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*Let* $\hom_B(G,H)$ *be the Bethe approximation of* $\hom(G,H)$*. There exists a bipartite graph* $G$ *and a simple graph* $H$*.*

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# Some problems

# Spanning trees

**Theorem**

*Let $G$ be a $d$–regular graph on $n$ vertices. Let $\tau(G)$ be the number of spanning trees. Then*

$$\tau(G) \leq \frac{c \log n}{n} \left( \frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}} \right)^n .$$

**Problem**

*Can we prove this theorem (possibly without the subexponential term) using that*

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# Bethe approximation for the Tutte polynomial

**Problem**

*Can we compute the Bethe approximation of the Tutte polynomial of a graph $G$?*

**Thank you for your attention!**