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On Toeplitz' conjecture from 1911

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In 1911 Otto Toeplitz made a still unproved conjecture: Let J be a closed Jordan curve in the plane. Then J contains four points which forms the vertices of a square. Many special cases of the conjecture have been proved (see the survey by Matschke in Notices of the AMS 61, 2014). In particular it holds for polygons.

Petterson, Östergård and Tverberg presented (in DCG 51, 2014) a stronger conjecture. It deals with a polygon of a particular type, P, which is drawn on $2(n + 1)$ horizontal and vertical lines that define an $n \times n$ "chessboard", consisting of $n \times n$ squares of side length 1. The bounded part of the plane, as defined by P, contains a maximal open square consisting of $m \times m$ of the 1×1–squares. Then the stronger conjecture says that one can find a square with vertices at corners of the 1×1–squares and area at least $2k \times k$ or $2k \times (k + 1) + 1$, according to whether m equals $2k$ or $2k + 1$. The stronger conjecture was then verified for $k = 1, 2, ..., 6$, using a computer. In this talk I'll verify another special case of it. Let the maximal open square (with horizontal and vertical sides) contained in the bounded part of the plane, (as defined by P) have area 1 or 4. Then it holds for *all* values of n. The argument can apparently be carried out for area 9, too, but it is too involved at this moment. Anyhow, the Toeplitz' conjecture (published in Über einige Aufgaben der Analysis situs, Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn, 4 (1911) 197) remains a really intriguing one.