THE ROYAL
SWEDISH
ACADEMY OF
SCIENCES

INSTITUT
MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

On Banach bundles and operator-valued Baker functions

M. Dupré, J. Glazebrook and E. Previato

REPORT No. 12, 2006/2007

ISSN 1103-467X
ISRN IML-R- -12-06/07- -SE

On Banach bundles and operator-valued Baker functions

Maurice J. Dupré
Department of Mathematics, Tulane University
New Orleans, LA 70118 USA
mdupre@tulane.edu

James F. Glazebrook
(Primary Inst.)
Department of Mathematics and Computer Science
Eastern Illinois University
600 Lincoln Ave., Charleston, IL 61920–3099 USA
jfglazebrook@eiu.edu
(Adjunct Faculty)
Department of Mathematics
University of Illinois at Urbana–Champaign
Urbana, IL 61801, USA

Emma Previato
Institut Mittag-Leffler
S-182 62 Djursholm, Sweden
permanent:
Department of Mathematics and Statistics
Boston University
Boston, MA 02215–2411, USA
ep@bu.edu

Abstract

The Baker and tau functions were introduced in the framework of an infinite dimensional Grassmannian as key concepts in the theory of the KP hierarchy and integrable systems. We show how several features of this theory can be extended when polarized separable Hilbert spaces are generalized to a class of complementizable Hilbert modules. We present operator-valued Baker and tau functions, in the latter case using a cross-ratio approach. We consider aspects of the Sato–Segal–Wilson construction which are related to the Taylor joint spectrum and quantized connections.

Mathematics Subject Classification (2000): 35Q53, 46L08, 58B25, 81R15

Keywords : Hilbert module, Laurent series generator, Burchnall-Chaundy ring, Baker function, tau function, matrix C*–algebra.

Contents

1 Introduction

The Burchnall–Chaundy ring of formal pseudodifferential operators plays a significant role in the theory of the Kadomtsev–Petviashvili (KP) hierarchy and related integrable systems. A fundamental work of Sato [45] provides a universal setting suited to such rings of operators and integrable equations, where solutions of the latter correspond to points in an infinite dimensional Grassmannian. An analytic theory of the Grassmannian as related to the Korteweg-de Vries equation (KdV) was given by Segal and Wilson [46] (which Krichever [27], Mulase [38], Previato and Wilson [42] among others extend to vector Grassmannians) and leads to valuable techniques for studying many types of integrable systems and infinite–dimensional Lie algebras (see e.g. [34, 37, 41]). A select class of eigenfunctions for this ring of operators involves the concept of a *Baker function* which affords a notable relationship with the multipurpose τ–function [45, 46]. In [20], we created some generalizations by considering an operator–valued function approach to this theory in the setting of certain types of Banach Grassmannians. In particular, we were able to produce the general notion of an *operator-valued Baker function* which is applicable when the ring of operators is extended to accommodate various algebra class coefficients. Then we established in [20] a representation of a commutative ring $\mathbb{A}$ of such operators as a commutative subring of a certain Banach algebra A via conjugation by a formal integral operator.

Here we explore some further developments of this theory by introducing several topics in a way that is partly expository, and partly establishes links with other constructions, when (separable) Hilbert spaces are replaced by certain *Hilbert modules*. The first round of topics presented include the *Taylor joint spectrum* of a finite number of commuting operators in A and the Grassmann manifold aspect of several related constructions. Then we describe the *operator cross ratio* and a generalization of the τ–function following a noteworthy paper of Zelikin [54] that can be worked into the context of [20]. As motivation for Zelikin's work in the scalar case, we report on the papers [32, 33] which link the Segal–Wilson theory to that of *twistor spaces*. Since [54] involves the geometry of spaces of polarizations (on a Hilbert space) we included the work of Spera and Wurzbacher [48, 49] which provides an alternative representation for the homogeneous spaces in question.

We then show how the development of [20] lends itself to the context of $(\mathsf{M}, \mathsf{M}^{\times})$ theory of Helton [24], which transitions scalar to matrix coefficients. Furthermore, we examine rings $\mathbb{A}$ of such operators that admit (matrix) C*–algebra coefficients, and by using the methods of [20, 45], arrive at the more general result that such rings also conjugate into certain Banach algebras as commutative subrings, the essential idea for it going back to Krichever [27]. Finally, we discuss following Connes [10, 11] how *quantized connections* can be related to the objects in these notes.

2 Algebraic preliminaries

2.1 The Grassmannian over a semigroup

To be general, let A be a monoidal (multiplicative) semigroup with group of units denoted by $G(A)$. Let

$$P(A) := \{p \in A \ : \ p^2 = p\} \ , \tag{2.1}$$

that is, $P(A)$ is the set of idempotent elements in A (for suitable A, we can regard elements of $P(A)$ as projections). Recall that the right Green's relation is $p\mathcal{R}q$ if and only if $pA = qA$ for $p, q \in A$.

Let $\mathrm{Gr}(A) = P(A)/\mathcal{R}$ be the set of equivalence classes in $P(A)$ under $\mathcal{R}$. As the set of such equivalence classes, $\mathrm{Gr}(A)$ will be called *the Grassmannian of* A. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi \ : P(A) \longrightarrow \mathrm{Gr}(A) \ . \tag{2.2}$$

Let $h : A \longrightarrow B$ be a semigroup homomorphism. Then it is straightforward to see that the diagram below is commutative :

$$\begin{array}{ccc} P(A) & \xrightarrow{P(h)} & P(B) \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \mathrm{Gr}(A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(B) \ . \end{array} \tag{2.3}$$

2.2 The spaces $V(p, A)$ and $\mathrm{Gr}(p, A)$

Definition 2.1. We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$ and $vuv = v$, in which case we call v a *relative inverse* (or *pseudoinverse*) for u . In general such a relative inverse is not unique. We take $W(A)$ to denote the set (or space, if A has a topology) of all partial isomorphisms of A .

If $u \in W(A)$ has a relative inverse v, then clearly $v \in W(A)$ with relative inverse u, and it is easy to see that both vu and uv belong to $P(A)$. Although v is not uniquely determined by u alone, it is uniquely determined once u, vu and uv are all specified [17].

If $p \in P(A)$, then we take $W(p, A) \subset W(A)$ to denote the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $vu = p$. Likewise, $W(A, q)$ denotes the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $uv = q$, so that we have $W(A, q) = W(q, A^{op})$. Now for $p, q \in P(A)$, we set

$$\begin{aligned} W(p, A, q) &= W(p, A) \cap W(A, q) \\ &= \{ \ u \in qAp : \exists \ v \in pAq \ , \ vu = p \text{ and } uv = q \ \} \ . \end{aligned} \tag{2.4}$$

Recall that two elements $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A . For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$, whereby it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.2. Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$ where p and q are similar.

Let $V(A)$ denote the space of all proper partial isomorphisms of A . Observe that $G(A)V(A)$ and $V(A)G(A)$ are both subsets of $V(A)$. In the following we set $G(p) = G(pAp)$.

If $p \in P(A)$, then we denote by $V(p, A)$ the space of all proper partial isomorphisms of A having a relative inverse $v \in W(q, A, p)$ for some $q \in \mathrm{Sim}(p, A)$. With reference to (2.4) this condition is expressed by

$$V(p, A) := \bigcup_{q \in \mathrm{Sim}(p,A)} W(p, A, q) \ . \tag{2.5}$$

Notice $V(p, A) \subset V(A) \cap W(p, A)$, but equality may not hold. Clearly, we have $G(A) \cdot p \subset V(p, A)$ and just as in [17] it can be shown that equality holds if A is a ring. The image of $\mathrm{Sim}(p, A)$ under the map Π defines the space $\mathrm{Gr}(p, A)$ viewed as the Grassmannian naturally associated to $V(p, A)$.

For a given unital semigroup homomorphism $h : A \longrightarrow B$, there is a restriction of (2.3) to a commutative diagram :

$$\begin{array}{ccc} V(p,A) & \xrightarrow{V(p,h)} & V(q,B) \\ \Pi_A \big\downarrow & & \big\downarrow \Pi_B \\ \mathrm{Gr}(p,A) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(q,B) \end{array} \tag{2.6}$$

where for $p \in P(A)$, we have set $q = h(p) \in P(B)$. Observe that in the general semigroup setting, $V(p,A)$ properly contains $G(A)p$. In fact, if $p \in P(A)$, then $V(p,A) = G(A)G(pAp)$ (see [19] Lemma 2.3.1).

Henceforth we shall be restricting to the case where A and B are Banach(able) algebras. Let $H(p)$ denote the isotropy subgroup for this left–multiplication. We have then the coset space representation $\mathrm{Gr}(p,A) = G(A)/G(\Pi(p))$ where $G(\Pi(p))$ denotes the isotropy subgroup of $\Pi(p)$. Then there is the inclusion of subgroups $H(p) \subset G(\Pi(p)) \subset G(A)$, resulting in a fibering $V(p,A) \longrightarrow \mathrm{Gr}(p,A)$ given by the exact sequence

$$G(\Pi(p))/H(p) \hookrightarrow G(A)/H(p) \longrightarrow G(A)/G(\Pi(p)) \ , \tag{2.7}$$

generalizing the well–known *Stiefel bundle* construction in finite dimensions. For a Banach algebra A, constructions leading to spaces similar to $\mathrm{Gr}(A)$ and $\mathrm{Gr}(p,A)$ can be found in e.g. [12, 40].

2.3 Analytic embeddings

Analytic embeddings of manifolds such as $V(p,A)$ and $\mathrm{Gr}(p,A)$ were studied in [20, 19]. A main result is the following:

Proposition 2.1. ([19] Proposition 4.4.3) *Let $\Phi : G(A) \to G(B)$ be an analytic embedding of Banach Lie groups and $p \in P(A)$ and $q \in P(B)$. Suppose the map Φ is equivariant with respect to a homomorphism $\phi_p : G(\Pi(p)) \longrightarrow G(\Pi(q))$, and that Φ is transversal to $G(B)_q$. If the map $g \longrightarrow \Pi_B(\Phi(g)q)$ is an open map onto its image in $\mathrm{Gr}(q,B)$ then there exists an induced analytic embedding of Grassmannians $\mathrm{Gr}(p,h) : \mathrm{Gr}(p,A) \longrightarrow \mathrm{Gr}(q,B)$.*

For the purpose of the present paper we are interested in the specific construction of rank 1 projections in $P(B)$ with respect to the commuting diagram (2.6). We invoke the development of [20] §7.3 and introduce multiplicative subsemigroups denoted S and T such that $V(p,A) \subset S \subseteq A$, where S consists of elements of A admitting a determinant (cf the 'admissible bases' of [46]), and let T be the induced the image of S under the resulting map *det*, so that $det(V(p,A)) \subset T \subseteq B$. This provides an analytic multiplicative subsemigroup homomorphism $h : S \longrightarrow T$ induced by *det*, where for suitable $\hat{p} \in \mathrm{Sim}(p,A)$, $q = h(\hat{p})$, we have a commutative diagram with vertical maps inclusions

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ \big\uparrow & & \big\uparrow \\ V(p,A) & \xrightarrow{V(p,h)=det} & V(q,B) \end{array} \tag{2.8}$$

where $q = h(\hat{p})$ is identified with the rank 1 projection in $P(B)$ induced by the map *det* .

2.4 The spatial correspondence

If $\boldsymbol{\mathcal{A}}$ is a given topological algebra and E is some $\boldsymbol{\mathcal{A}}$–module, then $A = \mathcal{L}_{\boldsymbol{\mathcal{A}}}(E)$ could be taken to be the ring of $\boldsymbol{\mathcal{A}}$–linear transformations of E . An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E . In order to understand the relationship between spaces such as $\mathrm{Gr}(p, A)$ and the usual Grassmannians of subspaces (of a vector space E), we will describe a 'spatial correspondence'.

Given a topological algebra $\boldsymbol{\mathcal{A}}$, suppose E is an $\boldsymbol{\mathcal{A}}$–module admitting a decomposition

$$E = F \oplus F^c \ , \qquad F \cap F^c = \{0\} \ , \tag{2.9}$$

where F, F^c are closed subspaces of E . We have already noted $A = \mathcal{L}(E)$ as the ring of linear transformations of E . Here $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{array}{ccc} V(p,\mathcal{L}(E)) & \xrightarrow{\ \varphi\ } & V(p,E) \\ {\scriptstyle \Pi}\big\downarrow & & \big\downarrow{\scriptstyle \Pi} \\ \mathrm{Gr}(p,\mathcal{L}(E)) & \xrightarrow{\ =\ } & \mathrm{Gr}(F,E) \end{array} \tag{2.10}$$

where $V(p, E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F . In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a bijective correspondence with those of $\mathrm{Gr}(F, E)$. The spatial correspondence is thus a convenient way of encoding the geometric/vector space structure of the latter into the former.

2.5 Fredholm operators on Banach spaces

The next step involves introducing from [52] the notion of compact and Fredholm operators between complex Banach spaces E and E' . Let $\mathcal{K}(E, E')$ denote the compact operators. The more general meaning of a Fredholm operator is here based on the notion of 'right and left aggregation'; we refer to [52] for details. An operator $T \in \mathrm{Fred}(E)$ is stable under compact perturbations and admits a well–defined *index* given by $\mathrm{Ind}(T) = \dim \mathrm{Ker}\ T - \mathrm{codim}\ \mathrm{Im}\ T$. The index $\mathrm{Ind}(T)$ is constant on connected components, is invariant under compact perturbations, and satisfies $\mathrm{Ind}(T_1T_2) = \mathrm{Ind}(T_1) + \mathrm{Ind}(T_2)$. Moreover, there is an induced homomorphism $\mathrm{Ind} : \mathrm{Fred}(E) \to \mathbb{Z}$.

Let E be a complex Banach space admitting a decomposition of the type (2.9). Let

$$\widehat{G} \subset \{ \begin{bmatrix} T_1 & *_1 \\ *_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F) \ , \ T_2 \in \mathrm{Fred}(F^c)\} \ , \tag{2.11}$$

be a Banach Lie group that generates a Banach algebra A acting on E, but with possibly a different norm. Here, $*_i$ for $i = 1, 2$ represent operators on E of some specified class.

Suppose that $\widehat{G}$ acts analytically on $\mathrm{Gr}(A)$ with a typical orbit denoted by $\widehat{\mathrm{Gr}}(A)$. Fixing $p \in P(A)$, let $\widehat{\mathrm{Gr}}(p, A) = \widehat{\mathrm{Gr}}(A) \cap \mathrm{Gr}(p, A)$, and let $\widehat{\mathrm{Gr}}_\alpha(A)$ denote a connected component of $\widehat{\mathrm{Gr}}(A)$ for which $\mathrm{Ind}\ T_1 = \alpha$. Accordingly, we define $\widehat{\mathrm{Gr}}_\alpha(p, A) = \widehat{\mathrm{Gr}}_\alpha(A) \cap \mathrm{Gr}(p, A)$. Observe that the restriction $V_\alpha(p, A) = V(p, A)|\ \widehat{\mathrm{Gr}}_\alpha(p, A)$, thus provides a framing for elements of $\widehat{\mathrm{Gr}}_\alpha(p, A)$.

2.6 The range of choices for the semigroup

The semigroup A can be taken to be a subsemigroup of a ring of operators, or, of the partial isomorphisms of a topological algebra [51]. These include certain Fréchet algebras (e.g. [22]), as well as Banachable algebras, a class of topological algebras whose underlying vector space is a Banach space which holds for any Banach algebra and for certain Jordan–Lie algebras as studied in e.g. [30]. For A a Banach algebra properties of the map (2.2) were considered in [40]. A survey of properties of infinite dimensional homogeneous spaces modeled on a Banach algebra A can be found in [2] (cf. also [28]).

If $\boldsymbol{\mathcal{A}}$ is an algebra, then we can form the matrix algebra $\mathcal{M}_n(\boldsymbol{\mathcal{A}})$ of all $n \times n$ matrices with entries in $\boldsymbol{\mathcal{A}}$, where operations are defined just as for scalar matrices. If $\boldsymbol{\mathcal{A}}$ is a *–algebra, then so too is $\mathcal{M}_n(\boldsymbol{\mathcal{A}})$, with involution given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$. In particular, if $\boldsymbol{\mathcal{A}}$ is a C*–algebra, then there exists a unique norm on $\mathcal{M}_n(\boldsymbol{\mathcal{A}})$ making it a C*–algebra (for an account of these facts, see e.g. [15, 35].

Remark 2.1. Suppose $\mathbb{B}$ is a commutative ring of certain pseudodifferential operators. In [20], the rings $\mathbb{B} \otimes \boldsymbol{\mathcal{A}}$ were considered where $\boldsymbol{\mathcal{A}}$ is a (unital) commutative *–algebra, as well as for certain complex vector spaces V, the algebras $A = \mathcal{L}(V \otimes \boldsymbol{\mathcal{A}})$.

Given a unital C*–algebra $\boldsymbol{\mathcal{A}}$ one way consider the standard (free countable dimensional) Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ over $\boldsymbol{\mathcal{A}}$ as defined by

$$H_{\boldsymbol{\mathcal{A}}} = \{\{\zeta_i\} \ , \ \zeta_i \in \boldsymbol{\mathcal{A}} \ , \ i \geq 1 \ : \sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \boldsymbol{\mathcal{A}} \ \} \cong \oplus \boldsymbol{\mathcal{A}}_i \ , \tag{2.12}$$

where each $\boldsymbol{\mathcal{A}}_i$ represents a copy of $\boldsymbol{\mathcal{A}}$.

Suppose H is a separable Hilbert space (we will assume 'separability' for Hilbert spaces throughout). We can form the algebraic tensor product $H \otimes_{\text{alg}} \boldsymbol{\mathcal{A}}$ on which there is an $\boldsymbol{\mathcal{A}}$–valued inner product

$$\langle x \otimes \zeta \ , \ y \otimes \eta \rangle \ = \ \langle x, y \rangle \ \zeta^* \eta \ , \qquad x, y \in H \ , \ \zeta, \eta \in \boldsymbol{\mathcal{A}} \ . \tag{2.13}$$

Thus $H \otimes_{\text{alg}} \boldsymbol{\mathcal{A}}$ becomes an inner product $\boldsymbol{\mathcal{A}}$–module whose completion is denoted by $H \otimes \boldsymbol{\mathcal{A}}$. Given an orthonormal basis for H, we have the following identification (unitary equivalence) given by $H \otimes \boldsymbol{\mathcal{A}} \approx H_{\boldsymbol{\mathcal{A}}}$ (see e.g. [29]).

We continue with the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ and consider decompositions of the type (2.9) where we say that $H_{\boldsymbol{\mathcal{A}}}$ is *complementizable* (see e.g. [31]) if we have a pair of submodules $(\mathsf{H}_1, \mathsf{H}_2)$, such that

$$H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_1 \oplus \mathsf{H}_2 \ , \ \text{and } \mathsf{H}_1 \cap \mathsf{H}_2 = \{0\} \ . \tag{2.14}$$

This motivates calling the pair $(\mathsf{H}_1, \mathsf{H}_2)$ a *polarization of* $H_{\boldsymbol{\mathcal{A}}}$. A main example is when we have a unitary $\boldsymbol{\mathcal{A}}$–module map J satisfying $J^2 = 1$, there is an induced eigenspace decomposition $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_-$, for which $\mathsf{H}_- \cong \mathsf{H}_+$. We also have the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ as described in [20] (generalizing that of $\boldsymbol{\mathcal{A}} = \mathbb{C}$ in [41]). For suitable choice of the projection $p \in P(A)$, the spaces $\widehat{\mathrm{Gr}}(p, A)$ have the structure of a Fredholm Grassmannian of the type studied for various assignments of $\boldsymbol{\mathcal{A}}$, (see e.g. [20, 26, 34, 41, 46], §2.5, and §2.8 below). When the off–diagonal elements (denoted $*_1, *_2$) in (2.11) are taken to be compact operators, these spaces are sometimes called the 'restricted Grassmannians'. In particular, for $A = \mathcal{L}_J(H)$ (here $\boldsymbol{\mathcal{A}} = \mathbb{C}$), the above spatial correspondence identifies the restricted Grassmannian $\widehat{\mathrm{Gr}}(H)$ with $\widehat{\mathrm{Gr}}(p, A)$, for suitable $p \in P(A)$. Observe that the former admits various interesting sub–Grassmannians $\widehat{\mathrm{Gr}}_\sharp(H)$ as in [41, 45, 46]. Thus, for suitable subalgebras $A_\sharp \subset A$ and projections $p_\sharp \in P(A_\sharp)$, we have the further identification $\widehat{\mathrm{Gr}}_\sharp(H) \cong \mathrm{Gr}(p_\sharp, A_\sharp)$.

Remark 2.2. The use of this notation can be clarified by reading $\sharp$, to be for example, one of $0, 1, \omega, \infty$ as in [41](p. 104).

Remark 2.3. In the case of Hilbert *–algebras, there is a generalization of the p-th Schatten ideals $\mathcal{L}_p(H_{\boldsymbol{\mathcal{A}}})$ [47]. The case where $\boldsymbol{\mathcal{A}}$ is a commutative Hilbert *–algebra is relevant to von Neumann algebras (see e.g. [10]), and one may there deal with a continuous trace algebra.

Remark 2.4. At least in the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, the manifold $\widehat{\mathrm{Gr}}(H) = \widehat{\mathrm{Gr}}(p, A)$ admits a suitable Hermitian structure and in particular that of an infinite dimensional Kähler manifold within a class of more general (infinite dimensional) flag varieties whose theory is described in [23].

2.7 Laurent series generator

Definition 2.3. [3] Let A be a unital Fréchet algebra. An invertible element $\zeta \in G(A) \subset A$ is said to be a *Laurent series generator for* A if each $a \in A$ is expressible as

$$a = \sum_{i=-\infty}^{\infty} a_i \, \zeta^i \ , \tag{2.15}$$

for scalars a_i, and the series converges absolutely with respect to each continuous seminorm on A . We say that A has the *unique expression property* if such a representation is always unique.

As shown in [3], such algebras A with a Laurent series generator include algebras which are isomorphic to various types of function algebras defined on S^1 or on the annulus.

2.8 Cyclic vectors and the big cell

Suppose now that A is a unital Banach algebra with Laurent series generator ζ (with the unique expression property) acting cyclically on a complex Banach space E . We observe that the idempotents (projections) of A under the action form certain subspaces of E . Keeping in mind the development of §2.5, we assume a decomposition as in (2.9), $E = F \oplus F^c$ where the closed (splitting) subspaces F and F^c are specified as follows. Let ϕ be a cyclic vector for the above action. Recall that this means clos $[A\phi] = E$.

Remark 2.5. Let H be a Hilbert space, and consider a weakly closed C*–subalgebra $\mathfrak{W}^*$ of $\mathcal{L}(H)$ consisting of operators. If H is separable and $\mathfrak{W}^*$ is maximal abelian on H, then $\mathfrak{W}^*$ always admits a cyclic vector [15] (Theorem 4.65).

Next, let us set:

(i) F to be the closed linear span of all vectors $\zeta^i \phi$ for $i \geq 0$, and

(ii) F^c to be the closed linear span of $\zeta^i \phi$ for $i < 0$.

With regards to (2.11), we now take compact operators $*_1 \in \mathcal{K}(F^c, F)$ and $*_2 \in \mathcal{K}(F, F^c)$.

Suppose that a fixed $p \in P(A)$ acts as the projection of E on F along F^c . We denote by $\widehat{\mathrm{Gr}}(p, A)$ the Grassmannian consisting of subspaces $W \in \mathrm{Gr}(F, E)$ where $W = r(E)$ for $r \in P(A)$ such that :

(1) the projection $p_1 = pr \ : \ W \longrightarrow F$ is in $\mathrm{Fred}(E)$, and

(2) the projection $p_2 = (1-p)r \ : \ W \longrightarrow F^c$ is in $\mathcal{K}(E)$.

Alternatively, for (2) we may take projections $q \in P(A)$ such that for the fixed $p \in P(A)$, the difference $q - p \in \mathcal{K}(E)$. Further, we define the *big cell* of $\widehat{\mathrm{Gr}}(p, A)$ as the collection of all subspaces W of E such that the projection $p_1 \in \mathrm{Fred}(E)$ is an isomorphism.

3 Representation of the Burchnall–Chaundy ring in a Banach algebra

3.1 The Burchnall–Chaundy ring and the formal Baker function

Let $\mathbb{B}$ denote the algebra of analytic functions $U \longrightarrow \mathbb{C}$ where U is a connected open neighbourhood of the origin in $\mathbb{C}$. The (generally noncommutative) algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in $\mathbb{B}$, consists of expressions

$$\sum_{i=0}^{N} a_i \, \partial^i \ , \ (a_i \in \mathbb{B} \ , \quad \text{for some } N \in \mathbb{Z}) \ . \tag{3.1}$$

Here $\partial \equiv \partial/\partial x$ and the a_i can be regarded as multiplication operators for which multiplication is defined by

$$[\partial, a] = \partial a - a\partial = d(a) \equiv \partial a/\partial x \ . \tag{3.2}$$

Quite often we consider commutative subrings $\mathbb{A}$ of $\mathbb{B}[\partial]$, but more generally we pass to the algebra $\mathbb{B}[\partial^{-1}]$ of formal pseudodifferential operators with coefficients in $\mathbb{B}$. This algebra is obtained from $\mathbb{B}[\partial]$ by formally inverting the operator ∂ (see e.g. [42, 46]). We consider an operator L of order 1 in a commutative subring $\mathbb{A} \subset \mathbb{B}[\partial^{-1}]$, so that L is given by

$$L = \partial + \sum_{i > -\infty}^{-1} a_i \, \partial^i \ . \tag{3.3}$$

The formalism of [45] ensures that the correspondence '$(\frac{\partial}{\partial x})^{-1} \leftrightarrow$ multiplication by z', realizes commutative subrings $\mathbb{A} \subset \mathbb{B}[\partial^{-1}]$ as subrings of $\mathbb{C}[[x]][z^{-1}]$.

3.2 The local quintuple

Now $X = \text{Spec } \mathbb{A}$ is an algebraic curve and we recall from [42, 46] that in this setting there is the following associated quintuple of data $(X, x_\infty, V, z, \varphi)$ which we describe as follows: $V \longrightarrow X$ is a rank r holomorphic vector bundle, x_∞ is a smooth point of X, z the inverse of a local parameter on X at x_∞, where z is used to identify a neighborhood of x_∞ in X with a neighborhood of the disk $D_\infty = \{z : |z| \geq 1\}$ in the Riemann sphere, and φ is an analytic trivialization of V over some neighborhood of $D_\infty \subset X$. Subsequently, we consider L^2–boundary values of V over X/D_∞ and φ identifies sections of V over S^1 with $\mathbb{C}^r$–valued functions. Thus we arrive at the (separable) Hilbert space $H = L^2(S^1, \mathbb{C}^r)$ as part of the essential set–up with a special case of (2.9), that is, on setting $F = H_+$ and $F^c = H_-$, we have as before a polarization

$$H = H_+ \oplus H_- \ , \text{ and } H_+ \cap H_- = \{0\} \ , \tag{3.4}$$

and for which $A = \mathcal{L}_J(H)$. Further, the quintuple $(X, x_\infty, V, z, \varphi)$ corresponds, *de facto*, to a point $W \in \widehat{\text{Gr}}(H) \cong \widehat{\text{Gr}}(p, A)$ (cf. [27, 38]). Following [42, 46] the properties of the projections p_1 and p_2 in §2.8 apply, and the kernel (respectively, cokernel) of the projection $p_1 : W \longrightarrow H_+$ is naturally isomorphic to $H^0(X, \mathcal{V}_\infty)$ (respectively, $H^1(X, \mathcal{V}_\infty)$) where $\mathcal{V}_\infty = V[-x_\infty]$ denotes the sheaf whose sections are those of V that vanish at x_∞ .

Remark 3.1. In [42, 46] (for $\boldsymbol{\mathcal{A}} = \mathbb{C}$) the resulting Grassmannian $\widehat{\text{Gr}}(H) \cong \widehat{\text{Gr}}(p, A)$ is such that the Fredholm index Ind $p_1 = 0$, and the subspaces W of H corresponding to the above quintuple belong to $\widehat{\text{Gr}}(H)$ if and only if $\chi(\mathcal{V}_\infty) = 0$, that is, if the degree of V is rg_X where g_X denotes the arithmetic genus of X . The 'big cell' is characterized just it was before.

By means of tensoring of coefficients by a Hilbert *–algebra $\boldsymbol{\mathcal{A}}$ we can make a straightforward modification of the previous discussion by taking $H_{\boldsymbol{\mathcal{A}}} = L^2(S^1, \mathbb{C}^r) \otimes \boldsymbol{\mathcal{A}}$, and a polarization as in (2.14). In particular, we have a Krichever–type map

$$(X, x_\infty, V, z, \varphi) \ \longrightarrow \ \text{points } W \in \widehat{\text{Gr}}(p, A) \ . \tag{3.5}$$

3.3 A formal integral operator

Here we set $\boldsymbol{\mathcal{A}} = \mathbb{C}$. Relative to $W \in \widehat{\text{Gr}}(H_{\boldsymbol{\mathcal{A}}}) = \widehat{\text{Gr}}(p, A)$, we set

$$B_W = \{f(z) = \sum_{s=-\infty}^{N} c_s z^{-s} : s \in \mathbb{N}, \ c_s \in \mathbb{C}\} \ , \tag{3.6}$$

which (see [46]) contains the coordinate ring of the curve $X \setminus \{x_\infty\}$. Following [42, 46] there exists an operator $K \in \mathbb{B}[\partial^{-1}]$ given by

$$K = 1 + \sum_{s=1}^{\infty} a_s(x) \ \partial^{-s} \ , \tag{3.7}$$

such that each $L \in \mathbb{A}$ satisfies the conjugation property $L = K(\partial)K^{-1}$. At the same time K can be regarded as a formal integral operator and the space of these we denote by $\mathcal{K}$. Under the above correspondence the (formal) Baker function ψ_W is defined as $\psi_W = Ke^{xz}$ and in fact there is an embedding [45]:

$$\begin{aligned} \widehat{\mathrm{Gr}}(H) = \widehat{\mathrm{Gr}}(p, A) &\longrightarrow \mathcal{K} \\ W &\mapsto K = e^{-xz}\psi_W \ . \end{aligned} \tag{3.8}$$

The main point is that the function ψ_W will be an eigenfunction for L, that is :

$$L\psi_W = z\ \psi_W \ , \tag{3.9}$$

and accordingly

$$\psi_W(x, z) = (1 + \sum_{s=1}^{\infty} a_s(x)\ z^{-s})\ e^{xz} \ . \tag{3.10}$$

(This is actually the 'stationary' Baker function since we have suppressed the time variables.) By means of conjugation by the (formal) integral operator K and applying the generalized Sato correspondence, the following was established:

Theorem 3.1. [20] *The subring* $\mathbb{A}$ *of* $\mathbb{B}[\partial]$ *conjugates into the Banach algebra* $A = \mathcal{L}_J(H)$ *as a commutative subring up to constant coefficient operators.*

A similar development of ideas, as shown in [20], can be applied to the more general case for $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ and $\boldsymbol{\mathcal{A}}$ is a (unital) C*–algebra, and $\mathbb{A} \subset \mathbb{B}[\partial^{-1}] \otimes \boldsymbol{\mathcal{A}}$. This we do in §9 (Theorem 9.1). A more general description of ψ_W as an operator–valued function is in progress.

4 A spectral correspondence

4.1 The Taylor joint spectrum

To further discuss the possible spectra of commuting subrings of operators in $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$, we recall the *Taylor joint spectrum* [21, 50] and outline the construction following [13]. Let $\Lambda \equiv \Lambda(\mathbf{e}) \equiv \Lambda_n[\mathbf{e}]$ be the exterior algebra on n generators $e_1, \ldots, e_n$ with identity $e_0 \equiv 1$. Otherwise said, Λ is the algebra of forms in $e_1, \ldots, e_n$ with complex coefficients, subject to the relation $e_ie_j + e_je_i = 0$ (where $1 \leq i, j \leq n$). Let $C_i : \Lambda \longrightarrow \Lambda$ denote the creation operator defined by $C_i\xi := e_i\xi$, where $\xi \in \Lambda$, and $1 \leq i \leq n$. If we decree $\{e_{i_1}, \ldots, e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}$ to be an orthonormal basis, then the exterior algebra Λ becomes a Hilbert space admitting an orthogonal decomposition $\Lambda = \oplus_{k=1}^{n}\Lambda^k$, where $\dim \Lambda^k = \binom{n}{k}$. Thus each $\xi \in \Lambda$ admits a unique orthogonal decomposition $\xi = e_i\xi' + \xi''$, where ξ' and ξ'' have no e_i contribution. It then follows that $C_i^*\xi = \xi'$. In fact, each C_i is a partial isometry satisfying $C_i^*C_j + C_jC_i^* = \delta_{ij}$, for $1 \leq i, j \leq n$.

Let E be a normed space, let $T \equiv (T_1, \ldots, T_n)$ be a commuting n–tuple of bounded operators on E and set $\Lambda(E) := E \otimes_{\mathbb{C}} \Lambda$. One defines $D_T : \Lambda(E) \longrightarrow \Lambda(E)$ by $D_T := \sum_{i=1}^{N} T_i \otimes C_i$. Clearly, we have $D_T^2 = 0$, so that Im $D_T \subseteq$ Ker D_T . The commuting n–tuple T is said to be non–singular on E if Im $D_T =$ Ker D_T . *The Taylor (joint) spectrum* $\sigma_{\mathsf{T}}(T, E)$ of T on E is the set

$$\sigma_{\mathsf{T}}(T, E) := \{\lambda \in \mathbb{C}^n : T - \lambda \text{ is singular}\} \ . \tag{4.1}$$

The decomposition $\Lambda = \oplus_{k=1}^{n}\Lambda^k$ induces a cochain complex $K^\bullet(T, E)$ known as *the Koszul complex* associated to T on E, given by

$$K^\bullet(T, E)\ : 0 \to \Lambda^0(E) \xrightarrow{D_T^0} \cdots \xrightarrow{D_T^{n-1}} \Lambda^n(E) \to 0 \ , \tag{4.2}$$

where D_T^k denotes the restriction of D_T to the subspace $\Lambda^k(E)$. Hence, in terms of the Koszul complex, the Taylor spectrum is expressed by

$$\sigma_{\mathsf{T}}(T, E) := \{\lambda \in \mathbb{C}^n : K^\bullet(T - \lambda, E) \text{ is not exact}\} . \tag{4.3}$$

In [50] it was shown that if E is a Banach space, then $\sigma_{\mathsf{T}}(T, E)$ is compact, non–empty, and contained in $\sigma_{\mathsf{J}}(T)$, *the joint algebraic spectrum of* T relative to the commutant of T, that is, $(T)' := \{B \in \mathcal{L}(E) : BT = TB\}$.

The commuting n–tuple T is said to be *Fredholm* on E if the associated Koszul complex $K^\bullet(T, E)$ has finite dimensional cohomology spaces, that is dim $H^*(K^\bullet(T, E)) < \infty$. In which case we have

$$\sigma_{\mathsf{T}}(T, E) := \{\lambda \in \mathbb{C}^n : T - \lambda \text{ is not Fredholm}\} , \tag{4.4}$$

and the *Fredholm index* is defined as the Euler Characteristic $\chi(K^\bullet(T, E))$ of the complex $K^\bullet(T, E)$ (cf. [21]).

4.2 Correspondence of spectra

We next recall Theorem 3.1 which represents $\mathbb{A}$ as a commutative subring of the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$. The following is almost immediate:

Theorem 4.1. *In relationship to a certain commuting n–tuple of operators $T \equiv (T_1, \ldots, T_n)$ in the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$, there exists a correspondence of spectra of rings*

$$\text{Spec } \mathbb{A} \longrightarrow \sigma_{\mathsf{J}}(T) \tag{4.5}$$

Proof. This follows essentially from the representation $\mathsf{s} : \mathbb{A} \longrightarrow A$ (see Theorem 3.1). Effectively, if $T_i = \mathsf{s}(L_i)$, where $L_i \in \mathbb{A}$, for $1 \leq i \leq n$, then we have

$$\mathsf{s}([L_i, L_j]) = [\mathsf{s}(L_i), \mathsf{s}(L_j)] = [T_i, T_j] = 0 , \tag{4.6}$$

since the image of s is a commutative subring of A . Thus we obtain a commuting n–tuple $T \equiv (T_1, \ldots, T_n)$, where each $T_i \in A$, and for which we have the Taylor spectrum $\sigma_T(T, E)$ where E is taken to be the underlying Banach space of A . Hence in this case $\sigma_T(T, E)$ maps into the joint algebraic spectrum $\sigma_{\mathsf{J}}(T)$. □

4.3 Relationship with spectral flow

Observe that on S^1 we have the standard Dirac operator given by $D = -\iota \frac{d}{dx}$ where x here denotes an angular variable. We obtain a periodic 1–parameter family of self–adjoint operators $D_t = D + t$, with parameter $t \in S^1$, and with periodicity expressed by $D_{t+1} = e^{-\iota x}\ D_t\ e^{\iota x}$. The eigenvalues λ_j are functions of t and when t goes once around S^1, the $\{\lambda_j\}$ (as a set) have to return to their original position. But there may be a shift $\lambda_j \mapsto \lambda_j + \mathsf{a}$, for $\mathsf{a} \in \mathbb{Z}$. This latter integer is called *the spectral flow* of the family [1]. Proceeding to the 2–torus $\mathbb{T}^2 = S^1 \times S^1$, we form the Dirac operator $\widetilde{D} = D_t + \frac{\partial}{\partial t}$ on $\mathbb{T}^2$. The spectral flow is then defined as the index Ind $\widetilde{D} \in \mathbb{Z}$.

Note also that we have a *spectral triple* $(\mathcal{F}, H, D)$ in the sense of [10], where $\mathcal{F}$ is the commutative C*–algebra $\mathcal{F} = C^\infty(S^1)$, and for which the Gelfand spectrum simply yields $\mathsf{Spec}\ \mathcal{F} = S^1$ (cf also [44]).

It is worth noting that following [24] the multiplication operators $g(x,\zeta) = e^{\ell(x,\zeta)}$ (for suitable functions ℓ, see below) define a 1–parameter (KP) flow on the *shift invariant subspaces* $W_{(\mathsf{a})} \in \widehat{\mathrm{Gr}}(H) \cong \widehat{\mathrm{Gr}}(p,A)$ as labeled by a Fredholm index $\mathsf{a} \in \mathbb{Z}$. The latter is the index of the projection $p_1 \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$ characterizing W; this index is zero if W is in the big cell, as we have pointed out. We conjecture a numerical relationship between this index and the Euler characteristic $\chi(K^\bullet(T,E))$ of the Koszul complex of some corresponding n–tuple of commuting Fredholm operators as was realized in the representation $\mathbb{A} \longrightarrow A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$.

5 The Baker function in the Banach space setting

Let D be the closed unit disc centered at the origin in $\mathbb{C}$ which contains the spectrum of the generator ζ in §2.7. Recalling $\mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$, we define $\Gamma_+(A)$ to be the group of (invertible) holomorphic maps $g : D \longrightarrow G(A)$, such that $g(0) = \mathbf{1}$, and define an action of $\Gamma_+(A)$ on F by $g \cdot v = g(\zeta)v$ where $g(\zeta)$ is given by the holomorphic functional calculus. Effectively, $\Gamma_+(A)$ is the flow of multiplication operators as given by

$$\Gamma_+(A) = \{\exp(\sum_a t_a z^a)\} \ , \tag{5.1}$$

(cf. [32, 46]). We also define $\Gamma_-(A)$ as the group of holomorphic maps of the form $g(\frac{1}{z})$ where $g \in \Gamma_+(A)$, and in fact [46] $\Gamma_-(A)$ acts freely and transitively on $\widehat{\mathrm{Gr}}(F,E) = \widehat{\mathrm{Gr}}(p,A)$, where $p \in \mathrm{Sim}(p_1, A)$ for some $p_1 \in P(A)$.

Next we consider subspaces $W \in \widehat{\mathrm{Gr}}(p,A)$ of the form

$$W = g h_g F \ , \tag{5.2}$$

with $g \in \Gamma_+(A)$ and $h_g \in \Gamma_-(A)$. Also for $g \in \Gamma_+(A)$, we consider projections

$$p_1^g \ : \ g^{-1}(W) \longrightarrow F \ , \ p_1^g \in \mathrm{Fred}(E) \ , \tag{5.3}$$

and define

$$\Gamma_+^W = \{g \in \Gamma_+(A) : p_1^g \text{ is an isomorphism}\} \ . \tag{5.4}$$

Definition 5.1. *The operator–valued Baker function* ψ_W associated to the subspace $W \in \widehat{\mathrm{Gr}}(p,A)$ in (5.2), is defined formally as :

$$\begin{aligned} \psi_W &= (p_1^g)^{-1}(\mathbf{1}) = W g(\zeta) \\ &= (\sum_{s=0}^{\infty} a_s(g)\ \zeta^{-s})\ g(\zeta) \ , \end{aligned} \tag{5.5}$$

where $g \in \Gamma_+^W$ and the a_s are analytic $\boldsymbol{\mathcal{A}}$–valued operator/matrix functions on Γ_+^W extending to meromorphic functions on all of $\Gamma_+(A)$ (cf. [20]).

By including an extra parameter x, we realize $g(\zeta)$ as $g(x,\zeta) = e^{\ell(x,\zeta)}$, where $\ell(x,\zeta)$ is an A–valued function, so that ψ_W in (5.5) is more explicitly expressed by

$$\psi_W(x,\zeta) = (\sum_{s=0}^{\infty} a_s(x)\ \zeta^{-s})\ e^{\ell(x,\zeta)} \ . \tag{5.6}$$

Effectively, the nature of the A–valued function $g(\zeta)$, will be determined by the specific cases to which the Baker function will be applied.

6 The operator cross ratio and generalized tau-function

Taking $\boldsymbol{\mathcal{A}}$ to be a commutative Hilbert*–algebra so that $H_{\boldsymbol{\mathcal{A}}}$ is a Hilbert *–module (see Remark 2.3) we continue with the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ and consider decompositions of the type (2.14) for which we have a pair of submodules $(\mathcal{H}_1, \mathcal{H}_2)$, such that $H_{\boldsymbol{\mathcal{A}}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. Thus consider submodules $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \subset H_{\boldsymbol{\mathcal{A}}}$ and a pair of polarizations $\Pi_{12} : \mathcal{H}_1 \oplus \mathcal{H}_2 = H_{\boldsymbol{\mathcal{A}}}$, and $\Pi_{34} : \mathcal{H}_3 \oplus \mathcal{H}_4 = H_{\boldsymbol{\mathcal{A}}}$. Let $P_i \in P(A)$ be projections such that Im $P_i = \mathcal{H}_i$, for $1 \leq i \leq 4$. Following [54] we define the *operator cross ratio* (with respect to the polarizations Π_{12}, Π_{34}) to be

$$\mathrm{CR}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = (P_1 - P_2)^{-1}(P_2 - P_3)(P_3 - P_4)^{-1}(P_4 - P_1) \ , \tag{6.1}$$

which corresponds to the composition

$$\mathcal{H}_1 \overset{P_3|\mathcal{H}_1}{\longrightarrow} \mathcal{H}_3 \overset{P_1|\mathcal{H}_3}{\longrightarrow} \mathcal{H}_1 \ , \tag{6.2}$$

leading to the identity on $\mathcal{H}_1$. Now suppose that the $\mathcal{H}_i$, for $i = 1, 2$, are fixed subspaces transformed into each other via an operator in the unitary group $\mathrm{U}(\mathrm{H}_{\boldsymbol{\mathcal{A}}})$ of module operations, and let $\mathcal{K}_j$, $j = 1, 2, 3$, denote their complements so that $\mathcal{H}_i \oplus K_j = H_{\boldsymbol{\mathcal{A}}}$. Since there is no essential algebraic change in generalizing from polarized Hilbert spaces to polarized Hilbert modules, we have following [54](Lemma 4) the relation:

$$\mathrm{CR}(\mathcal{H}_1, \mathcal{K}_1, \mathcal{H}_2, \mathcal{K}_2)\ \mathrm{CR}(\mathcal{H}_1, \mathcal{K}_3, \mathcal{H}_2, \mathcal{K}_1)\ \mathrm{CR}(\mathcal{H}_1, \mathcal{K}_2, \mathcal{H}_2, \mathcal{K}_3) = \mathrm{Id} \ . \tag{6.3}$$

Next we consider the universal bundle $\mathsf{U} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$ along with a local trivialization. We fix a subspace $W_+ \in \widehat{\mathrm{Gr}}(p, A)$. A local chart $\mathcal{U}_V$ in U is defined via a choice of a complementary subspace $V \in H_{\boldsymbol{\mathcal{A}}}$ for W_+ so that $H_{\boldsymbol{\mathcal{A}}} = W_+ \oplus V$. On a coordinate $x \in W$, the transitions from the chart $\mathcal{U}_{V_1}$ to $\mathcal{U}_{V_2}$ is deduced from (6.3) to be the cross ratio $\mathrm{CR}(W, V_1, W_+, V_2)$ [54] (p. 42). Thus the transitions between charts define endomorphisms $\mathrm{CR}(W, V_1, W_+, V_2)$ of the subspace W, that is, the transition functions for the universal bundle $\mathsf{U} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$. In turn, we obtain a cocycle $\{\mathrm{CR}\} \in H^1(\widehat{\mathrm{Gr}}(p, A), \mathrm{End}(\mathsf{U}))$.

Let $\mathsf{P} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$ be the associated principal G–bundle of $\mathsf{U} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$, where G denotes the Banach Lie structural group of the latter. Letting $\mathfrak{X}(\mathsf{P})$ denote the bundle of G–invariant vector fields on $T\mathsf{P}$ and $Y^v\mathsf{P}$ the (vertical) tangent bundle along the fibres, then we have the Atiyah sequence over $\widehat{\mathrm{Gr}}(p, A)$

$$0 \longrightarrow T^v\mathsf{P} \longrightarrow \mathfrak{X}(\mathsf{P}) \longrightarrow T\mathsf{P} \longrightarrow 0 \tag{6.4}$$

where the composition $\mathfrak{X}(\mathsf{P}) \overset{g}{\longrightarrow} T\mathsf{P} \overset{h}{\longrightarrow} \mathfrak{X}(\mathsf{P})$, for which $g \circ h = \mathrm{Id}$, defines a connection on P .

The above construction yields for local charts and transitions between them

$$\begin{aligned} u_i &: U_i \times G \longrightarrow \mathsf{P}|U_i \\ u_j &: U_j \times G \longrightarrow \mathsf{P}|U_j \\ \phi_{ij} &= u_i^{-1} u_j = \mathrm{CR}(W, V_i, W_+, V_j) \end{aligned} \tag{6.5}$$

Thus relative to transformations $a_i : \mathfrak{X}(\mathsf{P})|V_i \longrightarrow T\mathsf{P}|V_j$, we deduce on overlaps $V_{ij} = V_i \cap V_j$, a cocycle

$$a_{ij} = (a_j - a_i) : \mathfrak{X}(\mathsf{P})|V_{ij} \longrightarrow T\mathsf{P}|V_{ij} \ . \tag{6.6}$$

6.1 The space of polarizations

Let $(\mathsf{H}_+, \mathsf{H}_-)$ and $(\mathsf{K}_+, \mathsf{K}_-)$ be two such polarizations so that $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_- = \mathsf{K}_+ \oplus \mathsf{K}_-$, whereby the projections parallel to H_- and K_- are isomorphisms of the spaces H_+ and K_+ respectively. Further, if we take $\mathsf{H}_\pm \in \widehat{\mathrm{Gr}}(p, A)$ and for some $q \in P(A)$, we take $\mathsf{K}_\pm \in \widehat{\mathrm{Gr}}(q, A)$, then under these specified conditions the latter can be regarded as the 'dual Grassmannian' of $\widehat{\mathrm{Gr}}(p, A)$. Let us denote this dual Grassmannian by $\widehat{\mathrm{Gr}}^*(p, A)$. Then the space R of such polarizations can be regarded as a subspace

$$R \subset \widehat{\mathrm{Gr}}(p, A) \times \widehat{\mathrm{Gr}}^*(p, A) \ . \tag{6.7}$$

Let $\mathsf{H}_\pm$ and $\mathsf{K}_\pm$ be 'coordinatized' via maps $P_\pm : \mathsf{H}_\pm \longrightarrow \mathsf{H}_\mp$, and $Q_\mp : \mathsf{K}_\pm \longrightarrow \mathsf{K}_\mp$, respectively. Following [54] (Proposition 2), we can consider the composite map

$$\mathsf{H}_+ \overset{\mathsf{K}_-}{\longrightarrow} \mathsf{K}_+ \overset{\mathsf{H}_-}{\longrightarrow} \mathsf{H}_+ \ , \tag{6.8}$$

as represented by an operator cross–ratio

$$\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = (P_-P_+ - 1)^{-1}(P_-Q_+ - 1)(Q_-Q_+ - 1)^{-1}(Q_-P_+ - 1) \ . \tag{6.9}$$

Observe that we have a (holomorphic) universal vector bundle $\mathsf{V} \longrightarrow R$ where to each point $\mathcal{P} = (\mathsf{H}_+, \mathsf{H}_-) \in R$, the fibre H_+ is assigned. At the same time we also have the associated principal $\mathrm{GL}(\mathsf{H}_+)$–bundle $\mathsf{Q} \longrightarrow R$, on which there is a connection induced by the operator cross ratio. Again we follow the development of [54]§3 in order to describe this connection and its curvature as adapted to the present situation. We fix a point $\mathcal{P} = (\mathsf{H}_+, \mathsf{H}_-) \in R$, we consider a pair of local sections α, β of Q, which are related as follows

$$\alpha = \beta \mathfrak{T} \ , \qquad \beta = \alpha \mathfrak{T}^{-1} \ . \tag{6.10}$$

Next let $\nabla_\pm$ denote covariant differentiation with respect to the direction $\mathsf{H}_\pm$. The local sections α, β have the property that:

(a) α is covariantly constant along $\{\mathsf{H}_+\} \times \widehat{\mathrm{Gr}}^*(p, A)$, with respect to fixed H_+ .

(b) β is covariantly constant along $\widehat{\mathrm{Gr}}(p, A) \times \mathsf{H}_-$ with respect to fixed H_- .

(c) Properties (a) and (b) imply that $\nabla_-\alpha = 0, \nabla_+\beta = 0$, and $\nabla_+\alpha = \beta\nabla_+\mathfrak{T} = \alpha\mathfrak{T}^{-1}\nabla_+\mathfrak{T}$.

Setting $\omega_+ = \mathfrak{T}^{-1}\nabla_+\mathfrak{T}$, we obtain a flat connection on Q [32]. We have the exterior covariant derivative $d = \partial_+ + \partial_-$, where $\partial_\pm$ denotes the covariant derivative along $\mathsf{H}_\pm$. Straightforward calculations as in [54]§3 yield the following:

$$\begin{aligned} &\partial_+\omega_+ = 0 \\ &\partial_-\omega_+ = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+ \ . \end{aligned} \tag{6.11}$$

The curvature form is then given by

$$\Omega = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+ \ . \tag{6.12}$$

Thus $\mathfrak{T}$ as an operator valued function $\mathfrak{T} : \mathsf{U} \longrightarrow \boldsymbol{\mathcal{A}}$, defines a holomorphic symplectic structure on U. This permits the definition of the τ function as $\partial_+\partial_- \log \tau = \Omega$.

6.2 The example of trace class operators

An alternative, but equivalent, operator description leading to $\mathfrak{T}$ above can be obtained following [32]. Suppose $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in R$ are such that H_+ is the graph of a linear map $S : \mathsf{K}_+ \longrightarrow \mathsf{K}_-$ and H_- is the graph of a linear map $T : \mathsf{K}_- \longrightarrow \mathsf{K}_+$. Then on $H_{\boldsymbol{\mathcal{A}}}$ we consider the identity map $\mathsf{H}_+ \oplus \mathsf{H}_- \longrightarrow \mathsf{K}_+ \oplus \mathsf{K}_-$ in the block form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} , \tag{6.13}$$

where $a : \mathsf{H}_+ \longrightarrow \mathsf{K}_+,\ d : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are zero–index Fredholm operators, and $b : \mathsf{H}_+ \longrightarrow \mathsf{K}_+$, $c : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are in $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ (the compact operators), such that $S = ca^{-1}$ and $T = bd^{-1}$. The next matter is to consider the operator $1 - ST = 1 - ca^{-1}bd^{-1}$. In particular, with a view to defining a generalized determinant leading to *an operator–valued tau–function*, we need to consider cases where ST is taken to be of *trace class*.

(a) When $\boldsymbol{\mathcal{A}} = \mathbb{C}$ as in [32, 54], we take b, c to be Hilbert–Schmidt operators. Then ST is of trace–class, the operator $(1 - ST)$ is essentially $\mathcal{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-)$ above, and the tau–function is defined as

$$\tau(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = \det(1 - ST) = \det(1 - ca^{-1}bd^{-1}) \ . \tag{6.14}$$

Recall that in this situation we have the Plücker embedding $\widehat{\mathrm{Gr}}(p, A) \longrightarrow \mathbb{P}(\Lambda_+)$ (projective Fock space) [46], and the determinant line bundle $\mathsf{Det} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$ is induced by the pull–back of the universal line bundle over $\mathbb{P}(\Lambda_+)$ under this embedding. Then as pointed out in [32, 54], the function τ as a holomorphic section of Det, corresponds to a Hermitian holomorphic structure on Det following which the curvature Ω of the (flat) connection ω_+ is given by $\Omega = \partial_+ \partial_- \log \tau$ (cf. [6, 9]).

(b) In view of Remark 2.3, we may take the operators b, c as belonging to the class $\mathcal{L}_2(H_{\boldsymbol{\mathcal{A}}})$ [47] that generalizes the operators of Hilbert–Schmidt class on a Hilbert space. Then ST is of class $\mathcal{L}_1(H_{\boldsymbol{\mathcal{A}}})$ (generalizing trace–class) and $\tau(\mathsf{H}_+, \mathsf{K}_-, \mathsf{K}_+, \mathsf{K}_-)$ is definable when the operator $(1 - ST)$ admits a determinant in a suitable sense.

Recall that in §3.2 we presented quintuples of the form $(X, x_\infty, V, z, \varphi)$ where $V \longrightarrow X$ is taken to be a holomorphic Hermitian vector bundle and we recall that $X = \mathrm{Spec}\ \mathbb{A}$. We take $H_{\boldsymbol{\mathcal{A}}} = L^2(S^1, \mathbb{C}^r) \otimes \boldsymbol{\mathcal{A}}$ and $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$, and also recall that in (3.5) the quintuple $(X, x_\infty, V, z, \varphi)$ corresponds to a subspace $W \in \widehat{\mathrm{Gr}}(p, A)$. We further consider a (Hermitian) connection ∇_V on $V \longrightarrow X$. The main point here is that over the space of $\bar{\partial}$–operators (relative to ∇_V) there is defined a determinant line bundle with its Quillen connection and a canonical holomorphic section. We refer to [33] for a discussion of these latter topics, and more especially for the following. The τ–function arises in this context as an orbit in the phase space under the action of a subgroup $\mathcal{G}$ of the full complex gauge group $\mathcal{G}^c$ of V (which acts on the space of connections on V). A cocycle interpretation can be given as follows. Consider vector fields $u, v \in \mathrm{Lie}(\mathcal{G})$. Then the contracted product

$$\mathsf{i}_v\ \mathsf{i}_u\ d(d \log \tau) = c(u, v) \ , \tag{6.15}$$

yields a Lie algebra–valued cocycle for the central extension of $\mathrm{Lie}(\mathcal{G})$ into the space of C^∞ functions on the space of unitary connections (viz $\bar{\partial}$–operators) on (Hermitian) holomorphic vector bundles over certain domains with boundary in $\mathbb{C}$ such as the space D_∞ included in the data of §3.2 . As described in [33], this is part of the twistor interpretation of the main concepts of [41, 46] where subspaces $W \in \widehat{\mathrm{Gr}}(p, A)$ correspond to 'twistor lines'. We will say more about the twistorial aspect in the next section.

6.3 The Pfaffian line bundle on the space of polarizations

Now we will look closer at the geometric structure of the space of polarization R in relationship to almost complex structures. We follow mainly [48, 49] and make the modifications for Hilbert modules $H_{\boldsymbol{\mathcal{A}}}$ over a commutative Hilbert *–algebra $\boldsymbol{\mathcal{A}}$, as we have assumed. Firstly, given a polarization $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_-$, we regard the latter as specified by an element $\mathbb{F} \in \mathrm{End}(H_{\boldsymbol{\mathcal{A}}})$ given by, for $p_\pm \in P(A)$,

$$\mathbb{F} = p_+ - p_- = 2p_+ - 1 \ . \tag{6.16}$$

We also recall that $\mathrm{Gr}(H_{\boldsymbol{\mathcal{A}}}, \mathsf{H}_+) = \widehat{\mathrm{Gr}}(p, A)$, and here we take $p_W - p_+ \in \mathcal{L}_2(H_{\boldsymbol{\mathcal{A}}})$ (generalized Hilbert–Schmidt operators), where $\mathrm{Im}\ p_W = W$.

Remark 6.1. For a (separable) complex Hilbert space H, the *Canonical Anticommutation Relations* (CAR) algebra $\mathfrak{A}(H)$ is the universal C*–algebra generated by the well–known creation and annihilation operators (as in e.g. [39] – see also the Appendix). In view of the Gelfand–Neumark–Segal theorem, quasifree representations with respect to Hermitian projections P_1, P_2, are expressed in terms of triples of the type $(\pi_{P_1}, H_{P_1}, \xi_{P_1}), (\pi_{P_2}, H_{P_2}, \xi_{P_2})$ where ξ_{P_1}, ξ_{P_2} are the respective associated cyclic vectors. As pointed out in [48, 49], for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, the Powers–Størmer theorem states that the representations π_{P_1} and π_{P_2} are unitarily equivalent if and only if $P_1 - P_2 \in \mathcal{L}_2(H)$.

Given then $(H_{\boldsymbol{\mathcal{A}}}, \mathbb{F})$ as above, we now regard $H_{\boldsymbol{\mathcal{A}}}$ as a *real* Hilbert module and then complexify so as to obtain the Hilbert module $H^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}}$. Next we make the assignment

$$\mathsf{H}_+ \mapsto W = \mathsf{H}_+ \oplus \overline{\mathsf{H}}_- \ . \tag{6.17}$$

The polarization determined by $\mathbb{F}$ induces an almost complex structure $J \in \mathrm{End}(H_{\boldsymbol{\mathcal{A}}})$ given by $J = \iota\mathbb{F}$, for which $J^2 = -\mathrm{Id}$. Now recalling that in the restricted Grassmannian $\widehat{\mathrm{Gr}}(p, A)$ we have the Banach algebra $A = \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ and suitable $p \in P(A)$, then likewise we can express $\widehat{\mathrm{Gr}}(H^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}}, W)$ in terms of the Banach algebra $A' = \mathcal{L}(H^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})$ and suitable $q' \in P(A')$. Thus we have the Segré–type embeddings

$$\begin{aligned} R \subset \widehat{\mathrm{Gr}}(p, A) \times \widehat{\mathrm{Gr}}^*(p, A) &\stackrel{i}{\longrightarrow} \widehat{\mathrm{Gr}}(q', A') \\ P_{W_1} \mapsto P_{W_1} &\oplus \overline{(1 - P_{W_1})} \end{aligned} \tag{6.18}$$

Effectively, the space R when viewed as a space of almost complex structures, may be represented as a homogeneous space

$$R = \mathrm{O(H^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})/U(W)} \subset \widehat{\mathrm{Gr}}(\mathrm{p, A}) \times \widehat{\mathrm{Gr}}^*(\mathrm{p, A}) \ , \tag{6.19}$$

where 'O' and 'U' denote the groups of orthogonal and unitary module operators, respectively.

6.4 Remark on the twistor interpretation

Observe that this apparently generalizes the situation where one has a (real) $2N$–dimensional Hermitian manifold M for which there is the *twistor space* $\mathcal{Z}$ parametrizing locally definable almost complex structures on M . We thus obtain a twistor fibration

$$\mathrm{SO(2N)/U(N)} \hookrightarrow \mathcal{Z} \xrightarrow{\mathsf{p}} \mathrm{M} \ , \tag{6.20}$$

where a typical fibre is the Hermitian symmetric space $\mathrm{SO(2N)/U(N)} \subset \mathrm{G_N}(\mathbb{C}^{2N})$. Applied to the quintuple $(X, x_\infty, V, z, \varphi)$ we have from (3.5) the associated subspace $W \in \widehat{\mathrm{Gr}}(p, A)$ which as in [33] corresponds to a representation of a holomorphic vector bundle $\mathfrak{E} \longrightarrow \mathcal{Z}$ restricted to a 'twistor line' $\mathfrak{E}|\mathsf{p}^{-1}(m)$, for $m \in M$, and for which the flow of multiplication operators Γ_+^W corresponds to holomorphic symmetries.

6.5 The Pffafian line bundle

In the case of $\boldsymbol{\mathcal{A}} = \mathbb{C}$ considered in [48, 49], the space R is the 'isotropic Grassmannian' describing the $\mathrm{O}(\mathrm{H}^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})$–orbit of the anti–Fock states. It is worth remarking that in this case, any two such almost complex structures can be related via a rotation in the orthogonal group O(H) and the Shale–Stinespring theorem implies that such a rotation induces a C*–automorphism of the CAR algebra $\mathfrak{A}(W)$. We expect the same to apply for generally for the orthogonal group $\mathrm{O}(\mathrm{H}^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})$ where $\boldsymbol{\mathcal{A}}$ is some commutative *–algebra to which we now return. Commencing from the universal bundle $\mathsf{U}' \longrightarrow \widehat{\mathrm{Gr}}(q', A')$, we set $\mathsf{L} = \det(i^*\mathsf{U}')|R$, granted 'det' is defined. Then following [5], the *Pfaffian line bundle* $\mathsf{Pf} \longrightarrow R$ is given by

$$\mathsf{Pf} = \mathsf{L}^{1/2} \longrightarrow R = \mathrm{O}(\mathrm{H}^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})/\mathrm{U(W)} \ , \tag{6.21}$$

and for $\boldsymbol{\mathcal{A}} = \mathbb{C}$ this construction can be used in re–formulating the well–known Fock–Anti Fock space correspondence [48, 49].

7 Relation between the operator tau and Baker functions

Without loss of generality let us take $H = L^2(S^1, \mathbb{C}^r)$ and $\boldsymbol{\mathcal{A}}$ a commutative Hilbert *–algebra so that again $H_{\boldsymbol{\mathcal{A}}} = H \otimes \boldsymbol{\mathcal{A}} \cong L^2(S^1, \boldsymbol{\mathcal{A}})$ is a Hilbert *–module. We further consider polarizations $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_- = \mathsf{K}_+ \oplus \mathsf{K}_-$, for which

$$\mathsf{H}_+ = \sum_{k \geq 0} a_k z^k \ , \ \mathsf{H}_- = \sum_{k<0} a_k z^k \ , \tag{7.1}$$

where the a_k are analytic $\boldsymbol{\mathcal{A}}$–valued operator functions. We recall the flow of multiplication operators $\Gamma_+(A) = \{\exp(\sum_a t_a z^a)\}$ as in (5.1).

Before proceeding further we make a certain assumption concerning embeddings of $\widehat{\mathrm{Gr}}(p, A)$ where in referring to Proposition 2.1, we take B as the Banach algebra $B = \mathcal{L}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$, and $q \in B$ is similar to a rank 1 projection. Then on applying the spatial correspondence and the discussion on analytic embeddings in §2.3, we consider a projective embedding $\mathfrak{i} : \widehat{\mathrm{Gr}}(p, A) \longrightarrow \mathbb{P}(\mathfrak{H})$. Given such an embedding, we consider the tautological (universal) line bundle $\mathsf{L} \longrightarrow \mathbb{P}(\mathfrak{H})$, and then obtain via pull–back to $\widehat{\mathrm{Gr}}(p, A)$ a natural line bundle $\widehat{\mathsf{L}} \longrightarrow \widehat{\mathrm{Gr}}(p, A)$ where $\widehat{\mathsf{L}} = \mathfrak{i}^* \mathsf{L}$. Reasonably, we assume the existence of a global analytic section $\sigma : \widehat{\mathrm{Gr}}(p, A) \longrightarrow \widehat{\mathsf{L}}$ (this is realized in the case of the Plücker embedding as in [46] whereby $\widehat{\mathsf{L}} = \mathsf{Det}$). Whereas we do not require σ to be equivariant with respect to the action of Γ_+ on $\widehat{\mathsf{L}}$, we do require W to be transverse to H_- when viewed as the graph of a map $U : \mathsf{H}_+ \longrightarrow \mathsf{H}_-$. Also in view of the spatial correspondence, the section σ corresponds to an operator of trace class on $A = \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$.

In this case the operator cross–ratio definition of the tau function, in relationship to the big cell, which is interpreted as a twistor space in [32], takes the following form, obtained by modifying the constructions of [32] for the Hilbert module $H_{\boldsymbol{\mathcal{A}}}$. For each $W \in \widehat{\mathrm{Gr}}(p, A)$, the tau function of W is the analytic (holomorphic) operator–valued function $\tau_W : \Gamma_+^W \longrightarrow \boldsymbol{\mathcal{A}}$, defined by

$$\tau_W(g) = \frac{\sigma(g^{-1}W)}{g^{-1}\sigma(W)} , \tag{7.2}$$

and given W is transversal to H_-, we take $g^{-1} \in \Gamma_+^W$ as expressed by

$$g^{-1} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} . \tag{7.3}$$

Following the development of [32], we have

$$\tau_W(g) = \det(1 + a^{-1}b\, U) , \tag{7.4}$$

whenever 'det' on a trace–class operator on A is definable.

With the above conditions observed we proceed to establishing a relationship between the operator tau and Baker functions essentially by the means of [46]. Firstly, for $\zeta \in \mathbb{C}$, we set $q_\zeta = 1 - z\zeta^{-1} \in \Gamma_+^W$, for $|\zeta| > 1$.

Theorem 7.1. *Under the above hypotheses, we have the following relationship between the operator tau and Baker functions relative to big cell subspaces $W \in \widehat{\mathrm{Gr}}(p, A)$:*

$$\psi_W(g, \zeta) = \frac{\tau_W(g \cdot q_\zeta)}{\tau_W(g)} . \tag{7.5}$$

Proof. Following [46]§5, we start by expressing q_ζ^{-1} in the form

$$q_\zeta^{-1} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} , \tag{7.6}$$

where $b : \mathsf{H}_- \longrightarrow \mathsf{H}_+$, is given by $z^{-k} \longrightarrow \zeta^{-k} q_\zeta^{-1}$. Thus the operator ab^{-1} is of rank 1 (i.e. similar to a rank 1 projection), takes $f \in \mathsf{H}_+$ to the constant $f(\zeta)$. Thus $a^{-1}b\, U$ is of rank 1 (i.e. similar to a rank 1 projection), and we have

$$\begin{aligned} \tau_W(q_\zeta) &= \det(1 + a^{-1}b\, U) \\ &= 1 + \mathrm{Tr}\ (a^{-1}b\, U) \\ &= 1 + f_0(z) . \end{aligned} \tag{7.7}$$

Using (7.2), the right hand side of (7.5) is easily seen to be

$$\frac{\tau_W(g\cdot q_\zeta)}{\tau_W(g)} = \frac{\sigma(q_\zeta^{-1}g^{-1}W)}{q_\zeta^{-1}\sigma(g^{-1}W)} = \tau_{g^{-1}W}(q_\zeta)\ . \tag{7.8}$$

Now we recall that the operator Baker function

$$\psi_W(g,\zeta) = \sum_{s=0}^{\infty} a_s\zeta^{-s} = 1 + \sum_{s=1}^{\infty} a_s\zeta^{-s}\ , \tag{7.9}$$

for which the boundary values $|\zeta| \longrightarrow 1$ in the transverse space $g^{-1}W$. Now for such an element we deduce from the above details that

$$\tau_{g^{-1}W}(q_\zeta) = 1 + f_0(z)|g^{-1}W\ , \tag{7.10}$$

in the transverse space where for the last expression we have

$$1 + f_0(z)|g^{-1}W = 1 + \sum_{s=1}^{\infty} a_s\zeta^{-s}\ , \tag{7.11}$$

and together with (7.8) this last expression implies

$$\psi_W(g,\zeta) = \frac{\tau_W(g\cdot q_\zeta)}{\tau_W(g)}\ . \tag{7.12}$$

□

We had mentioned in §6.2 that the flatness of the connection ω_+ defines τ by $\Omega = \partial_+\partial_- \log\tau$. Note that this is the Čech version of the connection, as opposed to the Dolbeault version of [33], encoded by the (holomorphic) Birkhoff factorization of an $\mathrm{SL}(n,\mathbb{C})$-valued map over $\mathbb{P}\mathfrak{H} \times S^1$, whereby the cocycle becomes the standard cocycle γ of the loop algebra, with exterior covariant derivative $d\gamma = \Omega$.

The Pfaffian line bundle revisited

Recall that we have the Pfaffian line bundle $\mathsf{Pf} = \mathsf{L}^{1/2} \longrightarrow R = \mathrm{O}(\mathrm{H}^{\mathbb{C}}_{\boldsymbol{\mathcal{A}}})/\mathrm{U(W)}$. In a similar way to the previous description, we may consider a (canonical) section $\hat{\tau}$ of $\mathsf{Pf} \longrightarrow R$, and for $W \in R$, consider a flow of multiplication operators Ξ_W inducing C*–automorphisms of the CAR algebra $\mathfrak{A}(W)$. For suitable $g, h \in \Xi_W$, the operator–valued function $\hat{\psi}_W = \hat{\tau}(g\cdot h)\hat{\tau}(g)^{-1}$ may be considered, as well as the curvature $\widehat{\Omega}$ of a suitable (not necessarily flat) connection $\widehat{\nabla}$ on Pf . We expect a general relationship:

$$\partial_+\partial_- \log\hat{\psi}_W(g,\zeta) = \widehat{\Omega}(g\cdot h) - \widehat{\Omega}(g)\ , \tag{7.13}$$

along with another cocycle on a space of $\bar{\partial}$–operators

$$\mathsf{i}_v\mathsf{i}_u\widehat{\Omega}(g\cdot h) - \mathsf{i}_v\mathsf{i}_u\widehat{\Omega}(g) = \hat{c}(g\cdot h)(u,v) - \hat{c}(g)(u,v)\ . \tag{7.14}$$

8 Burchnall–Chaundy with matrix coefficients

8.1 Forward and backward periodic flags

In [46] the Hilbert flag manifolds $\mathcal{F}(k,n)$ were introduced (see also [23] for a comprehensive treatment). To an extent, the construction of these spaces was generalized in [24] in order to accommodate filtered subspaces of operator and matrix valued functions in analogs of the well known L^p and H^p (Hardy) spaces, for $1 \leq p \leq \infty$. Initially, one considers periodic flags $\mathsf{M}_1 \supset \mathsf{M}_2 \supset \ldots \supset \mathsf{M}_n$, in spaces such as $L^2(S^1, \mathbb{C}^n)$ and then introduces a subspace $\mathsf{M} \subset E = L^2(S^1, \mathcal{M}_{k\times n})$ expressible as

$$\mathsf{M} = [\mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_n] \ , \tag{8.1}$$

so that M consists of all E–valued functions such that the j-th column is an element of the subspace $\mathsf{M}_j \subset L^2(S^1, \mathbb{C}^k)$.

This is the initial setting for a series of so–called '$(\mathsf{M}, \mathsf{M}^\times)$ Theorems' in [24] to which we refer for details and explanation of terms. A brief account goes as follows. The subspace M (respectively, $\mathsf{M}^\times$) of E can be specified by *forward* (respectively, *backward*) periodic flags and the *block invariance* of M (respectively, the **–block invariance* of $\mathsf{M}^\times$). Let H^∞_L (respectively, H^∞_U) denote the matrix valued functions f in $\mathsf{H}^\infty(S^1, \mathcal{M}_{n\times n})$ such that $f(0)$ belongs to the algebra of lower (respectively, upper) triangular matrices. We recall here one of the main results of [24] :

Theorem 8.1. [24] (Th. 7.4) *Let $E = L^2(S^1, \mathcal{M}_{k\times n})$. Suppose that M is a block invariant and $\mathsf{M}^\times$ is a *–block invariant subspace of E such that*

$$E = \mathsf{M} \oplus \mathsf{M}^\times \ , \qquad \mathsf{M} \cap \mathsf{M}^\times = \{0\} \ . \tag{8.2}$$

Set $\nu_j = \dim \mathsf{M}_j/\mathsf{M}_{j+1}$, for $1 \leq j \leq n$ (here $\mathsf{M}_{n+1} = z\mathsf{M}_1$, and M_j is the set of all vector functions occurring in the j-th column of M). Then there is an associated multiplicity $N(\underline{\nu}) = k$ and a matrix function $g \in L^2(D, \mathcal{M}_{k\times k})$ such that

$$\mathsf{M} = \mathrm{clos}\ g\ \mathsf{H}^\infty_L \ , \ \textit{and}\ \ \mathsf{M}^\times = \mathrm{clos}\ g\ \overline{\mathsf{H}^\infty_U} \ , \tag{8.3}$$

(here the closures are taken in E). The matrix function g is uniquely determined up to an invertible $\underline{\nu} \times \underline{\nu}$ block diagonal right factor.

Remark 8.1. We observe that Theorem 8.1 is essentially a Grassmannian formalism via projection of the periodic flag. Thus on setting $A = \mathcal{L}(E)$, it can be regarded as a statement relative to $\mathrm{Gr}(p, A)$. Note also that in the formal development of the operator–valued Baker function, the holomorphic and meromorphic properties relating to the matrix/operator function g could be relaxed, and in the same way equations (5.2)–(5.5) could be re–formulated with F replaced by M (and F^c by $\mathsf{M}^\times$). Again, decreeing the big cell as that for which $p_1^g : g^{-1}(W) \longrightarrow \mathsf{M}$ is an isomorphism, we uncover the more general Baker function $\psi_W = (p_1^g)^{-1}(\mathbf{1})$. If indeed the function g is e.g. smooth or analytic (holomorphic as in [46]), then one can replace H^∞_L by H^2_L, etc. and dispense with 'clos' in Theorem 8.1. We refer to [24] (and references therein) for further specialities of this type of theorem.

In returning to the development of §5 where $g \in \Gamma_+(A)$ is holomorphic, etc., we then deduce from Theorem 8.1 that

$$E = g\ \mathsf{H}^2_L \oplus g\ \overline{\mathsf{H}^2_U} \ . \tag{8.4}$$

This analytic side of the picture can be related to commutative subrings $\mathbb{A}$ of $\mathbb{B}[\partial]$, where now $\mathbb{B}[\partial^{-1}]$ is taken to be the algebra of formal pseudodifferential operators with coefficients in a *complex matrix algebra* $\underline{\mathbb{B}}$. So we may consider an operator $L \in \mathbb{A} \subset \mathbb{B}[\partial^{-1}]$ as given by

$$L = \partial + \sum_{i > -\infty}^{-1} \mathsf{a}_i \, \partial^i \ , \tag{8.5}$$

where the a_i belong to some associated complex matrix ring. Applying the generalized Sato correspondence as in [20] §6, immediately leads to the following :

Theorem 8.2. *The subring $\mathbb{A}$ of $\mathbb{B}[\partial]$ conjugates into the Banach algebra $A = \mathcal{L}(E)$ as a commutative subring up to constant coefficient operators.*

8.2 Applications

Integrable matrix hierarchies, on the other hand, have been extensively studied (cf. [8, 14, 25] for surveys of theoretical nature), together with their reductions, which can be viewed as periodic flags in a vector Grassmannian. Upon our enhancing these Grassmannians by Banachization of the coefficients, we can ask for a relationship between the Riccati equation which unfolds mKdV into KdV, e.g., and the Riccati flow on the Grassmannian which naturally pertains to the cross-ratio [53].

9 The matrix C*–algebra case following Krichever

Let $\boldsymbol{\mathcal{A}}$ now be a (unital) commutative C*–algebra. Following [27], we consider a system of nonlinear equations in the coefficients of the operators

$$L_1 = \sum_{\alpha=0}^{n} u_\alpha(x) \ \frac{d^\alpha}{dx^\alpha} \ , \ L_2 = \sum_{\beta=0}^{n} v_\beta(x) \ \frac{d^\beta}{dx^\beta} \ , \tag{9.1}$$

which is equivalent to the commutativity condition $[L_1, L_2] = 0$ Here we take $u_\alpha, v_\beta \in \mathcal{M}_n(\boldsymbol{\mathcal{A}})$ as matrix C*–algebra valued functions (suitably normalized). In this setting the leading coefficients are treated as similar to scalar case. These are given by constant invertibles $u_n^{ij}(x) = c_i \delta_{ij} \ \mathbf{1}$ and $v_m^{ij}(x) = b_i \delta_{ij} \ \mathbf{1}$. Also, for those pairs (i, j) for which $c_i = c_j$, we set $u_{n-1}^{ij}(x) = 0$. Let $C = \{(i, j)\}$ denote the set of such pairs.

Thus we will be considering a commutative subring of operators $\mathbb{A} \subset \mathbb{B}[\frac{d}{dx}] \otimes \boldsymbol{\mathcal{A}}$. As far as the choice of the (unital) Banach algebra A is concerned, the previous general development of ideas affords us some scope. Typically, one might consider the Hilbert modules $H_{\boldsymbol{\mathcal{A}}}$ discussed in §2.6 and then take $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$; there is not much loss of generality in assigning A accordingly. In keeping with the development and application here of §2.8 and §5, let E denote the underlying Banach space of A and let $\zeta \in G(A) \subset A$, be a Laurent series generator for A .

Similar to the case for the original Baker function ψ_W, the commutativity of the operators L_1, L_2, is seen to correspond to the existence of a class of joint eigenfunctions. In view of §2.8 and §5 we consider formal solutions to the equation

$$L_1 \ \psi_W(x, \zeta) = \zeta^n \ \psi_W(x, \zeta) \ u_n \ , \tag{9.2}$$

where for $W \in \widehat{\mathrm{Gr}}(p, A)$, $W = gh_g F$, the function ψ_W is defined in (5.5) of Definition 5.1 and where we now we set $g(x, \zeta) = \exp[\zeta(x - x_0)]$, so that (5.6) becomes

$$\psi_W(x, \zeta) = (\sum_{s=0}^{\infty} a_s(x)\ \zeta^{-s})\ e^{\zeta(x-x_0)}\ . \tag{9.3}$$

Here the $a_s(x)$ are A–valued matrix functions which can be determined successively by equation coefficients of ζ^{-s} in (9.2). This results in the following :

Lemma 9.1. *There exists a unique formal solution of* (9.2) *given by* $\psi_W(x, \zeta, x_0)$ *which satisfies the normalization conditions* $a_0^{ij} = \delta_{ij}\ \mathbf{1}$, $a_s^{ij}(x_0) = 0$, $s \geq 1$.

Proof. Following [27], equating coefficients of ζ^{-s} on both sides of (9.2), yields the equation

$$\sum_{\alpha=0}^{n} \sum_{k=0}^{\alpha} C_\alpha^k\ \frac{\partial^{\alpha-1}}{\partial x^{\alpha-1}}\ a_{s+k} = a_{s+n}\ u_n\ , \tag{9.4}$$

which can be seen to be expressible in the form

$$0 = [u_n, a_{n+s}] + n u_n\ \frac{\partial}{\partial x}\ a_{n+s-1} + (\text{terms in } a_j\ ,\ j < n + s - 1)\ . \tag{9.5}$$

Thus from the s-th equation, we can determine an element $a_{s+n}^{ij}(x)$ for which $(i, j) \notin C$, and elements $\frac{\partial}{\partial x}\ a_{s+n-1}^{ij}$ if $(i, j) \in C$. □

Lemma 9.2. *The series* (9.3) *is a solution of* (9.2) *if and only if it can be represented in the form*

$$\psi_W(x, \zeta) = \psi_W(a, \zeta, x_0)\ \Lambda(\zeta, x_0)\ , \tag{9.6}$$

where the series

$$\Lambda(\zeta, x_0) = \sum_{s=0}^{\infty} \lambda_s(x_0)\ \zeta^{-s}\ , \tag{9.7}$$

admits non–zero A–valued matrix functions $\lambda_s^{ij}(x_0)$ *only for pairs* $(i, j) \in C$.

Proof. We verify that the series of (9.6) satisfies (9.2). Effectively,

$$\begin{aligned} L_1\ \psi_W(x, \zeta) &= \zeta^n \psi_W(a, \zeta, x_0)\ u_n\ \Lambda(\zeta, x_0) \\ &= \zeta^n \psi_W(a, \zeta, x_0)\ \Lambda(\zeta, x_0)\ u_n \\ &= \zeta^n \psi_W(x, \zeta)\ u_n\ , \end{aligned} \tag{9.8}$$

since from the key observation of (9.5), it can be deduced that $[\Lambda(\zeta, x_0), u_n] = 0$. □

Next, we assume that all the $c_i \neq c_f$ if $i \neq j$, so that $C = \{(i, i)\}$.

Theorem 9.1. *Let* $\mathbb{A} \subset \mathbb{B}[\frac{d}{dx}] \otimes \boldsymbol{\mathcal{A}}$ *be a subring of operators which commute with a given operator. Then* $\mathbb{A}$ *conjugates into the Banach algebra* A *as a commutative subring up to constant coefficient operators.*

Proof. Again we follow [27] with the necessary modifications. If $[L_1, L_2] = 0$, then we have

$$\begin{aligned} L_1 L_2 \ \psi_W(x,\zeta,x_0) &= L_2 L_1 \ \psi_W(x,\zeta,x_0) \\ &= \zeta^n L_2 \ \psi_W(x,\zeta,x_0) \ u_n \ , \end{aligned} \tag{9.9}$$

which shows that the series $L_2 \ \psi_W(x,\zeta,x_0)$ satisfies (9.2). On the other hand, if this is the case we also have from (9.9)

$$L_2 \psi_W(x,\zeta,x_0) = \psi_W(x,\zeta,x_0)\Lambda(\zeta,x_0) \ . \tag{9.10}$$

Now $\psi_W(x,\zeta,x_1)e^{\zeta(x_1-x_0)}$ is of the form (9.3) and also satisfies (9.2), so that we obtain

$$\psi_W(x,\zeta,x_1)e^{\zeta(x_1-x_0)} = \psi_W(x,\zeta,x_0)\Theta(\zeta,x_0) \tag{9.11}$$

where the latter leads to

$$\begin{aligned} \Lambda(\zeta,x_1) &= \Theta^{-1}(\zeta,x_0)\Lambda(\zeta,x_0)\Theta(\zeta,x_0) \\ &= \Lambda(\zeta,x_0) \ , \end{aligned} \tag{9.12}$$

where we note that both series admit diagonal A–valued matrices as coefficients. This proves the conjugacy part. Observe also that (for sufficiency), since we have

$$L_1 L_2 \ \psi_W(x,\zeta,x_0) = \zeta^n \psi_W(x,\zeta,x_0)\Lambda(\zeta) \ u_n \ , \tag{9.13}$$

it follows that the condition $[L_1, L_2] \ \psi_W(x,\zeta,x_0) = 0$, suffices for $[L_1, L_2] = 0$.

Finally, we show that the ring of operators which commute with the given operator, is indeed commutative. Suppose that $[L_1, L_2] = 0$ and $[L_1, L_3] = 0$, then we have

$$\begin{aligned} L_2 \ \psi_W(x,\zeta,x_0) &= \psi_W(x,\zeta,x_0)\Lambda_1(\zeta) \\ L_3 \ \psi_W(x,\zeta,x_0) &= \psi_W(x,\zeta,x_0)\Lambda_2(\zeta) \\ 0 &= [L_2, L_3] \ \psi_W(x,\zeta,x_0) \ , \end{aligned} \tag{9.14}$$

which implies $[L_2, L_3] = 0$. □

10 Quantized connections

We recall the commutative subring of operators $\mathbb{A}$ with (smooth) matrix C*–algebra coefficients. Following a similar development to [42, 46], we associate the commutative subring of operators $\mathbb{A}$ with (smooth) matrix C*–algebra coefficients and for which $X = \mathrm{Spec}(\mathbb{A})$ is a complete irreducible complex curve over $\mathbb{C}$. We next discuss some analytic data over X, in part generalizing that of [27, 42, 46].

Commencing with a holomorphic vector bundle $V \longrightarrow X$, we tensor with a complex C*–algebra $\boldsymbol{\mathcal{A}}$ so that $\mathcal{V} = V \otimes \boldsymbol{\mathcal{A}} \longrightarrow X$ becomes a holomorphic Banach bundle (for a description of holomorphic Banach bundles, see e.g. [7, 18]). Let x_∞ be a smooth point of X and z the inverse of a local parameter on X at x_∞, where z is used to identify a neighborhood of x_∞ in X with a neighborhood of the disk $D_\infty = \{z : |z| \geq 1\}$ in the Riemann sphere. We assume an analytic trivialization φ of $\mathcal{V}$ over some neighborhood of $D_\infty \subset X$. The next step is to consider L^2–boundary values of $\mathcal{V}$ over X/D_∞ and use φ to identify sections of $\mathcal{V}$ over S^1 with matrix–valued complex functions in $\boldsymbol{\mathcal{A}}$. We thus obtain the Hilbert module $H_{\boldsymbol{\mathcal{A}}} = L^2(S^1, \mathcal{M}_{k\times n}) \otimes \boldsymbol{\mathcal{A}}$, from which we form the Banach algebra $A = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ as before.

We proceed to relate this data to points in the Grassmannian $\mathrm{Gr}(p,A)$, but in this case we introduce the idea of a *quantized connection*. Consider the (smooth) C*–algebra $\mathcal{F} = C^{\infty}(S^1, \mathcal{M}_{m\times n})$ along with a $*$–representation $\Pi : \mathcal{F} \longrightarrow A$. To $\mathcal{F}$ we associate a *Fredholm module* $(H_{\boldsymbol{\mathcal{A}}}, T)$, where $T \in A$ is a self–adjoint operator satisfying $T^2 = 1$, and where for $a \in \mathcal{F}$,

$$[T, \Pi(a)] \in \mathcal{K}(H_{\boldsymbol{\mathcal{A}}}) \ , \tag{10.1}$$

(for properties of compact and Fredholm operators over Hilbert modules, see e.g. [4]).

Next let $\mathcal{E} \to \mathcal{F}$ be a finite projective module. Following [10, 11], quantized k–forms on $\mathcal{F}$ consist of the set

$$\Omega^k_{\mathcal{F}} = \{\sum a^0 da^1 \cdots da^k \ , \ a^j \in \mathcal{F} \ , \ da = J[T, \Pi(a)] \ \} \ , \tag{10.2}$$

recalling that $J \in \mathrm{End}(H_{\boldsymbol{\mathcal{A}}})$ satisfies $J^2 = -\mathrm{Id}$. A quantized connection ∇ on $\mathcal{F}$ is given by a linear map

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{F}} \Omega^1_{\mathcal{F}} \ , \tag{10.3}$$

satisfying the Leibniz relation

$$\nabla(\xi \cdot u) = (\nabla \xi)u + \xi \otimes du \ , \tag{10.4}$$

where $\xi \in \mathcal{E}$, $u \in \mathcal{F}$, and as before, $du = J[T, \Pi(u)]$. Let $(\mathcal{E}, \nabla_\alpha) \to \mathcal{F}$ be a finite projective module with its quantized connection $\nabla_\alpha = \nabla \cdot \mathbf{1} = \alpha \in \Omega^1_{\mathcal{F}}$. Similar to [11](Th. 9), the following proposition relates the flatness of ∇_α to points in the Grassmannian $\widehat{\mathrm{Gr}}(p,A)$ and the invariant subspaces of projections in $H_{\boldsymbol{\mathcal{A}}}$.

Proposition 10.1. *Let $(H_{\boldsymbol{\mathcal{A}}}, T)$ be a Fredholm module over $\mathcal{F}$ as above. Further, let $(\mathcal{E}, \nabla_\alpha) \longrightarrow \mathcal{F}$ be a finite projective module with $\alpha \in \Omega^1_{\mathcal{F}}$ and consider the projection $p = \frac{1}{2}(1+T) \in P(A)$. Then*

(1) $q = p - \frac{1}{2}J\alpha \in \widehat{\mathrm{Gr}}(p,A)$ if and only if ∇_α is a flat connection (i.e. has zero–curvature).

(2) p and q are mutually orthogonal projections if and only if $\alpha p = p\alpha$, i.e. α commutes with p .

(3) Let $W = \mathrm{Im}\ p \subset H_{\boldsymbol{\mathcal{A}}}$. Then W is a invariant subspace for q if and only if W is an invariant subspace for α .

Proof. Firstly, from (10.1) and (10.2) (for $k = 1$) we see that $\frac{1}{2}J\alpha \in \mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$. From the description of $\widehat{\mathrm{Gr}}(p,A)$ in §2.8 we show that the flatness of ∇_α is equivalent to $q \in P(A)$. Using the definition of q and p, it is easily checked that

$$q^2 = p - \frac{1}{2}J\alpha - \frac{1}{4}(J(T\alpha + \alpha T) + \alpha^2) \ . \tag{10.5}$$

Also, the curvature R_∇ of ∇_α is computed to be

$$R_\nabla = J(\alpha T + T\alpha) + \alpha^2 \ , \tag{10.6}$$

so it is clear that $R_\nabla = 0$ if and only if $q^2 = q = p - \frac{1}{2}J\alpha$, and this proves (1) (cf [11]Th. 9).

For (2), the condition $pq = qp = 0$ implies

$$p = \frac{1}{2}J\ p\alpha = \frac{1}{2}J\ \alpha p\ , \tag{10.7}$$

and the conclusion follows. Recall (cf [15]§4) that if W is an invariant subspace for q, then we must have $pqp = qp$. On checking both sides we have

$$\begin{aligned} pqp &= p^3 - \frac{1}{2}J\ p\alpha p = p - \frac{1}{2}J\ p\alpha p\ , \\ qp &= p - \frac{1}{2}J\ \alpha p\ . \end{aligned} \tag{10.8}$$

So (3) follows when $p\alpha p = \alpha p$, which implies that W is an invariant subspace for α . □

11 Appendix: Representation on Fock space and the CAR algebra

For H a separable (complex) Hilbert space with fixed choice of unitary structure, consider the associated Fock space

$$F(H) = \bigoplus_{n\geq 0} \Lambda^n(H)\ , \qquad \Lambda^0(H) = \mathbb{C}\ ,\ \Lambda^1(H) = H\ , \tag{11.1}$$

with vacuum vector Υ corresponding to $1 \in \mathbb{C}$. For each $v \in H$, we define a *creation* operator $c(v) \in \mathcal{L}(F(H))$ as follows. For $\zeta_1, \ldots, \zeta_n \in H$,

$$c(v)(\zeta_1 \wedge \ldots \wedge \zeta_n) = v \wedge \zeta_1 \wedge \ldots \wedge \zeta_n\ , \qquad c(v)(\alpha\Upsilon) = \alpha v\ . \tag{11.2}$$

Correspondingly, we define an *annihilation* operator $a(v) \in \mathcal{L}(F(H))$ as follows:

$$a(v)\Upsilon = 0\ , \qquad a(v)(w) = \ < w, v > \Upsilon\ ,\ w \in \Lambda^1(H)\ , \tag{11.3}$$

and for $w_0, \ldots, w_n \in H$,

$$a(v)(w_0 \wedge \ldots \wedge w_n) = \sum_{j=0}^{n} (-1)^j < w_j, v > w_0 \wedge \ldots \wedge \hat{w}_j \wedge \ldots \wedge w_n\ , \tag{11.4}$$

with $\bigcap \mathrm{Ker}\ a(v) = \mathbb{C} \cdot \Upsilon$, the orbit of the vacuum state. The operators $c(v)$ and $a(v)$ satisfy the well known anti–commutation relations

$$\begin{aligned} \{c(v), a(w)\} &= \ < v, w > I\ , \\ \{c(v), c(w)\} &= \ \{a(v), a(w)\} = 0\ , \end{aligned} \tag{11.5}$$

leading to the CAR algebra $\mathfrak{A}(H)$. For each $v \in H$, define $\pi(v) \in \mathcal{L}(F(H))$ by the self–adjoint operator $\pi(v) = c(v) + a(v)$ which satisfies $\pi(v)^2 = \|v\|^2 I$. If $\mathrm{Cl}(H)$ denotes the Clifford C*–algebra of H, then there is a unique (isometric) map $\pi_F : \mathrm{Cl}(H) \to \mathcal{L}(F(H))$ which is universal with respect to the diagram

$$\begin{array}{ccc} \mathrm{Cl}(H) & \xrightarrow{=} & \mathrm{Cl}(H) \\ \uparrow & & \downarrow \pi_F \\ H & \xrightarrow{\pi} & \mathcal{L}(F(H)) \end{array} \tag{11.6}$$

and for which $\pi_F = \pi$ (see e.g. [39]). Next, consider now the following closed orthogonal subspaces of $F(H)$,

$$F^+(H) = \mathrm{clos}\{\bigoplus_{m\geq 0} \Lambda^{2m}(H)\} \ , \qquad F^-(H) = \mathrm{clos}\{\bigoplus_{m\geq 0} \Lambda^{2m+1}(H)\} \ , \tag{11.7}$$

together with a symmetry $S \in \mathcal{L}(F(H))$ satisfying

$$S|F^\pm(H) = \pm I \ , \qquad F^\pm(H) = (I \pm S)F(H) \ . \tag{11.8}$$

A grading automorphism $\gamma : \mathrm{Cl}(H) \to \mathrm{Cl}(H)$, yields a decomposition

$$\mathrm{Cl}(H) = \mathrm{Cl}^+(H) \oplus \mathrm{Cl}^-(H) \ , \qquad \gamma| \ \mathrm{Cl}^\pm(H) = \pm I \ . \tag{11.9}$$

If $a \in \mathrm{Cl}(H)$, then $\pi(\gamma a) = S\pi(a)S$. In particular, it is straightforward to see that if $a \in \mathrm{Cl}^+(H)$, then $\pi(a)$ commutes with S and so leaves the closed subspaces $F^\pm(H)$ invariant. From this, it can be deduced that there are the even (+) and odd (−) Fock representations

$$\pi^\pm \ : \ \mathrm{Cl}^+(H) \longrightarrow \mathcal{L}(F^\pm(H)) \ , \tag{11.10}$$

that are both irreducible. The double commutant $\pi(\mathrm{Cl}^+(H))''$ generated by the image of $\mathrm{Cl}^+(H)$ under π, is a von Neumann algebra in $\mathcal{L}(F(H))$ for which $\pi(\mathrm{Cl}^+(H))'' = \{S\}'$.

Acknowledgements. E.P. most gratefully acknowledges support by the Institut Mittag-Leffler (Djursholm, Sweden) during her participation in the *Moduli Spaces* Program, spring 2007.

References

[1] M. F. Atiyah: Eigenvalues of the Dirac operator. *Lect. Notes in Math.* **1111**, 251–260, Springer Verlag 1985.

[2] D. Beltiţă: *Smooth homogeneous structures in operator theory*, Monographs and Surveys in Pure and Appl. Math. **137**, Chapman and Hall/CRC, Boca Raton Fl. 2006.

[3] S. J. Bhatt, H.V. Dedania and S. R. Patel: Fréchet algebras with a Laurent series generator and the annulus algebras. *Bull. Austral. Math. Soc.* **65** No. 3, (2002), 371–383.

[4] B. Blackadar: *K–Theory for Operator Algebras*. Springer Verlag 1986.

[5] D. Borthwick: The Pfaffian line bundle. *Commun. Math. Phys.* **149** (1992) (3), 463–493

[6] R. Bott and S. S. Chern: Hermitian vector bundles and the equidistribution theory of the zeros of their holomorphic sections. *Acta Math.* **114** (1965), 71–112.

[7] L. Bungart: On analytic fiber bundles I. Holomorphic fiber bundles with infinite dimensional fibers. *Topology* **7** (1968), 55–68.

[8] I. Cherednik: *Basic methods of soliton theory*. Translated from the Russian by Takashi Takebe. Advanced Series in Mathematical Physics, 25. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[9] S. S. Chern: *Complex Manifolds Without Potential Theory* (2nd Ed.) Springer Universitext 1979.

[10] A. Connes: *Noncommutative Geometry.* Academic Press 1994.

[11] A. Connes: The action functional in non–commutative geometry. *Commun. Math. Phys.* **117** (1988), 673–683.

[12] G. Corach, H. Porta and L. Recht: Differential geometry of systems of projections in Banach algebras. *Pacific J. Math.* **143** (2), (1990), 209–228.

[13] R. Curto: Taylor joint spectrum. *Encyclopedia of Mathematics*, Suppl. III. Edited by M. Hazewinkel. D. Reidel Publishing Co., Dordrecht, 2001.

[14] L. A. Dickey: *Soliton equations and Hamiltonian systems.* Second edition. Advanced Series in Mathematical Physics, 26. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[15] R. G. Douglas: *Banach Algebra Techniques in Operator Theory* (2nd Ed.). Graduate Texts in Mathematics **179**, Springer Verlag, New York–Berlin, 1998.

[16] M. J. Dupré and R. M. Gillette: *Banach Modules and Automorphisms of C*-Algebras.* Pitman Research Notes in Math. **92**, Pitman Publ. New York 1983.

[17] M. J. Dupré and J. F. Glazebrook: The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* **41** No. 3, (2001), 264–287.

[18] M. J. Dupré and J. F. Glazebrook: Holomorphic framings for projections in a Banach algebra. *Georgian Math. J.* **(**2002) No. 3, 481–494.

[19] M. J. Dupré and J. F. Glazebrook: Relative inversion and embeddings. *Georgian Math. J.* **11** No. 3, (2004), 425–448.

[20] M. J. Dupré, J. F. Glazebrook and E. Previato: A Banach algebra version of the Sato Grassmannian and commutative rings of differential operators. *Acta Applicandae Math.* **92** (3) (2006), 241–267.

[21] J. Eschmeier and M. Putinar: *Spectral Decompositions and Analytic Sheaves.* LMS Monographs New Series **10**, Clarendon Press, Oxford UK, 1996.

[22] B. Gramsch: Relative Inversion in der Störungstheorie von Operatoren und Ψ–Algebren. *Math. Ann.* **269**, (1984), 27–71.

[23] G. F. Helminck and A. G. Helminck: Hilbert flag varieties and their Kähler structure. *J. Phys. A: Math. Gen.* **35** (2002), 8531–8550.

[24] J. W. Helton: *Operator Theory, Analytic Functions, Matrices and Electrical Engineering.* CBMS Reg. Conf. Ser. **68**, Amer. Math. Soc., Rhode Island 1987.

[25] V. G. Kac and D. H. Peterson: Lectures on the infinite wedge-representation and the MKP hierarchy. *Systèmes dynamiques non linéaires: intégrabilité et comportement qualitatif*, 141–184, Sém. Math. Sup., 102, Presses Univ. Montréal, Montreal, QC, 1986.

[26] G. Khimshiashvili: Homotopy classes of elliptic transmission problems over C*–algebras. *Georgian Math. J.* **5** No. 5, (1998), 453–468.

[27] I. M. Krichever: Integration of nonlinear equations by the methods of algebraic geometry. *Funkcional. Anal. i Priložen.* **11** No. 1 (1977), 15–31.

[28] A. Kriegl and P. W. Michor: *The Convenient Setting of Global Analysis. Math. Surveys and Monographs* **53**, *Amer. Math. Soc.* 1997.

[29] E. C. Lance: *Hilbert C*–Modules. London Math. Soc. Lect. Notes* **210**, Cambridge Univ. Press, 1995.

[30] N. P. Landsman: *Mathematical Topics between Classical and Quantum Mechanics.* Springer Verlag, New York, 1998.

[31] V. M. Manuilov and E. V. Troitsky: *Hilbert C*–modules*, Trans. Math. Monographs **226**, Amer. Math. Soc., 2005.

[32] L. J. Mason, M. A. Singer and N. M. J. Woodhouse: Tau functions and the twistor theory of integrable systems. *J. Geom. and Phys.* **32**, (2000), 397–430.

[33] L. J. Mason, M. A. Singer and N. M. J. Woodhouse: Tau functions, twistor theory and quantum field theory. *Commun. Math. Phts.* **230**, (2002), 389–420.

[34] J. Mickelsson: *Current Algebras and Groups.* Plenum Press, New York–London, 1989.

[35] G. J. Murphy: *C*–Algebras and Operator Theory.* Academic Press 1990.

[36] A. Mishchenko and A. Fomenko: The index of elliptic operators over C*–algebras. (Russian) *Izv. Akad. Nauk SSSR, Se. Mat.* **43**, (1979), 831–859.

[37] T. Miwa, M. Jimbo and E. Date: *Solitons–Differential equations, symmetries and infinite dimensional algebras. Cambridge Tracts in Math.* **135**, Cambridge Univ. Press, 2000.

[38] M. Mulase: Category of vector bundles on algebraic curves and infinite dimensional grassmannians. *Internat. J. Math.* **1**(3) (1990), 293–342.

[39] R.J. Plymen and P.L. Robinson : *Spinors in Hilbert Space*, Cambridge Tracts in Math. **114**, Cambridge University Press 1994.

[40] H. Porta and L. Recht: Spaces of projections in a Banach algebra. *Acta Cient. Venez.* **38**, (1987), 408–426.

[41] A. Pressley and G. Segal: *Loop Groups and their Representations.* Oxford Univ. Press, 1986.

[42] E. Previato and G. Wilson: Vector bundles over curves and solutions of the KP equations. *Proc. Sympos. Pure Math.* **49** Part 1., *Amer. Math. Soc.*, 1989, 553–569.

[43] E. Previato: Multivariable Burchnall–Chaundy theory, *Philos. Trans. Roy. Soc. London*, to appear.

[44] A. Rennie and J. C. Várilly: Reconstruction of manifolds in noncommutative geometry, arxiv:math.OA/0610418

[45] M. Sato: The KP hierarchy and infinite dimensional Grassmann manifolds. Ibid [42].

[46] G. Segal and G. Wilson: Loop groups and equations of KdV type. *Publ. Math. IHES* **61**, (1985), 5–65.

[47] J. F. Smith: The p-classes of a Hilbert module, *Proc. Amer. Math. Soc.* **36** (2) (1972), 428–434.

[48] M. Spera: A C*–algebraic approach to determinants and Pfaffians. *Acta Cosmologica* Fasc. XXI-2, (1995), 203–208.

[49] M. Spera and T. Wurzbacher: Determinants, Pfaffians and quasi–free representations of the CAR algebra, *Rev. Math. Phys.* **10** (5) (1998), 705–721.

[50] J. L. Taylor: A joint spectrum for several commuting operators. *J. Funct. Anal.* **6** (1970), 172–191.

[51] L. Waelbroeck: *Topological Spaces and Algebras. Lect. Notes in Math.* **230**, Springer Verlag, Berlin, 1971.

[52] M. G. Zaidenberg, S. G. Krein, P. A. Kushment and A. A. Pankov: Banach bundles and linear operators. *Russian Math. Surveys* **30** No. 5, (1975), 115–175.

[53] M. I. Zelikin: On the theory of the matrix Riccati equation. *Mat. Sb.* **182** (1991), no. 7, 970–984; translation in *Math. USSR-Sb.* **73** (1992), no. 2, 341–354.

[54] M. I. Zelikin: Geometry of operator cross ratio. *Math. Sbornik* **197** (1) (2006), 39–54.