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EXTENSION OF SYMMETRIES ON EINSTEIN MANIFOLDS WITH BOUNDARY

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Abstract. We investigate the isometry extension property for Einstein metrics on manifolds with boundary; namely when Killing fields of the boundary metric extend to Killing fields of any filling Einstein metric. Applications to Bartnik's static extension conjecture are also discussed.

1. Introduction.

Let M^{n+1} be a compact $(n+1)$-dimensional manifold-with-boundary, and suppose g is a (Riemannian) Einstein metric on M, so that

$$Ric_g = \lambda g, \tag{1.1}$$

for some constant $\lambda \in \mathbb{R}$. The metric g induces a Riemannian boundary metric γ on ∂M. In this paper we consider the issue of whether isometries of the boundary structure $(\partial M, \gamma)$ necessarily extend to isometries of any filling Einstein manifold (M, g).

In general, without any assumptions, this isometry extension property will not hold. It is false for instance if ∂M is not connected. For example, let $M = S^3 \setminus (B_1 \cup B_2)$, where B_i are a pair of disjoint round 3-balls in S^3 endowed with a round metric; then a generic pair of Killing fields X_i on $S_i^2 = \partial B_i$ does not extend to a Killing field on M. Also, setting $M = T^3 \setminus B$ where B is a round 3-ball in a flat 3-torus T^3, one sees again that Killing fields on ∂M do not extend to Killing fields on T^3. This is due to the fact that $\pi_1(\partial M)$ does not surject onto $\pi_1(M)$. Both situations above can be remedied by making the topological assumption

$$\pi_1(M, \partial M) = 0, \tag{1.2}$$

so we will usually assume (1.2).

However, this condition is still not sufficient. Consider for example the flat product metric on $S^1 \times \mathbb{R}^2$. Let σ be any simple closed curve in $\mathbb{R}^2$ and let $T_\sigma = S^1 \times \sigma \subset S^1 \times \mathbb{R}^2$. Then T_σ bounds a compact domain $M \subset S^1 \times \mathbb{R}^2$, diffeomorphic to a solid torus. Any such T_σ is flat with respect to the induced metric, and so has a pair of orthogonal Killing fields. One of these, that tangent to the S^1 factor, clearly extends to a Killing field of M, (in fact $S^1 \times \mathbb{R}^2$). However, whenever σ is not a round circle in $\mathbb{R}^2$, (so that σ has non-constant geodesic curvature), the orthogonal Killing field on (T_σ, γ) tangent to σ does not extend as a Killing field to M.

Very similar examples are easily constructed via the Hopf fibration in the sphere $\mathbb{S}^3$, with M again a solid torus in S^3, as first pointed out to the author by H. Rosenberg [20], cf. [8] and references therein for detailed discussion. Similar examples also occur in hyperbolic space $\mathbb{H}^3$, and in higher dimensions by taking products.

The first main result describes at least one circumstance where the isometry extension property does hold. Let H denote the mean curvature of ∂M in (M, g).

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Theorem 1.1. *Let g be a $C^{m,\alpha}$ Einstein metric on M, $m \geq 3$, with $\lambda \leq 0$, with induced boundary metric γ on ∂M, and suppose* (1.2) *holds. Then any Killing field X on $(\partial M, \gamma)$ for which $X(H) = 0$, extends uniquely to a Killing field on (M, g).*

It follows for instance that for $H = const$, the identity component $Isom_0(\partial M, \gamma)$ of the isometry group of $(\partial M, \gamma)$ embeds in the isometry group of any $\lambda \leq 0$ Einstein filling metric (M, g):

$$Isom_0(\partial M, \gamma) \hookrightarrow Isom_0(M, g),$$

or equivalently, such isometries of the boundary extend to isometries of any Einstein filling metric. We expect this result also holds for $\lambda > 0$, cf. Remark 5.2.

A simple consequence of Theorem 1.1 is for example the following rigidity result.

Corollary 1.2. *Let g be a $C^{3,\alpha}$ Einstein metric on M^{n+1} which induces the round metric γ_{+1} on the boundary $\partial M = S^n$, $n \geq 2$. If $\pi_1(M) = 0$, $\lambda \leq 0$ and $H = const$, then modulo rescalings, (M, g) is isometric to a standard round ball in a simply connected space form $\mathbb{H}^{n+1}$ or $\mathbb{R}^{n+1}$.*

There are natural analogs of these results valid for exterior domains. Thus, let M^{n+1} be an open or non-compact manifold with compact "inner" boundary and with a finite number of non-compact ends. Metrically, consider complete metrics g on M which are asymptotically (locally) flat on each end. In this context, Theorem 1.1 also holds for Einstein metrics, cf. Proposition 5.4.

Another motivation for this paper is to prove similar results for the static vacuum Einstein equations, in particular in relation to a conjecture of Bartnik, the so-called static extension conjecture [5, 6]. Such metrics correspond to pairs $(g, u) \in Met^{m,\alpha}(M) \times C^{m,\alpha}_{+}(M)$ for which the warped product metric

$$g_N = u^2 d\theta^2 + g_M,$$

on $N = S^1 \times M$ is Ricci-flat, (or more generally Einstein), on N.

We then have the following uniqueness or rigidity result.

Theorem 1.3. *Suppose (M^{n+1}, g_M, u) is a complete solution to the static vacuum Einstein equations, with compact boundary ∂M, satisfying* (1.2)*, and with a finite number of asymptotically Euclidean ends.*

Suppose $(\partial M, \gamma_M)$ is a round metric on S^n, and the mean curvature H of ∂M satisfies $H = const \neq 0$. Then (M, g_M, u) is a domain in the standard spherically symmetric Schwarzschild metric g_m, of mass $m \in \mathbb{R}$, given by

$$g_m = (1 - \frac{2m}{r^{n-1}})^{-1} dr^2 + r^2 g_{S^n(1)}, \quad r^{n-1} \geq \max(2m, 0).$$

As is explained in more detail in §5, Theorem 1.3 may be viewed as an analog, and in a certain sense a strengthening, of the well-known black hole uniqueness theorem for static vacuum solutions, cf. [14], [9], [13]. We also prove results analogous to those above for the linearized Einstein (or static vacuum) equations; this leads to an improvement of a result of P. Miao [16] on the existence of static vacuum solutions with prescribed boundary data near standard flat data, cf. Corollary 6.9 and Remark 6.10.

We point out that the results above remain valid without the hypothesis (1.2) provided (M, g) is embedded as a domain in a complete, simply connected Einstein manifold $(\hat{M}, \hat{g})$. It should also be noted that the isometry extension property is false for isometries not contained in $Isom_0(\partial M, \gamma)$. As a simple example, consider a flat metric on a solid torus $M = D^2 \times S^1$ of the form

$$g_0 = dr^2 + r^2 d\theta_1^2 + d\theta_2^2,$$

for $r \in [0, 1]$. Then interchanging the two circles parametrized by θ_1 and θ_2 is an isometry of the boundary, which does not extend to an isometry of the solid torus. Of course ∂M has constant mean curvature in (M, g_0).

The proofs of the results above follow from a study of the global properties of the space of Einstein metrics g on M. As shown in [2], the moduli space $\mathcal{E}$ of such metrics is a smooth Banach manifold, for which the (Dirichlet) map to the boundary metrics

$$\Pi : \mathcal{E} \to Met(\partial M), \quad \Pi(g) = g_{T(\partial M)}, \tag{1.3}$$

is C^∞ smooth, cf. Theorem 2.1. The main results are then quite simple to prove when the metric (M, g) is non-degenerate, in the strong sense that the derivative $D\Pi$ of Π at g has trivial kernel, cf. Remark 3.3. They also hold, with somewhat more involved proofs, when $D\Pi$ has no cokernel, (or when $ImD\Pi$ is dense in $TMet(\partial M)$). In both of the situations above, the results hold without any condition on the mean curvature, i.e. without assuming $X(H) = 0$. The proof of Theorem 1.1 and Corollary 1.2 in general follows from a careful study of the Bianchi-gauged Einstein operator.

A brief survey of the contents of the paper is as follows. In §2, we introduce the basic setting and structural results on the space of Einstein metrics, needed for the work to follow. Section 3 studies elliptic boundary value problems and Fredholm properties of the boundary map Π in (1.3). Section 4 relates the isometry extension property with the (linearized) constraint equations induced by the Einstein equations on ∂M. In §5, we prove Theorem 1.1 and Corollary 1.2, while the static vacuum Einstein equations are then studied in Section 6.

2. The Space of Einstein Metrics

As above, let M denote a connected, compact, oriented $(n+1)$-dimensional manifold with compact, non-empty boundary ∂M. Consider the Banach space

$$Met(M) = Met^{m,\alpha}(M) \tag{2.1}$$

of Riemannian metrics on M which are $C^{m,\alpha}$ smooth up to ∂M. Here m is any fixed integer with $m \geq 2$, including $m = \infty$, (giving a Fréchet space), and $\alpha \in (0, 1)$. Let

$$\mathbb{E} = \mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M) \tag{2.2}$$

be the subset of Einstein metrics on M, $C^{m,\alpha}$ smooth up to ∂M, with

$$Ric_g = \lambda g, \tag{2.3}$$

for λ arbitrary, but fixed, (so that $\mathbb{E} = \mathbb{E}(\lambda)$); Ric_g is the Ricci curvature of g. The smoothness index (m, α) will occasionally be suppressed from the notation when its exact value is unimportant.

The space $\mathbb{E}^{m,\alpha}(M) \subset Met^{m,\alpha}(M)$ is invariant under the action of the group $\mathcal{D}_1 = \mathcal{D}_1^{m+1,\alpha}$ of orientation preserving $C^{m+1,\alpha}$ diffeomorphisms of M equal to the identity on ∂M. This action is free, (since any such isometry equal to the identity on ∂M is necessarily the identity), and well-known to be proper. The moduli space $\mathcal{E} = \mathcal{E}^{m,\alpha}(M)$ of Einstein metrics on M is defined to be the quotient

$$\mathcal{E} = \mathbb{E}/\mathcal{D}_1. \tag{2.4}$$

One has a natural Dirichlet boundary map

$$\Pi : \mathbb{E} \to Met(\partial M); \; \Pi(g) = \gamma = g|_{T(\partial M)}. \tag{2.5}$$

which clearly descends to a map

$$\Pi : \mathcal{E} \to Met(\partial M); \quad \Pi([g]) = \gamma. \tag{2.6}$$

We note the following result, proved in [2].

Theorem 2.1. *Suppose $\pi_1(M, \partial M) = 0$ and $m \geq 3$. Then the space $\mathcal{E}$ is a C^∞ smooth Banach manifold, (Fréchet manifold when $m = \infty$), and the boundary map Π is C^∞ smooth.*

Theorem 2.1 is proved by a suitable application of the implicit function theorem. Strictly speaking, this result is not needed for the proof of the main results in the Introduction; however it places the arguments to follow in a natural context.

Consider the Einstein operator

$$E : Met(M) \to S_2(M), \tag{2.7}$$

$$E(g) = Ric_g - \lambda g,$$

where $S_2(M)$ is the space of symmetric bilinear forms on M. The linearization of E is given by

$$L_E(k) = 2\frac{d}{dt}(Ric_{g+tk} - \lambda(g + tk))|_{t=0} = D^*Dk - 2R(k) - 2\delta^*\beta(k); \tag{2.8}$$

here $\delta^*X = \frac{1}{2}\mathcal{L}_X g$, $\beta(k) = \delta(k) + \frac{1}{2}dtrk$ is the Bianchi operator with respect to g and $R(h)$ is the action of the curvature tensor on symmetric bilinear forms k, cf. [7] for instance.

The tangent space $T_g\mathbb{E}$ is given by $KerL_E$. The derivative of the Dirichlet boundary map Π in (2.5) acts on forms k satisfying $L_E(k) = 0$ and is given by

$$D\Pi_g(k) = k^T|_{\partial M}, \tag{2.9}$$

where k^T is the tangential projection or restriction of k to $T(\partial M)$. Thus k^T is the variation of the boundary metric $\gamma = \Pi(g)$. It will also be important to consider the variation of the 2[nd] fundamental form A of ∂M in M. Thus, analogous to (2.6), one has a natural Neumann boundary map

$$\Pi_N : \mathcal{E} \to S^2(\partial M), \quad \Pi_N([g]) = A. \tag{2.10}$$

This is well-defined, since A is invariant under the action of $\mathcal{D}_1$. Note also that Π_N maps $\mathcal{E}^{m,\alpha}$ to $S_2^{m-1,\alpha}(\partial M)$. To compute the derivative of Π_N, let $g_s = g + sk$ be a variation of g. Since $A = \frac{1}{2}\mathcal{L}_N g$, one has $2A'_k \equiv 2\frac{d}{ds}A_{g_s}|_{s=0} = (\mathcal{L}_N g)' = \mathcal{L}_N k + \mathcal{L}_{N'} g$. A simple computation gives $N' = -k(N)^T - \frac{1}{2}k_{00}N$, where $k(N)^T$ is the component of $k(N)$ tangent to ∂M and $k_{00} = k(N, N)$. Thus

$$A'_k = (D\Pi_N)(k) = \frac{1}{2}(\mathcal{L}_N k + \delta^* V), \tag{2.11}$$

where $V = 2N' = -2k(N)^T - k_{00}N$.

The kernel of $D\Pi$ in (2.5) or (2.9) consists of forms k satisfying $L_E(k) = 0$ and $k^T = 0$ on ∂M, while the kernel of $D\Pi_N$ in (2.10) consists of such forms satisfying $(A'_k)^T = 0$ at ∂M. Thus, if both conditions hold,

$$k^T = 0, \ (A'_k)^T = 0 \ \text{ at } \ \partial M, \tag{2.12}$$

then (M, g) is both Dirichlet and Neumann degenerate, i.e. a singular point of each boundary map. We note that each of the conditions in (2.12) is gauge-invariant, i.e. invariant under the addition of terms of the form δ^*Z with $Z = 0$ on ∂M. Of course any form k satisfying $k = \nabla_N k = 0$ at ∂M satisfies (2.12). Changing such k by arbitrary such gauge transformations shows that (2.12) is equivalent to the statement that k is pure gauge, to first order at ∂M, i.e.

$$k = \delta^* Z + O(t^2), \tag{2.13}$$

near ∂M, with $Z = 0$ on ∂M, where $t(x) = dist_g(x, \partial M)$.

The natural or geometric Cauchy data for the Einstein equations (2.3) on M at ∂M consist of the pair (γ, A). If k is an infinitesimal Einstein deformation of (M, g), so that $L_E(k) = 0$, then the induced variation of the Cauchy data on ∂M is given by k^T and $(A'_k)^T$. It is natural to expect that an Einstein metric g is uniquely determined in a neighborhood of ∂M, up to isometry, by the Cauchy data (γ, A), i.e. one should have a suitable unique continuation property for Einstein metrics. Similarly, one would expect this holds for the linearized Einstein equations. The next result, proved in [1], confirms this expectation.

Theorem 2.2. *Let $g \in \mathbb{E}^{m,\alpha}$, $m \geq 3$, and suppose k is an infinitesimal Einstein deformation which is both Dirichlet and Neumann degenerate, so that $L_E(k) = 0$ and (2.12) holds. Then k is pure gauge near ∂M, i.e.*

$$k = \delta^* Z \quad \text{near} \quad \partial M, \tag{2.14}$$

with $Z = 0$ on ∂M.

As is well-known, the operator E is not elliptic, due its covariance under diffeomorphisms: one has $L_E(\delta^* Y) = 0$, for any vector field Y on M, at an Einstein metric. We will require ellipticity at several points and so need a choice of gauge to break the diffeomorphism invariance of the Einstein equations. In view of (2.8), the simplest and most natural choice for the work to follow is the Bianchi gauge. Thus, let $\widetilde{g}$ be a fixed (background) metric in $\mathbb{E}$. The associated Bianchi-gauged Einstein operator is given by the C^∞ smooth map

$$\Phi_{\widetilde{g}} : Met^{m,\alpha}(M) \to S_2^{m-2,\alpha}(M), \tag{2.15}$$

$$\Phi(g) = \Phi_{\widetilde{g}}(g) = Ric_g - \lambda g + \delta_g^* \beta_{\widetilde{g}}(g),$$

where $\beta_{\widetilde{g}}(g)$ is the Bianchi operator with respect to $\widetilde{g}$, while δ^* is taken with respect to g. Although $\Phi_{\widetilde{g}}$ is defined for all $g \in Met(M)$, we will only consider it acting on g near $\widetilde{g}$.

The linearization of Φ at $\widetilde{g} = g$ is given by

$$L : T_{\widetilde{g}} Met(M) \to S_2(M), \tag{2.16}$$

$$L(h) = 2(D\Phi)_{\widetilde{g}}(h) = D^* D h - 2R(h).$$

The operator L is formally self-adjoint and is clearly elliptic. Comparing (2.7) and (2.15), the relation between L and the linearization $L_E = E'$ of the Einstein operator E in (2.8) is given by

$$L_E = L - 2\delta^* \beta. \tag{2.17}$$

In §3, we will consider elliptic boundary value problems for the operator Φ.

Clearly $g \in \mathbb{E}$ if $\Phi_{\widetilde{g}}(g) = 0$ and $\beta_{\widetilde{g}}(g) = 0$, so that g is in the Bianchi gauge with respect to $\widetilde{g}$. Given $\widetilde{g}$, let $Met_C(M) = Met_C^{m,\alpha}(M)$ be the space of $C^{m,\alpha}$ smooth Riemannian metrics on M in Bianchi gauge with respect to $\widetilde{g}$ at ∂M:

$$Met_C(M) = \{g \in Met(M) : \beta_{\widetilde{g}}(g) = 0 \text{ at } \partial M\}. \tag{2.18}$$

Let

$$Z_C = \{g \in Met_C(M) : \Phi(g) = 0\} \tag{2.19}$$

be the 0-set of Φ and let $\mathbb{E}_C \subset Z_C$ be the subset of Einstein metrics g in Z_C.

To justify the use of Φ, one needs to show that the opposite inclusion holds, so that $\mathbb{E}_C = Z_C$. This has already been done in [2] and we summarize the results here.

Lemma 2.3. **(i).** *For g in $Met^{m,\alpha}(M)$, one has*

$$T_g Met^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = Ker\delta \oplus Im\delta^*, \tag{2.20}$$

where δ^* acts on $\chi_1^{m-1,\alpha}$, the space of $C^{m-1,\alpha}$ vector fields on M which vanish on ∂M.

(ii). *For $\widetilde{g} \in \mathbb{E}^{m,\alpha}$ and g in $Met^{m,\alpha}(M)$ close to $\widetilde{g}$, one has*

$$T_g Met^{m-2,\alpha}(M) \simeq S_2^{m-2,\alpha}(M) = Ker\beta \oplus Im\delta^*, \tag{2.21}$$

where β is the Bianchi operator with respect to $\widetilde{g}$. If $g \in \mathbb{E}^{m,\alpha}$, then (2.21) holds with m in place of $m - 2$.

(iii). *Any metric $g \in Z_C$ near $\widetilde{g} \in \mathbb{E}^{m,\alpha}$ is Einstein, and in Bianchi gauge with respect to $\widetilde{g}$, i.e.*

$$\beta_{\widetilde{g}}(g) = 0. \tag{2.22}$$

Similarly, if $k \in Met_C(M)$ is an infinitesimal deformation of $\widetilde{g}$ in Z_C, i.e. $L(k) = 0$, then k is an infinitesimal Einstein deformation and $\beta(k) = 0$.

Lemma 2.3 implies that $\mathbb{E}_C = Z_C$ near $\widetilde{g}$, and at least infinitesimally $\mathbb{E}_C$ is a local slice for the action of the diffeomorphism group $\mathcal{D}_1$ on $\mathbb{E}$. In fact, it is shown in [2] that $\mathbb{E}_C$ is a local slice for the action of $\mathcal{D}_1$.

The next result is a preliminary version of Theorem 1.1.

Proposition 2.4. *Let $g \in \mathbb{E}^{m,\alpha}$, $m \geq 3$, and suppose X is a Killing field on $(\partial M, \gamma)$ such that*

$$(\mathcal{L}_X A)^T = 0 \ \text{ at } \ \partial M, \tag{2.23}$$

If $\pi_1(M, \partial M) = 0$, then X extends to a Killing field on (M, g).

Proof: Since $\gamma \in Met^{m,\alpha}(\partial M)$, the Killing field X is $C^{m+1,\alpha}$ smooth on ∂M. By Lemma 2.3, X may be uniquely extended to a vector field X on M so that

$$\beta\delta^* X = 0 \ \text{ on } \ M. \tag{2.24}$$

Since $g \in \mathbb{E}^{m,\alpha}$, the solution X is then $C^{m+1,\alpha}$ up to ∂M. Hence the form $\kappa = \delta^* X$ is $C^{m-1,\alpha}$ up to ∂M and is an infinitesimal Einstein deformation in Bianchi gauge, i.e. $L(\kappa) = L_E(\kappa) = 0$ with $\beta(\kappa) = 0$. Note that by construction, $\kappa \in K$ so that $\kappa^T = 0$ at ∂M.

Next, note that

$$\mathcal{L}_X A = 2A'_\kappa. \tag{2.25}$$

Namely, since $\kappa = \frac{1}{2}\mathcal{L}_X g$, as in (2.11) one has $A'_\kappa = \frac{1}{4}\mathcal{L}_N\mathcal{L}_X g + \frac{1}{2}\mathcal{L}_{N'}g = \frac{1}{2}\mathcal{L}_X A + \frac{1}{4}\mathcal{L}_{[N,X]}g + \frac{1}{2}\mathcal{L}_{N'}g$. It is easy to verify that $2[N, X] = -N'$, which gives (2.24).

It then follows from (2.23) and Theorem 2.2 that the form κ on M is pure gauge near ∂M, i.e. there exists a vector field Z defined in a neighborhood Ω of ∂M, with $Z = 0$ at ∂M, such that

$$\kappa = \delta^* Z \ \text{ on } \ \Omega. \tag{2.26}$$

Of course the vector fields X and Z can only differ by a Killing field in Ω.

It then follows from a basically standard analytic continuation argument in the interior of M, cf. [15, §.VI.6.3] for instance, that the vector field Z may be extended so that (2.26) holds on all of M. A detailed proof of this is also given in [2, Lemma 2.6]. This analytic continuation argument requires the topological hypothesis (1.2) to obtain a well-defined, (single-valued) vector field Z on M. Moreover, since ∂M is connected, the condition $Z = 0$ on ∂M remains valid in the analytic continuation.

Since one has $\beta\delta^* Z = 0$ on M with $Z = 0$ on ∂M, it follows from Lemma 2.3 that $Z = \kappa = 0$ on M. This implies that $\delta^* X = 0$ on M, i.e. X has been extended to a Killing field on (M, g). ■

If, in place of the condition $\pi_1(M, \partial M) = 0$, one assumes that (M, g) is embedded as a domain in a complete, simply connected Einstein manifold $(\hat{M}, \hat{g})$, then the same argument as above shows that the vector field $Y = X - Z$ is a Killing field on $\Omega \subset M \subset \hat{M}$. It then follows directly from analytic continuation, cf. [15, §VI.6.4], that Y extends uniquely to a Killing field on all of $\hat{M}$, which proves Theorem 1.1 in this case also.

Remark 2.5. We point out that Proposition 2.4 also shows that if k is an infinitesimal Einstein deformation of (M, g) satisfying (2.12) and $\pi_1(M, \partial M) = 0$, then $k = 0$. The proof is the same as above.

3. Elliptic Boundary Value Problems for the Einstein Equations

In this section, we consider elliptic boundary value problems for the Bianchi-gauged Einstein operator Φ in (2.15) and the Fredholm properties of the Dirichlet boundary map Π in (2.6).

Recall that the kernel of the linearized operator L in (2.16) forms the tangent space $T_g Z_C$, ($g = \widetilde{g}$ here), and by Lemma 2.3,

$$T_g Z_C = T_g \mathbb{E}_C, \tag{3.1}$$

so that the kernel also represents the space of (non-trivial) infinitesimal Einstein deformations in Bianchi gauge. The natural Dirichlet-type boundary conditions for Φ are

$$\beta_{\widetilde{g}}(g) = 0, \quad g^T = \gamma \ \text{ at } \ \partial M. \tag{3.2}$$

However, contrary to first impressions, the operator Φ with boundary conditions (3.2) does not form a well-defined elliptic boundary value problem, (for g near $\widetilde{g}$). This is due to the well-known constraint equations, induced by the Gauss and Gauss-Codazzi equations on ∂M:

$$\delta(A - H\gamma) = -Ric_g(N, \cdot) = 0, \tag{3.3}$$

$$|A|^2 - H^2 + s_\gamma = s_g - 2Ric_g(N, N) = (n-1)\lambda. \tag{3.4}$$

Here H is the mean curvature of ∂M in M, while s denotes the scalar curvature.

As will be seen in §4, the momentum or vector constraint (3.3) is an important issue in the study of the isometry extension or rigidity results discussed in the Introduction. On the other hand, the Hamiltonian or scalar constraint (3.4) is important in understanding the Fredholm properties of the boundary map Π in (2.6). Thus for $g \in \mathbb{E}^{m,\alpha}$, one has $A \in S_2^{m-1,\alpha}(\partial M)$ so that (3.4) implies that $s_\gamma \in C^{m-1,\alpha}(\partial M)$. However, the space of metrics $\gamma \in Met^{m,\alpha}(\partial M)$ for which $s_\gamma \in C^{m-1,\alpha}(\partial M)$ is of infinite codimension in $Met^{m,\alpha}(\partial M)$. It follows that the linearization of the boundary map Π has infinite dimensional cokernel, at least when $m < \infty$, and so Π is never Fredholm. Hence, the boundary conditions (3.2) for the operator Φ are not elliptic.

Remark 3.1. It is worthwhile to understand situations where the linearization $D\Pi$ has infinite dimensional kernel and cokernel, even in the C^∞ case. Let

$$K = K_g = Ker D_g \Pi. \tag{3.5}$$

Via the slice representation $Z_C = \mathbb{E}_C \subset \mathbb{E}$ at $\widetilde{g} = g$, K consists of forms κ such that

$$L(\kappa) = 0 \text{ and } \beta_g(\kappa) = 0 \text{ on } M, \text{ with } \kappa^T = 0 \text{ on } \partial M. \tag{3.6}$$

Consider then the intersection $K \cap Im\delta^*$. Let Y be a vector field at ∂M, (not necessarily tangent to ∂M), and extend Y to a vector field on M to be the unique solution to the equation $\beta(\delta^* Y) = 0$ with the given boundary value, cf. Lemma 2.3. Then $L(\delta^* Y) = 0$ and the boundary condition $k^T = (\delta^* Y)^T = 0$ is equivalent to the equation

$$(\delta^* Y^T)^T + \langle Y, N \rangle A = 0 \ \text{ at } \ \partial M. \tag{3.7}$$

In particular if δ^*_T is the restriction of δ^* to vector fields tangent to ∂M at ∂M, then $K \cap Im\delta^*_T$ is isomorphic to the space of Killing fields on $(\partial M, \gamma)$.

On the other hand, if ∂M is totally geodesic on some open set $U \subset \partial M$, i.e. $A = 0$ on U, then the system (3.7) has solutions of the form $Y = fN$, for *any* f with $supp\, f \subset U$, so that $K \cap Im\delta^*$ is infinite dimensional. Such vector fields Y are infinitesimal isometries *at*, (as opposed to *on*), ∂M, in that they preserve the metric γ on ∂M to first order. Of course in general such Y do not extend to a Killing field on (M, g); see also Remark 4.3 for further discussion and examples. This behavior is classically very well-known in the context of surfaces embedded in $\mathbb{R}^3$, cf. [22].

A similar phenomenon holds for the cokernel. Thus, suppose $(\partial M, \gamma)$ is totally geodesic in a domain $U \subset \partial M$. Consider the linearization $s'_\gamma(h)$, for $h \in Im(D\Pi)$. By differentiating the scalar constraint (3.4) in the direction h, one sees that $s'_\gamma(h) = 0$ on U, for any such h. It follows that $ImD\Pi$ has infinite codimension, even in the C^∞ case, in such situations. The same argument and conclusion holds if $A = 0$ at just one point in ∂M.

Very little seems to be understood in characterizing the situations where K is finite dimensional or $K = 0$. Again, this is the case even in the classical setting of closed surfaces embedded in $\mathbb{R}^3$.

The discussion above implies there is no natural elliptic boundary value problem for the Einstein equations associated with Dirichlet boundary values. To obtain an elliptic problem, one needs to add either gauge-dependent terms or terms depending on the extrinsic geometry of ∂M in (M, g). To maintain a determined boundary value problem, one then has to subtract part of the intrinsic Dirichlet boundary data on ∂M.

There are several ways to carry this out in practice, but we will concentrate on the following situation. Let B be a $C^{m,\alpha}$ positive definite symmetric bilinear form on ∂M. In place of prescribing the boundary metric g^T on ∂M, only g^T modulo B will be prescribed. Thus, let $\pi_B : Met^{m,\alpha}(\partial M) \to Met^{m,\alpha}(\partial M)/B$, be the natural projection and set $\pi_B(\gamma) = [\gamma]_B$. In place of the second equation in (3.2), we impose

$$[g^T]_B = [\gamma]_B. \tag{3.8}$$

For example, when B equals the boundary metric g^T, one is prescribing the conformal class $[g^T]$ of the boundary metric g^T. Another natural choice is $B = A$, the 2^{nd} fundamental form of ∂M, when ∂M is convex, i.e. $A > 0$. For regularity purposes, we assume that $B \in S_2^{m,\alpha}(\partial M)$, so that $B = A$ is not actually allowed. Instead, one may take B to be a $C^{m.\alpha}$ smoothing of A.

The simplest gauge-dependent term one can add to (3.2) is the equation $g(\widetilde{N}, \widetilde{N}) = \gamma_{00}$, where $\widetilde{N}$ is the unit normal with respect to $\widetilde{g}$, while the simplest extrinsic geometric scalar is H, the mean curvature of ∂M in (M, g). The following result is proved in [2].

Proposition 3.2. *The Bianchi-gauged Einstein operator Φ with boundary conditions either*

$$\beta_{\widetilde{g}}(g) = 0, \quad [g^T]_B = [\gamma]_B, \quad H = h \ \text{ at } \ \partial M, \tag{3.9}$$

or

$$\beta_{\widetilde{g}}(g) = 0, \quad [g^T]_B = [\gamma]_B, \quad g(\widetilde{N}, \widetilde{N}) = \gamma_{00} \ \text{ at } \ \partial M, \tag{3.10}$$

is an elliptic boundary value problem of Fredholm index 0.

Given $\widetilde{g} \in \mathbb{E}^{m,\alpha}$, and given a choice of B as in (3.8), let $Met_B^{m,\alpha}(\partial M) = Met^{m,\alpha}(\partial M)/B$ be the space of equivalence classes of $C^{m,\alpha}$ metrics on ∂M (mod B), with natural projection or quotient map

$$\pi_B : Met^{m,\alpha}(\partial M) \to Met_B^{m,\alpha}(\partial M).$$

It follows from Proposition 3.2 and Lemma 2.3 that the map

$$\widetilde{\Pi}_H : \mathbb{E}_C \to Met_B^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{3.11}$$

$$\widetilde{\Pi}_H(g) = ([g^T]_B, H),$$

is Fredholm, of index 0, for g near $\widetilde{g}$. The same holds for the map $\hat{\Pi}(g) = ([g^T]_B, g(\widetilde{N}, \widetilde{N}))$, and in the following one may work with either of the maps $\widetilde{\Pi}_H$ or $\hat{\Pi}$; one obtains the same results for either one. However, to be concrete, we work with $\widetilde{\Pi}_H$.

Let π_1 be projection on the first factor in (3.11), and let $\Pi_0 = \pi_1 \circ \widetilde{\Pi}_H$. Thus $\Pi_0 = \pi_B \circ \Pi$. In analogy to (3.5), let

$$\widetilde{K}_H = KerD\widetilde{\Pi}_H, \tag{3.12}$$

where the derivative is taken at $g = \widetilde{g}$. In contrast to K in (3.5), $\widetilde{K}_H$ is always finite dimensional. One might call an Einstein metric $g \in \mathbb{E}$ *non-degenerate* if

$$\widetilde{K}_H = 0. \tag{3.13}$$

Thus, g is non-degenerate if and only if g is a regular point of the boundary map $\widetilde{\Pi}_H$ in which case $\widetilde{\Pi}_H$ is a local diffeomorphism near g.

Remark 3.3. It is worth pointing out that if (M, g) is strongly non-degenerate, in the sense that $K = 0$ in (3.5), then the results in §1 are easy to prove and hold in general, without any other assumptions, (such as the condition on H or on $\pi_1(M, \partial M)$). To see this, let ϕ_s be a local curve of $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ with $\phi_0 = id$ such that $\frac{d}{ds}\phi_s|_{s=0} = X$. If X is a Killing field on $(\partial M, \gamma)$, then

$$\phi_s^*\gamma = \gamma + O(s^2).$$

The curve $g_s = \phi_s^* g$ is a smooth curve in $\mathbb{E}$, and by construction, one has $[h] = [\frac{dg_s}{ds}] \in KerD\Pi$, for Π as in (2.6). One may then alter the diffeomorphisms ϕ_s by composition with diffeomorphisms $\psi_s \in \mathcal{D}_1^{m+1,\alpha}$ if necessary, so that $\kappa = \frac{d\psi_s^*(g_s)}{ds} \in K_g$, where $K = K_g$ is the kernel in (3.5) and $[h] = [\kappa]$. Thus

$$\kappa = \delta^* X',$$

where $X' = \frac{d(\phi_s \circ \psi_s)}{ds}$ is $C^{m+1,\alpha}$ smooth up to $\bar{M}$. Note that $X' = X$ at ∂M. If $K_g = 0$, then this gives

$$\delta^* X' = 0 \text{ on } M,$$

so that X' is a Killing field on (M, g). Thus, any Killing field on $(\partial M, \gamma)$ extends to a Killing field on (M, g), as claimed.

It follows that if this general isometry extension property fails, then the Dirichlet boundary map Π in (3.5) is necessarily degenerate.

Remark 3.4. Although currently the cokernel of $D\Pi$ remains hard to understand, cf. Remark 3.1, it is not difficult to describe the cokernel of $D\widetilde{\Pi}_H$. Thus, define

$$\widetilde{C} = \{((\mathcal{L}_N\kappa)_B^T, N(H'_\kappa)) : \kappa \in \widetilde{K}_H\}, \tag{3.14}$$

so that $\widetilde{C}$ represents Neumann-type data associated with the Dirichlet data in (3.9).

Note that $\widetilde{C} \subset S_B^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M)$, where $S_B^{m,\alpha}(\partial M) = T_g Met_B^{m,\alpha}(\partial M) \simeq S^{m,\alpha}(\partial M)/B$. Namely, for $\kappa \in \widetilde{K}_H$, one has $L(\kappa) = 0$ on M together with the elliptic boundary conditions $\beta(\kappa) = 0$, $\kappa_B^T = 0$, and $H'_\kappa = 0$ on ∂M. Since g is $C^{m,\alpha}$ up to ∂M, elliptic boundary regularity applied to this system gives $\kappa \in C^{m+1,\alpha}$, (cf. [11, 18]), so that $\mathcal{L}_N\kappa \in S_2^{m,\alpha}(\partial M)$ and $N(H'_\kappa) \in C^{m-1,\alpha}(\partial M)$.

It is then not difficult to prove (although we will not give the proof here), that the space $\widetilde{C}$ is a slice for $CokerD\widetilde{\Pi}_H$ in $S_B^{m,\alpha}(\partial M) \times C^{m,\alpha}(\partial M)$, so that

$$S_B^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M) = ImD\widetilde{\Pi}_H \oplus \widetilde{C}. \tag{3.15}$$

By restricting to the first factor, it follows immediately from (3.15) that

$$S_B^{m,\alpha}(\partial M) = ImD\Pi_0 \oplus \widetilde{S}, \tag{3.16}$$

where $\widetilde{S} = \{(\mathcal{L}_N\kappa)_B^T : \kappa \in \widetilde{K}_H\}$ and Π_0 is defined preceding (3.12).

Remark 3.5. One may use the diffeomorphism group to pass from the space $\mathbb{E}_B$ of Bianchi-gauged Einstein metrics to the full space $\mathbb{E}$, thus passing from $\widetilde{\Pi}_H$ to the more natural Dirichlet boundary map Π. In more detail, the image $\mathcal{H} = D\Pi(\mathbb{E}_B) \subset TMet^{m,\alpha}(\partial M)$ projects onto a space of finite codimension in $S_B^{m,\alpha}(\partial M)$ by (3.16). The full image $D\Pi(\mathbb{E})$ then consists of the span $\langle \mathcal{H}, Im\delta^* \rangle$, where δ^* acts on all vector fields at ∂M, not necessarily tangent to ∂M. It is an interesting question to understand when this space, or possibly its closure, is of finite codimension in $TMet^{m,\alpha}(\partial M)$. This corresponds roughly to Π being Fredholm.

4. Isometry Extension and the Divergence Constraint

This section is not actually needed for the proof of Theorem 1.1 and so may be skipped if preferred. Nevertheless, the results here are of interest in their own right, and shed a useful perspective on the general problem.

By Proposition 2.4, the basic issue is to understand when a Killing field on $(\partial M, \gamma)$ preserves the 2$^{\text{nd}}$ fundamental form A of ∂M in M. We begin with the following identity on $(\partial M, \gamma)$, which holds on any closed oriented Riemannian manifold.

Proposition 4.1. *Let X be a Killing field on $(\partial M, \gamma)$. Suppose τ is a divergence-free symmetric bilinear form on $(\partial M, \gamma)$. Then*

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = -2 \int_{\partial M} \langle \delta' \tau, X \rangle dV_\gamma, \tag{4.1}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X and $\delta' = \frac{d}{ds}\delta_{\gamma+sh}$ is the variation of the divergence on $(\partial M, \gamma)$ in the direction $h \in S_2(\partial M)$.

Proof: Since the flow of X preserves γ, one has

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle dV_\gamma = - \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV_\gamma. \tag{4.2}$$

Next, setting $\gamma_s = \gamma + sh$, the divergence theorem applied to the 1-form $\tau(X)$ on ∂M gives

$$0 = \int_{\partial M} \delta_{\gamma_s}(\tau(X)) dV_{\gamma_s} = \int_{\partial M} \langle \delta_{\gamma_s} \tau, X \rangle dV_{\gamma_s} - \tfrac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X \gamma_s \rangle dV_{\gamma_s}, \tag{4.3}$$

where the second equality is a simple computation from the definitions; the inner products are with respect to γ_s. Taking the derivative with respect to s at $s = 0$ and using the facts that X is a Killing field on ∂M and $\delta\tau = 0$, it follows that

$$\int_{\partial M} \langle \delta' \tau, X \rangle dV - \tfrac{1}{2} \int_{\partial M} \langle \tau, \mathcal{L}_X h \rangle dV = 0.$$

Combining this with (4.2) then gives (4.1). ∎

For later purposes, we note that if τ as above is divergence-free but X is not assumed to be a Killing field, then (4.1) is replaced by the following, cf. also (4.18)-(4.19) below:

$$\int_{\partial M} \langle \mathcal{L}_X \tau, h \rangle = -2 \int_{\partial M} \langle \delta'_h(\tau), X \rangle + \int_{\partial M} (\langle \delta^* X, \tau \rangle trh + \delta X \langle \tau, h \rangle + 4 \langle \tau \circ \delta^* X, h \rangle - 4 \langle \tau \circ h, \delta^* X \rangle). \tag{4.4}$$

We now examine the right side of (4.1) in connection with the divergence constraint (3.3); of course (3.3) implies the form $\tau = A - H\gamma$ is divergence-free on ∂M.

We first discuss the general perspective. As discussed in §2, one may view the pair (γ, A) as Cauchy data for the Einstein equations (2.3) at ∂M. The data (γ, A) are then formally freely specifiable subject to the constraints (3.3)-(3.4). Let $\mathcal{T}$ be the space of pairs (γ, τ) with τ divergence-free with respect to γ; here $\gamma \in Met^{m,\alpha}(\partial M)$, $\tau \in S_2^{m-1,\alpha}(\partial M)$. The space $\mathcal{T}$ is naturally a vector bundle over $Met^{m,\alpha}(\partial M)$,

$$\pi : \mathcal{T} \to Met^{m,\alpha}(\partial M), \tag{4.5}$$

with π the projection on the first factor. Let also $\mathcal{F} \subset \mathcal{T}$ be the subset of pairs satisfying the scalar constraint equation (3.4). When expressed in terms of $\tau = A - H\gamma$, (3.4) is equivalent to

$$|\tau|^2 - (tr\tau)^2 + s_\gamma = 0.$$

Pairs $(\gamma, \tau) \in \mathcal{F}$ determine formal solutions of the Einstein equations near ∂M. More precisely, let (t, x^i) be geodesic boundary coordinates for (M, g), so that by the Gauss Lemma, the metric g has the form

$$g = dt^2 + g_t, \tag{4.6}$$

where $t(x) = dist_g(x, \partial M)$ and g_t is the induced metric on the level set $S(t)$ of t. Pulling back by the flow lines of ∇t, g_t may be viewed as a curve of metrics on ∂M, and one may formally expand g_t in its Taylor series:

$$g_t \sim \gamma - tA - \tfrac{1}{2}t^2 \dot{A} + \cdots , \tag{4.7}$$

where $\dot{A} = \nabla_N A = -\nabla_T A$, $T = \nabla t = -N$. As noted above, the terms (γ, A) are freely specifiable, subject to the constraints (3.3)-(3.4). All the higher order terms in the expansion (4.7) are then determined by γ and A. To see this, one first uses the standard Riccati equation

$$\nabla_T A + A^2 + R_T = 0, \tag{4.8}$$

where $R_T(X, Y) = \langle R(X, T)T, Y\rangle$, cf. [19]. A standard formula gives $\nabla_T A = \mathcal{L}_T A - 2A^2$. Also, by the Gauss equation, the curvature term R_T may be expressed as

$$R_T = Ric^T - Ric_{int} + HA - A^2,$$

where $H = trA$, Ric_{int} is the intrinsic Ricci curvature of $S(t)$ and Ric^T is the tangential part, (tangent to $S(t)$), of the ambient Ricci curvature. Substituting in (4.8) gives

$$\ddot{g} = -2Ric^T + 2Ric_{int} + 4A^2 - 2HA. \tag{4.9}$$

For Einstein metrics satisfying (2.3), the right side of (4.9) involves only the first order t-derivatives of the metric g. Thus, repeated differentiation of (4.9) shows that all derivatives $g_{(k)} = \mathcal{L}_T^k g$ are determined at the boundary M by the Cauchy data (γ, A), so that (γ, A) determines the formal Taylor expansion of the curve g_t in (4.6) at $t = 0$.

The Cauchy-Kovalevsky theorem implies that if (γ, τ) are real-analytic forms on ∂M, then the formal series (4.7) converges to g_t, so that one obtains an actual Einstein metric g as in (4.6), defined in a neighborhood of ∂M. Of course, such metrics will not in general extend to globally defined Einstein metrics on M.

Now the right side of (4.1) is closely related to the linearization of the divergence constraint. Thus, if (γ_s, τ_s) is a curve in $\mathcal{T}$ with tangent vector $(\gamma', \tau') = (h, \tau')$ at $s = 0$, then one has

$$\delta'(\tau) + \delta(\tau') = 0, \tag{4.10}$$

where δ' is defined as in (4.1); this is the linearized divergence constraint.

Lemma 4.2. *If the derivative $D\pi$ in* (4.5) *is surjective at (γ, τ), $\tau = A - H\gamma$, then*

$$\mathcal{L}_X A = 0 \quad \text{on} \quad \partial M, \tag{4.11}$$

for any Killing field X on $(\partial M, \gamma)$. Conversely, if (4.11) *holds for all such Killing fields X, then $D\pi$ is surjective.*

Proof: This result follows easily from Proposition 4.1, with $\tau = A - H\gamma$. Thus, (4.10) gives $\delta'(A - H\gamma) = -\delta(\tau')$, for the variation δ' of δ in any direction $h \in T_\gamma Met(\partial M)$, for some τ'. Hence, (4.1) gives

$$\mathcal{F}(h) = \int_{\partial M} \langle \mathcal{L}_X \tau, h\rangle = -2\int_{\partial M} \langle \delta(\tau'), X\rangle = 2\int_{\partial M} \langle \tau', \delta^* X\rangle = 0, \tag{4.12}$$

since X is a Killing field on $(\partial M, \gamma)$. Since h is arbitrary, this implies that

$$\mathcal{L}_X \tau = 0,$$

on ∂M, and (4.11) follows by taking the trace of this equation. The same proof also gives the converse as well, using the splitting (4.13) below. ■

Thus, given $g \in \mathbb{E}$ and its corresponding 2^{nd} fundamental form A, giving the pair (γ, A) at ∂M, a fundamental issue is whether $D\pi$ is surjective at (γ, A), i.e. whether the linearized divergence constraint (4.10) is solvable, for any variation h of γ on ∂M, (or for a space of variations dense in $S_2(\partial M)$ in the L^2 norm). One cannot expect that this holds at a general pair $(\gamma, \tau) \in \mathcal{T}$. Namely, for any compact manifold ∂M, one has

$$\Omega^1(\partial M) = Im\delta \oplus Ker\delta^*, \tag{4.13}$$

where Ω^1 is the space of $(C^{m-1,\alpha})$ 1-forms on ∂M. Thus, solvability at (γ, τ) in general requires that

$$\delta'(\tau) \in Im\delta = (Ker\delta^*)^{\perp}. \tag{4.14}$$

Of course $Ker\delta^*$ is exactly the space of Killing fields on $(\partial M, \gamma)$, and so this space serves as a potential obstruction space.

Obviously, π is locally surjective when $(\partial M, \gamma)$ has no Killing fields. On the other hand, it is easy to construct examples where $(\partial M, \gamma)$ does have Killing fields and π is not locally surjective.

Example 4.3. Let $(\partial M, \gamma)$ be a flat metric on the n-torus T^n; for example $\gamma = d\theta_1^2 + \cdots + d\theta_n^2$. Let $\tau = f(\theta_1)d\theta_2^2$, (for example). Then $\delta\tau = 0$, for any C^1 function $f(\theta_1)$. The pair (γ, τ) is in $\mathcal{T}$, and in fact in $\mathcal{F} \subset \mathcal{T}$. Letting X be the Killing field ∂_{θ_1}, one has $\mathcal{L}_X\tau \neq 0$ whenever f is non-constant, so that by the converse of Lemma 4.2, π is not locally surjective at such (γ, τ).

If (γ, τ) above are real-analytic, then $(\partial M, \gamma)$ is the boundary metric of an Einstein metric defined on a thickening $\partial M \times I$ of ∂M. Of course in general, such thickenings will not extend to Einstein metrics on a compact manifold bounding ∂M.

To obtain examples on compact manifolds, one may use the examples of $\mathbb{R}^2 \times S^1$ or S^3 discussed in the Introduction. Here one has an infinite dimensional space of isometric embeddings of a flat torus in $\mathbb{R}^2 \times S^1$ or S^3 for which Killing fields on the boundary do not extend to Killing fields on $\mathbb{R}^2 \times S^1$ or S^3.

Now clearly $D\pi$ is surjective onto $ImD\Pi$, since $ImD\Pi$ consists of variations of the boundary metric determined by global variations of the Einstein metric g on M which of course satisfy (4.10). Hence if $D\Pi$ is onto, or has dense range in $S_2(\partial M)$, then Lemma 4.2 holds, i.e. (4.11) holds; compare with Remark 3.3. On the other hand, the examples above show that whether (4.11) holds or not must depend either on global properties of (M, g) or extrinsic properties of $\partial M \subset M$.

We close this section with a discussion of the Einstein-Hilbert action and its relation to (4.1). This is of independent interest, but will also be used in §5. The Einstein-Hilbert action with Gibbons-Hawking-York boundary term on M is

$$I(g) = I_{EH}(g) = -\int_M (s_g - 2\Lambda)dV_g - 2\int_{\partial M} H dv_\gamma, \tag{4.15}$$

where $\Lambda = \frac{n-1}{2}\lambda$, cf. [12]. The 1^{st} variation of I in the direction h is given by

$$\frac{d}{dr}I(g + rh) = \int_M \langle E_g, h\rangle dV_g + \int_{\partial M} \langle \tau_g, h\rangle dv_\gamma, \tag{4.16}$$

where E is the Einstein tensor, $E_g = Ric_g - \frac{s}{2}g + \Lambda g$ and $\tau = A - H\gamma$ is as above. Here and below, all parameter derivatives are taken at 0. Einstein metrics with $Ric_g - \lambda g = 0$ are critical points of I, among variations vanishing on ∂M. Consider a 2-parameter family of metrics $g_{r,s} = g + rh + sk$ where $E_g = 0$. Then

$$\frac{d^2}{dsdr}I(g_{r,s}) = \frac{d^2}{drds}I(g_{r,s}). \tag{4.17}$$

Computing the left side of (4.17) by taking the derivative of (4.16) in the direction k gives

$$\frac{d^2}{dsdr}I(g_{r,s}) = \int_M \langle E'(k), h\rangle dV_g + \int_{\partial M}\langle \tau'_k + a(k), h\rangle dv_\gamma. \tag{4.18}$$

Since $E_g = 0$, there are no further derivatives of the bulk integral in (4.16). Also, $a(k) = -2\tau\circ k + \frac{1}{2}(tr_\gamma k)\tau$ arises from the variation of the metric and volume form in the direction k; by definition $(\tau\circ k)(V,W) = \frac{1}{2}\{\langle\tau(V), k(W)\rangle + \langle\tau(W), k(V)\rangle\}$.

Similarly, for the right side of (4.17) one has

$$\frac{d^2}{drds}I(g_{r,s}) = \int_M \langle E'(h), k\rangle dV_g + \int_{\partial M}\langle \tau'_h + a(h), k\rangle dv_\gamma. \tag{4.19}$$

In particular, suppose k_D is an infinitesimal Einstein deformation in the kernel k_D from (3.5), so that $k_D|_{\partial M} = k^T = 0$. If h is an infinitesimal Einstein deformation, $h\in T\mathbb{E}$, with h purely tangential at ∂M, i.e. $h(N,\cdot) = 0$, then (4.17)-(4.19) gives,

$$\int_{\partial M}\langle \tau'_{k_D}, h\rangle dv_\gamma = \int_{\partial M}\langle \tau'_h, k_D\rangle dv_\gamma = 0. \tag{4.20}$$

By (4.18), one thus has

$$I''(k_D, h) = 0,$$

on-shell. Note this computation recaptures (4.12) when $k_D = \delta^*X = \frac{1}{2}\mathcal{L}_X g$.

For later purposes, we note that a standard computation, [7], gives, (for (M,g) Einstein),

$$2E'(h) = D^*Dh - 2R(h) - D^2 trh - \delta\delta h\, g + \Delta trh\, g + \tfrac{s}{n+1} trh\, g - 2\delta^*\delta h. \tag{4.21}$$

5. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1 and Corollary 1.2. As noted toward the end of the previous section, one needs to use global arguments to prove Theorem 1.1. We do this by studying global properties of the linearized operator L from (2.16) and its relation to the Bianchi gauge.

Suppose X is a vector field on and tangent to ∂M. Then by Lemma 2.3, X may be extended uniquely to a vector field on M as a solution to the equation

$$\beta\delta^*X = \tfrac{1}{2}(D^*DX - Ric(X)) = 0. \tag{5.1}$$

Thus $\kappa = \delta^*X$ is in Bianchi gauge and for L as in (2.16), one has

$$L(\delta^*X) = 2\beta\delta^*X = 0.$$

Lemma 5.1. *For X as above, suppose X is a Killing field on $(\partial M, \gamma)$ and $X(H) = 0$. (Thus, $\kappa^T = 0$ and $H'_\kappa = 0$ at ∂M). If $\lambda \leq 0$, then on (M,g), one has*

$$\delta X = \delta\delta^*X = 0. \tag{5.2}$$

Proof: The proof is a computation comparing two different points of view. To begin, by (4.18)-(4.19), one has

$$\int_M \langle E'(h), k\rangle + \int_{\partial M}\langle\tau'_h + a(h), k\rangle = \int_M\langle E'(k), h\rangle + \int_{\partial M}\langle \tau'_k + a(k), h\rangle.$$

For $k = \kappa = \delta^*X$, one has $E'(\kappa) = 0$, so that

$$\int_M\langle E'(h), \kappa\rangle = \int_{\partial M}\langle\tau'_\kappa + a(\kappa), h\rangle - \int_{\partial M}\langle\tau'_h + a(h), \kappa\rangle. \tag{5.3}$$

Now choose $h = \phi g$, where ϕ is arbitrary. First we observe both a terms vanish. Thus

$$\langle a(\kappa), \phi g\rangle = -2\langle\tau\circ\kappa, \phi g\rangle + \tfrac{1}{2}tr_\gamma\kappa\langle\tau, \phi g\rangle = 0,$$

since τ is pure tangential and κ vanishes tangentially. Similarly, for the same reasons,

$$\langle a(\phi g), \kappa\rangle = -2\langle \tau \circ \phi g, \kappa\rangle + \tfrac{1}{2}tr_{\gamma}\phi g\langle \tau, \kappa\rangle = 0.$$

Next,

$$\langle \tau'_{\phi g}, \kappa\rangle = \langle A'_{\phi g} - H'_{\phi g}\gamma - H\gamma'_{\phi g}, \kappa\rangle = \langle A'_{\phi g}, \kappa\rangle,$$

since $\gamma' = \phi\gamma$, (γ' is the variation of the induced metric on ∂M), and $\langle \kappa, \gamma\rangle = 0$. Also, as in (2.11),

$$2A'_{\phi g} = \nabla_N \phi g + 2A \circ \phi g - 2\delta^*(\phi g(N)^T) - \delta^*(\phi N). \tag{5.4}$$

The second term here vanishes when paired with κ and the third term also vanishes, since $g(N)^T = 0$. The last term is $\delta^*(\phi N) = \phi A + d\phi \cdot N$, so that $\langle \delta^*(\phi N), \kappa\rangle = \kappa(N, d\phi)$. This gives

$$2\langle \tau'_{\phi g}, \kappa\rangle = N(\phi)tr\kappa - \kappa(N, d\phi).$$

Observe that $\kappa(N, d\phi) = N(\phi)tr\kappa + \langle \kappa(N), d^T\phi\rangle$, since $\kappa_{00} = tr\kappa$. Hence,

$$2\int_{\partial M}\langle \tau'_{\phi g}, \kappa\rangle = -\int_{\partial M}\phi\delta^T(\kappa(N)^T).$$

Combining the computations above, (5.3) becomes

$$\int_M \langle E'(\phi g), \kappa\rangle = \int_{\partial M}\langle \tau'_{\kappa}, \phi g\rangle + \tfrac{1}{2}\int_{\partial M}\phi\delta^T(\kappa(N)^T).$$

Since $tr(\tau'_{\kappa}) = -\frac{1}{2}(n-1)X(H) = 0$, it follows that

$$\int_M \langle E'(\phi g), \kappa\rangle = -\tfrac{1}{2}\int_{\partial M}\phi\delta^T(\kappa(N)^T). \tag{5.5}$$

Now from (4.21), one computes, term-by-term, $2E'(\phi g) = -(\Delta\phi)g - \frac{2s}{n+1}\phi g - (n+1)D^2\phi - \Delta\phi\, g + (n+1)\Delta\phi\, g + s\phi g + 2D^2\phi$, which simplifies to $2E'(\phi g) = (n-1)\Delta\phi\, g + \frac{n-1}{n+1}s\phi\, g - (n-1)D^2\phi$. Recalling that $\lambda = \frac{s}{n+1}$, (5.5) becomes

$$\int_M (\Delta\phi + \lambda\phi)tr\kappa = \int_M \langle D^2\phi, \delta^* X\rangle + \frac{1}{n-1}\int_{\partial M}\phi\delta^T(\kappa(N)^T). \tag{5.6}$$

We now compute the terms in (5.6) via a different method. Thus, a standard formula gives, for any function ϕ, $\delta(D^2\phi) = -d(\Delta\phi) - Ric(d\phi)$. Hence,

$$\int_M \langle D^2\phi, \delta^* X\rangle = -\int_M \langle d(\Delta\phi), X\rangle + Ric(d\phi, X) + \int_{\partial M} D^2\phi(X, N).$$

Integrating the bulk term by parts, using $\langle X, N\rangle = 0$ at ∂M and the fact that $tr\kappa = -\delta X$, gives

$$\int_M \langle D^2\phi, \delta^* X\rangle = \int_M (\Delta\phi + \lambda\phi)tr\kappa + \int_{\partial M} D^2\phi(X, N).$$

We claim that the boundary term here vanishes. To see this, one has $D^2\phi(X, N) = \langle \nabla_X d\phi, N\rangle = XN(\phi) - \langle A(X), d\phi\rangle$. The first term here integrates to 0, since $\delta^T X = 0$ on ∂M. For the second term, $\int_{\partial M}\langle A(X), d\phi\rangle = \int_{\partial M}\phi\delta(A(X)) = \int_{\partial M}\phi(\delta A)(X) + \phi\langle A, \delta^* X\rangle = \int_{\partial M}\phi(\delta A)(X)$, since X is Killing on ∂M. Finally, the divergence constraint (3.3) gives $(\delta A)(X) = (\delta H\gamma)(X) = -X(H) = 0$, as claimed.

Thus we obtain

$$\int_M \langle D^2\phi, \delta^* X\rangle = \int_M (\Delta\phi + \lambda\phi)tr\kappa. \tag{5.7}$$

Comparing this with (5.6), using the fact that ϕ is arbitrary, it follows that

$$\delta^T(\kappa(N)^T) = 0. \tag{5.8}$$

We note that simple calculation shows that the two terms above involving $X(H)$ actually cancel, so that (5.8) holds in fact without this hypothesis.

Next, since $\beta(\kappa)(N) = 0$, computing directly from the definition of β gives $-N(\kappa_{00}) + \delta^T(\kappa(N)^T) - \kappa_{00}H + \frac{1}{2}N(tr\kappa) = 0$, so that $2\delta^T(\kappa(N)^T) + N(tr\kappa) = 2N(\kappa_{00}) + 2H\kappa_{00}$.

Also, using the fact that $\kappa^T = 0$ together with (5.4), one computes $0 = H'_\kappa = trA'_\kappa = tr[\nabla_N\kappa + 2\kappa \circ A - 2\delta^*(\kappa(N)^T) - \kappa_{00}A - d\kappa_{00} \cdot N] = N(tr\kappa) + 2\delta(\kappa(N)^T) - \kappa_{00}H - N(\kappa_{00})$. Note that $\delta((\kappa(N)^T)) = \delta^T((\kappa(N)^T))$, since $\kappa(N)^T$ is tangential. Combining these computations gives

$$N(tr\kappa) + 2\delta^T(\kappa(N)^T) = 0,$$

and hence via (5.8),

$$N(tr\kappa) = 0. \tag{5.9}$$

Since $L(\kappa) = 0$, one has $trL(\kappa) = \Delta tr\kappa + \frac{2s}{n+1}tr\kappa = 0$, i.e.

$$\Delta tr\kappa + 2\lambda tr\kappa = 0. \tag{5.10}$$

When $\lambda \leq 0$, it then follows from the Hopf maximum principle that $tr\kappa = const$. Since $\int_M tr\kappa = -\int_M \delta X = \int_{\partial M}\langle X, N\rangle = 0$, this gives $tr\kappa = 0$, which proves the result. ■

Remark 5.2. If $\lambda > 0$, then the same argument shows that if 2λ is not a Neumann eigenvalue of the Laplacian on (M, g), then Lemma 5.1 holds. We expect that Lemma 5.1 holds for general Einstein metrics, but do not pursue this further here.

Given Lemma 5.1, the starting point is to consider the pairing

$$\int_M \langle L(h), \kappa\rangle = \int_M \langle L(h), \delta^* X\rangle,$$

for L as in (2.16). Integrating by parts, this gives

$$\int_M \langle L(h), \kappa\rangle = \int_M \langle \delta(L(h)), X\rangle + \int_{\partial M} L(h)(X, N).$$

Now by Lemma 5.1, $0 = \int_M\langle L(h), \delta X g\rangle = \int_M trL(h) \cdot \delta X = \int_M\langle d(trL(h)), X\rangle - \int_{\partial M} trL(h)\langle X, N\rangle$. The boundary term here vanishes, since X is tangent to ∂M. It then follows, (from the definition of β), that

$$\int_M \langle L(h), \kappa\rangle = \int_M \langle \beta(L(h)), X\rangle + \int_{\partial M} L(h)(X, N). \tag{5.11}$$

Let

$$\mathcal{H} = \{h : \int_M \langle \beta(L(h)), X\rangle = 0\}. \tag{5.12}$$

This is a closed hypersurface in $S_2^{m,\alpha}(M)$, $m \geq 3$.

The main claim is that the operator L is surjective when restricted to $\mathcal{H}$. On $\mathcal{H}$, from (5.12) one has

$$\int_M \langle L(h), \kappa\rangle = \int_{\partial M} L(h)(X, N). \tag{5.13}$$

If for $h \in \mathcal{H}$, $L(h)$ can be prescribed arbitrarily, there exists, (for instance), $h_x \in \mathcal{H}$ such that $L(h_x)$ is of compact support and approximates arbitrarily chosely a δ-form at any given $x \in M$. Hence, $L(h_x) = 0$ on ∂M and it follows from (5.13) that $\kappa(x) \sim 0$. Since x and the approximation are arbitrary, this gives $\kappa = \delta^* X = 0$, which proves Theorem 1.1.

To prove the main claim, first note that $L|_{\mathcal{H}}$ has closed range. Namely, the operator L is elliptic, and one can add elliptic boundary data as in (3.9). Suppose then that $h_i \in \mathcal{H}$ and that the sequence $q_i = L(h_i)$ converges to a limit q in $S^{m-2,\alpha}(M)$. For h_i satisfying (3.9) for instance, and orthogonal to the kernel of L (if any) for such boundary data, elliptic boundary regularity implies that $\{h_i\}$ converges to a limit $h \in S^{m,\alpha}(M)$. The subspace $\mathcal{H}$ is closed, and hence $h \in \mathcal{H}$ with $L(h) = q$.

The operator L is surjective on the full domain $S^{m,\alpha}(M)$. (This is basically obvious, and is proved in [2], in the course of the proof of Theorem 2.1 in §2). Hence, if $L_{\mathcal{H}}$ is not surjective, there exists a form $k \in S^{m-2,\alpha}(M)$ in the target such that

$$\int_M \langle L(h), k\rangle = 0 \tag{5.14}$$

for all $h \in \mathcal{H}$. One needs then to prove that necessarily $k = 0$.

Integrating (5.14) by parts gives

$$0 = \int_M \langle L(h), k\rangle = \int_M \langle h, L(k)\rangle + \int_{\partial M} [\langle \nabla_N k, h\rangle - \langle \nabla_N h, k\rangle], \tag{5.15}$$

again for all $h \in \mathcal{H}$. We claim that the bulk and boundary terms on the right in (5.15) vanish separately. For this, it suffices to show that any form h of compact support in M is in $\mathcal{H}$. To see this, from the Bianchi identity one has $\beta L_E = 0$, and so (2.17) gives $\beta L(h) = 2\beta\delta^*(\beta(h)) = (D^*D - Ric)(\beta(h))$. Integrating then the defining property (5.12) for $\mathcal{H}$ by parts gives

$$\int_M \langle \beta(L(h)), X\rangle = \int_M \langle (D^*D - Ric)(\beta(h)), X\rangle$$

$$= \int_M \langle \beta(h), (D^*D - Ric)(X)\rangle + \int_{\partial M} \langle \nabla_N X, \beta(h)\rangle - \langle \nabla_N \beta(h), X\rangle.$$

By (5.1), $(D^*D - Ric)(X) = 2\beta\delta^* X = 0$, so that $\mathcal{H}$ is equivalently defined by the property

$$\mathcal{H} = \{h : \int_{\partial M} \langle \nabla_N X, \beta(h)\rangle - \langle \nabla_N \beta(h), X\rangle = 0\}. \tag{5.16}$$

In particular, since $\mathcal{H}$ is thus in fact "supported at ∂M", any form $h \in S^{m,\alpha}(M)$ of compact support in M is contained in $\mathcal{H}$. Hence, (5.15) implies that

$$L(k) = 0 \quad \text{on} \quad M. \tag{5.17}$$

A similar argument shows that

$$\delta k = 0 \quad \text{on} \quad M. \tag{5.18}$$

Namely, one has $L(\delta^* Z) = \delta^* Y$, where $Y = 2\beta\delta^* Z$. The operator $\beta\delta^*$ is surjective on vector fields Z with $Z = 0$ on ∂M, (by Lemma 2.3), and the condition $\delta^* Z \in \mathcal{H}$ is the statement $\int_M \langle \beta\delta^* Y, X\rangle = 0$. As above, this is satisfied by any Y of compact support in M. Hence, for any such Y, one has

$$0 = \int_M \langle L(\delta^* Z), k\rangle = \int_M \langle \delta^* Y, k\rangle = \int_M \langle Y, \delta k\rangle + \int_{\partial M} k(Y, N) = \int_M \langle Y, \delta k\rangle.$$

Since Y of compact support is arbitrary, this gives (5.18).

Returning to (5.14), it follows then from (5.15), (5.17) and (5.18) that

$$L(k) = 0, \ \ \delta k = 0, \ \text{ and } \int_{\partial M} \langle \nabla_N k, h\rangle - \langle \nabla_N h, k\rangle = 0, \tag{5.19}$$

for all $h \in \mathcal{H}$. As in (2.18), let

$$T_g Met_C^{m,\alpha} = \{h \in S^{m,\alpha}(M) : \beta(h) = 0 \text{ on } \partial M\}.$$

Then it suffices to show that L is surjective on the space

$$\mathcal{H}' = \mathcal{H} \cap T_g Met_C^{m,\alpha} = \{h : \beta(h) = 0 \text{ at } \partial M \text{ and } \int_{\partial M} \langle \nabla_N \beta(h), X\rangle = 0\}.$$

Observe that $\mathcal{H}'$ involves only the 1st derivative of $\beta(h)$ in the normal direction, or a 2nd derivative of h in mixed tangential/normal direction, at ∂M. Thus, within $\mathcal{H}'$, the full 1-jet consisting of h and $\nabla_N h$ may be freely prescribed on ∂M, subject to the single constraint that $\beta(h) = 0$. This is exactly the situation analysed in detail in [2] in connection with the proof of Theorem 2.1. There it is shown that the condition from (5.19)

$$\int_{\partial M} \langle \nabla_N k, h \rangle - \langle \nabla_N h, k \rangle = 0, \tag{5.20}$$

for all h such that $\beta(h) = 0$ at ∂M, implies that

$$k^T = 0 \text{ and } (A'_k)^T = 0 \ \text{ at } \ \partial M. \tag{5.21}$$

Note that the derivation of (5.21) from (5.20) involves only the 1st order jet of h at ∂M, and so is independent of the condition $\int_{\partial M} \langle \nabla_N \beta(h), X \rangle = 0$ for $\mathcal{H}'$.

Thus, it follows from (5.19) and (5.21) that the form k in (5.14) satisfies

$$L(k) = 0, \ \beta(k) = 0, \ \text{ and } \ k^T = 0, \ (A'_k)^T = 0 \ \text{ at } \ \partial M. \tag{5.22}$$

(We again refer to [2] for the proof that (5.18) implies that also $\beta(k) = 0$). Thus, k is an infinitesimal Einstein deformation, with vanishing geometric Cauchy data at ∂M. It follows then from Theorem 2.2, as in the proof of Proposition 2.4, that

$$k = 0 \ \text{ on } \ M.$$

This proves that $L|_{\mathcal{H}}$ is indeed surjective, and the result $\kappa = \delta^* X = 0$ follows discussed as above. Hence, any Killing field X on $(\partial M, \gamma)$ such that $X(H) = 0$ and $\pi_1(M, \partial M) = 0$ extends to a Killing field on (M, g). ■

Remark 5.3. The proof of Theorem 1.1 above shows that, when for instance $H = const$ at ∂M, one has

$$K \cap Im\delta^* = 0, \tag{5.23}$$

where δ^* acts on vector fields X tangent to ∂M, and $K = KerD\Pi$, as in (3.5).

However, for instance in dimension 3, all Einstein deformations are pure gauge, i.e. of the form $\delta^* V$, for some vector field V, not necessarily tangent to ∂M, cf. Remark 3.1. Hence, if Π is degenerate at some constant curvature metric (M^3, g), i.e. $K = K_g \neq 0$ and again $H = const$ at ∂M, then

$$K \cap \delta^* V \neq 0,$$

for general V at ∂M. The condition $H = const$ is necessary here, cf. Example 4.3.

Proof of Corollary 1.2.

Theorem 1.1 implies that the isometry group $SO(n+1)$ of S^n extends to a group of isometries of the Einstein manifold (M^{n+1}, g). This reduces the Einstein equations to a simple system of ODE's, (the metric g is of cohomogeneity 1), and it is standard that the only smooth solutions are given by constant curvature metrics, cf. [7] for example. ■

The same proof shows that if $(\partial M, \gamma)$ is homogeneous, then any Einstein filling metric (M, g) is of cohomogeneity 1. Such metrics have been completely classfied in many situations, cf. [7] for further information.

We complete this section with a brief discussion of exterior boundary value problems. Thus, suppose M^{n+1} is an open manifold with compact "inner" boundary ∂M and with a finite number of ends, each (locally) asymptotically flat. Topologically, each end is of the form $(\mathbb{R}^k \setminus B) \times T^{n+1-k}$, or a quotient of this space by a finite group of isometries. Here T^{n+1-k} is the $(n+1-k)$-torus, and we assume $3 \leq k \leq n+1$. Assume also, as usual, that $\pi_1(M, \partial M) = 0$. An Einstein metric is asymptotically locally flat (ALF) if it decays to a flat metric on each end at a rate $r^{-(k-2)}$, (the decay rate of the Greens function for the Laplacian), where r is the distance from a fixed point.

It is proved in [2] that the analog of Theorem 2.1 holds, namely the space of asymptotically locally flat Einstein metrics on an exterior domain M is a smooth Banach manifold, for which the Dirichlet boundary map is C^∞ smooth. Lemma 2.3 also holds in this context. All of the remaining results in §2 - §4 concern issues at or near ∂M, and it is straightforward to verify that their proofs carry over to this exterior context without change. In particular the analog of Theorem 1.1 holds:

Proposition 5.4. *Let g be a $C^{m,\alpha}$ Ricci-flat metric on an exterior domain M, $m \geq 3$, with a finite number of locally asymptotically flat ends. Suppose also* (1.2) *holds. Then any Killing field X on $(\partial M, \gamma)$ for which $X(H) = 0$, extends uniquely to a Killing field on (M, g).*

■

6. The Static Vacuum Einstein Equations

In this section, we study the isometry extension property for general solutions of the static vacuum equations and relate this to a conjecture of Bartnik [5, 6] on static vacuum extensions.

Given a compact Riemannian manifold (M^{n+1}, g_M), with non-empty boundary, form $N = S^1 \times M$ and set

$$g_N = u^2 d\theta^2 + g_M, \tag{6.1}$$

where u is a positive function on M. The static vacuum Einstein equations are the equations

$$Ric_{g_N} = 0. \tag{6.2}$$

When expressed on (M, g_M), these are equivalent to the system:

$$uRic_{g_M} = D^2 u, \quad \Delta u = 0, \tag{6.3}$$

where the Hessian and Laplacian are taken with respect to g_M.

Let $Met_S(N) = Met_S^{m,\alpha}(N)$ be the space of $C^{m,\alpha}$ static metrics on N, i.e. metrics of the form (6.1). One has $Met_S^{m,\alpha}(N) \simeq Met^{m,\alpha}(M) \times C_+^{m,\alpha}(M)$, where $C_+^{m,\alpha}(M)$ is the space of positive $C^{m,\alpha}$ functions on M. The space $\mathbb{E}_S(N)$ of static Einstein (Ricci-flat) metrics on N is equivalent to the space of pairs $g_N = (g_M, u) \in Met(M) \times C_+(M)$ as in (6.1) satisfying the equations (6.3). As before, one has a natural (Dirichlet) boundary map

$$\Pi : \mathbb{E}_S \to Met_S(\partial N), \quad \Pi(g) = g^T|_{\partial N}. \tag{6.4}$$

As in §2, consider the Bianchi-gauged Einstein operator on N,

$$\Phi : Met_S^{m,\alpha}(N) \to S_2^{m-2,\alpha}(N) \tag{6.5}$$

$$\Phi_{\widetilde{g}}(g_N) = Ric_{g_N} + 2\delta^*_{g_N}\beta_{\widetilde{g}}(g_N),$$

where $\beta_{\widetilde{g}}$ is the Bianchi operator with respect to a background metric $\widetilde{g} \in \mathbb{E}_S(N)$. It is straightforward to verify that all of the results of §2 hold for the spaces $Met_S(N)$, $\mathbb{E}_S(N)$ and $\mathcal{E}_S(N)$. In particular:

Proposition 6.1. *Suppose $\pi_1(N, \partial N) = 0$ and $m \geq 3$. Then the space $\mathcal{E}_S^{m,\alpha}$ of static Einstein metrics on N is a C^∞ smooth Banach manifold,* (*Fréchet manifold when $m = \infty$*), *for which the boundary map Π is C^∞ smooth.*

Proof: This is proved in [2], using the relation of the static vacuum equations to the Einstein equations coupled to a scalar field. Alternately, it is easy to see that this result may be proved in exactly the same way as the proof of Theorem 2.1 in [2], but working equivariantly with respect to the free isometric S^1 action. ■

The analog of Proposition 3.2 is the following:

Proposition 6.2. *Near any given background solution $\widetilde{g} \in \mathbb{E}_S$, the operator $\Phi = \Phi_{\widetilde{g}}$ in (6.5) with either of the boundary conditions:*

$$\beta_{\widetilde{g}}(g_N) = 0, \quad g_N|_{T(\partial M)} = \gamma_M, \quad H_M = h \ \text{ at } \ \partial M, \ \text{ or} \tag{6.6}$$

$$\beta_{\widetilde{g}}(g_N) = 0, \quad g_N|_{T(\partial M)} = \gamma_M, \quad g_N(\widetilde{N}, \widetilde{N}) = \gamma_{00} \ \text{ at } \ \partial M, \tag{6.7}$$

is an elliptic boundary value problem of index 0.

Here the induced metric γ_M is in $Met^{m,\alpha}(\partial M)$ while the mean curvature H_M of ∂M in (M, g_M) is in $C^{m-1,\alpha}(\partial M)$. Note that the potential u does not enter this boundary data and so is formally undetermined at ∂M; it plays the same role as the trace (with respect to B) in §3, which is also undetermined.

Proof: This result is very similar to previous work in [21] and [16], but since the result is needed strongly below and since there is no complete proof in the literature, we provide the details here.

It suffices to show that the leading order part of the linearized operators form an elliptic system. The leading order symbol of $L = D\Phi$ is given by

$$\sigma(L) = -|\xi|^2 I, \tag{6.8}$$

where I is the $Q \times Q$ identity matrix, with $Q = ((n+1)(n+2)/2) + 1$; Q is the sum of the dimension of the space of symmetric bilinear forms on $\mathbb{R}^{n+1}$, together with the extra vertical S^1 direction. For static metrics, all components of the metric are locally functions on $\mathbb{R}^{n+1}$, and all derivatives in the vertical S^1 direction are trivial. In the following, the subscript 0 represents the direction normal to ∂M in M, (or ∂N in N), subscript 1 denotes the vertical direction, tangent to S^1, while indices 2 through $n+1$ represent the directions tangent to ∂M. Note that one has $h_{1\alpha} = 0$, for all $\alpha \neq 1$. The positive roots of (6.8) are $i|\xi|$, with multiplicity Q at $\xi \in T^*(\mathbb{R}^{n+1})$.

Writing $\xi = (z, \xi_i)$, $i = 0, 2, \ldots, n+1$, (as above $\xi_1 = 0$), the symbols of the leading order terms in the boundary operators are given by:

$$-2izh_{0k} - 2i\sum_{j\geq 2}\xi_j h_{jk} + i\xi_k trh = 0, \quad k \geq 2,$$

$$-2izh_{00} - 2i\sum_{k\geq 2}\xi_k h_{0k} + iztrh = 0,$$

$$h^T = (\gamma')^T$$

$$h_{00} = \omega \ \text{ or } \ H'_h = \omega,$$

where $h^T \in S_2(\partial M)$. This gives $(n+1) + \frac{n(n+1)}{2} + 1 = Q$ boundary equations, as required. Ellipticity requires that the operator defined by the boundary symbols above has trivial kernel when z is set to the root $i|\xi|$. Carrying this out then gives the system

$$2|\xi|h_{0k} - 2i\sum_{j\geq 2}\xi_j h_{jk} + i\xi_k trh = 0, \quad k \geq 2, \tag{6.9}$$

$$2|\xi|h_{00} - 2i\sum_{k\geq 2}\xi_k h_{0k} - |\xi| trh = 0, \tag{6.10}$$

$$h_{11} = \phi, \quad h^T = 0, \tag{6.11}$$

$$h_{00} = 0 \ \text{ or } \ H'_h = 0, \tag{6.12}$$

where ϕ is an undetermined function.

Multiplying (6.9) by $i\xi_k$ and summing gives, via (6.11),

$$2|\xi|i\sum_{k\geq 2}\xi_k h_{0k} = |\xi|^2 trh.$$

Substituting (6.10) on the term on the left above then gives

$$2|\xi|^2 h_{00} - 2|\xi|^2 trh = 0.$$

Since $trh = h_{00} + \phi$, it follows that $\phi = 0$.

Now if $h_{00} = 0$, then $trh = 0$ and (6.9) then gives $h_{0k} = 0$, so that $h = 0$, as required. If instead one uses the second condition $H'_M = 0$ in (6.12), a simple computation shows that to leading order, $2H'_h = tr_{\partial M}(\nabla_N h - 2\delta^*(h(N)^T))$, which has symbol $\sum_{k\geq 2}(izh_{kk} - 2i\xi_k h_{0k})$. Setting this to 0 at the root $z = i|\xi|$ gives

$$\sum_{k\geq 2}(|\xi|h_{kk} + 2i\xi_k h_{0k}) = 0.$$

Via (6.11), this gives $-2i\sum\xi_k h_{0k} = 0$, and substituting this in (6.10) and using the fact that $\phi = 0$ gives

$$2|\xi|h_{00} - |\xi|h_{00} = 0,$$

so that $h_{00} = 0$, and thus from the work above, $h = 0$, again as required.

Finally, in the setting of Proposition 3.2, it is shown in [2] that the data (6.6) or (6.7) may be continuously deformed through elliptic boundary data, to elliptic boundary data for which L is self-adjoint and so of index 0. The proof in [2] carries over to this setting with only minor change, and so we refer to [2] for further details. The homotopy invariance of the index then completes the proof. ∎

In analogy to (3.11), consider the smooth boundary map

$$\Pi_H : \mathcal{E}_S^{m,\alpha}(N) \to Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{6.13}$$
$$\Pi_H(g_N) = (\gamma_M, H),$$

where $H = H_M$. Proposition 6.2 implies that the map Π_H is Fredholm of index 0, and so in particular the linearization $D\Pi_H$ has finite dimensional kernel and cokernel. Let

$$K_H = Ker D\Pi_H, \tag{6.14}$$

so that, when viewed as forms on M, K_H consists of pairs (κ, u') with $\kappa^T = 0$ and $H'_\kappa = 0$ at ∂M and satisfying the linearized static vacuum equations (6.3).

Remark 6.3. It is instructive here to consider the scalar constraint (3.4) in this context. First, on ∂N, the scalar constraint for g_N as in (6.1)-(6.2) is

$$|A|^2 - H^2 + s_{\gamma_{\partial N}} = 0, \tag{6.15}$$

where A, H and $s_{\gamma_{\partial N}}$ are the data associated with the metric g_N at ∂N. Thus, for $g_N \in \mathbb{E}_S^{m,\alpha}(N)$, one has $|A|^2$, $H^2 \in C^{m-1,\alpha}(\partial N)$ and hence $s_{\gamma_{\partial N}} \in C^{m-1,\alpha}(\partial N)$. Standard formulas give $s_{\gamma_{\partial N}} = s_{\gamma_{\partial M}} - 2\frac{\Delta u}{u}$. Thus, although the regularity of $g_N \in Met^{m,\alpha}(N)$ shows that each term $s_{\gamma_{\partial M}}$ and $\frac{\Delta u}{u}$ is only in $C^{m-2,\alpha}(\partial N)$, the constraint above implies that the difference is in the smoother space $C^{m-1,\alpha}(\partial N)$. This is all of course consistent with the boundary map Π_H in (6.13) being Fredholm.

The same analysis holds for the static vacuum Einstein equations (6.3) on (M, g_M). In this case, the scalar constraint reads, (since $s_{g_M} = 0$),

$$|A|^2 - H^2 + s_{\gamma_{\partial M}} = -2Ric(N,N) = -2u^{-1}D^2u(N,N). \tag{6.16}$$

Here $|A|^2$, $H^2 \in C^{m-1,\alpha}(\partial M)$, while $s_{\gamma_{\partial M}} \in C^{m-2,\alpha}(\partial M)$. On the other hand, $u^{-1}D^2u \in C^{m-2,\alpha}(\partial M)$, and so the equation above expresses again the fact that the difference is in the smoother space $C^{m-1,\alpha}(\partial M)$.

By Theorem 1.1, if $Z(H_N) = 0$ and Z is Killing on the boundary, then Z extends to a Killing field on any Einstein filling. Thus for instance if ∂N is static and $Z(H_N) = 0$ at ∂N, then any Einstein filling (N, g) also has an isometric S^1 action. Although such a filling is not automatically static, in many cases it is, cf. [4] for further analysis in this direction. Note moreover that a filling static Einstein metric is not necessarily strictly static, i.e. one does not necessarily have $u > 0$ globally.

In the case of static vacuum solutions N, it is useful to replace the condition $X(H_N) = 0$ by the condition $X(H_M) = 0$, where as above H_M is the mean curvature of ∂M in (M, g_M). This is carried out in the next result.

Proposition 6.4. *Let (N, g) be a static vacuum Einstein metric, with $\pi_1(N, \partial N) = 0$, and suppose X is a Killing field on ∂N. If $X(H_M) = 0$, then X extends to a Killing field on (N, g).*

Proof: The proof is similar to the proof of Theorem 1.1. We work throughout in the context of $(\partial N, \gamma_N)$. The 2nd fundamental form A of ∂N in N has the block form

$$A = \begin{pmatrix} N(\nu) & 0 \\ 0 & A_M \end{pmatrix}, \tag{6.17}$$

where A_M is the 2nd fundamental form of ∂M in (M, g_M) and $\nu = \log u$. Let $\tau = \tau_N = A - H\gamma_N$, so that τ has the block form

$$\tau = \begin{pmatrix} -H_M & 0 \\ 0 & A - H_M\gamma - N(\nu)\gamma \end{pmatrix}. \tag{6.18}$$

By assumption, $X(H_M) = 0$ and $X(u) = 0$ on ∂M, (since X is a Killing field on ∂N). Since $X(H) = X(H_M) + XN(\nu)$, it suffices by Theorem 1.1 to prove that $XN(\nu) = 0$. Without loss of generality, we assume that X is tangent to M.

As noted following (5.8), one has here also

$$\delta^T(\kappa(N)^T) = 0; \tag{6.19}$$

(the condition $X(H) = 0$ is not needed for the derivation of (5.8)). Let V be the unit vertical vector field on N; $V = Z/|Z|$, where Z is the static Killing field on N. Since the metric g_N is a warped product, so in particular $M \subset N$ is totally geodesic, one has

$$\nabla_H V = 0, \quad \text{and} \quad \nabla_V V = -d\nu, \tag{6.20}$$

for any horizontal vector H, (i.e. H tangent to M). For the computation to follow, we note that

$$\nabla_V \kappa = 0. \tag{6.21}$$

To prove (6.21), since Z is Killing on N, $\nabla_Z \kappa = \mathcal{L}_Z \delta^* X = \frac{1}{2}\mathcal{L}_Z\mathcal{L}_X g = \frac{1}{2}\mathcal{L}_X\mathcal{L}_Z g + \frac{1}{2}\mathcal{L}_{[Z,X]}g = \frac{1}{2}\mathcal{L}_{[Z,X]}g$. Thus, it suffices to show that $[Z, X] = 0$. Now the gauge condition $\beta\delta^* X = 0$ is geometric, and so invariant under isometries. Thus, X will be invariant under the isometric flow of Z on N provided it is so at the boundary, i.e. $[Z, X] = 0$ at ∂N. But this is a simple calculation. Namely, for any vector T tangent to ∂N, $\langle \nabla_X Z, T\rangle = -\langle \nabla_T Z, X\rangle = -T\langle Z, X\rangle + \langle Z, \nabla_T X\rangle = -\langle T, \nabla_Z X\rangle$, where the last equality follows from $\langle Z, X\rangle = 0$, i.e. X is tangent to M, together with the Killing property of X on ∂N. This shows that $\langle [Z, X], T\rangle = 2\langle \nabla_T X, Z\rangle = 0$, by (6.20) and the fact that $X(\nu) = 0$ on ∂N. A similar calculation shows that $\langle [Z, X], N\rangle = 0$ and hence $[Z, X] = 0$ at ∂N, proving (6.21).

It follows from (6.21) that, at ∂N, $0 = (\nabla_V \kappa)(N) = \nabla_V(\kappa(N)) - \kappa(\nabla_V N)$. Observe that $\kappa(\nabla_V N) = \kappa(A(V)) = N(\nu)\kappa(V)$. Since $\kappa^T = 0$ on ∂N, this gives, on ∂N,

$$\langle \nabla_V \kappa(N), V\rangle = 0. \tag{6.22}$$

It follows from (6.22) that $\langle \nabla_V \kappa(N)^T, V\rangle + \langle \nabla_V \kappa_{00} N, V\rangle = 0$, which gives $\langle \nabla_V \kappa(N)^T, V\rangle = -\langle \nabla_V(\kappa_{00}N), V\rangle = -\kappa_{00}\langle \nabla_V N, V\rangle = -\kappa_{00}N(\nu)$. Hence, via (6.19),

$$\delta_M(\kappa(N)^T) = \langle \nabla_V \kappa(N)^T, V\rangle = -\kappa_{00}N(\nu). \tag{6.23}$$

Given these preliminaries, we now apply the same analysis as in the proof of Lemma 5.1, beginning with (5.3), but with $h = \phi V \cdot V$, (in place of $h = \phi g$). First, one easily sees that the a terms in (5.3) vanish again. Next, via (5.4), and using (6.20) with $\kappa^T = 0$ on ∂N, one also easily sees that $\langle \tau'_{\phi V\cdot V}, \kappa\rangle = 0$. Thus, the analog of (5.3) becomes

$$\int_N \langle E'(\phi V \cdot V), \kappa\rangle = \int_{\partial N} \langle \tau'_\kappa, \phi V \cdot V\rangle.$$

To compute the right side, one has $H\langle \gamma', \phi V \cdot V\rangle = \phi H \kappa(V,V) = 0$, while $H'\langle \gamma, \phi V \cdot V\rangle = \frac{1}{2}\phi X N(\nu)$. Next, one computes term-by-term

$$2\langle A'_\kappa, \phi V \cdot V\rangle = \phi[NX(\nu) - 2\langle \nabla_V(\kappa(N)^T), V\rangle - \kappa_{00}N(\nu)].$$

Here we use $A(V,V) = N(\nu)$ and $\kappa(V,V) = X(\nu)$ as well as (6.20) and (6.21). By (6.23), the middle term on the right equals $2\kappa_{00}N(\nu)$. Combining the above then gives

$$2\langle \tau'_\kappa, \phi V \cdot V\rangle = \phi\{[N,X](\nu) + \kappa_{00}N(\nu)\} = 0.$$

To verify the last equality, since by (6.20) and (6.22), $\langle \kappa(N)^T, d\nu\rangle = 0$, one has $\langle \nabla_N X, d\nu\rangle = -\langle \nabla_{d\nu}X, N\rangle = -N(\nu)\langle \nabla_N X, N\rangle + \langle X, \nabla_{d\nu}N\rangle = -N(\nu)\kappa_{00} + \langle d\nu, \nabla_X N\rangle$, so that $[N,X](\nu) = -N(\nu)\kappa_{00}$, as claimed. Thus

$$\int_N \langle E'(\phi V \cdot V), \kappa\rangle = 0. \tag{6.24}$$

Next, we compute the E' integral, term-by-term, via (4.21). We will integrate all bulk terms involving derivatives of ϕ by parts, giving boundary terms at ∂N involving ϕ and $N(\phi)$. By (6.24), the sum of these boundary terms vanishes and this suffices for the proof. Thus, in the following we will consider only the form of the boundary terms, and ignore the bulk integrals.

To begin, for the first term in (4.21), using (6.20) one computes

$$\begin{aligned} D^*D(\phi V \cdot V) &= -\nabla_{e_i}\nabla_{e_i}(\phi V \cdot V) + \nabla_{\nabla_{e_i e_i}}(\phi V \cdot V) \\ &= -(\Delta\phi)V \cdot V - \nabla_V\nabla_V(\phi V \cdot V) - \nabla_{d\nu}(\phi V \cdot V) = \\ &-(\Delta\phi)V \cdot V + 2\phi((\nabla_V d\nu) \cdot V - 2d\nu \cdot d\nu) - d\nu(\phi)V \cdot V. \end{aligned}$$

Pairing this with κ gives $-(\Delta\phi)\kappa(V,V) + 2\phi\langle \nabla_V d\nu, \kappa(V)\rangle - 2\phi\kappa(d\nu, d\nu) - \kappa(V,V)\langle d\nu, d\phi\rangle$. Since $\kappa(V,V) = X(\nu)$, this integrates to, via integration-by-parts,

$$\int_N -\phi\Delta(X(\nu)) + 2\phi[X(\nu)|d\nu|^2 - \kappa(d\nu, d\nu)] + \phi\langle dX(\nu), d\nu\rangle + \int_{\partial N} \phi N(X(\nu)). \tag{6.25}$$

The other boundary terms vanish, since $X(\nu) = 0$ at ∂N; we have also used the fact that $\Delta\nu = 0$ on N in integrating by parts. For the second, curvature, term in (4.21), no derivatives of ϕ appear, so there are no boundary terms.

For the third term, one has

$$-\int_N \langle D^2\phi, \kappa\rangle = -\int_N \phi\delta\delta\kappa - \int_{\partial N} [\kappa(d\phi, N) + \phi(\delta\kappa)(N)]. \tag{6.26}$$

Here one has $\delta\kappa = -\frac{1}{2}dtr\kappa$, so $\delta\delta\kappa = \frac{1}{2}\Delta tr\kappa$ and $(\delta\kappa)(N) = -\frac{1}{2}N(tr\kappa)$. Note that $\kappa(d\phi, N) = \kappa_{00}N(\phi) + \langle (\kappa(N))^T, d\phi\rangle$, and the second term integrates to 0 by (6.19).

Next, for the fourth term, one has $\delta(\phi V \cdot V) = -\nabla_{e_i}(\phi V \cdot V)(e_i) = -e_i(\phi)(V \cdot V)(e_i) - 2\phi(\nabla_{e_i}V \cdot V)(e_i) = 2\phi d\nu$. So

$$\delta(\phi V \cdot V) = 2\phi d\nu.$$

The fourth term is then

$$(6.27)\qquad -2\int_N \delta(\phi d\nu)tr\kappa = -2\int_N \phi\langle d\nu, dtr\kappa\rangle + 2\int_{\partial N} \phi N(\nu)tr\kappa.$$

The fifth term is

$$(6.28)\qquad \int_N (\Delta\phi)tr\kappa = \int_N \phi\Delta tr\kappa + \int_{\partial N} tr\kappa N(\phi) - \phi N(tr\kappa).$$

Again, the sixth term gives no boundary term, while the seventh term is

$$(6.29)\qquad -4\int_N \langle\delta^*(\phi d\nu), \kappa\rangle = 2\int_N \langle\phi d\nu, dtr\kappa\rangle - 4\int_{\partial N} \phi\kappa(N, d\nu).$$

Here $\kappa(N, d\nu) = \langle\nabla_V\kappa(N), V\rangle = 0$, by (6.22).

Now collect the boundary terms above which have only an $N(\phi)$-factor. The sum of these in (6.26) and (6.28) must vanish, so that

$$\int_{\partial N} N(\phi)[-\kappa_{00} + tr\kappa] = 0.$$

This is of course trivial. Similarly, collecting the ϕ-boundary terms in (6.25)-(6.29) gives

$$\int_{\partial N} \phi[N(X(\nu)) + \tfrac{1}{2}N(tr\kappa) + 2N(\nu)tr\kappa - N(tr\kappa)] = 0,$$

and since ϕ is arbitrary,

$$(6.30)\qquad N(X(\nu)) + \tfrac{1}{2}N(tr\kappa) + 2N(\nu)tr\kappa - N(tr\kappa) = 0.$$

We now combine this with some further information. First, recall that $NX(\nu) = N(\kappa(V, V))$, while $N(\kappa(V, V)) - N(tr\kappa) = -N(tr_M\kappa)$. Thus $NX(\nu) - N(tr\kappa) = -N(tr_M\kappa)$ in (6.30). Next, since $(H_M)'_\kappa = tr_M(A'_\kappa) = 0$, one has

$$N(tr_M\kappa) + 2\delta_M(\kappa(N)^T) - \kappa_{00}H_M - N(\kappa_{00}) = 0.$$

Substituting this in (6.30) and using (6.23) gives

$$-2\kappa_{00}N(\nu) - \kappa_{00}H_M - N(\kappa_{00}) + \tfrac{1}{2}N(tr\kappa) + 2\kappa_{00}N(\nu) = 0.$$

As in the proof of Lemma 5.1, $(\beta(\kappa))(N) = 0$, so that

$$(6.31)\qquad \tfrac{1}{2}N(tr\kappa) = \kappa_{00}H_N + N(\kappa_{00}),$$

where we have also used (6.19). Comparing the two equations above gives

$$(6.32)\qquad \kappa_{00}N(\nu) = 0,$$

since $H_N = H_M + N(\nu)$. Via (6.30), this gives

$$(6.33)\qquad NX(\nu) = \tfrac{1}{2}N(tr\kappa).$$

Now the equation $H'_\kappa = \frac{1}{2}NX(\nu)$ gives, (using (6.19)),

$$N(tr\kappa) = \tfrac{1}{2}NX(\nu) + H_N\kappa_{00} + N(\kappa_{00}).$$

Comparing this with (6.31) shows that

$$NX(\nu) = N(tr\kappa),$$

and so via (6.33), $NX(\nu) = 0$. As discussed preceding (6.24), (6.32) implies that $[X, N](\nu) = 0$ at ∂N, and hence it follows that

$$XN(\nu) = 0,$$

which completes the proof. ∎

Next, we generalize Proposition 6.4 to the case where X is assumed to be Killing just on $(\partial M, \gamma_M)$ instead of on the full boundary $(N, \partial N)$.

Proposition 6.5. *Let (N, g) be a static vacuum Einstein metric with $\pi_1(N, \partial N) = 0$, and suppose $H_M = const \neq 0$ on ∂M. Then any Killing field on $(\partial M, \gamma_M)$ extends to a Killing field on (N, g_N).*

Proof: Again, we work throughout on (N, g_N). Let X be a Killing field on ∂M. A simple calculation then gives

$$\delta^* X = X(\nu) V \cdot V \quad \text{on} \quad \partial N, \tag{6.34}$$

where as before V is the unit vector along the S^1 fibers.

In this situation, we use the more general formula (4.4) in place of (4.1). Observe that Proposition 6.4 and Lemma 4.2 imply that the linearized divergence constraint is solvable in any direction $h \in T_{\gamma_N} Met(\partial N)$. Hence, for any variation h of the boundary metric γ_N of ∂N, one has

$$\int_{\partial N} \langle \mathcal{L}_X \tau, h \rangle = 2 \int_{\partial N} \langle \tau'_h, \delta^* X \rangle + \int_{\partial N} (\langle \delta^* X, \tau \rangle trh + \delta X \langle \tau, h \rangle + 4\langle \tau \circ \delta^* X, h \rangle - 4\langle \tau \circ h, \delta^* X \rangle). \tag{6.35}$$

As test form in (6.35), we just choose $h = \delta^* X$ on ∂N, so that by (6.34), h is purely vertical. One computes $\langle \delta^* X, \tau \rangle trh = -H_M X(\nu)^2$, while $\delta X \langle \tau, h \rangle = -X(\nu)(-H_M) X(\nu) = H_M X(\nu)^2$. Thus, the first two terms in the right-most integral in (6.35) cancel. Similarly, the last two terms also cancel. For the first term on the right in (6.35), since $\tau(V, V) = -H_M$, τ'_h is the variation of $-H$ in the direction $h = \delta^* X$. Since X is tangent to ∂M and H_M is constant, the flow of X preserves H_M, so that, (for this choice of h), one has $\langle \tau'_h, \delta^* X \rangle = 0$.

It follows then from (6.35) that

$$\int_{\partial N} X(\nu)(\mathcal{L}_X \tau)(V, V) = 0. \tag{6.36}$$

From (6.18), one computes $(\mathcal{L}_X \tau)(V, V) = -X(H_M) + 2H_M \langle [X, V], V \rangle$, while $[X, V] = X(\nu) V$. Since again H_M is constant, $(\mathcal{L}_X \tau)(V, V) = 2H_M X(\nu)$, and so (6.36) gives

$$\int_{\partial N} H_M X(\nu)^2 = 0.$$

Since $H_M = const. \neq 0$ by assumption, this implies $X(\nu) = 0$, so that by (6.34), X is a Killing field on ∂N. The result then follows from Proposition 6.4. ∎

Remark 6.6. (i). We expect Proposition 6.5 also holds when $H_M = 0$, but the proof will be more complicated. In any case, static vacuum Einstein metrics admitting a compact minimal hypersurface are rather rare, cf. [17] for example.

(ii). The Einstein equations (6.3) for static metrics g_N on $N = S^1 \times M$ with non-zero λ as in (1.1) have the form

$$Ric_{g_M} = u^{-1} D^2 u + \lambda g_N, \quad -u^{-1} \Delta u = \lambda,$$

on (M, g), where the Hessian and Laplacian are with respect to g_M. It is straightforward to check that all of the results above remain valid for static vacuum Einstein metrics with any $\lambda \leq 0$ and for any $\lambda \in \mathbb{R}$ modulo Lemma 5.1 or Remark 5.2.

Next, we turn to exterior boundary value problems and the proof of Theorem 1.3. Thus, as at the end of §4, suppose $N^{n+2} = S^1 \times M^{n+1}$ is an open manifold with compact boundary ∂M and with a finite number of (locally) asymptotically flat ends. We consider complete static Einstein metrics on N which are ALF at infinity. Note that consequently the potential function u tends to a constant at each end of N.

Proposition 6.7. *The space of asymptotically locally flat static vacuum Einstein metrics on an exterior region $N = S^1 \times M$, (with $\pi_1(N, \partial N) = 0$), is a smooth Banach manifold. The boundary map*

$$\Pi_H : \mathcal{E}_S^{m,\alpha} \to Met^{m,\alpha}(\partial M) \times C^{m-1,\alpha}(\partial M), \tag{6.37}$$
$$\Pi_H(g) = (\gamma, H),$$

$H = H_M$, is smooth and Fredholm, of Fredholm index 0.

Proof: This is proved in [2], in the same way as in Proposition 6.1. ■

As noted at the end of §4, all of the results of §3 - §5 hold in the context of asymptotically locally flat metrics with compact boundary, since all of these results take place locally at or near ∂N.

These results, and in particular Proposition 6.7, are relevant to the static extension conjecture of Bartnik [5, 6], described as follows. Let $M = \mathbb{R}^3 \setminus B$, where B is a 3-ball. Then conjecturally, given any boundary data (γ, H) on $S^2 = \partial M$, as in (6.37), there is a complete, asymptotically flat static vacuum solution (M, g_M) on M, realizing the given boundary data. In other words, the map Π_H is surjective. There are other more restrictive, and so perhaps more realistic, versions of this conjecture which assume for instance some positivity conditions on the boundary data, such as $H > 0$. Another natural question is the uniqueness of static vacuum extensions, i.e. the question of when Π_H is injective.

Proof of Theorem 1.3.

Suppose (M, g_M) is an asymptotically flat solution to the static vacuum Einstein equations with compact boundary ∂M, satisfying $\pi_1(M, \partial M) = 0$, and suppose further that γ_M is a round metric on a sphere S^n with mean curvature $H = const \neq 0$.

It then follows from Proposition 6.5 that any Killing field on S^n extends to a Killing field on (M, g_M), and hence (M, g_M) has an isometric $SO(n+1)$ action, (M is spherically symmetric). As in the proof of Corollary 1.2, the static vacuum Einstein equations then reduce to a system of ODE's, and it is straightforward to verify that the only smooth asymptotically flat solution is given by the Schwarzschild metric

$$g_m = (1 - \frac{2m}{r^{n-1}})^{-1} dr^2 + r^2 g_{S^n(1)},$$

of mass m, for some $m \in \mathbb{R}$, $r^{n-1} \geq \max(2m, 0)$. ■

Note that for the Schwarzschild metric, the sphere $S(r)$, given as the r-level set of the function r, has mean curvature

$$H = \frac{n}{r}\sqrt{1 - \frac{2m}{r^{n-1}}}.$$

In particular, if $(\partial M, \gamma_M) = (S^n, g_{+1})$, and $H = n$, then g_m is the flat metric on $\mathbb{R}^{n+1} \setminus B(1)$.

The black hole uniqueness theorem [14, 9, 13] states that a complete, asymptotically flat solution of the static vacuum Einstein equations which is smooth up to the horizon $\{u = 0\}$ is isometric to the Schwarzschild metric, (with $m > 0$). At the horizon, the equations (6.3) imply that

$$A = 0, \ u = 0, \ \text{and } N(u) = const. \tag{6.38}$$

The data (6.38) consist of more than half the full geometric Cauchy data $(\gamma, A, u, N(u))$ for the static vacuum equations (6.3); only the metric γ is not prescribed on ∂M. Thus, from the standpoint of elliptic systems, the boundary data (6.38) are overdetermined, although the static equations (6.3) are degenerate at the horizon. On the other hand, the data in Theorem 1.3 are determined elliptic data. From this point of view, Theorem 1.3 is a strengthening of black hole uniqueness.

Also, Theorem 1.3 holds in all dimensions, while the black hole uniqueness theorem fails in dimensions greater than 4, cf. [10] and references therein.

Next we prove that the linearized version of Proposition 6.5 also holds. As in (6.14), let

$$K_H = Ker D\Pi_H.$$

Proposition 6.8. *If (M, g, u) is a static vacuum solution with $H = const \neq 0$ at ∂M and $\pi_1(M, \partial M) = 0$, then K_H is invariant under $Isom_0(\partial M, \gamma)$, i.e. for any form $\alpha \in K_H$ and Killing field X on $(\partial M, \gamma)$,*

$$\mathcal{L}_X \alpha = 0, \tag{6.39}$$

on M, where X is extended uniquely to a Killing field on (M, g, u).

Proof: This follows quite directly from the proof of Proposition 6.5 by linearization. To see this, consider the curve of metrics $g_u = g + u\alpha$ on N. The curve g_u satisfies the static vacuum Einstein equations to first order in u at $u = 0$. Note that $\alpha \in K_H$ implies that $\gamma'_\alpha = 0$ and $H'_\alpha = 0$. In particular, $\alpha^T = 0$, where T denotes projection or restriction to $T(\partial M)$, and the mean curvature of ∂M is constant to first order in u. The only non-zero component of α at ∂N is then the vertical component, given by

$$\alpha(V, V) = 2\nu',$$

where ν' is the variation of the potential ν.

The proof now proceeds by linearizing the proofs of Propositions 6.5, 6.4 and Theorem 1.1. First observe that the analog of (6.34) holds, in that

$$\mathcal{L}_X \alpha = 2(\delta^*)' X = 2X(\nu')V \cdot V \quad \text{on} \quad \partial N.$$

Since the curve g_u is static Einstein to first order in u, as in (6.35) one has along g_u:

$$\int_{\partial N} \langle \mathcal{L}_X \tau, h \rangle dV_{g_u} = 2\int_{\partial N} \langle \delta_h \tau', \delta^* X \rangle dV_{g_u} + \int_{\partial N} (\langle \delta^* X, \tau \rangle trh + \delta X \langle \tau, h \rangle) dV_{g_u} + o(u), \tag{6.40}$$

where all metric quantities are evaluated on g_u; X is a fixed Killing field for γ_0. As before, evaluate (6.40) on $h = \delta^* X = \frac{1}{2}\mathcal{L}_X g$. Taking the derivative with respect to u at $u = 0$ gives, as in (6.36),

$$\int_{\partial N} X(\nu)(\mathcal{L}_X \tau')(V, V) = 0,$$

where τ' is the derivative of τ_{g_u} at $u = 0$. Proceeding then as in the proof of Proposition 6.5, and using the fact that $H' = 0$, gives, as before,

$$\int_{\partial N} HX(\nu')^2 = 0,$$

and hence $X(\nu') = 0$. Thus, it follows that

$$(\mathcal{L}_X \alpha)^T = 2(\frac{d}{du}(\delta^*_{g_u} X))^T = 0 \quad \text{at} \quad \partial N. \tag{6.41}$$

(This is the linearization of the statement that $(\mathcal{L}_X g)^T = 0$ at ∂N).

Next, as above, one carries out the proof of Proposition 6.4 with respect to g_u in place of g. This introduces error terms of order $o(u)$, which then vanish when taking the derivative. We leave it to the reader to follow through the calculations, when then give

$$XN(\nu') = 0 \quad \text{at} \quad \partial N.$$

Finally, one linearizes the proof of Lemma 5.1 and Theorem 5.1 in the same way, to see that $\frac{d}{du}\mathcal{L}_X g_u = 0$ on N, which implies (6.39). ■

Corollary 6.9. *The flat solution* $(\mathbb{R}^{n+1}\setminus B(1), g_0, 1)$ *of the static vacuum Einstein equations is non-degenerate, in the sense that at this metric,*

$$K_H = Ker D\Pi_H = 0. \tag{6.42}$$

The same result holds for the interior flat solution $(B^{n+1}(1), g_0, 1)$.

Proof: Proposition 6.8 implies that at any such solution, any $\kappa \in K_H$ is $SO(n+1)$ invariant, i.e. spherically symmetric. As in the proof of Theorem 1.3, the form κ then satisfies a system of ODE's, and it is simple to verify that this has no non-trivial solutions. ■

It follows that the boundary map Π_H is a diffeomorphism near the standard flat data (γ_{+1}, n) in $\mathbb{R}^{n+1}$. Namely, by Theorem 1.3, $\Pi_H^{-1}(\gamma_{+1}, n)$ is a single point, given by the standard flat solution as above. Corollary 6.9 then shows that the boundary data (γ_{+1}, n) are a regular value of Π_H and hence by the inverse function theorem, (or Proposition 6.7), Π_H is a diffeomorphism near the standard flat data. Thus, any given boundary data (γ, H) near (γ_{+1}, n) are realized uniquely as the boundary data of complete asymptotically flat static vacuum solutions.

We note that the same result, with the same proof, also holds for exterior regions of the Schwarzschild metric, for all non-critical values $m \neq m_0$ of the mass.

Remark 6.10. Corollary 6.9 generalizes a result of Miao [16], who proved this result for dimension $M = 3$, for boundary data and static vacuum solutions which are invariant under reflection in the standard coordinate planes in $\mathbb{R}^3$, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ invariant. It is useful to explain the differences in the approaches leading to these different results.

Thus, in [16], Miao essentially proves that the Bianchi-gauged Einstein operator Φ in (2.15), *together with* the boundary conditions (γ, H) imposed, has linearization giving an isomorphism at the standard flat solution, when all data satisfies the symmetry condition above. The result then follows from the inverse function theorem.

The approach taken here uses the fact that the operator Φ is a submersion at any (static) Einstein metric, leading via the implicit function theorem to the manifold theorem, Theorem 2.1. This involves no specification of boundary data. One then proves that the boundary map Π_H is a local diffeomorphism at the standard flat solution, when restricted to the manifold $\mathcal{E}_S$ of static vacuum solutions. This is of course different than requiring Φ is an isomorphism.

Loosely speaking, and in physics terminology, Miao works off-shell but with fixed boundary conditions, while the approach in this paper is on-shell, (i.e. on the space of solutions), but without fixed boundary conditions initially.

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