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FOLIATIONS BY STABLE SPHERES WITH CONSTANT MEAN CURVATURE FOR ISOLATED SYSTEMS WITH GENERAL ASYMPTOTICS

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ABSTRACT. We prove the existence and the uniqueness of a foliation by surfaces with constant mean curvature for asymptotically flat manifolds satisfying the Regge-Teitelboim condition at infinity. It is known that the center of mass is well-defined for manifolds satisfying this condition. We also show that the foliation is asymptotically concentric, and its geometric center is equal to the center of mass. The construction of the foliation generalizes the results of Huisken-Yau, Ye, and Metzger, where spherically asymptotically flat manifolds and their small perturbations were studied.

1. INTRODUCTION

Whether a foliation of constant mean curvature surfaces uniquely exists in an exterior region of an asymptotically flat manifold is a fundamental problem in general relativity. The significance of this problem is that the foliation provides an intrinsic geometric structure near infinity, supplies a definition of the center of mass in the setting of general relativity, and has a relation to the Hawking mass.

Currently, a widely-used definition of asymptotic flatness is expressed in terms of coordinates outside a compact set in the manifold and requires a suitable decay rate on the metric. The definition is convenient for calculation purposes, but it is unnatural and obscures interesting geometry and physics [Ya82, p.697]. In order to understand the canonical structure of asymptotically flat manifolds, Yau suggests that a constant mean curvature foliation is a promising description of asymptotic flatness[1]. Moreover, once the foliation exists and is unique, one can develop polar coordinates analogous to the polar coordinates in Euclidean space, and then a canonical concept of center of mass can been defined. On the other hand, the Hawking mass is a quantity introduced to capture the energy content of the region bounded

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[1]Bando, Kasue, and Nakajima [BKN89] provide another geometric description of the asymptotic flatness using curvature conditions.

by a two surface N which is defined as follows:

$$m_H(N) = \frac{|N|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \left(16\pi - \int_N H^2 \, d\sigma \right).$$

Christodoulou and Yau [CY86] have proven that the Hawking mass is nonnegative on a stable surface with constant mean curvature, and Bray [Br01] has shown that the Hawking mass is monotonically increasing along this foliation and converges to the ADM mass.

For the existence and the uniqueness of such a foliation, some results have been achieved for spherically asymptotically flat manifolds which are asymptotically flat manifolds with metrics of the form

$$\begin{aligned} g_{ij}(x) = \left(1 + \frac{2m}{|x|}\right)\delta_{ij} + p_{ij}, \\ p_{ij}(x) = O(|x|^{-2}), \partial^\alpha p_{ij}(x) = O(|x|^{-2-|\alpha|}). \end{aligned} \tag{1.1}$$

Huisken and Yau [HY96] proved the existence of the foliation assuming the metric is spherically asymptotically flat and showed the foliation is unique if each leaf is stable and lies outside a suitable compact set. Using the unique foliation, they defined the center of mass. Ye [Ye96] used a different approach and proved the existence of the foliation under the same assumption that the metric be spherically asymptotically flat, and the uniqueness of the foliation under slightly different conditions. A more general uniqueness result was proven by Qing and Tian [QT07]. Metzger [M07] generalized the previous results to manifolds whose metrics are small perturbations of spherically asymptotically flat metrics[2]. However, these results have been limited to asymptotically flat manifolds with special restrictions on the $|x|^{-1}$-term of the metrics. Especially, requiring manifolds to satisfy (1.1) corresponds to an artificial choice of the time-slice in the space-time. Furthermore, another notion of the center of mass is defined for asymptotically flat manifolds satisfying the Regge-Teitelboim condition, so it is desirable to generalize the previous results to this setting.

In this paper, we show that the foliation indeed exists in the exterior region of an asymptotically flat manifold satisfying the Regge-Teitelboim condition when the ADM mass is nonzero, and the foliation is unique under certain assumptions. Most importantly, we not only remove the strong condition on the $|x|^{-1}$-term of metrics, but also allow metrics to have more general decay rates. To clearly state the results, we first provide some definitions.

A three-manifold M with a Riemannian metric g and a two-tensor K is called a vacuum initial data set (M, g, K) if g and K satisfy the constraint

[2]Metzger considered a different foliation $\{\Sigma\}$ which is the constant θ foliation where the expansion $\theta = H \pm P$, H is the mean curvature of Σ, and $P = \mathrm{tr}^\Sigma K$. This foliation particularizes to the constant mean curvature foliation if one let $K = 0$.

equations

$$
\begin{aligned}
R_g - |K|_g^2 + (\mathrm{tr}_g(K))^2 = 0,\\
\mathrm{div}_g(K) - d(\mathrm{tr}_g(K)) = 0,
\end{aligned}
\tag{1.2}
$$

where R_g is the scalar curvature of M, and $\mathrm{tr}_g(K) = g^{ij}K_{ij}$. We use the Einstein summation convention and sum over repeated indices.

Definition 1.1. *Let $q \in (1/2, 1]$. We say (M, g, K) is asymptotically flat (AF) if it is a vacuum initial data set, and there exist coordinates $\{x\}$ outside a compact set, say B_{R_0}, in M such that*

$$
\begin{aligned}
g_{ij}(x) = \delta_{ij} + h_{ij}(x),\ h_{ij} = O(|x|^{-q}) \qquad & K_{ij}(x) = O(|x|^{-1-q})\\
g_{ij,k}(x) = O(|x|^{-1-q}) \qquad & K_{ij,k}(x) = O(|x|^{-2-q})\\
g_{ij,kl}(x) = O(|x|^{-2-q}) \qquad & K_{ij,kl}(x) = O(|x|^{-3-q}),
\end{aligned}
$$

and similarly on higher derivatives.

For AF manifolds, the ADM mass m is defined by

$$
m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{|x|=r} \sum_{i,j} \left(g_{ij,i} - g_{ii,j}\right)\nu_g^j \, d\sigma_g, \tag{1.3}
$$

where $\{|x| = r\}$ is the Euclidean sphere, ν_g is the unit outward normal vector field with respect to the metric g, and $d\sigma_g$ is the volume form of the induced metric from (M, g, K). Bartnik [B86] proves that the ADM mass is well-defined when the decay rate q is greater than $1/2$. Another equivalent definition of ADM mass is

$$
m = \frac{1}{16\pi} \lim_{r\to\infty} \int_{|x|=r} \left(Ric_{ij}^M - \frac{1}{2} R_g g_{ij} \right) (-2x^i)\nu_g^j \, d\sigma_g. \tag{1.4}
$$

Definition 1.2. *We say (M, g, K) is asymptotically flat satisfying the Regge-Teitelboim condition (AF-RT) if (M, g, K) is AF, and g, K satisfy these asymptotically even/odd conditions*

$$
\begin{aligned}
g_{ij}^{odd}(x) = O(|x|^{-1-q}) \qquad & K_{ij}^{even}(x) = O(|x|^{-2-q})\\
\left(g_{ij}^{odd}\right)_{,k}(x) = O(|x|^{-2-q}) \qquad & \left(K_{ij}^{even}\right)_{,k}(x) = O(|x|^{-3-q}),
\end{aligned}
$$

and on higher derivatives, where $f^{odd}(x) = f(x) - f(-x)$ and $f^{even}(x) = f(x) + f(-x)$, [RT]*. Notice that f^{odd} and f^{even} are only defined outside B_{R_0} in which the coordinates are defined.*

For (M, g, K) satisfying AF-RT, the center of mass $\mathcal{C}$ is defined by

$$
\begin{aligned}
\mathcal{C}^\alpha \;=\; \frac{1}{16\pi m} \lim_{r\to\infty} \Bigg(& \int_{|x|=r} x^\alpha (g_{ij,i} - g_{ii,j})\nu_g^j d\sigma_g \\
& - \int_{|x|=r} (h_{i\alpha}\nu_g^i - h_{ii}\nu_g^\alpha) \, d\sigma_g \Bigg).
\end{aligned}
\tag{1.5}
$$

It is noted that another notion of center of mass analogous to (1.4) using the three dimensional Einstein tensor and a Euclidean conformal Killing vector field has been studied and proven to be equivalent to $\mathcal{C}$ in [H08]. For the purpose of this paper, we only consider the above definition.

Our main theorems in this paper are the following:

Theorem 1. *If (M, g, K) is AF-RT with $q \in (1/2, 1]$, there exists a foliation by surfaces $\{\Sigma_R\}$ with constant mean curvature $H(\Sigma_R) = (2/R) + O(R^{-1-q})$ in the exterior region of M. Each leaf Σ_R is a $c_0 R^{1-q}$-graph over $S_R(\mathcal{C})$ and is strictly stable.*

The definition of *strictly stable* and *stable* are referred to Definition 3.4. Also, throughout this article, c and c_i are constants independent of R. For one single surface N, we have the following uniqueness result where the minimal radius is denoted by $\underline{r}(N) = \min\{|z| : z \in N\}$.

Theorem 2. *Assume (M, g, K) is AF-RT with $q \in (1/2, 1]$. There exists σ_1 so that if N has the following properties:*

(1) N is topologically a sphere
(2) N has constant mean curvature $H = H(\Sigma_R)$ where $R \geq \sigma_1$
(3) N is stable
(4) $\underline{r} \geq H^{-a}$ for some a satisfying $(9 - q)/(8 + 4q) < a \leq 1$

then $N = \Sigma_R$ for some R.

In Theorem 2, we do *not* assume that N is a leaf of some foliation. Thus, in the region $M \setminus B_{H^{-a}}(0)$, Σ_R is the only stable surface with constant mean curvature $H(\Sigma_R)$. In particular, $\{\Sigma_R\}$ is the only foliation by stable surfaces with constant mean curvature so that each leaf with mean curvature H lies inside $M \setminus B_{H^{-a}}(0)$. It is noted that when the decay rate $q = 1$, $a > 2/3$ which is exactly the restriction imposed in [M07], but the size of $B_{H^{-a}}(0)$ increases as q approaches to $1/2$. If we replace the condition on $\underline{r}(N)$ by the condition that the maximal radius $\bar{r}(N) = \max\{|z| : z \in N\}$ and $\underline{r}(N)$ are comparable, we derive a uniqueness result which holds outside a *fixed* compact set.

Theorem 3. *Assume (M, g, K) is AF-RT with $q \in (1/2, 1]$. There exist c_2 and σ_2 so that if N has the following properties:*

(1) N is topologically a sphere
(2) N has constant mean curvature $H = H(\Sigma_R)$ where $R \geq \sigma_2$
(3) N is stable
(4) $\bar{r} \leq c_2 (\underline{r})^{\frac{1}{a}}$ for some a satisfying $(9 - q)/(8 + 4q) < a \leq 1$

then $N = \Sigma_R$ for some R.

The article is organized as follows. In Section 2, an important identity relating the mean curvature to the center of mass (2.2) is derived. In Section 3, we prove the existence of the foliation (Theorem 3.1 and Theorem 3.7) and show its geometric center is equal to the center of mass (Corollary 3.3).

In Section 4, Theorem 2 and Theorem 3 are proven after certain a priori estimates are established.

2. Estimates on Surfaces Close to $S_R(p)$

The following three lemmas are computational results. Recall that $h_{ij} = g_{ij} - \delta_{ij}$ in Definition 1.1. The first lemma indicates an important identity relating the mean curvature to the center of mass $\mathcal{C}$. We obtain some estimates for Euclidean spheres $S_R(p) \equiv \{y : \sum_{l=1}^{3}(y^i - p^i)^2 = R^2\}$ in the second lemma and the analogous estimates for surfaces close to $S_R(p)$ in the third lemma.

Lemma 2.1. *Let H_S be the mean curvature of $S_R(p)$ and $d\sigma_e$ be the volume form of the standard sphere metric. Then*

$$
\begin{aligned}
(i) \quad H_S = \frac{2}{R} &+ \frac{1}{2} h_{ij,k}(y-p) \frac{(y^i - p^i)(y^j - p^j)(y^k - p^k)}{R^3} \\
&+ 2h_{ij}(y-p)\frac{(y^i - p^i)(y^j - p^j)}{R^3} - h_{ij,i}(y-p)\frac{y^j - p^j}{R} \\
&+ \frac{1}{2} h_{ii,j}(y-p)\frac{y^j - p^j}{R} - \frac{h_{ii}(y)}{R} + O(R^{-2-q}). \qquad (2.1)
\end{aligned}
$$

(ii) *For $\alpha = 1, 2, 3$,*

$$
\int_{S_R(p)} (y^\alpha - p^\alpha)\left(H_S - \frac{2}{R}\right) d\sigma_e = 8\pi m p^\alpha - 8\pi m \mathcal{C}^\alpha + O(R^{-q}). \qquad (2.2)
$$

Proof. Because a similar computation for spherically asymptotically flat manifolds could be found in [H08, Section 6], and the basic idea is exactly the same, we include only the sketch of the proof here.

Let ν_g denote the outward unit normal vector field on $S_R(p)$ with respect to the induced metric from g. Computing directly, we have

$$
\begin{aligned}
\nu_g \quad = \quad \frac{\nabla|y-p|}{|\nabla|y-p||_g} &= \left(1 + \frac{1}{2} h_{st}\frac{(y^s - p^s)(y^t - p^t)}{|y-p|^2}\right)\frac{y^l - p^l}{|y-p|}\frac{\partial}{\partial y^l} \\
&\quad - h_{kl}\frac{y^k - p^k}{|y-p|}\frac{\partial}{\partial y^l} + O(R^{-1-q}). \qquad (2.3)
\end{aligned}
$$

The mean curvature H_S of $S_R(p)$ is then equal to $\mathrm{div}_g \nu_g$, and a straightforward calculation gives us the identity (2.1).

For the second identity, we denote $f_1(y) = H_S - 2/R$. First we notice that the left hand side of (2.2) converges because AF-RT implies that the leading order term of $f_1(y)$ is even and vanishes after integrated with the odd function $y^\alpha - p^\alpha$. Then we only need to analyze the second order term of $f_1(y)$ which is exactly the case considered in [H08, Lemma 6.1]. By the divergence theorem and a density theorem proved in that article, we derive

$$\begin{aligned}
&\int_{S_R(p)} (y^\alpha - p^\alpha) \frac{1}{2} h_{ij,k}(y) \frac{(y^i - p^i)(y^j - p^j)(y^k - p^k)}{R^3} \, d\sigma_e \\
&= \frac{1}{2} \int_{S_R(p)} (y^\alpha - p^\alpha) h_{ij,i} \frac{y^j - p^j}{|y-p|} \, d\sigma_e - 2 \int_{S_R(p)} (y^\alpha - p^\alpha) h_{ij} \frac{(y^i - p^i)(y^j - p^j)}{|y-p|^3} \, d\sigma_e \\
&\quad + \frac{1}{2} \int_{S_R(p)} h_{ii} \frac{y^\alpha - p^\alpha}{|y-p|} \, d\sigma_e + \frac{1}{2} \int_{S_R(p)} h_{i\alpha} \frac{y^i - p^i}{|y-p|} \, d\sigma_e.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{S_R(p)} (y^\alpha - p^\alpha) \left(H_S - \frac{2}{R} \right) d\sigma_e \\
&= -\frac{1}{2} \left(\int_{S_R(p)} (y^\alpha - p^\alpha)(h_{ij,i} - h_{ii,j}) \frac{y^j - p^j}{|y-p|} \, d\sigma_e - \int_{S_R(p)} \left(h_{i\alpha} \frac{y^i - p^i}{|y-p|} - h_{ii} \frac{y^\alpha - p^\alpha}{|y-p|} \right) d\sigma_e \right) \\
&= \frac{1}{2} p^\alpha \int_{S_R(p)} (g_{ij,i} - g_{ii,j}) \frac{y^j - p^j}{|y-p|} \, d\sigma_e \\
&\quad - \frac{1}{2} \left(\int_{S_R(p)} y^\alpha (g_{ij,i} - g_{ii,j}) \frac{y^j - p^j}{|y-p|} \, d\sigma_e - \int_{S_R(p)} \left(h_{i\alpha} \frac{y^i - p^i}{|y-p|} - h_{ii} \frac{y^\alpha - p^\alpha}{|y-p|} \right) d\sigma_e \right) \\
&= 8\pi m p^\alpha - 8\pi m \mathcal{C}^\alpha + O(R^{-q}),
\end{aligned}$$

where we have used the definitions of the ADM mass (1.3) and the center of mass (1.5) in the last equality. □

In the following lemmas, c denotes a constant independent of R, and for any functions f on $S_R(p)$, we define $f^{odd}(y) = f(y) - f(-y+2p)$ and $f^{even}(y) = f(y) + f(-y+2p)$, where y and $-y+2p$ are antipodal points on $S_R(p)$. Also, we denote f^* to be the pullback of f defined by $f^*(x) = f(Rx+p)$, and f^* is a function on $S_1(0)$.

Lemma 2.2. *Let A_S be the second fundamental form on $(S_R(p), g|_S)$ where $g|_S$ is the induced metric on $S_R(p)$ from g, Δ_S be the Laplacian on $(S_R(p), g|_S)$. Let Δ_S^e be the Laplacian on $(S_R(p), g_e|_S)$ where g_e is the Euclidean metric on M, and $g_e|_S$ is the induced metric on $S_R(p)$ from g_e. Then*

(i) $|A_S|^2 = \dfrac{2}{R^2} + E_1$ *where* $|E_1| \le cR^{-2-q}$ *and* $|E_1^{odd}| \le cR^{-3-q}$.

(ii) *For any* $f \in C^{2,\alpha}(S_R(p))$,
$\Delta_S f = \Delta_S^e f + E_2$ *where* $|E_2| \le cR^{-2-q} \|f^*\|_{C^{2,\alpha}}$
and $|E_2^{odd}| \le c\left(R^{-3-q} \|f^*\|_{C^{2,\alpha}} + R^{-2-q} \|(f^*)^{odd}\|_{C^{2,\alpha}} \right)$.

(iii) $Ric^M(\nu_g, \nu_g) = E_3$ *where* $|E_3| \le cR^{-2-q}$ *and* $|E_3^{odd}| \le cR^{-3-q}$.

Proof. Let $\{u_1, u_2\}$ be local coordinates on an open set of $y \in S_R(p)$. The second fundamental form A_S is locally equal to

$$(A_S)_{ij} = -g\left(\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}, \nu_g\right).$$

First, notice that in the proofs of this and the next lemmas, we temporarily denote $g_{ab} = g(e_a, e_b)$ for $a, b \in \{1,2,3\}$ where $e_a = \frac{\partial}{\partial u_i}$ if $a = i \in \{1,2\}$ and $e_a = \nu_g$ if $a = 3$ (instead of the original meaning in Definition 1.1). Therefore, the above identity becomes

$$(A_S)_{ij} = -\Gamma^3_{ij} \tag{2.4}$$

because ν_g is a unit normal vector field. Also, because g is AF, $g(y) = g_e + h$ and $h = O(|y|^{-q})$. Around y, we have

$$\Gamma^3_{ij} = \frac{1}{2}\left(g_{i3,j} + g_{j3,i} - g_{ij,3}\right) = (\Gamma_e)^3_{ij} + |\partial h| \tag{2.5}$$

where we denote the difference of Γ^3_{ij} and $(\Gamma_e)^3_{ij}$ symbolically by $|\partial h|$. It means that the difference of Γ^3_{ij} and $(\Gamma_e)^3_{ij}$ can be bounded by some constant (depending only on g) multiplying ∂h and has the same asymptotically even/odd behavior as ∂h, where ∂h denotes the derivatives either in the tangential or in the normal directions. Notice that the derivatives in the tangential and normal directions *do not* change the asymptotic evenness of h, but only add one more decay rate. For example, $h = O(|y|^{-q})$ and $h^{odd} = O(|y|^{-1-q})$. Then $\partial h = O(|y|^{-1-q})$ and ∂h is still asymptotically even at the decay rate $(\partial h)^{odd} = O(|y|^{-2-q})$. In the following arguments, we will use similar notations to bound lower order terms for simplicity.

We substitute (2.5) back to (2.4) and derive

$$(A_S)_{ij} = (A_e)_{ij} + |\partial h|.$$

Therefore, if the principal curvature of $(S_R(p), g|_S)$ and the principal curvature of $(S_R(p), g_e|_S)$ are denoted by $(\lambda_S)_i$ and ${\lambda^e}_i$ respectively, the above identity says:

$$(\lambda_S)_i = {\lambda^e}_i + |\partial h| = \frac{1}{R} + |\partial h|. \tag{2.6}$$

Then

$$|A_S|^2 = (\lambda_S)_1^2 + (\lambda_S)_2^2 = \frac{2}{R^2} + \frac{1}{R}|\partial h| + |\partial h|^2.$$

We could conclude (i) by analyzing the error terms on the right hand side and by using the AF-RT condition.

Using $g = g_e + h$, the Laplacian in the local coordinates is

$$\begin{aligned} \Delta_S f &= \sqrt{g}^{-1}\frac{\partial}{\partial u_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial u_j}f\right) \\ &= \Delta^e_S f + \left(|h||\partial g||\partial f| + |h||\partial^2 f| + |\partial h||\partial f|\right). \end{aligned} \tag{2.7}$$

By the definition of f^*, $|\partial f(y)| = R^{-1}|\partial f^*(x)|$ and $|\partial^2 f(y)| = R^{-2}|\partial^2 f^*(x)|$, and then (ii) follows.

For (iii), notice that $|Ric^M(\nu_g,\nu_g)| = |D^2 g|$, where Dg denotes the usual derivatives of g in $\{\frac{\partial}{\partial x_i}\}$ directions as in Definition 1.1. Therefore, $|D^2 g| = O\left(|y|^{-2-q}\right)$ and $\left|(D^2 g)^{odd}\right| = \left|D^2\left(g^{odd}\right)\right| = O(|y|^{-3-q})$. □

In the previous lemma, we have shown that some quantities on $(S_R(p), g|_S)$ are close to those on $(S_R(p), g_e|_S)$ by using the AF-RT condition. In the following lemma, we will generalize the above results and prove that similar estimates also hold for surfaces which are cR^{1-q}-graphs over $S_R(p)$ for some constant c (recall $q \in (1/2, 1]$, the decay rate of the AF metrics).

Lemma 2.3. *Let N be a graph over $S_R(p)$ defined by*

$$N = \left\{z = y + \psi\nu_g : \psi \in C^{2,\alpha}\left(S_R(p)\right)\right\}.$$

Assume $\|\psi^*\|_{C^{2,\alpha}(S_1(0))} \le cR^{1-q}$ and $\|(\psi^*)^{odd}\|_{C^{2,\alpha}(S_1(0))} \le cR^{-q}$. Let μ_g be the outward unit normal vector field on N, A_N be the second fundamental form, and Δ_N be the Laplacian on $(N, g|_N)$. Then

(i) $\quad |A_N|^2 = \dfrac{2}{R^2} + E_1'$ *where* $|E_1'| \le cR^{-2-q}$ *and* $|(E_1')^{odd}| \le cR^{-3-q}$.

(ii) *For any $f \in C^{2,\alpha}(N)$, we let $\widetilde{f}(y) = f(\Psi(y))$, $\Psi(y) \equiv y + \psi\nu_g$, and $f^* = (\widetilde{f})^*$, the pull-back function defined on $S_1(0)$. Then*

$$(\Delta_N f)(\Psi(y)) = \Delta_S^e \widetilde{f}(y) + E_2' \quad \text{where } |E_2'| \le cR^{-2-q}\|f^*\|_{C^{2,\alpha}}$$
$$\text{and } |(E_2')^{odd}| \le c\left(R^{-2-2q}\|f^*\|_{C^{2,\alpha}} + R^{-2-q}\|(f^*)^{odd}\|_{C^{2,\alpha}}\right).$$

(iii) $\quad \left(Ric^M(\mu_g,\mu_g)\right)(\Psi(y)) = E_3'$

$$\text{where } |E_3'| \le cR^{-2-q} \text{ and } |(E_3')^{odd}| \le cR^{-3-q}.$$

Proof. Similarly as in the proof of Lemma 2.2, let $\{u_1, u_2\}$ be local coordinates on an open set U of $y \in S_R(p)$. Moreover, without loss of generality, we assume $\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \nu_g\}$ are orthonormal at y with respect to the metric g. Let $\{v_1, v_2\}$ be the corresponding local coordinates on $V = \Psi(U) \subset N$ and μ_g be the outward unit normal vector field on N with respect to g. Because M is AF, up to lower order terms, we have

$$\frac{\partial}{\partial v_i} = \frac{\partial}{\partial u_i} + (A_S)_{ij}\psi\frac{\partial}{\partial u_j} + \frac{\partial\psi}{\partial u_i}\nu_g \tag{2.8}$$

$$\mu_g = \nu_g + \psi H_S \nu_g - \sum_{i=1,2}\frac{\partial\psi}{\partial u_i}\frac{\partial}{\partial u_i} \tag{2.9}$$

where we parallel transport $\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \nu_g\right\}$ to $z = y + \psi\nu_g$ along the unique geodesic connecting y and z. In this proof, we denote

$$\bar{g}_{ab} = g(\bar{e}_a, \bar{e}_b) \qquad \text{where } \bar{e}_a = \tfrac{\partial}{\partial v_i} \text{ if } a = i \in \{1,2\} \text{ and } \bar{e}_3 = \mu_g$$
$$g_{ab} = g(e_a, e_b) \qquad \text{where } e_a = \tfrac{\partial}{\partial u_i} \text{ if } a = i \in \{1,2\} \text{ and } e_3 = \nu_g.$$

where g_{ab} is defined the same as in the proof of the previous lemma. By (2.8) and (2.9), we have for $i \in \{1,2\}, a, b \in \{1,2,3\}$

$$\begin{aligned}\bar{g}_{ia} &= g_{ia} + |\psi||A_S||g| + |\partial\psi||g| \\ \bar{g}_{ia,b} &= g_{ia,b} + |\partial\psi||A_S||g| + |\psi||\partial A_S||g| + |\psi||A||\partial g| \\ &\quad + |\partial^2\psi||g| + |\partial\psi|^2|\partial g|. \end{aligned} \tag{2.10}$$

To prove (i), notice that

$$(A_N)_{ij} = -g\left(\nabla_{\frac{\partial}{\partial v_i}} \frac{\partial}{\partial v_j}, \mu_g\right) = -\bar{\Gamma}^3_{ij}$$

and

$$\begin{aligned}\bar{\Gamma}^3_{ij} &= \frac{1}{2}\left(\bar{g}_{i3,j} + \bar{g}_{j3,i} - \bar{g}_{ij,3}\right) \\ &= \Gamma^3_{ij} + |\partial\psi||A_S||g| + |\psi||\partial A_S||g| + |\psi||A_S||\partial g| + |\partial^2\psi||g| + |\partial\psi|^2|\partial g|\end{aligned}$$

Therefore, by (2.6) and the previous two identities,

$$|A_N|^2 = |A_S|^2 + \frac{1}{R}\left(|\partial\psi||A_S||g| + |\psi||\partial A_S||g| + |\psi||A||\partial g| + |\partial^2\psi||g| + |\partial\psi|^2|\partial g|\right)$$

where the terms with the weakest decay rate of the error terms are, for instance,

$$\frac{1}{R}|\partial^2\psi||g| = O(R^{-2-q}).$$

Similarly, we could compute $(E_1')^{odd}$ and use Lemma 2.2(i) to conclude (i).

For (ii), the Laplacian in local coordinates is

$$\begin{aligned}\Delta_N f(z) &= \sqrt{\bar{g}}\frac{\partial}{\partial v_i}\left(\sqrt{\bar{g}}\bar{g}^{ij}\frac{\partial}{\partial v_j}f(z)\right) \\ &= \sqrt{g}\frac{\partial}{\partial u_i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial u_j}f(z)\right) + |\partial\psi||A_S||g||\partial f| \\ &\quad + |\psi||\partial A_S||g||\partial f| + |\psi||A_S||g||\partial g||\partial f| + |\psi||A_S||g||\partial^2 f|,\end{aligned}$$

and then

$$\begin{aligned}(\Delta_N f)(\Psi(y)) &= \Delta_S \widetilde{f}(y) + |\partial\psi||A_S||g||\partial\widetilde{f}| \\ &\quad + |\psi||\partial A_S||g||\partial\widetilde{f}| + |\psi||A_S||g||\partial g||\partial\widetilde{f}| + |\psi||A_S||g||\partial^2\widetilde{f}|\end{aligned}$$

where the terms with the weakest decay rate of the error terms are, for instance,

$$|\partial\psi||A_S||g||\partial\widetilde{f}(y)| \le R^{-1}|\partial\psi||A_S||g||\partial f^*(x)| \le CR^{-2-q}\|f^*\|_{C^{2,\alpha}}.$$

Then, (ii) follows from Lemma 2.2(ii) and similar estimates on $(E_2')^{odd}$.

Using Lemma 2.2(iii) and the identity

$$Ric^M(\mu_g,\mu_g)(z) = Ric^M(\nu_g,\nu_g) + |D^2 g||\psi||A_S| + |D^2 g||\partial\psi|,$$

we can conclude (iii).

□

3. Existence of the Foliation

In this section, we will use the implicit function theorem as in [Ye96] and prove the existence of the foliation by constant mean curvature surfaces, assuming the ADM mass $m \neq 0$. From our construction, each leaf of the foliation is close to some Euclidean sphere centered at p. We will also show that p converges to the center of mass $\mathcal{C}$.

Through this section, we assume (M, g) is AF-RT with $q \in (1/2, 1]$, and c and c_i denote constants independent of R. The first theorem states the existence of a surface with some given constant mean curvature.

Theorem 3.1 (Existence of the CMC surfaces). *There exist constants σ_0 and c_0 so that, for all $R > \sigma_0$, there is Σ_R with constant mean curvature $2/R + O(R^{-1-q})$, and Σ_R is a $c_0 R^{1-q}$-graph over $S_R(\mathcal{C})$, i.e. $\Sigma_R = \{y + \psi\nu_g : y \in S_R(\mathcal{C})\}$ with $\|\psi^*\|_{C^{2,\alpha}} \leq c_0 R^{1-q}$.*

Proof. Because the mean curvature of $S_R(p)$ is equal to some constant up to $O(R^{-1-q})$-terms (see (2.1)), we would like to construct Σ_R by perturbing $S_R(p)$ in the normal direction. However, contrast to the case in which (M, g) is spherically asymptotically flat, the mean curvature of $S_R(p)$ is *not* close to some constant enough to apply the implicit function theorem. Therefore, we will first construct the unique approximating spheres $\mathcal{S}(p, R)$ associated to $S_R(p)$ whose mean curvature is close to some constant up to $O(R^{-1-2q})$-terms. Then, the implicit function theorem on $\mathcal{S}(p, R)$ can be used to find Σ_R when the center p is correctly chosen.

Step 1. Constructing $\mathcal{S}(p, R)$:

Define $L_0 = -\Delta_0 - 2$ to be the Euclidean second variation operator on $S_1(0)$, where Δ_0 is the Laplacian with respect to the standard spherical metric on $S_1(0)$. It is known that L_0 has the kernel spanned by $\{x^1, x^2, x^3\}$ where x is the position vector, because the mean curvature is preserved by translations in the Euclidean space. Also notice that the L^2-complement $(\text{Range} L_0)^\perp = \text{span}\{x^1, x^2, x^3\}$ by the self-adjointness of L_0. Recall that $f_1(y) = H_S - (2/R)$ as in Lemma 2.1. Let $f_2(y - p)$ be the leading order term of $f_1(y)$ defined by

$$\begin{aligned} f_2(y-p) =& \frac{1}{2} h_{ij,k}(y-p) \frac{(y^i - p^i)(y^j - p^j)(y^k - p^k)}{R^3} \\ &+ 2h_{ij}(y-p)\frac{(y^i-p^i)(y^j-p^j)}{R^3} - h_{ij,i}(y-p)\frac{y^j - p^j}{R} \\ &+ \frac{1}{2} h_{ii,j}(y-p)\frac{y^j-p^j}{R} - \frac{h_{ii}(y-p)}{R} \end{aligned}$$

and then $f_1(y) - f_2(y - p) = O(R^{-2-q})$. For any $\phi \in C^{2,\alpha}(S_R(p))$, we consider the equation,

$$-\Delta_S^e \phi - \frac{2}{R^2}\phi = f_2(y-p) - \frac{\mathbf{A} \cdot (y-p)}{R^{3+q}} - \bar{f}_2, \tag{3.1}$$

where $\bar{f}_2 \equiv (4\pi R^2)^{-1} \int_{S_R(p)} f_2(y-p)\, d\sigma_e$, and $\mathbf{A}\cdot(y-p)R^{-3-q}$ is the correction term such that the right hand side of (3.1) is in the range. More explicitly, $\mathbf{A} = (A^1, A^2, A^3)$ and, for $\alpha = 1, 2, 3$,

$$A^\alpha = \frac{3}{4\pi} R^{-1+q} \int_{S_R(p)} (y^\alpha - p^\alpha) f_2(y-p)\, d\sigma_e. \tag{3.2}$$

Notice that $A^\alpha = O(1)$. Then there exists a unique solution $\phi_0 \in \mathrm{Ker}^\perp$. In order to obtain a scale-free (in R) estimate of ϕ_0, we scale and translate $y \in S_R(p)$ to $x \in S_1(0)$ by $y = Rx + p$ and have

$$L_0 \phi^* = R^2 \left(f_2^*(x) - \frac{\mathbf{A}\cdot x}{R^{2+q}} - \bar{f}_2 \right),$$

where $f_2^*(x) = f_2(Rx)$. By the Hölder estimate, because $\phi_0^* \in (\mathrm{Ker}L_0)^\perp$

$$\|\phi_0^*\|_{C^{2,\alpha}} \le c \left\| R^2 \left(f_2^*(x) - \frac{\mathbf{A}\cdot x}{R^{2+q}} - \bar{f}_2 \right) \right\|_{C^{0,\alpha}} \le \frac{c_0}{2} R^{1-q} \tag{3.3}$$

for some c_0 depending only on g and Dg. Moreover, $(\phi_0^*)^{odd}$ satisfies the following equation

$$L_0 (\phi_0^*)^{odd} = R^2 (f_2^*)^{odd} - \frac{2\mathbf{A}\cdot x}{R^q}.$$

Therefore, because $(\phi_0^*)^{odd} \in (\mathrm{Ker}L_0)^\perp$, by the Hölder estimate and the fact that f_2^* is asymptotically even, we have (by choosing c_0 larger if necessary)

$$\|(\phi_0^*)^{odd}\|_{C^{2,\alpha}} \le c \left\| R^2 (f_2^*)^{odd} - \frac{2\mathbf{A}\cdot x}{R^q} \right\|_{C^{0,\alpha}} \le c_0 R^{-q}. \tag{3.4}$$

Then we define

$$\mathcal{S}(p,R) = \{ y + \phi_0 \nu_g \}$$

where $\phi_0(y) = \phi_0^*\left((y-p)/R\right)$. In particular, $\mathcal{S}(p,R)$ is a graph over $S_R(p)$ with $\|\phi_0^*\|_{C^{2,\alpha}} \le 2^{-1} c_0 R^{1-q}$ and $\|(\phi_0^*)^{odd} \le c_0 R^{-q}$ which satisfies the conditions for N in Lemma 2.3.

Remark. The constant $\bar{f}_2$ in (3.1) is not necessary in this argument. However, the solution ϕ to (3.1) has zero mean value because of the presence of $\bar{f}_2$, and then the geometric radius $\mathcal{S}(p,R)$ is precisely R. In the case of spherically asymptotically flat manifolds, the right hand side of (3.1) is zero, and the only solution in the complement of the kernel is $\phi = 0$. Then $\mathcal{S}(p,R) = S_R(p)$.

<u>Step 2</u>. Calculating the mean curvature of $\mathcal{S}(p,R)$:

Denoting $H_S(\phi_0)$ the mean curvature of $\mathcal{S}(p,R)$, we use Taylor's theorem for mappings between two Banach spaces (cf. [MRA]) and have

$$H_S(\phi_0) = H_S(0) + dH_S(0)\phi_0 + \int_0^1 \left(dH_S(s\phi_0) - dH_S(0) \right) \phi_0 ds$$

where dH_S is the first Fréchet derivative in the ϕ_0-component, and $dH_S(0)$ is the linearized mean curvature operator on $S_R(p)$ defined by

$$dH_S(0) = \Delta_S + |A_S|^2 + Ric^M(\nu_g, \nu_g),$$

where Δ_S, A_S, and $Ric^M(\nu_g, \nu_g)$ are defined as in Lemma 2.2. The integral term above can be bounded by $\sup_{s\in[0,1]} \left|d^2 H_S(s\phi_0)\phi_0\phi_0\right|$ by the mean value inequality, and

$$d^2 H_S(s\phi_0)\phi_0\phi_0 = \frac{\partial^2}{\partial t^2} H_S(t\phi_0)\bigg|_{t=s}.$$

The left hand side is the second Fréchet derivative and the right hand side is the second derivative of the mean curvature of the surface $N_s \equiv \{y + s\phi_0\nu_g : y \in S_R(p)\}$. For R large, the unit outward normal vector field on N_s is close to ν_g, and a straightforward calculation gives us

$$\begin{aligned} \left|\frac{\partial^2}{\partial t^2} H_S(t\phi_0)\right| &\leq c\left(|R_{ijkl}|\,|A_{N_s}|\,|\phi_0|^2 + |A_{N_s}|\,|\phi_0||\partial^2\phi_0| + |A_{N_s}|^3\,|\phi_0|^2\right) \\ &\leq cR^{-3}\|\phi_0^*\|^2_{C^{2,\alpha}}. \end{aligned} \tag{3.5}$$

Noticing that $H_S(0)$ is the mean curvature of $S_R(p)$, by (2.1) in Lemma 2.1 and (3.1), we have

$$\begin{aligned} H_S(\phi_0) &= \frac{2}{R} + f_1(y) + \Delta_S\phi_0 + \left(|A_S|^2 + Ric^M(\nu_g,\nu_g)\right)\phi_0 \\ &\quad + \int_0^1 \left(dH_S\left(s\phi_0\right) - dH_S(0)\right)\phi_0\, ds \\ &= \frac{2}{R} + \bar{f}_2 + f_1(y) - f_2(y-p) + \frac{\mathbf{A}\cdot(y-p)}{R^{3+q}} + E_4, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} E_4 = (\Delta_S - \Delta_S^e)\phi_0 &+ \left(|A_S|^2 - \frac{2}{R^2} + Ric^M(\nu_g,\nu_g)\right)\phi_0 \\ &+ \int_0^1 \left(dH_S\left(s\phi_0\right) - dH_S(0)\right)\phi_0\, ds. \end{aligned}$$

By Lemma 2.2 and (3.5), the error term E_4 is bounded by

$$\begin{aligned} |E_4| &\leq c\left(R^{-2-q}\|\phi_0^*\|_{C^{2,\alpha}} + R^{-3}\|\phi_0^*\|^2_{C^{2,\alpha}}\right) \leq cR^{-1-2q} \\ |(E_4)^{odd}| &\leq c\Big(R^{-3-q}\|\phi_0^*\|_{C^{2,\alpha}} + R^{-2-q}\|(\phi_0^*)^{odd}\|_{C^{2,\alpha}} \\ &\qquad + R^{-4-q}\|\phi_0^*\|^2_{C^{2,\alpha}} + R^{-3}\|(\phi_0^*)^{odd}\|_{C^{2,\alpha}}\|\phi_0\|_{C^{2,\alpha}}\Big) \\ &\leq cR^{-2-2q}. \end{aligned} \tag{3.7}$$

In the last inequalities, we have used (3.3) and (3.4). Therefore, we derive $H_S(\phi_0) = (2/R) + \bar{f}_2 + O(R^{-1-2q})$.

Step 3. Constructing the CMC surfaces:

We will construct a surface Σ_R with constant mean curvature by using the normal perturbations of $\mathcal{S}(p,R)$. In the following, we suppress the index R of Σ_R when it is clear from context. We denote the mean curvature of the normal graph ψ over $\mathcal{S}(p,R)$ by $H_{\mathcal{S}}(p,R,\psi)$. By Taylor's theorem, for any $\psi \in C^{2,\alpha}(\mathcal{S}(p,R))$ with $\|\psi^*\|_{C^{2,\alpha}} \le c$ and a parameter λ which is for the moment arbitrary (we choose λ to be a negative power in R below),

$$\begin{aligned} H_{\mathcal{S}}(p,R,\lambda\psi) &= H_{\mathcal{S}}(p,R,0) + \Delta_{\mathcal{S}}\lambda\psi + \left(|A_{\mathcal{S}}|^2 + Ric^M(\mu_g,\mu_g)\right)\lambda\psi \\ &\quad + \int_0^1 \left(dH_{\mathcal{S}}(p,R,s(\lambda\psi)) - dH_{\mathcal{S}}(p,R,0)\right)\lambda\psi\, ds \qquad (3.8) \end{aligned}$$

where $\Delta_{\mathcal{S}}, A_{\mathcal{S}}$, and μ_g are defined as in Lemma 2.3 where $N = \mathcal{S}(p,R)$, and $\widetilde{\psi}$ and ψ^* are denoted correspondingly. By (3.6) and (3.8) (notice that $H_S(\phi_0) = H_{\mathcal{S}}(p,R,0) =$ mean curvature of $\mathcal{S}(p,R)$), solving

$$H_{\mathcal{S}}(p,R,\lambda\psi) = \frac{2}{R} + \bar{f}_2 \qquad (3.9)$$

is equivalent to solving ψ to the following equation:

$$\begin{aligned} 0 &= f_1(y) - f_2(y-p) + \frac{\mathbf{A}\cdot(y-p)}{R^{3+q}} + E_4 + \Delta_{\mathcal{S}}\lambda\psi + \left(|A_{\mathcal{S}}|^2 + Ric^M(\mu_g,\mu_g)\right)\lambda\psi \\ &\quad + \int_0^1 \left(dH_{\mathcal{S}}(p,R,s(\lambda\psi)) - dH_{\mathcal{S}}(p,R,0)\right)\lambda\psi\, ds. \\ &= f_1(y) - f_2(y-p) + \frac{\mathbf{A}\cdot(y-p)}{R^{3+q}} + E_4 + \Delta^e_S(\lambda\widetilde{\psi}) + \frac{2}{R^2}\lambda\widetilde{\psi} + E_5 \qquad (3.10) \end{aligned}$$

where

$$\begin{aligned} E_5(y) &= \lambda(\Delta_{\mathcal{S}}\psi)\circ\Psi(y) - \lambda\Delta^e_S\widetilde{\psi} \\ &\quad + \left(|A_{\mathcal{S}}|^2(\Psi(y)) - \frac{2}{R^2} + Ric^M(\mu_g,\mu_g)\right)\lambda\widetilde{\psi} \\ &\quad + \int_0^1 \left(dH_{\mathcal{S}}(p,R,s(\lambda\psi)) - dH_{\mathcal{S}}(p,R,0)\right)\lambda\psi\, ds \end{aligned}$$

and $\Psi(y) = y + \phi_0\nu_g$. Using Lemma 2.3 and (3.5), we have

$$\begin{aligned} |E_5| &\le c\left(\lambda R^{-2-q}\|\psi^*\|_{C^{2,0}} + \lambda^2 R^{-3}\|\psi^*\|^2_{C^{2,0}}\right) \\ |(E_5)^{odd}| &\le c\Big(\lambda R^{-2-2q}\|\psi^*\|_{C^{2,\alpha}} + \lambda R^{-2-q}\left\|(\psi^*)^{odd}\right\|_{C^{2,\alpha}} \\ &\quad + \lambda^2 R^{-4-q}\|\psi^*\|^2_{C^{2,\alpha}} + \lambda^2 R^{-3}\|\psi^*\|_{C^{2,\alpha}}\left\|(\psi^*)^{odd}\right\|_{C^{2,\alpha}}\Big). \end{aligned}$$

We pull back (3.10) on $S_1(0)$,

$$L_0\psi^* = \lambda^{-1}R^2\left(f_1^*(x) - f_2^*(x) + \frac{\mathbf{A}\cdot x}{R^{2+q}} + E_4^* + E_5^*\right).$$

where $f_1^*(x) = f_1(Rx+p)$ and E_4^*, E_5^* are denoted similarly, but $f_2^*(x) = f_2(Rx+p-p) = f_2(Rx)$. We choose

$$\lambda = R^{-a} \text{ for a fixed } a \in (1-q, q). \qquad (3.11)$$

This interval is non-empty because $q > 1/2$. Then the above identity becomes

$$L_0\psi^* = R^{2+a}\left(f_1^*(x) - f_2^*(x) + \frac{\mathbf{A}\cdot x}{R^{2+q}} + E_4^* + E_5^*\right) \equiv F(p, R, \psi^*) \qquad (3.12)$$

In order to find a solution ψ^* to the above equation, a necessary condition is that $F(p,R,\psi^*)$ lies inside RangeL_0. Using $m \neq 0$, we will show this can be achieved by correctly choosing $p = p(R,\psi^*)$. First, by the definition of $\mathbf{A}$ in (3.2), we have

$$\begin{aligned}
&\int_{S_1(0)} x^\alpha \left(f_1^*(x) - f_2^*(x) + \frac{\mathbf{A}\cdot x}{R^{2+q}} + E_4^* + E_5^*\right) d\sigma_e \\
=& \int_{S_1(0)} x^\alpha f_1^*(x)\, d\sigma_e + \int_{S_1(0)} x^\alpha \left(E_4^* + E_5^*\right) d\sigma_e \\
=& \int_{S_R(p)} \frac{y^\alpha - p^\alpha}{R} f_1(y) R^{-2}\, d\sigma_e + \int_{S_1(0)} x^\alpha \left(E_4^* + E_5^*\right) d\sigma_e
\end{aligned}$$

Using (2.2) in Lemma 2.1, the first integral is equal to

$$\int_{S_R(p)} \frac{y^\alpha - p^\alpha}{R} f_1(y) R^{-2}\, d\sigma_e = 8\pi m\left(p^\alpha - \mathcal{C}^\alpha\right) R^{-3} + O(R^{-3-q}).$$

To deal with the error term E_4^*, by the asymptotically even/odd properties of E_4^* and (3.7),

$$\begin{aligned}
\left|\int_{S_1(0)} x^\alpha E_4^*\, d\sigma_e\right| &\leq \left|\int_{S_1(0)\cap\{x^\alpha\geq 0\}} x^\alpha (E_4^*)^{odd}\, d\sigma_e\right| \\
&\leq \left|\int_{S_R(p)\cap\{y^\alpha - p^\alpha \geq 0\}} \frac{(y^\alpha - p^\alpha)}{R} (E_4)^{odd} R^{-2}\, d\sigma_e\right| \\
&\leq c \sup_{S_R(p)} \left|(E_4)^{odd}\right| \leq cR^{-2-2q}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\left|\int_{S_1(0)} x^\alpha E_5^*\, d\sigma_e\right| &\leq c \sup_{S_R(p)} \left|E_5^{odd}\right| \\
&\leq cR^{-2-q-a}\left(\|\psi^*\|_{C^{2,\alpha}} + \|\psi^*\|^2_{C^{2,\alpha}}\right). \qquad (3.13)
\end{aligned}$$

Therefore, because $m \neq 0$, we can choose

$$p(R,\psi^*) = \mathcal{C} + E_6, \qquad (3.14)$$

with

$$\begin{aligned}
|E_6| &\leq c\left(R^{-q} + R^3\left(\sup\left|E_4^{odd}\right| + \sup\left|E_5^{odd}\right|\right)\right) \\
&\leq cR^{-\epsilon}\left(1 + \|\psi^*\|_{C^{2,\alpha}} + \|\psi^*\|^2_{C^{2,\alpha}}\right)
\end{aligned}$$

for some $\epsilon > 0$ because $a \in (1-q, q)$. Then $F(p(R,\psi^*), R, \psi^*) \in$ RangeL_0 for any R large and $\psi^* \in C^{2,\alpha}(S_1(0))$.

Remark. The lower bound $a > 1 - q$ is used in estimating E_6 so that E_5 has the stronger decay rate than the term $p - \mathcal{C}$ in $F(p, R, \psi)$. The upper bound $a < q$ is used so that $F(p, E, \psi)$ is bound by a negative power in R, so we are able to iterate the equation (3.12) as demonstrated in the next paragraph.

To complete the proof, we will use iteration to find a fixed point ψ^* of (3.12). Given any ψ_1^* with $\|\psi_1^*\|_{C^{2,\alpha}} \leq 1$, there exists a unique $C^{2,\alpha}$ function $\psi_2^* \in \mathrm{Ker} L_0^{\perp}$, so that

$$L_0 \psi_2^* = F\left(p(R, \psi_1^*), R, \psi_1^*\right)$$

and by the Hölder inequality,

$$\|\psi_2^*\|_{C^{2,\alpha}(S_1(0))} \leq c\|F\|_{C^{0,\alpha}(S_1(0))} \leq cR^{a-q} \leq cR^{-\epsilon_2}$$

for some $\epsilon_2 > 0$. Choosing R large enough (independent of ψ_1^*), we have $\|\psi_2^*\|_{C^{2,\alpha}} \leq 1$. Continuing the iteration, we obtain a sequence of functions $\{\psi_k^*\}$ inside a unit ball of $C^{2,\alpha}(S_1(0))$ and

$$L_0 \psi_k^* = F\left(p(R, \psi_{k-1}^*), R, \psi_{k-1}^*\right).$$

By the Arzela-Ascoli theorem, there is a subsequence $\psi_{k_j}^* \to \psi_0^*$ in $C^{2,\mu}$ for some $\mu \in (0, \alpha)$, and then ψ_0^* is a solution to

$$L_0 \psi_0^* = F\left(p(R, \psi_0^*), R, \psi_0^*\right).$$

Therefore, by letting $\psi_0(z) = \psi_0^*\left(\frac{z - \nu_g \phi_0 - p}{R}\right)$, $\lambda\psi_0$ is a solution to the identity (3.9). Geometrically, the identity indicates that the graph over $\mathcal{S}(p, R)$

$$\Sigma = \left\{ y + \phi_0 \nu_g + \frac{\psi_0}{R^a} \mu_g \right\}$$

has constant mean curvature $(2/R) + \bar{f}_2$. Because μ_g is close to ν_g by (2.9), and $p = \mathcal{C} + O(R^{-\epsilon})$, for R large, we can rearrange and write Σ as a graph over $S_R(\mathcal{C})$

$$\Sigma_R = \left\{ y + \psi \nu_g : \psi \in C^{2,\alpha}(S_R(\mathcal{C})) \right\}$$

with $\|\psi^*\|_{C^{2,\alpha}} \leq c_0 R^{1-q}$.

□

After constructing the family of surfaces with constant mean curvature, the geometric definition of center of mass in [HY96] generalizes.

Definition 3.2. *Let $z \in \Sigma_R$ be the position vector. The center of mass of (M, g, K) is defined by, for $\alpha = 1, 2, 3$,*

$$\mathcal{C}_{HY}^{\alpha} = \lim_{R \to \infty} \frac{\int_{\Sigma_R} z^\alpha \, d\sigma_e}{\int_{\Sigma_R} d\sigma_e}.$$

The following corollary says that $\mathcal{C}_{HY}$ is equal to $\mathcal{C}$, and it generalizes the result in [H08, Theorem 2].

Corollary 3.3. *Assume (M, g, K) is AF-RT with $q \in (1/2, 1]$. Then $\mathcal{C}_{HY}$ converges and is equal to $\mathcal{C}$.*

Proof. Let Φ be the diffeomorphism from $S_R(p)$ to Σ_R defined by $\Phi(y) = y + \phi_0\nu_g + R^{-a}\psi_0\mu_g$. Then by the definition and the area formula,

$$\begin{aligned}\frac{\int_{\Sigma_R} z^\alpha \, d\sigma_e}{\int_{\Sigma_R} d\sigma_e} &= \frac{\int_{S_R(p)} \left(y^\alpha + \phi_0\nu_g^\alpha + R^{-a}\psi_0\mu_g^\alpha\right) J\Phi \, d\sigma_e}{\int_{S_R(p)} J\Phi \, d\sigma_e} \\ &= p^\alpha + \frac{\int_{S_R(p)} O(R^{1-2q}) \, d\sigma_e}{\int_{S_R(p)} d\sigma_e}.\end{aligned}$$

After taking limits and using (3.14), we prove the corollary. □

Next, we prove that the family of surfaces with constant mean curvature $\{\Sigma_R\}$ form a smooth foliation. In order to use the inverse function theorem for the mean curvature map, we would like to estimate the eigenvalues of its linearized operator.

Definition 3.4 (Stability). *A hypersurface N in M is called stable if the second variation operator $L_N := -\Delta_N - \left(|A_N|^2 + Ric^M(\mu_g, \mu_g)\right)$ has the non-negative lowest eigenvalue μ_0 among functions with zero mean value, i.e.*

$$\int_N uL_N u \, d\sigma \geq \mu_0 \int_N u^2 \, d\sigma \geq 0 \qquad \text{for all non-zero } u \text{ with } \int_N u \, d\sigma = 0.$$

N is called strictly stable, if μ_0 is strictly positive.

Lemma 3.5. *Let N be a cR^{1-q}-graph over $S_R(p)$ defined as in Lemma 2.3. For R large, N is strictly stable with the lowest eigenvalue $\mu_0 = (6m/R^3) + O(R^{-2-2q})$.*

Proof. By Lemma 2.3,

$$\begin{aligned}&\left|(L_N f)(\Psi(y)) - \left(-\Delta_S^e f(\Psi(y)) - \frac{2}{R^2} f(\Psi(y))\right)\right| \\ &\leq c\left(R^{-2-q} + R^{-2-q}\|f^*\|_{C^{2,\alpha}(S_1(0))}\right).\end{aligned}$$

For R large, the lowest eigenfunctions f for L_N are equal to the lowest eigenfunctions of $-\Delta_S^e - \frac{2}{R^2}$ which are in span$\{y^1 - p^1, y^2 - p^2, y^3 - p^3\}$, up to lower terms $O(R^{1-q})$. Therefore, without loss of generality, we only need to estimate the following integral

$$\int_N \left(\frac{z^\alpha - p^\alpha}{R}\right) L_N \left(\frac{z^\alpha - p^\alpha}{R}\right) d\sigma$$

and prove it is greater than some suitable positive constant.

Let μ_{Lap} be the first eigenvalue of $-\Delta_N$ among functions with zero mean value. We follow the proof of Lichnerowicz's theorem on the lower bound of μ_{Lap}, but modify it for our setting (cf. [C, Chapter III.4]). For any compact manifold N without boundary, and any functions f on N, the Bochner-Lichnerowicz identity says

$$\frac{1}{2}\Delta_N|\nabla^N f|^2 = (\mathrm{Hess}_N f)^2 + \langle \nabla^N f, \nabla^N \Delta_N f\rangle + Ric^N(\nabla^N f, \nabla^N f).$$

Because N is two-dimensional in our case, $Ric^N(\nabla^N f, \nabla^N f) = K|\nabla^N f|^2$ where K is the Gauss curvature of N. Moreover, let f be the first eigenfunction of $-\Delta_N$, i.e. $-\Delta_N f = \mu_{Lap} f$, we have

$$\frac{1}{2}\Delta_N|\nabla^N f|^2 \geq \frac{(\Delta_N f)^2}{2} - \mu_{Lap}|\nabla^N f|^2 + K|\nabla^N f|^2$$

with equality if and only if $\mathrm{Hess}_N f = cI$ for some constant c, where I is the identity matrix. After integrating the above inequality, we derive

$$\mu_{Lap}^2 \int_N f^2\, d\sigma \geq \int_N 2K|\nabla^N f|^2\, d\sigma. \tag{3.15}$$

Let $(\lambda_N)_i$ be the principal curvature of N. By the Gauss equation,

$$\begin{aligned}
2K &= H^2 - |A|^2 - 2Ric^M(\mu_g,\mu_g) + R_g \\
&= 2(\lambda_N)_1(\lambda_N)_2 - 2Ric^M(\mu_g,\mu_g) + R_g \\
&= 2\left(\frac{1}{R} + \left((\lambda_N)_1 - \frac{1}{R}\right)\right)\left(\frac{1}{R} + \left((\lambda_N)_2 - \frac{1}{R}\right)\right) - 2Ric^M(\mu_g,\mu_g) + R_g \\
&= \frac{2H}{R} - \frac{2}{R^2} - 2Ric^M(\mu_g,\mu_g) + O(R^{-2-2q}) \\
&= \frac{2}{R^2} + \frac{2}{R}\left(H - \frac{2}{R}\right) - 2Ric^M(\mu_g,\mu_g) + O(R^{-2-2q}).
\end{aligned}$$

Notice that in the second last equality, we have used $(\lambda_N)_i = (1/R) + O(R^{-1-q})$, $H = (2/R) + O(R^{-1-q})$ as calculated in the proof of Lemma 2.3, and $R_g = O(R^{-2-2q})$ by the constraint equations. Moreover, by (2.3) and (2.9), the normal vector μ_g is close to the Euclidean normal vector on the sphere, so $|N|$ is close to the area of the Euclidean sphere,

$$|N| = 4\pi R^2 + O(R^{2-q}). \tag{3.16}$$

We substitute K calculated above into (3.15) and have

$$\begin{aligned}
\mu_{Lap}^2 \int_N f^2\, d\sigma \;\; &\geq \;\; \frac{2}{R^2}\mu_{Lap}\int_N f^2\, d\sigma + \frac{2}{R}\int_N \left(H - \frac{2}{R}\right)|\nabla^N f|^2\, d\sigma \\
&\quad -2\int_N Ric^M(\mu_g,\mu_g)|\nabla^N f|^2\, d\sigma \\
&\quad + \int_N O(R^{-2-2q})|\nabla^N f|^2\, d\sigma
\end{aligned} \tag{3.17}$$

with equality if $f = (z^l - p^l/R) + O(R^{-q})$, and therefore, $\mu_{Lap} = (2/R^2) + O(R^{-2-q})$. Let $u_l = (z^l - p^l)/R$ and $c_l = \int_N u_l^2\, d\sigma$,

$$c_l = \frac{4\pi R^2}{3} + O(R^{2-q}) \quad \text{and} \quad |\nabla^N u_l|^2 = \frac{1 - u_l^2}{R^2} + O(R^{-2-q}).$$

Replacing f by u_l and dividing (3.17) by μ_{Lap} and c_l, we derive, for $l = 1, 2, 3$,

$$\begin{aligned}\mu_{Lap} &= \frac{2}{R^2} + \frac{c_l^{-1}}{R}\int_N \left(H - \frac{2}{R}\right)(1 - u_l^2)\,d\sigma \\ &\quad -c_l^{-1}\int_N Ric^M(\mu_g, \mu_g)(1 - u_l^2)\,d\sigma + O(R^{-2-2q}). \qquad (3.18)\end{aligned}$$

Let v be the lowest eigenfunction of L_N, i.e. $L_N v = \mu_0 v$,

$$\mu_0 \int_N v^2\,d\sigma = \int_N v L_N v \geq \mu_{Lap}\int_N v^2\,d\sigma - \int_N |A_N|^2 v^2\,d\sigma - \int_N Ric^M(\mu_g, \mu_g) v^2\,d\sigma$$

with equality if $v = u_l + O(R^{-q})$. Therefore, noticing that

$$\begin{aligned}|A_N|^2 &= (\lambda_N)_1^2 + (\lambda_N)_2^2 = \left(\frac{1}{R} + \left((\lambda_N)_1 - \frac{1}{R}\right)\right)^2 + \left(\frac{1}{R} + \left((\lambda_N)_2 - \frac{1}{R}\right)\right)^2 \\ &= \frac{2}{R^2} + \frac{2}{R}\left(H - \frac{2}{R}\right) + O(R^{-2-2q}) \qquad (3.19)\end{aligned}$$

and substituting μ_{Lap} by (3.18), we have

$$\begin{aligned}\mu_0 &= \mu_{Lap} - \frac{2}{R^2} - \frac{c_l^{-1}}{R}\int_N \left(H - \frac{2}{R}\right) 2u_l^2\,d\sigma - c_l^{-1}\int_N Ric^M(\mu_g, \mu_g) u_l^2\,d\sigma + O(R^{-2-2q}) \\ &= \frac{c_l^{-1}}{R}\int_N \left(H - \frac{2}{R}\right)(1 - 3u_l^2)\,d\sigma - c_l^{-1}\int_N Ric^M(\mu_g, \mu_g)\,d\sigma + O(R^{-2-2q}).\end{aligned}$$

Because $c_1 = c_2 = c_3$ up to lower order terms and $\sum_l u_l^2 = 1$ up to lower order terms, we sum the above identity over l, divide it by 3, and derive

$$\mu_0 = -\frac{3}{4\pi R^2}\int_N Ric^M(\mu_g, \mu_g)\,d\sigma + O(R^{-2-2q}).$$

Using the definition of the ADM mass (1.4) (it is known that $\{|x| = r\}$ can be replaced by more general surfaces including N) and the fact that the scalar curvature R_g is of lower order by the constraint equations, for R large,

$$\begin{aligned}&\int_N Ric^M(\mu_g, \mu_g)\,d\sigma = \int_N Ric^M_{ij}\left(\frac{z^i - p^i}{R}\right)\mu_g^j\,d\sigma \\ &= -\frac{1}{2R}\int_N Ric^M_{ij}(-2z^i)\mu_g^j\,d\sigma + O(R^{-1-q}) = -\frac{8\pi m}{R} + O(R^{-1-q}).\end{aligned}$$

Therefore,

$$\mu_0 = -\frac{3}{4\pi R^2}\left(\frac{-8\pi m}{R}\right) + O(R^{-2-2q}) = \frac{6m}{R^3} + O(R^{-2-2q}).$$

□

To prove that L_N is invertible, the similar calculations in [HY96] could be applied in our setting.

Lemma 3.6. *Let N be a cR^{-1-q}-graph over $S_R(p)$ defined as in Lemma 2.3. For R large, L_N is invertible and $L_N^{-1} : C^{0,\alpha}(N) \to C^{2,\alpha}(N)$ with $|L_N^{-1}| \le cm^{-1}R^3$.*

Proof. Let η_0 be the lowest eigenvalue of L_N with the corresponding eigenfunction h_0 among all functions (not necessarily with zero mean value). First notice that by (3.19)

$$\begin{aligned}\eta_0 &= \min_{\{\|u\|_{L^2}=1\}} \int_N -u\Delta_N u - \left(|A_N|^2 + Ric^M(\mu_g,\mu)\right) u^2 \, d\sigma \\ &\ge -\frac{2}{R^2} + O(R^{-2-q}).\end{aligned}$$

On the other hand, if we replace u by the constant $|N|^{-1/2}$, we obtain

$$\eta_0 \le -\frac{2}{R^2} + O(R^{-2-q}),$$

and then

$$\eta_0 = -\frac{2}{R^2} + O(R^{-2-q}). \tag{3.20}$$

We would like to derive a L^2-estimate on the difference of h_0 and its mean value $\bar{h}_0 \equiv |N|^{-1}\int_N h_0 \, d\sigma$.

$$\begin{aligned}& -\Delta_N(h_0 - \bar{h}_0) - \left(|A_N|^2 + Ric^M(\mu_g,\mu_g)\right)(h_0 - \bar{h}_0) \\ =\; & \eta_0(h_0 - \bar{h}_0) + \left(\eta_0 + |A_N|^2 + Ric^M(\mu_g,\mu_g)\right)\bar{h}_0 \end{aligned} \tag{3.21}$$

Multiplying the above identity by $(h_0 - \bar{h}_0)$ and integrating it over N, we get

$$\begin{aligned}\int_N |\nabla^N(h_0 - \bar{h}_0)|^2 \, d\sigma &= \int_N \left(\eta_0 + |A_N|^2 + Ric^M(\mu_g,\mu_g)\right)(h_0 - \bar{h}_0)^2 \, d\sigma \\ &\quad + \int_N (\eta_0 + |A_N|^2 + Ric^M(\mu_g,\mu_g))(h_0 - \bar{h}_0)\bar{h}_0 \, d\sigma.\end{aligned}$$

As shown in the previous lemma, $\mu_{Lap} = 2/R^2 + O(R^{-2-q})$ for functions with zero mean value. Also, by (3.19) and (3.20), $\eta_0 + |A_N|^2 = O(R^{-2-q})$ pointwisely. Then

$$\begin{aligned}&\left(\frac{2}{R^2} + O\left(R^{-2-q}\right)\right)\int_N |h_0 - \bar{h}_0|^2 \, d\sigma \\ &\le cR^{-2-q}\int_N |h_0 - \bar{h}_0|^2 \, d\sigma + cR^{-2-q}\int_N |h_0 - \bar{h}_0||\bar{h}_0| \, d\sigma \\ &\le cR^{-2-q}\int_N |h_0 - \bar{h}_0|^2 \, d\sigma + cR^{-2-q}\left(\epsilon R^q \int_N |h_0 - \bar{h}_0|^2 \, d\sigma + \frac{c(\epsilon)}{R^q}\int_N |\bar{h}_0|^2 \, d\sigma\right).\end{aligned}$$

For ϵ small enough, we obtain

$$\int_N |h_0 - \bar{h}_0|^2 \, d\sigma \le cR^{-2q}\int_N |\bar{h}_0|^2 \, d\sigma. \tag{3.22}$$

Let η_1 be the next eigenvalue with the corresponding eigenfuction h_1. We will show that η_1 is positive and $\eta_1 \geq (6m/R^3) + O(R^{-2-2q})$. First,

$$0 = \int_N h_0 h_1 \, d\sigma = \int_N (h_0 - \bar{h}_0)(h_1 - \bar{h}_1) \, d\sigma + \int_N \bar{h}_0 h_1 \, d\sigma.$$

Then

$$\left| \int_N h_1 \, d\sigma \right| \leq |\bar{h}_0|^{-1} \left(\int_N (h_0 - \bar{h}_0)^2 \, d\sigma \right)^{\frac{1}{2}} \left(\int_N (h_1 - \bar{h}_1)^2 \, d\sigma \right)^{\frac{1}{2}}. \quad (3.23)$$

Because $L_N h_1 = \eta_1 h_1$,

$$\begin{aligned} \int_N (h_1 - \bar{h}_1) L_N (h_1 - \bar{h}_1) \, d\sigma = & \int_N \eta_1 (h_1 - \bar{h}_1)^2 \, d\sigma \\ & + \int_N \bar{h}_1 (h_1 - \bar{h}_1) \left(|A_N|^2 + Ric^M(\mu_g, \mu_g) \right) \, d\sigma. \end{aligned}$$

The left hand side is bounded below by $\mu_0 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma$. Therefore, by Lemma 3.5, (3.19), (3.22), and (3.23)

$$\begin{aligned} & \left(\frac{6m}{R^3} + O\left(R^{-2-2q}\right) \right) \int_N (h_1 - \bar{h}_1)^2 \, d\sigma \\ \leq & \eta_1 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma + |N|^{-1} \left| \int_N h_1 \, d\sigma \right| \left| \int_N \left(|A_N|^2 + Ric^M(\mu_g, \mu_g) \right) (h_1 - \bar{h}_1) \, d\sigma \right| \\ \leq & \eta_1 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma + c|N|^{-1} R^{-2-q} \left| \int_N h_1 \, d\sigma \right| \int_N |h_1 - \bar{h}_1| \, d\sigma \\ \leq & \eta_1 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma + c|N|^{-\frac{1}{2}} R^{-2-q} \left| \int_N h_1 \, d\sigma \right| \left(\int_N |h_1 - \bar{h}_1|^2 \, d\sigma \right)^{\frac{1}{2}} \\ \leq & \eta_1 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma + c|N|^{-\frac{1}{2}} R^{-2-q} |\bar{h}_0|^{-1} \left(\int_N (h_0 - \bar{h}_0)^2 \, d\sigma \right)^{\frac{1}{2}} \left(\int_N (h_1 - \bar{h}_1)^2 \, d\sigma \right) \\ \leq & \eta_1 \int_N (h_1 - \bar{h}_1)^2 \, d\sigma + cR^{-2-2q} \int_N (h_1 - \bar{h}_1)^2 \, d\sigma. \end{aligned}$$

Therefore,

$$\eta_1 \geq \frac{6m}{R^3} + O\left(R^{-2-2q}\right)$$

and then L_N is invertible with $|L_N^{-1}| \leq cm^{-1} R^3$. □

In particular, the above lemma says that L_{Σ_R} is invertible for surfaces Σ_R with constant mean curvature constructed in Theorem 3.1. In the next theorem, we then use the invertibility of L_{Σ_R}, the estimates in the above two lemmas, and the inverse function theorem to show that $\{\Sigma_R\}$ form a smooth foliation.

Theorem 3.7 (Smooth Foliation)**.** *Let $\{\Sigma_R\}$ be the family of surfaces with constant mean curvature constructed in Theorem 3.1. Then $\{\Sigma_R\}$ form a smooth foliation in the exterior region of (M, g).*

Proof. Let $\mathcal{H} : C^{2,\alpha}(\Sigma_R) \to C^{0,\alpha}(\Sigma_R)$ be the mean curvature operator defined by

$$\mathcal{H}(u) = \text{ the mean curvature of } \{z + \mu_g u : z \in \Sigma_R\}.$$

Because $d\mathcal{H} = -L_{\Sigma_R}$ is a linear isomorphism by Lemma 3.6, $\mathcal{H}$ is a diffeomorphism from some neighborhood U_R of $0 \in C^{2,\alpha}$ to some neighborhood V_R of $\mathcal{H}(0)$ by the inverse mapping theorem. Moreover, $\{\Sigma_R\}$ vary smoothly in R. To show that $\{\Sigma_R\}$ form a foliation, we need to prove Σ_{R_1} and Σ_R have no intersection for any $R_1 \neq R$. First, when R_1 is close to R and Σ_{R_1} is the graph of u for $u \in U_R$, we show that u satisfying $\mathcal{H}(u) = \text{constant}$ has a sign. In the following, we suppress the index R and denote Σ_R simply by Σ. By the Taylor theorem, for any $u \in U_R$,

$$\mathcal{H}(u) = \mathcal{H}(0) - L_\Sigma u + \int_0^1 \left(d^2\mathcal{H}(su) - d\mathcal{H}(0)\right) u\, ds.$$

Because $\mathcal{H}(u)$ and $\mathcal{H}(0)$ are constants, we can rewrite the above identity as

$$L_\Sigma u = C_1 + \int_0^1 \left(d^2\mathcal{H}(su) - d\mathcal{H}(0)\right) u\, ds. \tag{3.24}$$

A direct calculation by integrating the above identity over Σ and by (3.19),

$$\begin{aligned} C_1 &= -|\Sigma|^{-1} \int_\Sigma \left(|A_\Sigma|^2 + Ric^M(\mu_g, \mu_g)\right) u\, d\sigma \\ &\quad -|\Sigma|^{-1} \int_\Sigma \int_0^1 \left(d^2\mathcal{H}(su) - d\mathcal{H}(0)\right) u\, ds d\sigma \\ &= -\frac{2}{R^2}\bar{u} + O(R^{-2-q}\|u\|_{C^{2,\alpha}}). \end{aligned} \tag{3.25}$$

In order to prove that $\sup_\Sigma |u - \bar{u}|$ is small comparing with $|\bar{u}|$, we decompose $u = h_0 + u_0$ where h_0 is the lowest eigenfunction of L_Σ and $\int_\Sigma h_0 u_0\, d\sigma = 0$, and we will prove h_0 has a sign and u_0 is small.

$$\sup_\Sigma |u - \bar{u}| \le \sup_\Sigma |h_0 - \bar{h}_0| + 2 \sup_\Sigma |u_0|.$$

By the standard De Giorgi-Nash-Moser theory, because $h_0 - \bar{h}_0$ satisfies the second order elliptic equation (3.21) and $\|h_0 - \bar{h}_0\|_{L^2}$ is controlled as in (3.22),

$$\begin{aligned} \sup_\Sigma |h_0 - \bar{h}_0| &\le cR^{-1}\|h_0 - \bar{h}_0\|_{L^2(\Sigma)} + cR|\bar{h}_0| \left\|\eta_0 + |A_\Sigma|^2 + Ric^M(\mu_g, \mu_g)\right\|_{L^4} \\ &\le cR^{-1-q}|N|^{1/2}|\bar{h}_0| + cR^{-\frac{1}{2}-q}|\bar{h}_0| \le cR^{-q}|\bar{h}_0| \end{aligned} \tag{3.26}$$

$$\begin{aligned} \sup_{x\neq y} \frac{|h_0(x) - h_0(y)|}{|x-y|^\alpha} &\le cR^{-\alpha}\left(\sup |h_0 - \bar{h}_0| + R|\bar{h}_0| \left\|\eta_0 + |A_\Sigma|^2 + Ric^M(\mu_g, \mu_g)\right\|_{L^4}\right) \\ &\le cR^{-\alpha-q}|\bar{h}_0| \end{aligned} \tag{3.27}$$

Therefore, h_0 has a sign when R large by (3.26). It remains to estimate $\sup_\Sigma |u_0|$ and prove that u_0 is too small to change the sign of u. By the definition of u_0, (3.20), (3.24), and (3.25),

$$\begin{aligned} L_\Sigma u_0 &= -\eta_0 h_0 - \frac{2}{R^2}\bar{h}_0 - \frac{2}{R^2}\bar{u}_0 + O(R^{-2-q}\|u\|_{C^{2,\alpha}}) \\ &= \frac{2}{R^2}(h_0 - \bar{h}_0) - \frac{2}{R^2}\bar{u}_0 + O(R^{-2-q}\|u\|_{C^{2,\alpha}}). \end{aligned}$$

Because L_Σ has no kernel, by the Hölder estimate, for R large,

$$\begin{aligned} \|u_0\|_{C^{2,\alpha}} &\leq c\left(\|h_0 - \bar{h}_0\|_{C^{0,\alpha}} + \|\bar{u}_0\|_{C^{0,\alpha}} + R^{-q}\|u\|_{C^{2,\alpha}}\right) \\ &\leq c\left(\|h_0 - \bar{h}_0\|_{C^{0,\alpha}} + |\bar{u}_0| + R^{-q}\|h_0\|_{C^{2,\alpha}}\right) \\ &\leq c\left(\|h_0 - \bar{h}_0\|_{C^{0,\alpha}} + |\bar{u}_0| + R^{-q}|\bar{h}_0|\right). \end{aligned} \tag{3.28}$$

Because $\int_\Sigma h_0 u_0 \, d\sigma = 0$, similarly as in (3.23), we have

$$\left|\int_\Sigma u_0 \, d\sigma\right| \leq 2|N||\bar{h}_0|^{-1} \sup_\Sigma |h_0 - \bar{h}_0| \sup_\Sigma |u_0|$$

and then by (3.26)

$$|\bar{u}_0| \leq 2|\bar{h}_0|^{-1} \sup_\Sigma |h_0 - \bar{h}_0| \sup_\Sigma |u_0| \leq cR^{-q} \sup_\Sigma |u_0|.$$

Then $|\bar{u}_0|$ could be absorbed into the left hand side of (3.28). For R large enough, by (3.27) we have

$$\sup |u_0| \leq \|u_0\|_{C^{2,\alpha}} \leq c\left(\|h_0 - \bar{h}_0\|_{C^{0,\alpha}} + R^{-q}|\bar{h}_0|\right) \leq cR^{-q}|\bar{h}_0|.$$

Concluding above estimates, for R large, $|u - \bar{u}| \leq 2^{-1}|\bar{u}|$, and then u has a sign.

We have proved that, in the neighborhood U_R where the inverse function theorem holds, any two surfaces with constant mean curvature have no intersection. Because the size U_R is independent of R by the bounds of $|d^2\mathcal{H}|$ and $|L_\Sigma^{-1}|$ (c.f. [MRA, Proposition 2.5.6]), we could inductively proceed the argument toward infinity and conclude that $\{\Sigma_R\}$ form a foliation in the exterior region. □

4. Uniqueness of the Foliation

In this section, we assume (M, g, K) is AF-RT with $q \in (1/2, 1]$, and Σ_R is the surface with constant mean curvature constructed in Theorem 3.1. Also recall that Σ_R is a $c_0 R^{1-q}$-graph over $S_R(\mathcal{C})$.

4.1. **Local Uniqueness.**

Theorem 4.1 (Local Uniqueness)**.** *Given any $c_1 \geq c_0$, there exists $\sigma_1 = \sigma_1(c_1)$ so that, for $R \geq \sigma_1$, if N is a $c_1 R^{1-q}$-graph over $S_R(\mathcal{C})$, is topologically a sphere, and has mean curvature equal to $H(\Sigma_R)$, then $N = \Sigma_R$.*

Proof. Because the normal vectors on $S_R(\mathcal{C})$ and Σ_R are close, we can write N as a graph over Σ_R and assume $N = \{z + v\mu_g : z \in \Sigma_R\}$. We first prove that there is a constant $\widetilde{c}_1$ so that if $\|v\|_{C^{2,\alpha}(\Sigma_R)} \leq 2\tilde{c}_1$, then $v \equiv 0$, by using the invertibility of L_{Σ_R}.

$$H_{\Sigma_R}(v) = H_{\Sigma_R}(0) - L_{\Sigma_R} v + \int_0^1 (dH_{\Sigma_R}(sv) - dH_{\Sigma_R}(0))\, v\, ds$$
$$\Longrightarrow \qquad L_{\Sigma_R} v = \int_0^1 (dH_{\Sigma_R}(sv) - dH_{\Sigma_R}(0))\, v\, ds.$$

Because $|L_{\Sigma_R}^{-1}| \leq cm^{-1}R^3$ by Lemma 3.6 and (3.5),

$$\begin{aligned} \|v\|_{C^{2,\alpha}(\Sigma_R)} &\leq cm^{-1}R^3 \left\| \int_0^1 (dH_{\Sigma_R}(sv) - dH_{\Sigma_R}(0))\, v\, d\sigma \right\|_{C^{0,\alpha}(\Sigma_R)} \\ &\leq cm^{-1}R^3 R^{-3} \|v\|^2_{C^{2,\alpha}(\Sigma_R)} \leq cm^{-1} \|v\|^2_{C^{2,\alpha}(\Sigma_R)} \end{aligned}$$

Choosing any $\widetilde{c}_1 < c^{-1}m/2$, we have

$$\|v\|_{C^{2,\alpha}(\Sigma_R)} \leq 2\widetilde{c}_1 \Longrightarrow v \equiv 0.$$

Recalling the construction in Theorem 3.1, Σ_R is a $O(R^{-a})$-graph over $\mathcal{S}(p,R)$ for some fixed $a \in (1-q,q)$ and $p = \mathcal{C} + O(R^{-\epsilon})$. For R large, $\mathcal{S}(\mathcal{C},R)$ is within $\widetilde{c}_1$-distance of Σ_R. Therefore, because the normal vectors of Σ_R and $\mathcal{S}(\mathcal{C},R)$ are close, for R large,

$$N \text{ is a } \widetilde{c}_1\text{-graph over } \mathcal{S}(\mathcal{C},R) \Longrightarrow \|v\|_{C^{2,\alpha}(\Sigma_R)} \leq 2\widetilde{c}_1 \Longrightarrow N = \Sigma_R. \quad (4.1)$$

By the assumption, N is the graph of u over $S_R(\mathcal{C})$ with $\|u\|_{C^{2,\alpha}} \leq c_1 R^{1-q}$. The mean curvature of $N = \{y + u\nu_g\}$ is

$$\begin{aligned} H_S(u) &= H_S(0) + \Delta_S u + \left(|A_S|^2 + Ric^M(\nu_g,\nu_g)\right) u \\ &\quad + \int_0^1 (dH_S(sv) - dH_S(0))\, v\, ds. \end{aligned} \quad (4.2)$$

Then let ϕ_0 be the function defined as in the proof of Theorem 3.1,

$$\begin{aligned} -\Delta_S^e(u-\phi_0) - \frac{2}{R^2}(u-\phi_0) &= f_1(y) - f_2(y-\mathcal{C}) + \frac{\mathbf{A}\cdot(y-\mathcal{C})}{R^{3+q}} + \bar{f}_2 \\ &\quad + (\Delta_S - \Delta_S^e)u + \left(|A_S|^2 - \frac{2}{R^2} + Ric^M(\nu_g,\nu_g)\right) u \\ &\quad + \int_0^1 (dH_S(su) - dH_S(0))\, u\, ds. \end{aligned} \quad (4.3)$$

We decompose u into the part perpendicular to the kernel and the part inside the kernel in the L^2-space,

$$u = u^\perp + \mathbf{B}\cdot(y-\mathcal{C})R^{-q}$$

where, for $\alpha = 1,2,3$,

$$B^\alpha = \frac{3R^{-4+q}}{4\pi}\int_{S_R(\mathcal{C})} u(y^\alpha - \mathcal{C}^\alpha)\, d\sigma_e \quad \text{and} \quad \int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) u^\perp\, d\sigma_e = 0.$$

Applying the Hölder estimate on (4.3), because $u^\perp - \phi_0$ is in the orthogonal complement of the kernel,

$$\|(u^\perp - \phi_0)^*\| \le cR^{-q}\left(1 + \|u^*\|_{C^{2,\alpha}}\right).$$

To estimate the part inside the kernel, we observe that if (4.3) is projected into the kernel, we have

$$\begin{aligned} 0 &= \int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) f_1 \, d\sigma_e \\ &\quad + \int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) \left((\Delta_S - \Delta_S^e) u^\perp + \left(|A_S|^2 - \frac{2}{R^2} + Ric^M(\nu_g, \nu_g) \right) u^\perp \right) d\sigma_e \\ &\quad - \int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) L_S \left(\mathbf{B} \cdot (y - \mathcal{C}) R^{-q} \right) d\sigma_e + E_7 \end{aligned}$$

where the error term $E_7 = \int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) \int_0^1 (dH_S(su) - dH_S(0))\, u \, ds \, d\sigma$. Because the sphere is centered at the center of mass, by Lemma 2.1,

$$\int_{S_R(\mathcal{C})} (y^\alpha - \mathcal{C}^\alpha) f_1 \, d\sigma_e = 8\pi m (\mathcal{C}^\alpha - \mathcal{C}^\alpha) + O(R^{-q}) = O(R^{-q}).$$

Then by the similar estimates as in Step 3 of the proof of Theorem 3.1, the second line and E_7 are of lower order. Moreover, using the eigenvalue estimate in Lemma 3.5 (the equality case)

$$8\pi m B^\alpha R^{1-q} = O(R^{-q}).$$

Therefore,

$$\begin{aligned} \|u - \phi_0\|_{C^{2,\alpha}} &\le \|u^\perp - \phi_0\|_{C^{2,\alpha}} + \left\| \mathbf{B} \cdot (y - \mathcal{C}) R^{-q} \right\|_{C^{2,\alpha}} \\ &\le cR^{-q}\left(1 + \|u\|_{C^{2,\alpha}}\right) + |\mathbf{B}| R^{1-q} \\ &\le cR^{-q} + c_1 R^{1-2q} + cR^{-q}. \end{aligned}$$

By choosing R large (depending on c_1), we have

$$\|u - \phi_0\|_{C^{2,\alpha}} \le \frac{\widetilde{c}_1}{2}.$$

Because $\mathcal{S}(\mathcal{C}, R) = \{y + \phi_0 \nu_g\}$ and also the normal vectors on $S_R(\mathcal{C})$ and on $\mathcal{S}(\mathcal{C}, R)$ are close enough, we could arrange so that N is a $\widetilde{c}_1$-graph over $\mathcal{S}(\mathcal{C}, R)$. Then by (4.1), $\Sigma_R' = \Sigma_R$. □

The above theorem says that for surfaces bounded away the Euclidean sphere centered at the center of mass, Σ_R is the only one with constant mean curvature $H(\Sigma_R)$. In particular, we can generalize the above statement for any constant mean curvature surfaces which are spherical.

Corollary 4.2. *Assume $|p - \mathcal{C}| \le \widetilde{c}_0 R^{1-q}$. Given any $\widetilde{c}_1 \ge \widetilde{c}_0 + c_0$, there exists $\sigma_1 = \sigma_1(\widetilde{c}_0, \widetilde{c}_1)$ so that, for $R \ge \sigma_1$, if N is a $\widetilde{c}_1 R^{1-q}$-graph over $S_R(p)$, is topologically a sphere, and has mean curvature equal to $H(\Sigma_R)$, then $N = \Sigma_R$.*

Proof. Assume N is a $\widetilde{c}_1 R^{1-q}$-graph over $S_R(p)$. Because the normal vectors on $S_R(p)$ and $S_R(\mathcal{C})$ are close and $|p-\mathcal{C}| \leq \widetilde{c}_0 R^{1-q}$, N is a $(\widetilde{c}_0+\widetilde{c}_1)R^{1-q}$-graph over $S_R(\mathcal{C})$ when R large. Then we can apply the previous theorem (by letting $c_1 = \widetilde{c}_0 + \widetilde{c}_1$) and derive that $N = \Sigma_R$. □

4.2. **A Priori Estimates.** For general surfaces with constant mean curvature, we would like to derive a priori estimates and show that they are close to the Euclidean spheres.

Let N be a surface with constant mean curvature H. Assume N is stable (Definition 3.4) and topologically a sphere. Let the minimum radius and the maximal radius of N denoted by $\underline{r}(N) = \min\{|z| : z \in N\}$ and $\bar{r}(N) = \max\{|z| : z \in N\}$ respectively. A is the second fundamental form of N, $\mathring{A} = A - \frac{1}{2}Hg|_N$ is the trace-free part of A, μ_g is the outward unit normal vector field on N, and Δ and ∇ are the Laplacian and the covariant derivative on N with respect to the induced metric $g|_N$. Moreover, to simplify the notations, we suppress the superscript M and denote R_{ijkl} or Riem the Riemannian curvature tensor and Ric the Ricci curvature tensor of (M,g) respectively.

The following Sobolev inequality holds and can be found in [HY96, Proposition 5.4].

Lemma 4.3 (Sobolev Inequality). *For $\underline{r}$ large, there is a constant c so that for any Lipschitz functions g on N,*

$$\left(\int_N |g|^2 \, d\sigma\right)^{\frac{1}{2}} \leq c \int_N |\nabla g| + H|g| \, d\sigma \tag{4.4}$$

Lemma 4.4. *Assume N is a hypersurface in M with constant mean curvature H. Also, M is topologically a sphere and stable. Then there is some constant c so that the following estimates hold.*

(1) For any $s > 2$,

$$\int_N |x|^{-s} \, d\sigma \leq c\underline{r}^{2-s}.$$

(2)

$$\left\| |\mathring{A}| \right\|_{L^2} \leq c\underline{r}^{-\frac{q}{2}}.$$

(3)

$$c^{-1} \leq H^2|N| \leq c.$$

Proof. Using the first variation formula as in [HY96, Lemma 5.2], for any $s > 2$,

$$\int_N |x|^{-s} \, d\sigma \leq c\underline{r}^{2-s} H^2 |N|.$$

As in [HY96, Proposition 5.3] and using that the Ricci curvature bounded by $|x|^{-2-q}$,

$$\int_N |\mathring{A}|^2 \, d\sigma \leq c\underline{r}^{-q} H^2 |N|.$$

If we can prove (3), especially the upper bound, then (1) and (2) follow directly from the above inequalities.

The lower bound in (3) can be derived by letting $|g| = H$ in the Sobolev inequality (4.4). For the upper bound, we first observe that the Gauss equation and the Gauss-Bonnet theorem imply

$$\begin{aligned}\int_N \frac{1}{2}H^2\, d\sigma &\leq \int_N 2K + |\mathring{A}|^2 - R_g + 2Ric(\mu_g,\mu_g)\, d\sigma \\ &\leq c + c\left(\int_N |\mathring{A}|^2 + |x|^{-2-q}\, d\sigma\right) \\ &\leq c + c\underline{r}^{-q} H^2 |N|.\end{aligned}$$

For $\underline{r}$ large, we can absorb the last term to the left hand side, and prove (3). □

By the Simons identity ([S68], [SSY75], and [M07]), for any hypersurface N in M, we have

$$\begin{aligned}\Delta A_{\alpha\beta} &= \nabla_\alpha\nabla_\beta H + H A_\alpha^\delta A_{\delta\beta} - |A|^2 A_{\alpha\beta} + A_\alpha^\delta R_{\epsilon\beta\epsilon\delta} + A^{\delta\epsilon} R_{\delta\alpha\beta\epsilon} \\ &\quad + \nabla_\beta\left(Ric_{\alpha k}\nu^k\right) + \nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right)\end{aligned}$$

where the Greek letters ranges over $\{1,2\}$, and the Latin letters ranges over $\{1,2,3\}$.

Lemma 4.5.

$$\left\||\mathring{A}|^2\right\|_{L^2} + \left\|\nabla|\mathring{A}|\right\|_{L^2} + \left\||\nabla\mathring{A}|\right\|_{L^2} + \left\|H|\mathring{A}|\right\|_{L^2} \leq c\underline{r}^{-1-q}.$$

Proof.

$$2|\mathring{A}|\Delta|\mathring{A}| + 2\left|\nabla|\mathring{A}|\right|^2 = \Delta|\mathring{A}|^2 = 2\mathring{A}^{\alpha\beta}\Delta\mathring{A}_{\alpha\beta} + 2|\nabla\mathring{A}|^2.$$

By the Cauchy-Schwarz inequality $|\nabla\mathring{A}|^2 \geq \left|\nabla|\mathring{A}|\right|^2$ and the following estimate (see [SY81] and [M07]),

$$\begin{aligned}|\nabla\mathring{A}|^2 - \left|\nabla|\mathring{A}|\right|^2 &\geq \frac{1}{17}|\nabla\mathring{A}|^2 - \frac{16}{17}\left(|\omega|^2 + |\nabla H|^2\right) \\ &\geq \frac{1}{34}|\nabla\mathring{A}|^2 + \frac{1}{34}\left|\nabla|\mathring{A}|\right|^2 - \frac{16}{17}\left(|\omega|^2 + |\nabla H|^2\right)\end{aligned}$$

we have

$$|\mathring{A}|\Delta|\mathring{A}| \geq \mathring{A}^{\alpha\beta}\Delta\mathring{A}_{\alpha\beta} + \frac{1}{34}|\nabla\mathring{A}|^2 + \frac{1}{34}\left|\nabla|\mathring{A}|\right|^2 - \frac{16}{17}\left(|\omega|^2 + |\nabla H|^2\right)$$

where $\omega = Ric(\cdot,\mu_g)^T$ denotes the tangential projection of $Ric(\cdot,\mu_g)$ on N.

Because N has constant mean curvature, we substitute the Simons identity into the above inequality and have

$$\begin{aligned}|\mathring{A}|\Delta|\mathring{A}| &\geq H\mathring{A}^{\alpha\beta}A_\alpha^\delta A_{\delta\beta} - |A|^2\mathring{A}^{\alpha\beta}A_{\alpha\beta} + \mathring{A}^{\alpha\beta}A_\alpha^\delta R_{\epsilon\beta\epsilon\delta} \\ &\quad + \mathring{A}^{\alpha\beta}A^{\delta\epsilon}R_{\delta\alpha\beta\epsilon} + \mathring{A}^{\alpha\beta}\nabla_\beta\left(Ric_{\alpha k}\nu^k\right) + \mathring{A}^{\alpha\beta}\nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right) \\ &\quad + \frac{1}{34}|\nabla\mathring{A}|^2 + \frac{1}{34}\left|\nabla|\mathring{A}|\right|^2 - \frac{16}{17}|\omega|^2. \qquad (4.5)\end{aligned}$$

Integrating $-|\mathring{A}|\Delta|\mathring{A}|$,

$$\begin{aligned}\int_N |\nabla|\mathring{A}||^2 + \frac{1}{34}|\nabla\mathring{A}|^2\,d\sigma &\le \int_N -H\mathring{A}^{\alpha\beta}A^{\delta}_{\alpha}A_{\delta\beta} + |A|^2\mathring{A}^{\alpha\beta}A_{\alpha\beta}\,d\sigma\\ &\quad - \int_N \mathring{A}^{\alpha\beta}A^{\delta}_{\alpha}R_{\epsilon\beta\epsilon\delta} + \mathring{A}^{\alpha\beta}A^{\delta\epsilon}R_{\delta\alpha\beta\epsilon}\,d\sigma\\ &\quad - \int_N \left(\mathring{A}^{\alpha\beta}\nabla_\beta\left(Ric_{\alpha k}\nu^k\right)\right) + \mathring{A}^{\alpha\beta}\nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right)\,d\sigma\\ &\quad - \int_N \frac{1}{34}\left|\nabla|\mathring{A}|\right|^2 - \frac{16}{17}|\omega|^2\,d\sigma.\end{aligned}$$

A direct calculation shows the first two terms on the right hand side

$$-H\mathring{A}^{\alpha\beta}A^{\delta}_{\alpha}A_{\delta\beta} + |A|^2\mathring{A}^{\alpha\beta}A_{\alpha\beta} = (|A|^2 - H^2)|\mathring{A}|^2 - H\mathring{A}^{\alpha\beta}\mathring{A}^{\delta}_{\alpha}\mathring{A}_{\delta\beta}.$$

The last term vanishes because M is two-dimensional. Therefore,

$$\begin{aligned}\int_N \frac{35}{34}|\nabla|\mathring{A}||^2 + \frac{1}{34}|\nabla\mathring{A}|^2\,d\sigma &\le \int_N (|A|^2 - H^2)|\mathring{A}|^2\,d\sigma\\ &\quad - \int_N \mathring{A}^{\alpha\beta}A^{\delta}_{\alpha}R_{\epsilon\beta\epsilon\delta} + \mathring{A}^{\alpha\beta}A^{\delta\epsilon}R_{\delta\alpha\beta\epsilon}\,d\sigma\\ &\quad - \int_N \left(\mathring{A}^{\alpha\beta}\nabla_\beta\left(Ric_{\alpha k}\nu^k\right)\right) + \mathring{A}^{\alpha\beta}\nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right)\,d\sigma\\ &\quad + \int_N \frac{16}{17}|\omega|^2\,d\sigma. \qquad (4.6)\end{aligned}$$

The second line can be bounded by

$$c\int_N |\mathring{A}|^2|\mathrm{Riem}| + |\mathring{A}|H|\mathrm{Riem}|\,d\sigma$$

Using integration by parts as in [M07], the third line can be bounded by $c\int_N |\omega|^2\,d\sigma$. We then use the stability to control the first line. Because N is stable, for any u with mean value $\bar{u}$,

$$\begin{aligned}\int_N |A|^2u^2\,d\sigma &\le \int_N |\nabla u|^2\,d\sigma + \int_N |A|^2(2u\bar{u} - \bar{u}^2)\,d\sigma - \int_N Ric(\mu_g,\mu_g)(u-\bar{u})^2\,d\sigma\\ &\le \int_N |\nabla u|^2\,d\sigma + \int_N \left(|\mathring{A}|^2 + \frac{1}{2}H^2\right)(2u\bar{u} - \bar{u}^2)\,d\sigma\\ &\quad +2\int_N |Ric|(u^2 + \bar{u}^2)\,d\sigma.\end{aligned}$$

Because $2u\bar{u} - \bar{u}^2 \le u^2$ and $|Ric(x)| \le c|x|^{-2-q}$, we let $u = |\mathring{A}|$ and rewrite the above inequality as follows:

$$\int_N \left(|A|^2 - \frac{1}{2}H^2\right)|\mathring{A}|^2\,d\sigma \le \int_N |\nabla|\mathring{A}||^2\,d\sigma + 2\bar{u}\int_N |\mathring{A}|^3 + 2\int_N |x|^{-2-q}\left(|\mathring{A}|^2 + \bar{u}^2\right)\,d\sigma.$$

Multiplying the above inequality by 69/68, adding it to (4.6), and moving the remaining H^2-term to the left,

$$\begin{aligned} & \int_N |\mathring{A}|^4 \, d\sigma + \int_N \left|\nabla|\mathring{A}|\right|^2 \, d\sigma + \int_N |\nabla \mathring{A}|^2 \, d\sigma + H^2 \int_N |\mathring{A}|^2 \, d\sigma \\ \leq \ & c\bar{u} \int_N |\mathring{A}|^3 \, d\sigma + c \int_N |x|^{-2-q} \left(|\mathring{A}|^2 + \bar{u}^2\right) \, d\sigma + c \int_N |\mathring{A}|H||x|^{-2-q} \, d\sigma \\ & + c \int_N |x|^{-4-2q} \, d\sigma. \end{aligned}$$

Because $\|\mathring{A}\|_{L^2} \leq c\underline{r}^{-\frac{q}{2}}$ as proven in Lemma 4.4(2), by the Hölder inequality and Lemma 4.4 (3),

$$\bar{u}^2 = |N|^{-2} \left(\int_N |\mathring{A}| \, d\sigma \right)^2 \leq |N|^{-1} \int_N |\mathring{A}|^2 \, d\sigma \leq c|N|^{-1} \underline{r}^{-q} \leq cH^2 \underline{r}^{-q}.$$

By Young's inequality,

$$c\bar{u} \int_N |\mathring{A}|^3 \, d\sigma \leq \frac{1}{4} \int_N |\mathring{A}|^4 \, d\sigma + c\underline{r}^{-q} H^2 \int_N |\mathring{A}|^2 \, d\sigma.$$

For $\underline{r}$ large enough, these two terms could be absorbed to the left hand side. Similarly, the rest of terms

$$\begin{aligned} c \int_N |x|^{-2-q} \left(|\mathring{A}|^2 + \bar{u}^2\right) \, d\sigma + c \int_N |x|^{-4-2q} \, d\sigma & \\ \leq \frac{1}{4} \int_N |\mathring{A}|^4 \, d\sigma + \underline{r}^{-2-q} \bar{u}^2 |N| + c\underline{r}^{-2-2q} & \\ c \int_N |\mathring{A}|H|x|^{-2-q} \leq \frac{1}{2} H^2 \int_N |\mathring{A}|^2 \, d\sigma + c\underline{r}^{-2-2q}. & \end{aligned}$$

We then derive

$$\left\||\mathring{A}|^2\right\|_{L^2} + \left\|\nabla|\mathring{A}|\right\|_{L^2} + \left\||\nabla \mathring{A}|\right\|_{L^2} + \left\|H|\mathring{A}|\right\|_{L^2} \leq c\underline{r}^{-1-q}.$$

□

4.3. **The Position Estimate.** In order to prove that N is close to some sphere pointwisely and is a nice graph, we would like to derive the pointwise estimate of $|\mathring{A}|$. We use the Moser iteration similarly as in [QT07].

Lemma 4.6. *For any function $u \geq 0$, $f \geq 0$, and h on N satisfying*

$$-\Delta u \leq fu + h \tag{4.7}$$

we have the pointwise control on u as follows:

$$\sup_N u \leq c(\|f\|_{L^2} + H + \underline{r}^{-1})(\|u\|_{L^2} + \underline{r}H^{-1}\|h\|_{L^2})$$

Proof. Replacing g by g^2 in the Sobolev inequality (4.4) and using the Hölder inequality, we derive a variant of the Sobolev inequality

$$\begin{aligned}\left(\int_N |g|^4\, d\sigma\right)^{\frac{1}{2}} &\leq c\left(\int_N |g||\nabla g|\, d\sigma + \int_N H|g|^2\, d\sigma\right)\\ \leq\ & c\left(\int_N |g|^2\, d\sigma\right)^{\frac{1}{2}}\left(\left(\int_N |\nabla g|^2\, d\sigma\right)^{\frac{1}{2}} + \left(\int_N H^2|g|^2\, d\sigma\right)^{\frac{1}{2}}\right) \quad (4.8)\end{aligned}$$

Let $k = \underline{r}\|h\|_{L^2}$, and $\widetilde{u} = u + k$. Then multiplying $\widetilde{u}^{p-1}$ on the both sides of (4.7)

$$-\widetilde{u}^{p-1}\Delta\widetilde{u} \leq f\widetilde{u}^p - kf\widetilde{u}^{p-1} + \frac{h}{\widetilde{u}}\widetilde{u}^p \leq f\widetilde{u}^p + \frac{h}{k}\widetilde{u}^p = \widetilde{f}\widetilde{u}^p$$

where $\widetilde{f} = f + k^{-1}h$. Integrating the above inequality, we have, for $p \geq 2$,

$$\begin{aligned}\int_N \left|\nabla(\widetilde{u}^{\frac{p}{2}})\right|^2 d\sigma &= \int_N \left|\frac{p}{2}\widetilde{u}^{\frac{p}{2}-1}\nabla\widetilde{u}\right|^2 d\sigma = \int_N \frac{p^2}{4}|\widetilde{u}|^{p-2}|\nabla\widetilde{u}|^2\, d\sigma\\ &= \frac{p^2}{4(p-1)}\int_N (p-1)\widetilde{u}^{p-2}|\nabla\widetilde{u}|^2\, d\sigma\\ &= \frac{p^2}{4(p-1)}\int_N -\widetilde{u}^{p-1}\Delta\widetilde{u}\, d\sigma\\ &\leq \frac{p}{2}\int_N \widetilde{u}^p\widetilde{f}\, d\sigma.\end{aligned}$$

We let g be $\widetilde{u}^{\frac{p}{2}}$ in (4.8) and substitute the gradient term by the above inequality,

$$\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2}} \leq c\left(\int_N \widetilde{u}^p\, d\sigma\right)^{\frac{1}{2}}\left(\left(\frac{p}{2}\int_N \widetilde{u}^p\widetilde{f}\, d\sigma\right)^{\frac{1}{2}} + \left(\int_N H^2\widetilde{u}^p\, d\sigma\right)^{\frac{1}{2}}\right).$$

By the Hölder inequality, the last two terms can be bounded by

$$\begin{aligned}\left(\frac{p}{2}\int_N \widetilde{u}^p\widetilde{f}\, d\sigma\right)^{\frac{1}{2}} &\leq \left(\frac{p}{2}\right)^{\frac{1}{2}}\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{4}}\left(\int_N \widetilde{f}^2\, d\sigma\right)^{\frac{1}{4}},\\ \left(\int_N H^2\widetilde{u}^p\, d\sigma\right)^{\frac{1}{2}} &\leq \left(\int_N H^4\, d\sigma\right)^{\frac{1}{4}}\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{4}}.\end{aligned}$$

Therefore, using the above inequalities and Young's inequality,

$$\begin{aligned}\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2}} \leq\ & \frac{1}{2}\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2}} + \frac{1}{4\epsilon_1}c\left(\frac{p}{2}\right)^{\frac{1}{2}}\left(\int_N \widetilde{u}^p\, d\sigma\right)\left(\int_N \widetilde{f}^2\, d\sigma\right)^{\frac{1}{2}}\\ & + \frac{c}{4\epsilon_2}\left(\int_N \widetilde{u}^p\, d\sigma\right)\left(\int_N H^4\, d\sigma\right)^{\frac{1}{2}}\end{aligned}$$

where $\epsilon_1 = (2c\sqrt{2p})^{-1}$ and $\epsilon_2 = (4c)^{-1}$. Therefore,

$$\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2}} \leq cp\left(\left(\left(\int_N \widetilde{f}^2\, d\sigma\right)^{\frac{1}{2}} + \left(\int_N H^4\, d\sigma\right)^{\frac{1}{2}}\right)\left(\int_N \widetilde{u}^p\, d\sigma\right)\right).$$

Then,

$$\left(\int_N \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2p}} \leq c^{\frac{1}{p}} p^{\frac{1}{p}} \left(\|\widetilde{f}\|_{L^2} + H\right)^{\frac{1}{p}} \left(\int_N \widetilde{u}^p\, d\sigma\right)^{\frac{1}{p}}.$$

Now letting $p = 2^i, i = 1, 2, 3, \ldots$, we then have

$$\left(\int_N \widetilde{u}^{2^{l+1}}\, d\sigma\right)^{2^{-l-1}} \leq \left(c\left(\|\widetilde{f}\|_{L^2} + H\right)\right)^{\sum_{i=1}^{l} 2^{-i}} 2^{\sum_{i=1}^{l}(i2^{-i})} \|\widetilde{u}\|_{L^2}.$$

Let $l \to \infty$,

$$\begin{aligned}
\sup_N u &\leq \sup_N \widetilde{u} \leq c\left(\|\widetilde{f}\|_{L^2} + H\right)\|\widetilde{u}\|_{L^2} \\
&\leq c\left(\|f\|_{L^2} + k^{-1}\|h\|_{L^2} + H\right)\left(\|u\|_{L^2} + kH^{-1}\right) \\
&\leq c\left(\|f\|_{L^2} + \underline{r}^{-1} + H\right)\left(\|u\|_{L^2} + \underline{r}H^{-1}\|h\|_{L^2}\right)
\end{aligned}$$

□

Corollary 4.7.

$$\sup|\mathring{A}| \leq c(\underline{r}^{-1-q} + H^{-1}\underline{r}^{-2-q}).$$

Furthermore, if $\underline{r}(N) \geq H^{-a}$ *for some fixed* a *with* $2/(2+q) < a \leq 1$*, then*

$$\sup|\mathring{A}| \leq cH^{1+\epsilon}$$

where $\epsilon = (2+q)a - 2$.

Proof. Calculating similarly as in (4.5), $|\mathring{A}|$ satisfies the following inequality

$$\begin{aligned}
-|\mathring{A}|\Delta|\mathring{A}| &\leq (|A|^2 - H^2)|\mathring{A}|^2 - \mathring{A}^{\alpha\beta}A^{\delta}_{\alpha}R_{\epsilon\beta\epsilon\delta} \\
&\quad -\mathring{A}^{\alpha\beta}A^{\delta\epsilon}R_{\delta\alpha\beta\epsilon} - \mathring{A}^{\alpha\beta}\nabla_\beta\left(Ric_{\alpha k}\nu^k\right) - \mathring{A}^{\alpha\beta}\nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right) \\
&\leq c\left(|\mathring{A}|^4 + |\mathring{A}|^2|x|^{-2-q} + |\mathring{A}|H|x|^{-2-q} + |\mathring{A}||x|^{-3-q}\right)
\end{aligned}$$

where we have used that $|R_{ijkl}| \leq c|x|^{-2-q}$ and $|\nabla R_{ijkl}| \leq c|x|^{-3-q}$. Therefore $\mathring{A}$ satisfies (4.7) with

$$\begin{aligned}
f &= c(|\mathring{A}|^2 + |x|^{-2-q}) \\
h &= c(H|x|^{-2-q} + |x|^{-3-q}).
\end{aligned}$$

By Lemma 4.4(1) and Lemma 4.5,

$$\|f\|_{L^2} \leq c\underline{r}^{-1-q}, \qquad \|h\|_{L^2} \leq c\underline{r}^{-2-q}.$$

Then by Lemma 4.5 and Lemma 4.6,

$$\begin{aligned}
\sup|\mathring{A}| &\leq c(\underline{r}^{-1-q} + H + \underline{r}^{-1})(H^{-1}\underline{r}^{-1-q}) \\
&\leq c(H + \underline{r})(H^{-1}\underline{r}^{-1-q}).
\end{aligned}$$

□

After we obtain the estimates on $|\mathring{A}|$, we first prove that N can be approximated by a sphere $S_{r_0}(p)$. The following lemma is essentially the same as in [HY96, Proposition 2.1], but we remove the conditions on $|\nabla \mathring{A}|$ and $\bar{r}$.

Lemma 4.8. *Let N satisfy the same assumptions as in Theorem 2. There exists p so that for all $z \in N$,*

$$|\lambda^e{}_i(z) - r_0^{-1}| \leq cH^{1+\epsilon} \tag{4.9}$$
$$|\lambda^e{}_i(z-p) - \nu_e(z)| \leq cH^{\epsilon} \tag{4.10}$$
$$|(z-p) - r_0\nu_e(z)| \leq cr_0H^{\epsilon} \leq cH^{-1+\epsilon} \tag{4.11}$$

where $r_0 = 2/H$, $\lambda^e{}_i(z)$ is the principal curvature, and $\nu_e(z)$ is the outward unit normal vector at z with respect to the Euclidean metric. Moreover, N is a graph over $S_{r_0}(p)$ so that

$$N = \left\{ z = y + \nu_g \phi : y \in S_{r_0}(p), \phi \in C^1(S_{r_0}(p)) \right\}$$

with $\|\phi^*\|_{C^1} \leq cH^{-1+\epsilon}$.

Proof. By Corollary 4.7, $\sup_N |\mathring{A}| \leq cH^{1+\epsilon}$. Because M is AF, for $\underline{r}$ large,

$$\begin{aligned} \sup_N |\mathring{A}^e| &\leq |\mathring{A}| + c|z|^{-1-q} \leq cH^{1+\epsilon} \\ |H^e - H| &\leq c\underline{r}^{-1-q} \leq cH^{1+\epsilon}. \end{aligned}$$

We would like to use the bound of these Euclidean quantities to show that N is close to some sphere in the Euclidean space. To derive (4.9), for any point $z \in N$,

$$\begin{aligned} \left|\lambda^e{}_i(z) - \frac{1}{2}H\right| &\leq \left|\lambda^e{}_i(z) - \frac{1}{2}H^e(z)\right| + \left|\frac{1}{2}H^e(z) - \frac{1}{2}H\right| \\ &\leq |\mathring{A}^e| + cH^{1+\epsilon} \leq cH^{1+\epsilon}. \end{aligned}$$

Let $r_0^{-1} = (1/2)H$, and then (4.9) follows. For (4.10) and (4.11), we derive the upper bound on the diameter of N which is defined by the *intrinsic* distance on N equipped with its induced metric from the Euclidean space. Using the Gauss equation for N inside the Euclidean space,

$$\begin{aligned} & \left| Ric^N_{ij} - \frac{1}{4}(H^e)^2\gamma_{jk} \right| \leq c\left(|H^e||\mathring{A}^e| + |\mathring{A}^e|^2\right) \leq cH^{2+\epsilon} \\ \Longrightarrow \quad & Ric^N \geq \frac{1}{8}H^2. \end{aligned}$$

for H small. The Bonnet-Myers theorem says that $\mathrm{diam}(N) \leq cH^{-1}$. Then, the same argument in [HY96, Proposition 2.1] holds.

To prove that N is a C^1-graph over $S_{r_0}(p)$, for any point $z \in N$, we define $\psi(y) = |z-y|$ where $y \in S_{r_0}(p)$ is the intersection of the ray $z-a$ and $S_{r_0}(p)$.

By (4.10) and assuming H is small, y is uniquely determined, because for all $z \in N$

$$\left| \frac{z-p}{r_0} - \nu_e \right| \leq \frac{1}{2}.$$

In particular, ν_e never becomes perpendicular to the radial direction, and then $N = \{z + \psi\nu_e : z \in S_{r_0}(p)\}$ is well-defined. To obtain the C^1 bound on ψ, by (4.10) and (4.11),

$$\begin{aligned}
\|\psi\|_{C^0} &\leq \sup_{z\in N} |z-y| \leq \sup_{z\in N} |(z-p)-(y-p)| = \sup_{z\in N} \left| z-p-r_0\frac{z-p}{|z-p|}\right| \\
&= \sup_{z\in N} |(z-p)-r_0\nu_e| + \sup_{z\in N}\left| r_0\nu_e - r_0\frac{(z-p)}{|z-p|}\right| \leq cH^{-1+\epsilon}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\partial\psi| &= |z-y|^{-1}\left|\langle \nabla^e(z-y), z-y\rangle\right| \leq |\nabla^e(z-p)-\nabla^e(y-p)| \\
&\leq r_0\left|\nabla_e\nu^e - \nabla^e\frac{(y-p)}{|y-p|}\right| + cr_0H^{-1+\epsilon} \\
&\leq r_0|\mathring{A}^e| + cr_0H^{-1+\epsilon} \\
&\leq cH^{\epsilon}
\end{aligned}$$

where we have used that the principal curvature of $S_{r_0}(p)$ is r_0^{-1}. Therefore, we conclude $\|\psi^*\|_{C^1} \leq cH^{-1+\epsilon}$. Moreover, because ν_e and ν_g are close,

$$N = \{z+\nu_e\psi : y \in S_{r_0}(p)\} = \{z + \nu_g\phi : y \in S_{r_0}(p)\}$$

for some ϕ satisfying $\|\phi^*\|_{C^1} \leq cH^{-1+\epsilon}$. □

However, in order to use the Taylor theorem as before, N should be a graph whose $C^{2,\alpha}$-norm is under control. Therefore, we have to derive the pointwise estimate on the $C^{1,\alpha}$-norm of $\mathring{A}$. A modified Moser iteration which involves a special choice of the cut-off functions as in [QT07] will be proved in the following for N satisfying the previous estimates.

Lemma 4.9. *For any function $u \geq 0$, $f \geq 0$, and h on N satisfying*

$$-\Delta u \leq fu + h \tag{4.12}$$

we have the pointwise control on u as follows:

$$\sup_N u \leq c\left((\|f\|_{L^2} + H)\|u\|_{L^2} + H^{-2}\|h\|_{L^2}\|f\|_{L^2}\right).$$

Remark. Compare this lemma with Lemma 4.6, and the undesirable term $H^{-1}\|h\|_{L^2}$ in estimating $|\nabla\mathring{A}|$ is removed.

Proof. We replace $k = H^{-1}\|h\|_{L^2}$ in Lemma 4.6 and define $\widetilde{u}$ and $\widetilde{f}$ the same as there. Let χ be a cut-off function on N. The same calculations in Lemma 4.6 give

$$\int_N \left|\nabla(\chi\widetilde{u}^{\frac{p}{2}})\right|^2 d\sigma \leq \frac{p}{2}\int_N \chi^2\widetilde{f}\widetilde{u}^p\, d\sigma + \int_N |\nabla\chi|^2\widetilde{u}^p\, d\sigma$$

and then

$$\left(\int_N \chi^4 \widetilde{u}^{2p}\, d\sigma\right)^{\frac{1}{2}} \le cp \left(\left(\left(\int_N \widetilde{f}^2\, d\sigma\right)^{\frac{1}{2}} + H + (\sup |\nabla \chi|)^{\frac{1}{2}} \right) \int_{\text{supp}(\chi)} \widetilde{u}^p\, d\sigma.$$

Let $p_i = 2^i, i = 1,2,3,\ldots$ and the cut-off functions supported on N be defined by,

$$\chi_i(z) = \begin{cases} 1 & \text{if } z \in B_{(1+2^{-i})\delta H^{-1}}(z_0) \\ 0 & \text{if } z \text{ outside } B_{(1+2^{-i+1})\delta H^{-1}}(z_0) \end{cases}$$

where $z_0 \in N$ arbitrary, and δ is specified later. Then

$$\begin{aligned} &\left(\int_{B_{(1+2^{-l})\delta H^{-1}}(z_0)} \widetilde{u}^{2^{1+l}}\, d\sigma\right)^{2^{-1-l}} \\ &\le c^{\sum_{i=1}^l 2^{-i}} 2^{\sum_{i=1}^l i 2^{-i}} \left(\|\widetilde{f}\|_{L^2}^{\sum_{i=1}^l 2^{-i}} + H^{\sum_{i=1}^l 2^{-i}} + \left(\frac{2^i}{\delta H^{-1}}\right)^{\sum_{i=1}^l 2^{-1-i}} \right) \|\widetilde{u}\|_{L^2(B_{2\delta H^{-1}}(z_0))}. \end{aligned}$$

Let $l \to \infty$,

$$u(z_0) + k = \widetilde{u}(z_0) \le c(\|f\|_{L^2} + k^{-1}\|h\|_{L^2} + H + \delta^{-1}H) \left(\|u\|_{L^2} + k \,|B_{2\delta H^{-1}}(z_0)|^{\frac{1}{2}} \right).$$

We now choose δ so that

$$c \left(\int_{B_{2\delta H^{-1}}(z_0)} H^2\, d\sigma \right)^{\frac{1}{2}} \le \frac{1}{8}.$$

Most importantly, δ can be chosen independent of H by using that M is AF and N is a graph over $S_{r_0}(p)$ whose C^1-norm is controlled (the area formula contains only the first derivative of the graph, but no higher derivatives) as follows:

$$\begin{aligned} \int_{B_{2\delta H^{-1}}(z_0)} H^2\, d\sigma &= \text{area}_g\,(B_{2\delta H^{-1}}(z_0))\, H^2 \\ &\le 2\ \text{area}_{g_e}\,(B_{2\delta H^{-1}}(z_0))\, H^2 \le c(\delta H^{-1})^2 H^2 \le c\delta^2 \end{aligned}$$

where we have used g is AF and the area formula for the (Euclidean) graph. Substituting k by $H^{-1}\|h\|_{L^2}$,

$$\begin{aligned} u(z_0) + H^{-1}\|h\|_{L^2} \le\ & \frac{1}{4} H^{-1}\|h\|_{L^2} \\ & + c\left((\|f\|_{L^2} + H)\|u\|_{L^2} + H^{-2}\|h\|_{L^2}\|f\|_{L^2} \right). \end{aligned}$$

Moving the first term on the right hand side to the left, we complete the proof. □

Corollary 4.10.

$$\sup_N |\nabla \mathring{A}| \le c(\underline{r}^{-2-2q} + H\underline{r}^{-1-q} + H^{-2}\underline{r}^{-4-2q}).$$

If $\underline{r}(N) \geq H^{-a}$ *for some fixed* a *with* $2/(2+q) < a \leq 1$*, then*

$$\sup_N |\nabla \mathring{A}| \leq cH^{2+\epsilon}.$$

Proof. Let $T_{\gamma\alpha\beta} = \nabla_\gamma \mathring{A}_{\alpha\beta}$. Because $|\nabla T|^2 \geq |\nabla|T||^2$ by the Cauchy-Schwarz inequality,

$$\begin{aligned} & 2|T|\Delta|T| + 2\left|\nabla|T|\right|^2 = \Delta|T|^2 = 2T^{\gamma\alpha\beta}\Delta T_{\gamma\alpha\beta} + 2|\nabla T|^2 \\ \Longrightarrow \quad & -|T|\Delta|T| \leq -T^{\gamma\alpha\beta}\Delta T_{\gamma\alpha\beta}. \end{aligned}$$

If we would like to derive an inequality for $\Delta|\nabla A|$, we only need to compute $\nabla^\gamma \mathring{A}^{\alpha\beta}\Delta\left(\nabla_\gamma \mathring{A}_{\alpha\beta}\right)$. Changing the order of differentiation,

$$\begin{aligned} \Delta(\nabla_\gamma \mathring{A}_{\alpha\beta}) &= \nabla_\gamma \Delta \mathring{A}_{\alpha\beta} + g^{\rho\delta}(\nabla_\epsilon \mathring{A}_{\alpha\beta}) R_{\delta\ \gamma\rho}^{\ \epsilon} + g^{\rho\delta}(\nabla_\delta \mathring{A}_{\epsilon\beta}) R_{\alpha\ \gamma\rho}^{\ \epsilon} \\ &\quad + g^{\rho\delta}(\nabla_\delta \mathring{A}_{\alpha\epsilon}) R_{\beta\ \gamma\rho}^{\ \epsilon} + g^{\rho\delta}\nabla_\rho\left(\mathring{A}_{\epsilon\beta} R_{\alpha\ \gamma\delta}^{\ \epsilon} + \mathring{A}_{\alpha\epsilon} R_{\beta\ \gamma\delta}^{\ \epsilon}\right). \end{aligned}$$

By the Simons identity,

$$\begin{aligned} & \left(\nabla^\gamma \mathring{A}^{\alpha\beta}\right)\nabla_\gamma \Delta \mathring{A}_{\alpha\beta} \\ = \ & H\left(\nabla^\gamma \mathring{A}^{\alpha\beta}\right)\nabla_\gamma\left(A_\alpha^\delta A_{\delta\beta}\right) - \left(\nabla^\gamma \mathring{A}^{\alpha\beta}\right)\nabla_\gamma\left(|A|^2 A_{\alpha\beta}\right) \\ & + \left(\nabla^\gamma \mathring{A}^{\alpha\beta}\right)\nabla_\gamma\left(A_\alpha^\delta R^M_{\epsilon\beta\epsilon\delta} + A^{\delta\epsilon}R_{\delta\alpha\beta\epsilon} + \nabla_\beta\left(Ric^M_{\alpha k}\nu^k\right) + \nabla^\delta\left(R_{k\alpha\beta\delta}\nu^k\right)\right) \\ \geq \ & -|\mathring{A}|^2|\nabla\mathring{A}|^2 - c\left(H|\nabla\mathring{A}|^2|A| + |\nabla\mathring{A}|^2|\mathrm{Riem}| + |\nabla\mathring{A}||\mathring{A}||\nabla\mathrm{Riem}| + |\nabla\mathring{A}|H||\nabla\mathrm{Riem}|\right. \\ & \left. + |\nabla\mathring{A}||\nabla^2\mathrm{Riem}| + |\nabla\mathring{A}|^2|\nabla\mathrm{Riem}|\right) \end{aligned}$$

Using $|\mathrm{Riem}| \leq c|x|^{-2-q}$, $|\nabla\mathrm{Riem}| \leq c|x|^{-3-q}$, $|\nabla^2\mathrm{Riem}| \leq c|x|^{-4-q}$, and combining the above two estimates,

$$\begin{aligned} & -|\nabla\mathring{A}|\Delta|\nabla\mathring{A}| \\ \leq \ & -\nabla^\gamma\mathring{A}^{\alpha\beta}\Delta\left(\nabla_\gamma\mathring{A}_{\alpha\beta}\right) \\ \leq \ & c\left(|\mathring{A}|^2|\nabla\mathring{A}|^2 + H^2|\nabla\mathring{A}|^2 + H|\mathring{A}||\nabla\mathring{A}|^2 + |\nabla\mathring{A}|^2|x|^{-2-q}\right. \\ & \left. + |\nabla\mathring{A}||\mathring{A}||x|^{-3-q} + |\nabla\mathring{A}|H||x|^{-3-q} + |\nabla\mathring{A}||x|^{-4-q} + |\nabla\mathring{A}|^2|x|^{-3-q}\right). \end{aligned}$$

Then $|\nabla\mathring{A}|$ satisfies (4.7) with

$$\begin{aligned} f &= c(|\mathring{A}|^2 + H^2 + H|\mathring{A}| + |x|^{-2-q} + |x|^{-3-q}) \\ h &= c(|\mathring{A}||x|^{-3-q} + H|x|^{-3-q} + |x|^{-4-q}). \end{aligned}$$

Then $\|f\|_{L^2} \leq c\underline{r}^{-1-q}$ and $\|h\|_{L^2} \leq c\underline{r}^{3-q}$. By Lemma 4.5 and Lemma 4.9, a direct calculation completes the proof. □

Similarly, we can derive the Hölder norm $\left[|\nabla\mathring{A}|\right]_\alpha \leq cH^{2+\epsilon+\alpha}$. Using the same argument in Lemma 4.8, we prove the following:

Corollary 4.11. *N is a graph defined by* $N = \{z + \nu_g\phi : y \in S_{r_0}(p)\}$ *with* $\|\phi^*\|_{C^{2,\alpha}} \leq cH^{-1+\epsilon} \leq cr_0^{1-\epsilon}$.

4.4. **Global Uniqueness.**

Proof of Theorem 2 . By the Taylor theorem and Corollary 4.11, the mean curvature H of N is equal to

$$H = H(S_{r_0}(p)) + \Delta_S\phi + \left(|A_S|^2 + Ric^M(\nu_g,\nu_g)\right)\phi + \int_0^1 (dH(s\phi) - dH(0))\,\phi\, ds.$$

Because Lemma 2.1 (i) and $H = 2/r_0$,

$$\begin{aligned} -\Delta_S^e(\phi-\phi_0) - \frac{2}{{r_0}^2}(\phi-\phi_0) &= f_1(y) - f_2(y-p) + \frac{\mathbf{A}\cdot(y-p)}{r_0^{3+q}} + \bar{f}_2 \\ &+ (\Delta_S - \Delta_S^e)\,\phi + \left(|A_S|^2 - \frac{2}{{r_0}^2} + Ric^M(\nu_g,\nu_g)\right)\phi \\ &+ \int_0^1 (dH(s\phi) - dH(0))\,\phi\, ds \end{aligned} \tag{4.13}$$

We decompose $\phi = \phi^\perp + r_0^{-\epsilon}\left(\mathbf{B}\cdot(y-p)\right)$, where

$$B^\alpha = \frac{3r_0^{-4+\epsilon}}{4\pi}\int_S \phi(y^\alpha - p^\alpha)\, d\sigma_e = O(1).$$

Because $\phi^\perp - \phi_0 \in \mathrm{Ker}L_0$, we have

$$\|\phi^\perp - \phi_0\|_{C^{2,\alpha}} \le c r_0^{1-2\epsilon}.$$

Moreover,

$$\begin{aligned} \|(\phi^\perp)^{odd}\|_{C^{2,\alpha}} &\le \|\phi_0^{odd}\|_{C^{2,\alpha}} \\ &\quad + cr_0^2\left(r_0^{-2-q} + r_0^{-2-q}\|\phi\| + r_0^{-4-q}\|\phi\|^2 + r_0^{-3}\|(\phi^\perp)^{odd}\|\|\phi\|\right) \\ &\le C(r_0^{-q} + r_0^{1-2\epsilon-q} + r_0^{1-4\epsilon}). \end{aligned}$$

Bootstrapping $\|(\phi^\perp)^{odd}\|_{C^{2,\alpha}}$, we derive

$$\|(\phi^\perp)^{odd}\|_{C^{2,\alpha}} \le cr_0^{1-2\epsilon-q}.$$

Integrating (4.13) with $y^\alpha - p^\alpha$ and using Lemma 2.1 for the first term,

$$\begin{aligned} 0 &= \int_{S_{r_0}(p)} (y^\alpha - p^\alpha) f_1(y)\, d\sigma_e + \int_{S_{r_0}(p)} (y^\alpha - p^\alpha) L_S r_0^{-\epsilon}\left(\mathbf{B}\cdot(y-p)\right) d\sigma_e \\ &\quad + \int_{S_{r_0}(p)} (y^\alpha - p^\alpha)\left((\Delta_S - \Delta_S^e)\phi^\perp - \left(|A_S|^2 - \frac{2}{r_0^2} + Ric^M(\nu_g,\nu_g)\right)\phi^\perp\right) d\sigma_e \\ &\quad + \int_{S_{r_0}(p)} (y^\alpha - p^\alpha)\int_0^1 (dH(s\phi) - dH(0))\,\phi\, ds\, d\sigma_e \\ &= 8\pi m(p^\alpha - \mathcal{C}^\alpha) + B^\alpha r_0^{2-\epsilon}\frac{6m}{r_0^3}\frac{4\pi r_0^2}{3} + cr_0^3\left(r_0^{-3-q}\|\phi\| + r_0^{-2-q}\|(\phi^\perp)^{odd}\|\right) \\ &\quad + cr_0^3\left(r_0^{-4-q}\|\phi\|^2 + r_0^{-3}\|(\phi^\perp)^{odd}\|\|\phi\|\right) \\ \Longrightarrow &\quad |p^\alpha - \mathcal{C}^\alpha| \le c\left(r_0^{1-\epsilon} + r_0^{1+(1-q-4\epsilon)}\right). \end{aligned}$$

Recall that $\epsilon = (2+q)a - 2$. If $a > (9-q)/(8+4q)$,

$$|p^\alpha - \mathcal{C}^\alpha| \le cr^{1-\epsilon_1} \qquad \text{and} \qquad |p^\alpha| \le cr^{1-\epsilon_1}$$

for some $\epsilon_1 > 0$. It shows that the center p doesn't drift away too much. Let z_0 be a point so that $\underline{r} = |z_0|$,

$$\underline{r} = |z_0| \ge |z_0 - p| - |p| \ge r_0 - cH^{-1+\epsilon} - cr_0^{1-\epsilon_1} \ge cr_0.$$

Therefore, we can replace the assumption $\underline{r} \ge H^{-a}$ by $\underline{r} \ge cr_0 \ge cH^{-1}$ in the above estimates and have

$$\sup_N |\mathring{A}| \le cr_0^{-1-q} \qquad \text{and} \qquad \sup_N |\nabla \mathring{A}| \le cr_0^{-2-q}.$$

They imply, particularly, N is a cr_0^{1-q}-graph over $S_{r_0}(p)$ and $|p - \mathcal{C}| \le cr_0^{1-q}$. Although H may not be exactly equal to $H(\Sigma_{r_0})$, we can choose r_1 so that $H = H(\Sigma_{r_1})$ with $r_1 = r_0 + O(r_0^{-q})$. Then we can apply the local uniqueness result of Corollary 4.2 by viewing N as a graph over $S_{r_1}(a)$ and conclude $N = \Sigma_{r_1}$. □

To prove a result of the uniqueness outside a *fixed* compact set, we replace the condition on $\underline{r}$ by the condition that $\bar{r}$ and $\underline{r}$ satisfy $\bar{r} \le c_2 \underline{r}^{a^{-1}}$ for any $(9-q)/(8+4q) < a \le 1$.

Proof of Theorem 3. If N lies completely outside $B_{H^{-a}}(0)$, by Theorem 3, $N = \Sigma_R$. We assume that $N \ne \Sigma_R$. Therefore $N \cap B_{H^{-a}}(0) \ne \phi$ for any $(9-q)/(8+4q) < a \le 1$, and then $\underline{r} \le H^{-a} \le 3R^a$ if R large enough because $H = (2/R) + O(R^{-1-q})$. On the other hand, at z_0 where $\bar{r} = |z_0|$,

$$\frac{2}{\bar{r}} \le H^e(N)(z_0) \le H + c|z_0|^{-1-q} \le \frac{2}{R} + cR^{-1-q} + c\bar{r}^{-1-q}.$$

For R large,

$$\frac{2}{\bar{r}} \le \frac{4}{R} + \frac{1}{\bar{r}}$$

and then $R/4 \le \bar{r}$. Therefore,

$$\frac{1}{4(3)^{\frac{1}{a}}} (\underline{r})^{\frac{1}{a}} \le \frac{1}{4(3)^{\frac{1}{a}}} (3cR^a)^{\frac{1}{a}} \le \frac{R}{4} \le \bar{r}.$$

Choosing any $c_2 < \frac{1}{4(3)^{\frac{3}{2}}}$, we obtain $c_2 \underline{r}^{\frac{1}{a}} < \bar{r}$ which contradicts to the assumption. Therefore, $N = \Sigma_R$. □

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