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**A symmetrization of the relativistic Euler
equations in several spatial variables**

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REPORT No. 9, 2008/2009, fall

ISSN 1103-467X

ISRN IML-R- -9-08/09- -SE+fall

A SYMMETRIZATION OF THE RELATIVISTIC EULER EQUATIONS IN SEVERAL SPATIAL VARIABLES

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ABSTRACT. We consider the Euler equations governing relativistic compressible fluids evolving in the Minkowski spacetime with several spatial variables. We propose a new symmetrization which makes sense for solutions containing vacuum states and, for instance, applies to the case of compactly supported solutions, which are important to model star dynamics. Then, relying on these symmetrization and assuming that the velocity does not exceed some threshold and remains bounded away from the light speed, we deduce a local-in-time existence result for solutions containing vacuum states. We also observe that the support of compactly supported solutions does not expand as time evolves.

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Date: December 8, 2008.

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2000 *AMS Subject Classification:* 35L65, 76N10. *Key Words:* relativistic fluid, Euler equations, vacuum, symmetric hyperbolic system.

Submitted to: *Kinetic and Related Models*.

1. INTRODUCTION AND MAIN RESULT

The dynamics of relativistic compressible fluids evolving in the Minkowski spacetime with n spatial variables is governed by the Euler equations (for instance [4])

$$(1.1) \quad \begin{aligned} \partial_t \left(\frac{\rho + \epsilon^2 p}{1 - \epsilon^2 |u|^2} - \epsilon^2 p \right) + \sum_{k=1}^n \partial_{x_k} \left(\frac{\rho + \epsilon^2 p}{1 - \epsilon^2 |u|^2} u_k \right) &= 0, \\ \partial_t \left(\frac{\rho + \epsilon^2 p}{1 - \epsilon^2 |u|^2} u_j \right) + \sum_{k=1}^n \partial_{x_k} \left(\frac{\rho + \epsilon^2 p}{1 - \epsilon^2 |u|^2} u_j u_k + p \delta_{jk} \right) &= 0. \end{aligned}$$

Here, ρ and $u = (u_j)_{1 \leq j \leq n}$ denote the mass density and the (n -dimensional) velocity vector of the fluid and functions of the variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, while the parameter $1/\epsilon$ represents the light speed and δ_{jk} denotes the Kronecker symbol. The range of physical interest for the unknown (ρ, u) is defined by

$$(1.2) \quad \rho \geq 0, \quad |u|^2 := \sum_{j=1}^n u_j^2 < \epsilon^{-2},$$

while the pressure $p = p(\rho)$ is assumed to satisfy

$$(1.3) \quad 0 \leq p'(\rho) < \epsilon^{-2}.$$

Under these conditions, it can be checked that the system of conservation laws (1.1) is symmetric hyperbolic as long as vacuum is avoided, i.e. under the restriction $\rho > 0$. That is, it can be written in the symmetric hyperbolic in the so-called entropy variables (Makino and Ukai [5, 6])

$$\partial_t W + \sum_{k=1}^n A_k(W) \partial_{x_k} W = 0,$$

where $W := (\rho, u) \in \mathbb{R}_+ \times B_{1/\epsilon}$ (the set $B_{1/\epsilon}$ being the open ball with radius $1/\epsilon$) and such that, for every unit vector $\nu = (\nu_k) \in \mathbb{R}^n$, the matrix $\sum_{k=1}^n \nu_k A_k(W)$ admits real eigenvalues and a basis of eigenvectors. It is also established in [5, 6]. the initial value problem with non-vacuum initial data, i.e.

$$(1.4) \quad (\rho, u)(0, \cdot) = (\bar{\rho}, \bar{u}),$$

when the initial data $(\bar{\rho}, \bar{u}) \in \mathbb{R}_+ \times B_{1/\epsilon}$ is bounded away from vacuum, admits a local-in-time solution.

In the present paper, we are interested in the symmetrization and the local-in-time existence for the relativistic fluid equations (1.1) when the initial data $(\bar{\rho}, \bar{u})$ take arbitrary values in $\mathbb{R}_+ \times B_{1/\epsilon}$ and are allowed to contain vacuum states. Some partial but pioneering results were obtained on this problem by Rendall [8] and Guo and Tahvildar-Zadeh [2]. For an overview of the standard theory, we refer to the relevant the chapter on the Euler equations in Choquet-Bruhat's book [1].

We emphasize that compactly supported solutions are important in the applications, for instance to model the dynamics of stars. However, the current existence theory does not cover this situation; indeed, the transformation proposed in [5, 6] does not apply for our purpose since the coefficients $A_k(W)$ therein blow-up near the vacuum. On the other hand, when $\epsilon = 0$, the system (1.1) reduces to the non-relativistic Euler equations, for which local existence of solutions, even solutions containing vacuum states, was established by Makino, Ukai, and Kawashima [7]. The objective of the present paper is precisely to provide a suitable generalization of the theory in [7] to encompass relativistic fluids. This will be achieved by introducing yet another symmetrization which significantly differs from previously proposed ones.

An outline of this paper is as follows. In Section 2, we begin our investigation of the Euler equations and derive a first version of our symmetrization, which is based on using as main unknowns “generalized Riemann invariants” and a “normalized velocity”. Then, in Section 3, we recall basic material on Lorentz transformations and establish a technical lemma. Next, in Section 4 we are in a position to establish the existence result of this paper. The main technical difficulty is to check a positive-definiteness property for the symmetric system. Finally, in Section 5 we conclude with some remark about the support of solutions.

2. SYMMETRIZATION OF THE RELATIVISTIC EULER EQUATIONS

Modified mass and velocity variables. We are going to define new variables defined by nonlinear transformations of the mass density and the norm of the velocity vector, which allow us to put the relativistic Euler equations in a symmetric form. To begin with, we introduce the *modified mass density* variable w by

$$(2.1) \quad \begin{aligned} w = w(\rho) &:= \int_0^\rho \frac{c(s)}{q(s)} ds, \\ c(\rho) &:= \sqrt{p'(\rho)}, \quad q(\rho) := \rho + \epsilon^2 p(\rho), \end{aligned}$$

where $c(\rho) := \sqrt{p'(\rho)}$ represents the sound speed in the fluid.

From now on, we assume that $w(\rho)$ defined above is *finite*. This is the case if, near the vacuum, the equation of state is asymptotic to the one of polytropic perfect fluids, i.e. $p(\rho) \sim k \rho^\gamma$ with $\gamma > 1$ and $k > 0$. In the special case $p(\rho) = k \rho^\gamma$ and when ϵ is taken to tend to 0, then the function $w(\rho)$ approaches $\rho^{(\gamma-1)/2}$ (up to a multiplicative constant), which precisely coincides with the function introduced in [7] in the non-relativistic case.

In addition, based on the norm $|u|$ of the velocity vector u , we define the *modified velocity scalar*

$$(2.2) \quad v = v(|u|) := \frac{1}{2\epsilon} \ln \left(\frac{1 + \epsilon |u|}{1 - \epsilon |u|} \right).$$

We also introduce the n -dimensional, *normalized velocity vector* and the associated *projection operator*

$$(2.3) \quad \tilde{u} := \frac{u}{|u|}, \quad E(u) := I - \tilde{u} \otimes \tilde{u},$$

respectively, where I denotes the $n \times n$ identity matrix. Observe that $E(u)$ is singular as a function of u , when u is close to origin; however, the map $|u|^2 E(u)$ is actually smooth.

In the rest of this section, (ρ, u) denotes a given smooth solution to (1.1).

Proposition 2.1 (Formulation in terms of the modified mass and velocity variables). *The relativistic Euler equations are equivalent to the following system in the variables (w, v, \tilde{u}) :*

$$\begin{aligned} (1 - \epsilon^4 |u|^2 c(\rho)^2) \partial_t w + (1 - \epsilon^2 c(\rho)^2) u \cdot \nabla w \\ + c(\rho) (1 - \epsilon^2 |u|^2) \tilde{u} \cdot \nabla v + c(\rho) |u| \nabla \cdot \tilde{u} &= 0, \\ (1 - \epsilon^4 |u|^2 c(\rho)^2) \partial_t v + c(\rho) (1 - \epsilon^2 |u|^2) \tilde{u} \cdot \nabla w \\ + (1 - \epsilon^2 c(\rho)^2) u \cdot \nabla v - \epsilon^2 c(\rho)^2 |u|^2 \nabla \cdot \tilde{u} &= 0, \\ (1 - \epsilon^2 |u|^2)^{-1} |u| (\partial_t \tilde{u} + u \cdot \nabla \tilde{u}) + c(\rho) E(u) \nabla w &= 0. \end{aligned}$$

Proof. Step 1. The first equation in (1.1) takes the form

$$\begin{aligned} \frac{q'(\rho)}{1 - \epsilon^2 |u|^2} \partial_t \rho + q(\rho) \frac{2\epsilon^2}{(1 - \epsilon^2 |u|^2)^2} u \cdot \partial_t u - \epsilon^2 p'(\rho) \partial_t \rho \\ + \sum_{k=1}^n \frac{q'(\rho)}{1 - \epsilon^2 |u|^2} u_k \partial_{x_k} \rho + q(\rho) \frac{2\epsilon^2}{(1 - \epsilon^2 |u|^2)^2} u_k u \cdot \partial_{x_k} u + \frac{q(\rho)}{1 - \epsilon^2 |u|^2} \partial_{x_k} u_k = 0, \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1 + \epsilon^2 |u|^2 (q'(\rho) - 1)}{1 - \epsilon^2 |u|^2} \partial_t \rho + \frac{q'(\rho)}{1 - \epsilon^2 |u|^2} u \cdot \nabla \rho \\ + \frac{\epsilon^2 q(\rho)}{(1 - \epsilon^2 |u|^2)^2} (\partial_t |u|^2 + u \cdot \nabla |u|^2) + \frac{q(\rho)}{1 - \epsilon^2 |u|^2} \nabla \cdot u = 0. \end{aligned}$$

By multiplying this equation by $w'(\rho)$ we find

$$\begin{aligned} \frac{1 + \epsilon^4 |u|^2 c(\rho)^2}{1 - \epsilon^2 |u|^2} \partial_t w + \frac{1 + \epsilon^2 c(\rho)^2}{1 - \epsilon^2 |u|^2} u \cdot \nabla w \\ + \frac{\epsilon^2 c(\rho)}{(1 - \epsilon^2 |u|^2)^2} (\partial_t |u|^2 + u \cdot \nabla |u|^2) + \frac{c(\rho)}{1 - \epsilon^2 |u|^2} \nabla \cdot u = 0. \end{aligned}$$

To rewrite the above equation in a more convenient form, we observe that

$$\frac{dw}{d|u|} = \frac{1}{1 - \epsilon^2 |u|^2},$$

and so, after further multiplication by $(1 - \epsilon^2 |u|^2)$, the equation for the modified mass density reads

$$(2.4) \quad \begin{aligned} (1 + \epsilon^4 |u|^2 c(\rho)^2) \partial_t w + (1 + \epsilon^2 c(\rho)^2) u \cdot \nabla w \\ + 2\epsilon^2 c(\rho) |u| (\partial_t v + u \cdot \nabla v) + c(\rho) \nabla \cdot u = 0. \end{aligned}$$

Step 2. Next, we expand the second equation in (1.1) and obtain

$$\begin{aligned} \partial_t \left(\frac{q(\rho)}{1 - \epsilon^2 |u|^2} \right) u + \frac{q(\rho)}{1 - \epsilon^2 |u|^2} \partial_t u \\ + u \nabla \cdot \left(\frac{q(\rho)}{1 - \epsilon^2 |u|^2} u \right) + \frac{q(\rho)}{1 - \epsilon^2 |u|^2} u \cdot \nabla u + p'(\rho) \nabla \rho = 0, \end{aligned}$$

which, after using the mass equation, yields

$$\frac{q(\rho)}{1 - \epsilon^2 |u|^2} (\partial_t u + u \cdot \nabla u) + p'(\rho) (\epsilon^2 \partial_t \rho u + \nabla \rho) = 0.$$

Multiplying by $1/q(\rho)$ we arrive at an equation for the velocity vector

$$(2.5) \quad (1 - \epsilon^2 |u|^2)^{-1} (\partial_t u + u \cdot \nabla u) + c(\rho) (\epsilon^2 \partial_t w u + \nabla w) = 0.$$

We now multiply (2.5) by the vector u itself, and obtain

$$(1 - \epsilon^2 |u|^2)^{-1} (\partial_t |u|^2 + u \cdot \nabla |u|^2) + 2c(\rho) (\epsilon^2 \partial_t w |u|^2 + u \cdot \nabla w) = 0,$$

which, after a further multiplication by $(2|u|)^{-1}$, becomes

$$(2.6) \quad \partial_t v + u \cdot \nabla v + c(\rho) (\epsilon^2 |u| \partial_t w + \tilde{u} \cdot \nabla w) = 0.$$

Step 3. To derive the equation for \tilde{u} we multiply (2.5) by the projection matrix $E(u)$ and obtain

$$(1 - \epsilon^2 |u|^2)^{-1} E(u) (\partial_t u + u \cdot \nabla u) + c(\rho) E(u) (\epsilon^2 \partial_t w u + \nabla w) = 0.$$

In view of the identities

$$\begin{aligned} E(u)u &= 0, \\ E(u) \partial_t u &= \partial_t u - \tilde{u} \partial_t |u| = |u| \partial_t \tilde{u}, \\ E(u)(u \cdot \nabla u) &= |u| u \cdot \nabla \tilde{u}, \end{aligned}$$

we arrive at the (third) equation for the normalized velocity \tilde{u} , as stated in the proposition.

Finally, we are in a position to return to the equation (2.4) and, by plugging (2.6) in (2.4), find

$$(1 - \epsilon^4 |u|^2 c(\rho)^2) \partial_t w + (1 - \epsilon^2 c(\rho)^2) u \cdot \nabla w + c(\rho) \nabla \cdot u = 0.$$

Hence, observing that

$$\nabla \cdot u = (1 - \epsilon^2 |u|^2) \tilde{u} \cdot \nabla v + |u| \nabla \cdot \tilde{u},$$

we obtain the desired equation for w , as stated in the proposition. In turn, we can also put (2.5) in the form stated in the proposition for the function v . \square

A symmetric hyperbolic formulation. At this juncture, it may be interesting to consider the one-dimensional case $n = 1$. Considering the result in Proposition 2.1 and setting $u_x := \nabla u$, etc. and then observing that (in the one-dimensional case) $E \equiv 0$ and $\tilde{u} \equiv 1$, we obtain the following form of the relativistic Euler equations in the variables (w, v)

$$\begin{aligned} (1 - \epsilon^4 u^2 c(\rho)^2) \partial_t w + (1 - \epsilon^2 c(\rho)^2) u w_x + c(\rho) (1 - \epsilon^2 u^2) v_x &= 0, \\ (1 - \epsilon^4 u^2 c(\rho)^2) \partial_t v + c(\rho) (1 - \epsilon^2 u^2) w_x + (1 - \epsilon^2 c(\rho)^2) u v_x &= 0, \end{aligned}$$

which, obviously, is a symmetric hyperbolic system. In particular, it is obvious that the coefficient $1 - \epsilon^4 u^2 c(\rho)^2$ remains bounded away from zero, provided the sound speed or the fluid velocity scalar (or both) remain bounded away from the light speed.

To derive our symmetrization in general dimension, we need two additional observations.

- First, since \tilde{u} has unit norm we can write

$$\nabla \cdot \tilde{u} = \nabla \cdot \tilde{u} - \tilde{u} \cdot \nabla \frac{|\tilde{u}|^2}{2} = \text{tr}(E(u) \nabla \tilde{u}),$$

where “tr” denotes the trace of a matrix. This allows us to rewrite the last term of the w -equation in Proposition 2.1, in the form

$$c(\rho) |u| \nabla \cdot \tilde{u} = c(\rho) |u| \text{tr}(E(u) \nabla \tilde{u}).$$

Interestingly enough, this term can now be viewed as the “symmetric counterpart” of the term

$$|u| c(\rho) E(u) \nabla w$$

already contained in the \tilde{u} -equation in Proposition 2.1.

- At this stage, only one term poses some problem if we are to reach the desired symmetric form, that is, the term

$$-\epsilon^2 c(\rho)^2 |u|^2 \nabla \cdot \tilde{u}$$

in the v -equation of Proposition 2.1. To compensate for this term, one would need to have the term $-\epsilon^2 c(\rho)^2 |u|^2 \nabla v$ in the \tilde{u} -equation, but it does not appear that a direct transformation could achieve this. So, we introduce a further transformation based on still some new unknowns:

$$(2.7) \quad z_{\pm} := v \pm w,$$

which we will refer to as the *generalized Riemann invariant variables*. According to Proposition 2.1, we have the equations

$$\begin{aligned} (1 - \epsilon^4 |u|^2 c(\rho)^2) \partial_t z_{\pm} + (1 - \epsilon^2 c(\rho)^2) (|u| \pm c(\rho)) \tilde{u} \cdot \nabla z_{\pm} \\ \pm (1 \mp \epsilon^2 c(\rho) |u|) c(\rho) |u| \nabla \cdot \tilde{u} = 0, \\ (1 - \epsilon^2 |u|^2)^{-1} |u|^2 (\partial_t \tilde{u} + u \cdot \nabla \tilde{u}) + \frac{1}{2} |u| c(\rho) E(u) (\nabla z_+ - \nabla z_-) = 0. \end{aligned}$$

Consequently, by combining together the above observations we arrive at the main conclusion of this section:

Proposition 2.2 (Symmetric form of the Euler equations). *In terms of the generalized Riemann invariant variables (z_+, z_-) and the normalized velocity \tilde{u} defined in (2.1), (2.2), (2.7), the relativistic Euler equations take the following symmetric form*

$$(2.8) \quad (1 + \epsilon^2 |u| c(\rho)) \partial_t z_+ + \frac{1 - \epsilon^2 c(\rho)^2}{1 - \epsilon^2 c(\rho) |u|} (|u| + c(\rho)) \tilde{u} \cdot \nabla z_+ + c(\rho) |u| \operatorname{tr}(E(\tilde{u}) \nabla \tilde{u}) = 0,$$

$$(2.9) \quad (1 - \epsilon^2 |u| c(\rho)) \partial_t z_- + \frac{1 - \epsilon^2 c(\rho)^2}{1 + \epsilon^2 c(\rho) |u|} (|u| - c(\rho)) \tilde{u} \cdot \nabla z_- - c(\rho) |u| \operatorname{tr}(E(\tilde{u}) \nabla \tilde{u}) = 0,$$

$$(2.10) \quad \frac{2|u|^2}{1 - \epsilon^2 |u|^2} (\partial_t \tilde{u} + u \cdot \nabla \tilde{u}) + c(\rho) |u| E(\tilde{u}) \nabla z_+ - c(\rho) |u| E(\tilde{u}) \nabla z_- = 0,$$

where the unknowns z_\pm are real-valued and \tilde{u} is a unit vector, $|\tilde{u}| = 1$.

Here, the quantity ρ must be regarded as a function of the variable $z_+ - z_- = 2w$, where w was defined earlier as a function of ρ . On the other hand, u is a function of $z_+ - z_-$, namely

$$\epsilon |u| = \frac{e^{\epsilon(z_+ + z_-)} - 1}{e^{\epsilon(z_+ + z_-)} + 1}$$

Observe that the above symmetric formulation does allow the density variable to vanish, since the coefficients above remain bounded as the density approaches the vacuum. However, since the coefficient in front of $\partial_t \tilde{u}$ in the third equation vanishes with u , we see that the above formulation requires the velocity u to be bounded away from the origin which, of course, is not a realistic assumption to put on general solutions with vacuum. In Section 4, however, we will discuss a reduction of the general initial value problem which ensures this condition after applying a well-chosen Lorentz transformation to an arbitrary solution.

Remark 2.3. *The assumed normalization $|\tilde{u}|^2 = 1$ could be relaxed in the formulation of the system. In fact, if this condition holds at the initial time, then it holds for all times, as is clear from the transport equation satisfied by $|\tilde{u}|$*

$$\partial_t |\tilde{u}| + u \cdot \nabla |\tilde{u}| = 0,$$

which follows from (2.10) multiplied by \tilde{u} .

3. PROPERTIES OF LORENTZ TRANSFORMATION

Transformation formulas. We will need to rely on the Lorentz invariance property of the relativistic Euler equations and, therefore, in the present section, we collect several technical results about Lorentz transformations.

For every $U \in \mathbb{R}^n$ with $U \neq 0$, we set $\tilde{U} := U/|U|$ and we decompose any vector $x \in \mathbb{R}^n$ in a unique way such that

$$x = x_{\parallel} \tilde{U} + x_{\perp}, \quad x_{\parallel} = x \cdot \tilde{U} \in \mathbb{R}, \quad x_{\perp} \cdot U = 0.$$

The *Lorentz transformation* $(t, x) \mapsto (t', x')$ associated with the vector U is then defined by

$$\begin{aligned} t' &= \gamma(U) (t - \epsilon^2 U \cdot x), \\ x'_{\parallel} &= \gamma(U) (x_{\parallel} - U_{\parallel} t), \\ x'_{\perp} &= x_{\perp}, \end{aligned}$$

where

$$\gamma(U) = \frac{1}{\sqrt{1 - \epsilon^2 |U|^2}}$$

is the so-called Lorentz factor. This transformation can be put in an equivalent form

$$(3.1) \quad \begin{aligned} t' &= \gamma(U) (t - \epsilon^2 U \cdot x), \\ x' &= -\gamma(U) U t + (I + (\gamma(U) - 1) \tilde{U} \otimes \tilde{U}) x. \end{aligned}$$

It may be also convenient to use the modified velocity scalar V associated with U (following the definition in the previous section) and given by

$$e^{\epsilon V} := \gamma(U) (1 + \epsilon |U|) = \left(\frac{1 + \epsilon |U|}{1 - \epsilon |U|} \right)^{1/2},$$

and to rewrite the Lorentz transformation as

$$\begin{aligned} (t' \pm \epsilon \tilde{U} \cdot x') &= e^{\mp \epsilon V} (t \pm \epsilon \tilde{U} \cdot x), \\ x'_{\perp} &= x_{\perp}. \end{aligned}$$

or, equivalently, as

$$(3.2) \quad \begin{aligned} t' &= \cosh(\epsilon V) t - \epsilon \sinh(\epsilon V) \tilde{U} \cdot x, \\ x' &= -\sinh(\epsilon V) t + \epsilon \cosh(\epsilon V) \tilde{U} \cdot x, \\ x'_{\perp} &= x_{\perp}. \end{aligned}$$

Recall also that Lorentz transformations together with spatial rotations form the so-called *Poincaré group of isometries*, characterized by the condition that the length element of the Minkowski metric is preserved, that is,

$$-\epsilon^{-2} t'^2 + x'_{\parallel}{}^2 + |x'_{\perp}|^2 = -\epsilon^{-2} t^2 + x_{\parallel}^2 + |x_{\perp}|^2.$$

Recall also that the relativistic Euler equations are invariant under Lorentz transformations which, for instance, can be checked by direct (tedious) calculations or from more abstract considerations. The following transformation rule will also be useful in the following section.

Lemma 3.1 (Velocity transformation formula). *Let $U \in \mathbb{R}^n$ with $U \neq 0$, and denote by u, u' the fluid velocity vectors in different coordinate systems $(t, x), (t', x')$ related by the Lorentz transformation (3.1). Then, the transformation law for these velocity vectors is*

$$(3.3) \quad \begin{aligned} u' &= \frac{1}{1 - \epsilon^2 U \cdot u} \left(-U + \left(\gamma(U)^{-1} I + (1 - \gamma(U)^{-1}) \tilde{U} \otimes \tilde{U} \right) u \right) \\ &=: \frac{1}{\epsilon} \Phi(\epsilon u, \epsilon U). \end{aligned}$$

Proof. The fluid velocity vector represents the velocity of a fictitious point-mass moving along with the fluid. Thus, $u = u(t, x)$ represents the velocity vector of a point-mass located at x at the time t , while $u' = u'(t', x')$ is the velocity of the *same* point-mass in the coordinate system (t', x') . Consequently, the vectors u and u' are given by

$$(3.4) \quad u := \frac{dx}{dt}, \quad u' := \frac{dx'}{dt'}.$$

Now, in view of (3.1),

$$(3.5) \quad \begin{aligned} u' &= \frac{dx'}{dt} \left(\frac{dt'}{dt} \right)^{-1} \\ &= \left(-\gamma(U)U + (I + (\gamma(U) - 1)\tilde{U} \otimes \tilde{U}) \frac{dx}{dt} \right) \left(\gamma(U) \left(1 - \epsilon^2 U \cdot \frac{dx}{dt} \right) \right)^{-1}, \end{aligned}$$

which, together with (3.4), yields (3.3). \square

A technical property on the Lorentz-transformed velocity. In the following, we will need to have a lower bound on the fluid velocity vector, so we establish here a preliminary estimate. Throughout, $\epsilon \in (0, 1)$ and all of the constants are *independent* of ϵ . Given $r_0 \in (0, 1)$, we define

$$B_{r_0} := \{\epsilon u \in \mathbb{R}^3 \mid \epsilon|u| \leq r_0\}.$$

Lemma 3.2 (Uniform bounds for the velocity). *Given any $r_0 \in (0, 1)$ and any vector $U \in \mathbb{R}^3$ satisfying $r_0 < \epsilon|U| < 1$, there exist positive constants $0 < \delta_1 < \delta_2 < 1$ depending only on r_0 and ϵU , such that the Lorentz transformed velocity (3.3) has a norm uniformly bounded away from, both, the origin and the light speed, i.e.*

$$\delta_1 \leq |\Phi(\epsilon u, \epsilon U)| \leq \delta_2$$

hold for any $\epsilon u \in B_{r_0}$.

We observe that the above statement is sharp, in the sense that the constants δ_1, δ_2 may approach the endpoints of the interval $(0, 1)$ when r_0 also approaches the endpoints of $(0, 1)$ or when U approaches the endpoints of the interval $(r_0, 1)$.

Proof. To simplify the notation, we introduce the new variables and function

$$X := \epsilon u, \quad Z := \epsilon U, \quad W(X, Z) := |\epsilon u'|^2 = |\Phi(\epsilon u, \epsilon U)|^2.$$

It is convenient to choose the coordinate system so that

$$Z = (r_1, 0, 0), \quad 0 < r_0 < r_1 < 1.$$

Setting $X = (X_1, X_2, X_3)$ and noting that $\tilde{Z} \otimes \tilde{Z}X = (X_1, 0, 0)$, we get

$$(3.6) \quad W(X, Z) = \left| \frac{1}{1 - X \cdot Z} \left(\gamma(U)^{-1} X + (1 - \gamma(U)^{-1}) \tilde{Z} \otimes \tilde{Z}X - Z \right) \right|^2 \\ = \frac{1}{(1 - r_1 X_1)^2} \left((X_1 - r_1)^2 + \gamma(U)^{-2} (X_2^2 + X_3^2) \right).$$

The norm $|Z| = r_1$ being fixed, we are going now to compute the extremum values of the function W within the domain $|X| \leq r_0$.

We first compute the minimum by noting that

$$W(X, Z) \geq \left(\frac{X_1 - r_1}{1 - r_1 X_1} \right)^2 =: g(X_1).$$

An obvious lower bound is $W(X, Z) \geq (r_1 - r_0)^2 (1 + r_1 r_0)^{-2}$. To obtain an optimal bound, we compute

$$g'(X_1) = 2 \frac{X_1 - r_1}{1 - r_1 X_1} \frac{1 - r_1^2}{(1 - r_1 X_1)^2},$$

which is negative for any $|X_1| \leq r_0 < r_1$. Consequently, the minimum is attained at $X_1 = r_0$, and

$$(3.7) \quad \epsilon |u'| = W(X, Z)^{1/2} \geq g(r_0)^{1/2} \\ = \frac{r_1 - r_0}{1 - r_0 r_1} =: \delta_1 > 0.$$

Next, to compute the maximum, we set $|X| = r \leq r_0$. Noting that $\gamma(U)^{-2} = 1 - |Z|^2 = 1 - r_1^2$, we get

$$(X_1 - r_1)^2 + \gamma(U)^{-2} (X_2^2 + X_3^2) \\ = (1 - X_1 r_1)^2 + X_1^2 - 1 + r_1^2 - r_1^2 X_1^2 + (1 - r_1^2) (X_2^2 + X_3^2) \\ = (1 - X_1 r_1)^2 + X_1^2 (1 - r_1^2) - (1 - r_1^2) + (1 - r_1^2) (X_2^2 + X_3^2) \\ = (1 - X_1 r_1)^2 - (1 - r_1^2) (1 - |X|^2) \\ = (1 - X_1 r_1)^2 - (1 - r_1^2) (1 - r^2)$$

and, therefore,

$$W(X, Z) = 1 - h(X_1, r), \quad h(X_1, r) = \frac{(1 - r_1^2)(1 - r^2)}{(1 - X_1 r_1)^2}.$$

Clearly, for each fixed $r \in (0, r_0]$, the function $h(X_1, r)$ attains its minimum value at the point $X_1 = -r$, so that

$$\begin{aligned} W(X_1, Z) &\leq 1 - h(-r, r) = \frac{(1 + rr_1)^2 - (1 - r_1^2)(1 - r^2)}{(1 + rr_1)^2} \\ &= \frac{(r + r_1)^2}{(1 + rr_1)^2} =: k(r). \end{aligned}$$

Since

$$k'(r) = 2 \frac{(r + r_1)(1 - r_1^2)}{(1 + rr_1)^3} > 0,$$

for all $r \geq 0$ we have

$$\begin{aligned} W(X, Z) &\leq k(r_0) = \frac{(r_0 + r_1)^2}{(1 + r_0 r_1)^2} \\ &\leq k(1) = 1. \end{aligned}$$

Finally, by choosing $\delta_2 := k(r_0)^{1/2}$ we obtain the desired inequality and the proof of the lemma is completed. \square

A symmetrization for non-relativistic fluids. For clarity, let us explain our strategy to avoid the zero velocity problem in the simpler case of the *non-relativistic* Euler equations

$$\begin{aligned} (3.8) \quad &\partial_t \rho + \sum_{k=1}^n \partial_{x_k} (\rho u_k) = 0, \\ &\partial_t (\rho u_j) + \sum_{k=1}^n \partial_{x_k} (\rho u_j u_k + p \delta_{jk}) = 0. \end{aligned}$$

Since this is just (1.1) for the limit case $\epsilon \rightarrow 0$, and since the symmetrization argument of Section 2 is still valid for this case, setting $\epsilon = 0$ in Proposition 2.2 yields a symmetrization of (3.8), in the form

$$\begin{aligned} (3.9) \quad &\partial_t z_+ + (|u| + c(\rho)) \tilde{u} \cdot \nabla z_+ + c(\rho) |u| \operatorname{tr}(E(\tilde{u}) \nabla \tilde{u}) = 0, \\ &\partial_t z_- + (|u| - c(\rho)) \tilde{u} \cdot \nabla z_- - c(\rho) |u| \operatorname{tr}(E(\tilde{u}) \nabla \tilde{u}) = 0, \\ &2|u|^2 (\partial_t \tilde{u} + u \cdot \nabla \tilde{u}) + c(\rho) |u| E(\tilde{u}) \nabla z_+ - c(\rho) |u| E(\tilde{u}) \nabla z_- = 0. \end{aligned}$$

This symmetrization still has the drawback of having a possibly vanishing coefficient (the velocity) in the third equation.

Now, recall that the non-relativistic Euler equations (3.8) are invariant under Galilean transformations and introduce the coordinate frame translated at some given velocity U . We denote the new variables and unknowns with the symbol \sharp , that is,

$$(3.10) \quad t^\sharp := t, \quad x^\sharp := x - Ut, \quad \rho^\sharp := \rho, \quad u^\sharp := u - U.$$

We first obtain, by virtue of the Galilean invariance,

$$(3.11) \quad \begin{aligned} \partial_t^\# \rho^\# + \sum_{k=1}^n \partial_{x_k}^\# (\rho^\# u_k^\#) &= 0, \\ \partial_t^\# (\rho^\# u^\#) + \sum_{k=1}^n \partial_{x_k}^\# (\rho^\# u_j^\# u_k^\# + p^\# \delta_{jk}) &= 0. \end{aligned}$$

and then we note that the following symmetrization of (3.11) can be deduced from the same argument as above:

$$(3.12) \quad \begin{aligned} \partial_t^\# z_+^\# + (|u^\#| + c(\rho^\#)) \tilde{u}^\# \cdot \nabla^\# z_+^\# + c(\rho^\#) |u| \operatorname{tr}(E(\tilde{u}^\#) \nabla^\# \tilde{u}^\#) &= 0, \\ \partial_t^\# z_-^\# + (|u^\#| - c(\rho^\#)) \tilde{u} \cdot \nabla z_-^\# - c(\rho^\#) |u^\#| \operatorname{tr}(E(\tilde{u}^\#) \nabla^\# \tilde{u}^\#) &= 0, \\ 2|u^\#|^2 (\partial_t \tilde{u}^\# + u^\# \cdot \nabla^\# \tilde{u}^\#) + c(\rho^\#) |u^\#| E(\tilde{u}^\#) \nabla^\# z_-^\# + c(\rho^\#) |u^\#| E(\tilde{u}^\#) \nabla^\# z_+^\# &= 0. \end{aligned}$$

Now, the advantage of the symmetrization (3.12), in comparison with (3.9), is obvious: In view of (3.10), $u^\#$ never vanishes as long as u remains bounded, provided the reference velocity U can be chosen so that, say, $2|u| \leq |U|$. In turn, Kato's theory ensures the local well-posedness for the system (3.12) and, as a consequence, for the original system (3.9).

In the next section, we will see that the same strategy works for the relativistic case provided Galilean transformations are replaced by Lorentz transformations.

4. LOCAL-WELL-POSEDNESS THEORY

Relying on the symmetric form discovered in Proposition 2.2 we are now ready to establish the main results of the present paper. We denote here by $H_{ul}^r(\mathbb{R}^n)$ the uniformly local Sobolev space of order $r \geq 0$. (Recall that, by definition, the Sobolev norm in these spaces is computed on unit balls with arbitrary center varying in \mathbb{R}^n .)

Theorem 4.1 (Local-in-time solutions in Sobolev spaces). *Consider the relativistic Euler equation for an equation of state $p = p(\rho)$ satisfying the hyperbolicity condition (1.3) together with the following condition near the vacuum*

$$(4.1) \quad \limsup_{\substack{\rho \rightarrow 0 \\ \rho > 0}} \frac{c(\rho)}{w(\rho)} < \infty.$$

For every constant $M > 0$, there exists $\kappa \in (0, 1)$ such that the following property holds. Given at $t = 0$ an initial data $\bar{\rho}, \bar{u}$ belonging to the Sobolev space H_{ul}^r with $r > 1 + n/2$ and satisfying the constraints

$$0 \leq \bar{\rho} \leq M, \quad \epsilon^2 |\bar{u}|^2 \leq \kappa,$$

there exists a unique local solution ρ, u to the corresponding initial-value problem, which is defined up to a some maximal time $T > 0$ and satisfies

$$\rho, u \in C([0, T], H_{ul}^r(\mathbb{R}^n)) \cap C^1([0, T], H_{ul}^{r-1}(\mathbb{R}^n))$$

and

$$\rho \geq 0, \quad \epsilon^2 |u|^2 < 1.$$

The rest of this section is devoted to the proof of Theorem 4.1.

Step 1. By assumption, the initial velocity scalar is bounded away from the light speed and, in consequence, thanks to Lemma 3.2, we can find a vector $U \in \mathbb{R}^n$ with sufficiently large norm $|U|$ such that the transformed fluid velocity \bar{u}' , defined as in (3.3) by

$$(4.2) \quad \begin{aligned} \bar{u}' &= \frac{1}{1 - \epsilon^2 U \cdot \bar{u}} \left(\gamma(U)^{-1} \bar{u} + (1 - \gamma(U)^{-1}) (\tilde{U} \otimes \tilde{U}) \bar{u} - U \right), \\ &= \frac{1}{\epsilon} \Phi(\epsilon \bar{u}, \epsilon U), \end{aligned}$$

is bounded and bounded away from the origin. Precisely, for some constants $0 < \bar{\delta}_1 < \bar{\delta}_2 < 1$ we have

$$(4.3) \quad \bar{\delta}_1 \leq \epsilon |\bar{u}'| \leq \bar{\delta}_2.$$

Step 2. Recall that the general theory established by Kato [3] covers symmetric hyperbolic systems of the form

$$(4.4) \quad A_0(W) \partial_t W + \sum_{j=1}^n A_j(W) \partial_j W = 0,$$

where the $(d \times d)$ -matrix fields A_0, A_j are real-valued and symmetric with regular coefficients, and the matrix A_0 is uniformly positive definite. For the system (2.8)–(2.10), we have $d = n + 2$, $W = (z_+, z_-, \tilde{u})^t$ (a column vector, the subscript “ t ” standing for transposition), and

$$(4.5) \quad A_0(W) = \begin{pmatrix} a_0 & 0 & 0 \\ 0 & b_0 & 0 \\ 0 & 0 & c_0 |u|^2 I \end{pmatrix}, \quad A_j(W) = \begin{pmatrix} a_1 \tilde{u}_j & 0 & a_2 |u| e_j \\ 0 & b_2 \tilde{u}_j & -a_2 |u| e_j \\ a_2 |u| e_j^t & -a_2 |u| e_j^t & c_0 |u|^2 u_j I \end{pmatrix},$$

where

$$(4.6) \quad \begin{aligned} a_0 &= 1 + \epsilon^2 |u| c(\rho), & b_0 &= 1 - \epsilon^2 |u| c(\rho), & c_0 &= \frac{2}{1 - \epsilon^2 |u|^2}, \\ a_1 &= \frac{1 - \epsilon^2 c(\rho)^2}{1 - \epsilon^2 c(\rho) |u|} (|u| + c(\rho)), & b_1 &= \frac{1 - \epsilon^2 c(\rho)^2}{1 + \epsilon^2 c(\rho) |u|} (|u| - c(\rho)), \\ a_2 &= c(\rho), & e_j &= (E_{j1}(u), E_{j2}(u), \dots, E_{jn}(u)), \end{aligned}$$

and we recall that $I = (\delta_{ij})$ denotes the n -dimensional identity matrix. Recall that we are interested in the initial-value problem associated with (4.4) where initial data are prescribed on the initial hyperplane

$$\mathcal{H}_0 : \quad t = 0.$$

First, observe that

$$(4.7) \quad \langle A_0(W) \xi, \xi \rangle = a_0 |\xi_1|^2 + b_0 |\xi_2|^2 + c_0 |u|^2 |\widehat{\xi}|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product in \mathbb{R}^{n+2} and

$$\xi = (\xi_1, \xi_2, \dots, \xi_{n+2}) = (\xi_1, \xi_2, \widehat{\xi}) \in \mathbb{R}^{n+2}, \quad \widehat{\xi} = (\xi_3, \dots, \xi_{n+2}) \in \mathbb{R}^n.$$

As was already noted before Remark 2.3 and is readily seen directly from (4.6), the matrix A_0 can be positive definite only if the velocity u never vanishes. In other words, provided the initial velocity \bar{u} is bounded away from 0, according to Kato's theory, a local-in-time solution exists and is unique in the uniformly local Sobolev space H_{ul}^s for $s > 1 + n/2$. As stated in Section 1, however, this lower bound on the velocity is not physically realistic.

On the other hand, from the physical view point, the zero velocity is not a special value and, in any case, one should be able to recover the strict positivity property for the matrix A_0 . At this juncture, recalling the strategy presented at the end of the preceding section for the non-relativistic Euler equations, we propose ourselves to apply a Lorentz transformation.

Using the Lorentz invariance property (Lemma 3.1) of the Euler equations, we see that the symmetric formulation (2.8)–(2.10) can be also expressed in the transformed coordinates (t', x') defined by (3.1), that is,

$$(4.8) \quad A_0(W') \partial_{t'} W' + \sum_{j=1}^n A_j(W') \partial_{x'_j} W' = 0,$$

where $W' = (z'_+, z'_-, \tilde{u}')$ is defined from the transformed unknowns (ρ', u') . (Of course, the mass density remains unchanged but is now regarded as a function of (t', x') .)

After this transformation, the expression (4.7) becomes

$$(4.9) \quad \langle A_0(W') \xi, \xi \rangle = a'_0 |\xi_1|^2 + b'_0 |\xi_2|^2 + c'_0 |u'|^2 |\widehat{\xi}|^2,$$

with a'_0, b'_0, c'_0 defined by (4.6) with (ρ, u) replaced by (ρ', u') . In view of the lower and upper bounds (4.3), we conclude that the transformed matrix $A_0(W')$ is *positive definite in the coordinate system (t', x')* . Hence, Kato's theory applies to the initial value problem for (4.8), *without* any assumption on the fluid velocity, but provided initial data are imposed on the initial hypersurface $t' = 0$.

In contrast to the non-relativistic case, however, this is not the end of our discussion, since $t' = 0$ is not the hypersurface of interest. This is due to the fact that, in the relativistic setting, the initial plane \mathcal{H}_0 is not preserved by the transformation (3.1).

Step 3. In fact, the initial hyperplane \mathcal{H}_0 is mapped, in the new coordinate system (t', x') , to the “oblique” hyperplane

$$\mathcal{H}'_0 : \quad t' = -\epsilon^2 U \cdot x'.$$

In order to prove local well-posedness for the oblique initial-value problem to (4.8) with data prescribed on \mathcal{H}'_0 , it is convenient to introduce a further change of coordinates

$$(4.10) \quad t'' = t' + \epsilon^2 U \cdot x', \quad x'' = x',$$

which maps the hyperplane \mathcal{H}'_0 to the hyperplane

$$\mathcal{H}''_0 : \quad t'' = 0.$$

This transformation puts the system (4.8) into the form

$$(4.11) \quad B_0(W'') \partial_{t''} W'' + \sum_{j=1}^n B_j(W'') \partial_{x''_j} W'' = 0,$$

where $W''(t'', x'') = W'(t', x')$ and the new matrix-coefficients are

$$(4.12) \quad \begin{aligned} B_0(W'') &= A_0(W') + \epsilon^2 \sum_{j=1}^n U_j A_j(W'), \\ B_j(W'') &= A_j(W'), \quad j = 1, \dots, n. \end{aligned}$$

(Note in passing that using a Lorentz transformation instead of (4.10) would be physically more natural, but would lead to precisely the same matrix B_0 and slightly more complicated expression.)

We are now going to establish that the matrix $B_0(W'')$ is positive definite for data that have bounded mass density and whose velocity scalar is bounded away from the light speed. Provided this is checked, Kato’s theory then applies to the initial value problem for (4.11) on the hyperplane \mathcal{H}''_0 , which is the one of interest.

At this juncture, it is worth recalling the standard fact that the definite-positivity property above implies that the oblique hyperplane \mathcal{H}'_0 is a non-characteristic hypersurface for the hyperbolic system (4.8) which also is sufficient to imply local well-posedness. Namely, the matrix $B_0(W'')$ can be written as

$$B_0(W'') = \sum_{a=0}^n \nu_a A_a(W''),$$

where the vector $\nu = (\nu_0, \dots, \nu_n) = (1, \epsilon^2 U) \in \mathbb{R}^{n+1}$ is normal to \mathcal{H}'_0 . Thus, the positivity property of $B_0(W'')$ implies also that $\det(B_0(W'')) \neq 0$.

Let us summarize the expressions we need here. An easy computation with (4.12) shows that

$$(4.13) \quad \begin{aligned} \langle B_0(W'') \xi, \xi \rangle &= (a'_0 + a'_1 \epsilon^2 (U \cdot \tilde{u}')) \xi_1^2 + 2a'_2 \epsilon^2 |u'| (U \cdot (E(\tilde{u}') \hat{\xi})) \xi_1 \\ &\quad + (b'_0 + b'_1 \epsilon^2 (U \cdot \tilde{u}')) \xi_2^2 - 2a'_2 \epsilon^2 |u'| (U \cdot (E(\tilde{u}') \hat{\xi})) \xi_2 \\ &\quad + c'_0 |u'|^2 (1 + \epsilon^2 (U \cdot u')) |\hat{\xi}|^2, \end{aligned}$$

where we recall that $\xi = (\xi_1, \xi_2, \dots, \xi_{n+2}) = (\xi_1, \xi_2, \hat{\xi}) \in \mathbb{R}^{n+2}$. The primed quantities a'_0 etc. are still defined by (4.6) but with u replaced by u' determined by the Lorentz transformation associated with the reference velocity U .

Hence, we can regard the expression

$$Q(\xi; \epsilon u, \epsilon c(\rho'), \epsilon U) := \langle B_0(W'') \xi, \xi \rangle$$

as a polynomial in ξ and a nonlinear function in $\epsilon u, \epsilon c(\rho'), \epsilon U$. As before, to simplify the notation, we introduce

$$X := \epsilon u, \quad Y := \epsilon c(\rho'), \quad Z := \epsilon U,$$

to deduce

$$\begin{aligned} Q(\xi; X, Y, Z) &= \left(a'_0 + a'_1 \tilde{\Phi}(X, Z) \cdot Z \right) \xi_1^2 + 2a'_2 |\Phi(X, Z)| \left(Z \cdot (E(\Phi(X, Z)) \hat{\xi}) \right) \xi_1 \\ &\quad + \left(b'_0 + b'_1 \tilde{\Phi}(X, Z) \cdot Z \right) \xi_2^2 - 2a'_2 |\Phi(X, Z)| \left(U \cdot (E(\tilde{\Phi}(X, Z)) \hat{\xi}) \right) \xi_2 \\ &\quad + c'_0 |\Phi(X, Z)|^2 (1 + (Z \cdot \Phi(X, Z))) |\hat{\xi}|^2, \end{aligned}$$

where (after appropriate rescaling in ϵ for the coefficients a'_1, b'_1, a'_2)

(4.14)

$$\begin{aligned} a'_0 &= 1 + |\Phi(X, Z)| Y, & b'_0 &= 1 - |\Phi(X, Z)| Y, & c'_0 &= \frac{2}{1 - |\Phi(X, Z)|^2}, \\ a'_1 &= \frac{1 - Y^2}{1 - Y |\Phi(X, Z)|} (|\Phi(X, Z)| + Y), & b'_1 &= \frac{1 - Y^2}{1 + Y |\Phi(X, Z)|} (|\Phi(X, Z)| - Y), \\ a'_2 &= Y, \end{aligned}$$

with $\tilde{\Phi}(X, Z) := \Phi(X, Z)/|\Phi(X, Z)|$. Replacing the coefficients a'_0, a'_1 , etc by their values, we finally obtain

$$(4.15) \quad Q(\xi; X, Y, Z) = Q_1 \xi_1^2 + Q_2 \xi_2^2 + Q_3 |\hat{\xi}|^2 + Q_{13}(\xi_1, \hat{\xi}) + Q_{23}(\xi_2, \hat{\xi}),$$

where

$$Q_1 := 1 + |\Phi(X, Z)|Y + (\tilde{\Phi}(X, Z) \cdot Z) \frac{1 - Y^2}{1 - Y|\Phi(X, Z)|} (|\Phi(X, Z)| + Y),$$

$$Q_2 := 1 - |\Phi(X, Z)|Y + (\tilde{\Phi}(X, Z) \cdot Z) \frac{1 - Y^2}{1 + Y|\Phi(X, Z)|} (|\Phi(X, Z)| - Y),$$

$$Q_3 := \frac{2|\Phi(X, Z)|^2}{1 - |\Phi(X, Z)|^2} \left(1 + (Z \cdot \Phi(X, Z))\right),$$

$$Q_{13}(\xi_1, \hat{\xi}) := 2Y |\Phi(X, Z)| \left(Z \cdot (E(\tilde{\Phi}(X, Z))\hat{\xi}) \right) \xi_1$$

$$Q_{23}(\xi_2, \hat{\xi}) := -2Y |\Phi(X, Z)| \left(Z \cdot (E(\tilde{\Phi}(X, Z))\hat{\xi}) \right) \xi_2,$$

Recall that $E(\tilde{\Phi}(X, Z)) = I - \tilde{\Phi}(X, Z) \otimes \tilde{\Phi}(X, Z)$. We are interested in the range where $X, Z \in \mathbb{R}^n$ have norm bounded away from 1, and Y remains in a bounded closed subset of $[0, 1)$.

It remains to check the following *purely algebraic* result.

Lemma 4.2 (Uniform positivity property). *For any given $Y_0 \in (0, 1)$, there exist $r_* \in (0, 1)$ and $Z \in \mathbb{R}^n, r_* < |Z| < 1$ such that*

$$(4.16) \quad Q(\xi, \xi; X, Y, Z) \geq c_0 |\xi|^2, \quad \xi \in \mathbb{R}^{n+2}$$

holds for all $Y \in [0, Y_0], |X| \leq r_*$.

Proof. In the below, we fix $Y_0 \in (0, 1)$ and $Y \in [0, Y_0]$ is an arbitrary number. It is convenient to choose the coordinate system so that

$$r_0 = |X|, \quad Z = (r_1, 0, 0), \quad 0 < r_0 < r_1 < 1.$$

Set

$$\Phi_* = \frac{r_1 - r_0}{1 - r_0 r_1}, \quad \Phi^* = \frac{r_0 + r_1}{1 + r_0 r_1}.$$

Then

$$0 < \Phi_* < \Phi^* < 1$$

and Lemma 3.2 says that

$$\Phi_* \leq |\Phi(X, Z)| \leq \Phi^*.$$

A simple computation shows that

$$Z \cdot \Phi(X, Z) = \frac{\tilde{Z} \cdot X - |Z|}{1 - X \cdot Z} |Z| = \frac{X_1 - r_1}{1 - X_1 r_1} r_1 =: R < 0,$$

which, together with $W(X, Z)$ in (3.6), leads to

$$\begin{aligned} S &:= Z \cdot \tilde{\Phi}(X, Z) = \frac{R}{W(X, Z)^{1/2}} \\ &= |Z| \frac{X_1 - r_1}{((X_1 - r_1)^2 + \gamma(U)^{-2}(X_2^2 + X_3^2))^2} \end{aligned}$$

Thus, we obtain

$$-r_1 = -|Z| \leq S \leq 0,$$

and the equality in the first inequality is realized for $X = (X_1, 0, 0)$.

As a consequence, we have,

$$\begin{aligned}
Q_1 &= \frac{1 - |\Phi(X, Z)|^2 Y^2 + S(1 - Y^2) (|\Phi(X, Z)| + Y)}{1 - Y|\Phi(X, Z)|} \\
&= \frac{(1 - |\Phi(X, Z)|^2) Y^2 + (1 - Y^2) (1 + S(|\Phi(X, Z)| + Y))}{1 - Y|\Phi(X, Z)|} \\
(4.17) \quad &\geq (1 - Y_0^2) (1 - r_1(\Phi^* + Y_0)) =: q_1.
\end{aligned}$$

Therefore, we see that if

$$(4.18) \quad k_0 := r_1(\Phi^* + Y_0) < 1,$$

then $q_1 > 0$ and so, $Q_1 > 0$.

Similarly,

$$\begin{aligned}
Q_2 &= \frac{1 - |\Phi(X, Z)|^2 Y^2 + S(1 - Y^2) (|\Phi(X, Z)| - Y)}{1 + Y|\Phi(X, Z)|} \\
&= \frac{(1 - |\Phi(X, Z)|^2) Y^2 + (1 - Y^2) (1 + S(|\Phi(X, Z)| - Y))}{1 + Y|\Phi(X, Z)|} \\
&\geq \frac{(1 - Y_0^2) (1 - r_1 \max(0, |\Phi(X, Z)| - Y))}{1 + Y_0 \Phi^*}.
\end{aligned}$$

Therefore, we have

$$Q_2 \geq \frac{(1 - Y_0^2)(1 - r_1 \Phi^*)}{1 + Y_0 \Phi^*} =: q_2 > 0,$$

since $|\Phi(X, Z)| - Y \leq |\Phi(X, Z)| \leq \Phi^*$.

On the other hand, clearly we have

$$Q_3 \geq \frac{2\Phi_*^2}{1 - \Phi_*^2} (1 - \Phi^* r_1) =: q_3 > 0,$$

and

$$\begin{aligned}
|Q_{13}(\xi_1, \widehat{\xi})| &\leq 2Y |\Phi(X, Z)| |Z| |\xi_1| |\widehat{\xi}| \leq 2Y_0 \Phi^* r_1 |\xi_1| |\widehat{\xi}|, \\
|Q_{23}(\xi_1, \widehat{\xi})| &\leq 2Y |\Phi(X, Z)| |Z| |\xi_2| |\widehat{\xi}| \leq 2Y_0 \Phi^* r_1 |\xi_1| |\widehat{\xi}|.
\end{aligned}$$

Set now

$$q_4 := Y_0 \Phi^* r_1$$

and define a quadratic formula of three variables $(x, y, z) \in \mathbb{R}^3$,

$$Q_*(x, y, z) = q_1 x^2 + q_2 y^2 + q_3 z^2 - 2q_4(x + y)z.$$

Then, we obtain

$$Q(\xi; X, Y, Z) \geq Q_*(|\xi_1|, |\xi_2|, |\widehat{\xi}|)$$

for any $\xi \in \mathbb{R}^{n+2}$; that is, Q is positive definite if so is Q_* .

Since $q_1, q_2 > 0$, we can write, for any $\kappa \in (0, 1)$,

$$\begin{aligned} Q_*(x, y, z) &= \kappa(q_1x^2 + q_2y^2) + (1 - \kappa)q_1 \left(x - \frac{q_4}{(1 - \kappa)q_1}z \right)^2 \\ &\quad + (1 - \kappa)q_2 \left(y - \frac{q_4}{(1 - \kappa)q_2}z \right)^2 + \left(q_3 - \frac{q_4^2}{(1 - \kappa)q_1} - \frac{q_4^2}{(1 - \kappa)q_2} \right) z^2, \end{aligned}$$

and since $\kappa \in (0, 1)$ is arbitrary, we can conclude that Q_* is positive definite if and only if

$$(4.19) \quad D_* := q_3 - \frac{q_4^2}{q_1} - \frac{q_4^2}{q_2} > 0.$$

Let $r_* \in (0, 1)$ be a number to be determined later. Let $a \in (1, 1/r_*)$ and set $r_1 = ar_*$. Then, observe that for any $r_0 \in [0, r_*]$,

$$\Phi_* = \frac{r_1 - r_0}{1 - r_0r_1} \geq \frac{(a - 1)r_*}{1 - ar_*^2}, \quad \Phi^* = \frac{r_0 + r_1}{1 + r_0r_1} \leq \frac{(a + 1)r_*}{1 + r_0r_1} \leq (a + 1)r_*.$$

Hence,

$$\begin{aligned} k_0 &= r_1(\Phi^* + Y_0) \leq ar_*((a + 1)r_* + Y_0) \leq a(a + 2)r_* =: k_1r_*, \\ q_1 &= (1 - Y_0^2)(1 - k_0) \geq (1 - Y_0^2)(1 - a(a + 2)r_*) =: K_1(r_*), \end{aligned}$$

and

$$\begin{aligned} q_2 &= \frac{(1 - Y_0^2)(1 - r_1\Phi^*)}{1 + Y_0r_1} \geq \frac{1}{2}(1 - Y_0^2)(1 - a(a + 1)r_*^2) =: K_2(r_*), \\ q_3 &= \frac{2\Phi_*^2}{1 - \Phi_*^2} \left(1 - \Phi^*r_1 \right) \geq \frac{2(a - 1)^2r_*^2(1 - a(a + 1)r_*^2)}{(1 - ar_*^2)^2 - (a - 1)^2r_*^2} =: K_3(r_*)r_*^2, \\ q_4 &= Y_0\Phi^*r_1 \leq Y_0(a + 1)ar_*^2 =: K_4r_*^2. \end{aligned}$$

Note that $k_1, K_4 > 0$ are independent of r_* and that all the above inequalities hold for all $r_0 \in [0, r_*]$. Observe that if r_* is sufficiently small, then $K_1(r_*), K_2(r_*), K_3(r_*)$ are positive and

$$(4.20) \quad D_* \geq \left(K_3(r_*) - K_4 \left(\frac{1}{K_1(r_*)} + \frac{1}{K_2(r_*)} \right) r_*^2 \right) r_*^2$$

holds for all r_0 . It is easy to see that when $r_* \rightarrow 0$, we have

$$k_0 \rightarrow 0, \quad K_1(r_*) \rightarrow 1 - Y_0^2,$$

and

$$K_2(r_*) \rightarrow \frac{1}{2}(1 - Y_0^2), \quad K_3(r_*) \rightarrow 2(a - 1)^2.$$

We can now conclude from (4.20) that *for all $a > 1$, there exists $r_* \in (0, 1)$ such that $ar_* < 1, k_0 < 1, D_* > 0$, which completes the proof of Lemma 4.2 with $r_1 = |Z| = ar_*$.* \square

In turn, Kato's theory guarantees the existence of a solution defined in a small neighborhood of this hyperplane \mathcal{H}_0'' . Making the transformation back to the original variables, we obtain a solution in a small neighborhood of the initial line $t = 0$. This completes the proof of Theorem 4.1.

Step 4. We need to show that the density ρ remains non-negative. In fact, since the solution remains smooth enough, say of class C^1 in space and continuous in time, we can define the characteristic curves by the Cauchy-Lipschitz theorem

$$\dot{y}(t) = u(t, y(t)), \quad t \geq 0.$$

Hence, by writing the w -equation as

$$\begin{aligned} & (\partial_t w + u \cdot \nabla w) \\ &= -\frac{c(\rho)}{1 - \epsilon^4 |u|^2 c(\rho)^2} \left((1 - \epsilon^2 |u|^2) (\epsilon^2 c(\rho) u \cdot \nabla w + \tilde{u} \cdot \nabla v) + |u| \nabla \cdot \tilde{u} \right), \end{aligned}$$

and integrating along the characteristics we obtain

$$\frac{d}{dt} (w(t, y(t))) = O(1) c(\rho(t, y(t))).$$

Here, the (bounded) quantity $O(1)$ involves only the sup-norm of first-order derivatives of the solution.

Now, in view of our assumption (4.1) we have

$$c(\rho) \leq C_R w(\rho)$$

on any compact set $\rho \in [0, R]$ and for some constant C_R . So, we deduce that

$$w(t, y(t)) = w(0, y(0)) e^{t O(1)},$$

along every characteristic, which, in particular, shows that w remains non-negative. This completes the proof of Theorem 4.1.

5. SUPPORT OF TAME SOLUTIONS

Following [7] we define a concept of solutions, in which the velocity vector is required to satisfy a (non-degenerate) evolution equation even in the presence of vacuum, as follows.

Definition 5.1 (Notion of tame solution). *A measurable map $(\rho, u) : [0, T] \times [0, \infty) \times [-\epsilon^{-2}, \epsilon^2]$ is called a tame solution of the relativistic Euler equations if*

- (ρ, u) is a solution of class C^1 of the Euler equations,
- w is also of class C^1 , and
- the equation $\partial_t u + u \cdot \nabla u = 0$ holds in the interior of the set $\{\rho = 0\}$.

Relying on this definition, we can establish that the support of a solution does not expand in time.

Theorem 5.2 (Property of the support of a tame solution). *Consider the relativistic Euler equation for an equation of state $p = p(\rho)$ satisfying the hyperbolicity condition (1.3) together with the following vacuum condition*

(4.1). If (ρ, u) is a tame solution of the relativistic Euler equations and has compact support, then its support does not expand in time, that is,

$$\text{supp}(\rho, u)(t) \subset \text{supp}(\bar{\rho}, \bar{u}), \quad t \in [0, T].$$

Proof. We now consider the relativistic Euler equations in the form

$$B_0(V) \partial_t V + \sum_{j=1}^n B_j(V) \partial_j V = 0,$$

where the map $V := (w, v, \tilde{u})$ is of class C^1 and satisfies

$$|\partial_t V| \leq C_1 \sup_j |B_0(V)^{-1} B_j(V)|.$$

Here, the constant C_1 depends on the sup norm of first-derivatives of the solution. In view of the explicit expressions of the matrices B_0 and B_j (see Section 2) we obtain

$$\sup_j |B_0(V)^{-1} B_j(V)| \leq C_2 |V|,$$

where we have used our assumption (4.1). Now, by Gronwall's lemma we get

$$|V(t, x)| \leq e^{Ct} |V(0, x)|, \quad t > 0, \quad x \in \mathbb{R}^n,$$

which completes the proof of the theorem. \square

ACKNOWLEDGMENTS

The first author (PLF) was partially supported by the Centre National de la Recherche Scientifique (CNRS) and the Agence Nationale de la Recherche (ANR) through the grant 06-2-134423 (MATH-GR). The research of the second author (SU) was supported by the Department of Mathematics and Liu Bie Ju Centre for Mathematical Sciences at the City University of Hong Kong. This work was initiated during a two-week visit of the first author in these institutions, and both authors are very grateful to Tong Yang for inviting them and particularly appreciated his hospitality.

This paper was completed when the first author was visiting the Mittag-Leffler Institute in the Fall 2008 during the Semester Program "Geometry, Analysis, and General Relativity" organized by L. Andersson, P. Chruściel, H. Ringström, and R. Schoen.

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