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REPORT No. 13, 2008/2009, fall

ISSN 1103-467X
ISRN IML-R- -13-08/09- -SE+fall

ON HIGHER DIMENSIONAL BLACK HOLES WITH ABELIAN ISOMETRY GROUP

by

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Abstract. — We consider $(n+1)$–dimensional, stationary, asymptotically flat, or Kaluza-Klein asymptotically flat black holes, with an abelian s–dimensional subgroup of the isometry group satisfying an orthogonal integrability condition. Under suitable regularity conditions we prove that the area of the group orbits is positive on the domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, vanishing only on the boundary $\partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ and on the "symmetry axis" $\mathscr{A}$. We further show that the orbits of the connected component of the isometry group are timelike throughout the domain of outer communications. Those results provide a starting point for the classification of such black holes. Finally, we show non-existence of zeros of static Killing vectors on degenerate Killing horizons, as needed for the generalisation of the static no-hair theorem to higher dimensions.

Contents

1. Introduction

In this work we study the global structure of stationary space-times with $s+1$, $s \geq 0$, commuting Killing vectors $K_{(\mu)}$, $\mu = 0, \dots s$, satisfying the *orthogonal integrability* condition:

$$\forall\ \mu = 0, \dots, s \qquad dK_{(\mu)} \wedge K_{(0)} \wedge \dots \wedge K_{(s)} = 0\ . \tag{1.1}$$

This class includes the Kerr metrics, the "black strings" Kerr$\times S^1$ and other abelian Kaluza-Klein black-holes as in [2], the Emparan-Reall "black rings" [15], a subset of the Myers-Perry black holes [25], as well as the Elvang-Figueras "black Saturns" [13].

The author is grateful to the Mittag-Leffler Institute, Djursholm, Sweden, for financial support and hospitality during part of work on this paper.

Condition (1.1) automatically holds in, say vacuum, $(n+1)$–dimensional space-times when $s = n-2$ and when the "axis" $\mathscr{A}$ defined below is non-empty [4] (compare [7, 14]). However, one might wish to consider metrics where (1.1) is imposed as a restrictive condition, without necessary assuming that $s = n-2$.

A prerequisite to the classification of the above geometries [2, 5] (see also [7]) is the understanding of the global structure of the domain of outer communications[(1)] $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$: one needs a product structure of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ with respect to the action of the stationary Killing vector field, information about $\pi_1(\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle)$, knowledge of the causal character of the orbits of the Killing vectors. The aim of this work is to settle some of those issues.

Specifically, one of the key issues is the analysis of the zero level set of the "area function" W, defined as

$$W := -\det\left(\mathfrak{g}(K_{(\mu)}, K_{(\nu)})_{\mu,\nu=0,\ldots,s}\right) .$$

Indeed the uniqueness theory of such black holes [2, 5] uses $\sqrt{W}$ as one of the coordinates on the quotient of the domain of outer communications by the isometry group. Clearly a function W changing sign would invalidate the whole approach. Our first main result, Theorem 5.1 below, asserts that, under suitable regularity and asymptotic conditions, the area function W vanishes within the domain of outer communications *only* on the *axis*

$$\mathscr{A} := \{p \in \mathscr{M} \mid Z(p) = 0\} , \tag{1.2}$$

where

$$Z := \det\left(\mathfrak{g}(K_{(i)}, K_{(j)})_{i,j=1,\ldots,s}\right) .$$

The proof relies heavily on the analysis in [7], as well as on the results in [9] which are reviewed in our context in Section 3.

Next, inspection of the uniqueness arguments in [22,24,27,28] shows that serious difficulties arise there if the orbits of the isometry group cease to be timelike on $\mathscr{A}$. The second main result of our work is Theorem 6.1 below, that this does not occur.

The simplest non-trivial abelian isometry Lie group is $\mathbb{R}$, then $s = 0$ and the orthogonal integrability condition (1.1) is known as the staticity condition. Now, the generalization of the uniqueness theory of static asymptotically flat black holes requires the non-vanishing of the static Killing vector on degenerate components of the event horizon. We prove this in Proposition 7.3; this is the third main result in this paper.

For reasons discussed in detail shortly, we work in the framework of manifolds which are asymptotically flat in a Kaluza-Klein sense, as defined below; manifolds which are asymptotically flat in the usual sense occur as a special case of our analysis.

2. Kaluza-Klein asymptotic flatness

Consider an $(n+1)$–dimensional space-time $(\mathscr{M}, \mathfrak{g})$ which is asymptotically flat in the usual sense, as e.g. in [10]. It follows from the analysis there that there exists a homomorphism from the connected component G_0 of the identity of the group of isometries of $(\mathscr{M}, \mathfrak{g})$ to a subgroup of the Lorentz group, constructed using the leading order behavior of the Killing vectors of $(\mathscr{M}, \mathfrak{g})$. Assuming that the ADM four-momentum of $(\mathscr{M}, \mathfrak{g})$ is timelike, arguments similar to those leading to

[(1)] See Section 2 and [7] for terminology and definitions.

(6.24) below show that the dimension of any commutative subgroup of G_0 does not exceed $n/2+1$, where $n/2$ arises from the rank of $SO(n)$, while "+1" comes from a possible time-translation. This implies that the hypothesis of asymptotic flatness is compatible with the condition $s = n-2$ only in space-dimension n equal to three and four.

However, in the context of Kaluza-Klein theories, there are situations of interest which are not asymptotically flat and to which the current analysis applies. A trivial example is given by space-times of the form $(\mathscr{M}\times S^1, \mathfrak{g}^{(n+1)}+dx^2)$, where $\mathfrak{g}^{(n+1)}$ is an $(n+1)$–dimensional asymptotically flat, say Ricci flat, metric (e.g. Kerr, or Myers-Perry, or Emparan-Reall). In this trivial product case the Einstein equations reduce to the ones for the quotient metric $(\mathscr{M}, \mathfrak{g}^{(n+1)})$, so there is no point in generalising. Now, one can imagine situations where the higher-dimensional metric asymptotes to a product solution, but does *not* lead to a metric satisfying the required hypotheses after passing to the quotient. For example, the quotient metric associated with a vacuum metric will not satisfy the positive energy condition in general. So there appears to be some interest to relax the asymptotic flatness condition, perhaps to show eventually that the resulting solutions must be trivial products.

With this motivation in mind, we shall say that $\mathscr{S}_{\rm ext}$ is a *Kaluza-Klein asymptotic end*, or *asymptotic end* for short, if $\mathscr{S}_{\rm ext}$ is diffeomorphic to $\big(\mathbb{R}^n\setminus \overline{B}(R)\big)\times N$, where $\overline{B}(R)$ is a closed coordinate ball of radius R, and N is a compact manifold. Let $\mathring{k}$ be a fixed Riemannian metric on N, and let $\mathring{g} = \delta\oplus\mathring{k}$, where δ is the Euclidean metric on $\mathbb{R}^n$.

We shall say that a Riemannian metric g on $\mathscr{S}_{\rm ext}$ is *Kaluza-Klein asymptotically flat*, or *KK–asymptotically flat* for short, if there exists $\alpha>0$ and $k\geq 1$ such that for $0\leq \ell\leq k$ we

$$\mathring{D}_{i_1}\ldots\mathring{D}_{i_\ell}(g_{jk}-\mathring{g}_{jk}) = O(r^{-\alpha-\ell})\ , \tag{2.1}$$

where $\mathring{D}$ denotes the Levi-Civita connection of $\mathring{g}$, and r is the radius function in $\mathbb{R}^n$, $r:=\sqrt{(x^1)^2+\ldots(x^n)^2}$, with the x^i's being any Euclidean coordinates of $(\mathbb{R}^n,\delta)$. We shall say that a general relativistic initial data set $(\mathscr{S}_{\rm ext},g,K)$ is *Kaluza-Klein asymptotically flat*, or *KK–asymptotically flat*, if $(\mathscr{S}_{\rm ext},g)$ is KK–asymptotically flat and if for $0\leq\ell\leq k-1$ we have

$$\mathring{D}_{i_1}\ldots\mathring{D}_{i_\ell}K_{jk} = O(r^{-\alpha-1-\ell})\ . \tag{2.2}$$

The above reduces to the usual notion of asymptotic flatness when N is a set containing one point. So an asymptotically flat initial data set is also KK–asymptotically flat.

Consider a space-time $\mathscr{M}$ containing a KK–asymptotically flat end $(\mathscr{S}_{\rm ext},K,g)$, and suppose that there exists on $\mathscr{M}$ a Killing vector field X with complete orbits. Then X will be called *stationary* if X approaches the timelike unit normal to $\mathscr{S}_{\rm ext}$ when one recedes to infinity along $\mathscr{S}_{\rm ext}$. $(\mathscr{M},\mathfrak{g})$ will then be called stationary. Such a space-time will then be called KK–asymptotically flat.

Similarly to the standard asymptotically flat case, we set

$$\mathscr{M}_{\rm ext} := \cup_{t\in\mathbb{R}}\phi_t[X](\mathscr{S}_{\rm ext})\ ,$$

where $\phi_t[X]$ denotes the flow of X. Assuming stationarity, the *domain of outer communications* is defined as in [7, 11]:

$$\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle := I^-(\mathscr{M}_{\rm ext})\cap I^+(\mathscr{M}_{\rm ext})\ .$$

3. Simple connectedness of the orbit manifold

In the current context, and in higher dimensions $n \geq 4$, simple connectedness holds for asymptotically flat globally hyperbolic domains of outer communications satisfying the null energy condition

$$R_{\mu\nu}Y^\mu Y^\nu \geq 0 \quad \text{for null} \quad Y^\mu \ . \tag{3.1}$$

Indeed, the analysis in [12, 17–19], carried-out there in dimension $3+1$, is independent of dimensions. However, as already discussed, asymptotic flatness imposes $n = 3$ or 4 if one wishes to derive, rather than impose, the orthogonal integrability condition (1.1). In any case, KK-asymptotically flat solutions will not be simply connected in general, as demonstrated by the Schwarzschild$\times\mathbb{T}^m$ "black branes".

Now, whenever simple connectedness fails, the twist potentials might fail to exist and the whole reduction process [2], that relies on their existence, breaks down. It turns out that the quotient space $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$ remains simply connected for KK–asymptotically flat models, which justifies existence of twist potentials whenever $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$ is a manifold. Moreover, simply connectedness of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$ will be used below to show that the area function has no zeros on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$.

Indeed, a variation upon the usual topological censorship arguments [12, 17, 19] gives:

Theorem 3.1 (**[9]**). — *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time satisfying the null energy condition, and containing a KK–asymptotically flat end $\mathscr{S}_{\rm ext}$. Suppose that $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and that there exists an action of $G = \mathbb{R}\times G_s$ on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ by isometries which, on $\mathscr{M}_{\rm ext} \approx \mathbb{R}\times\mathscr{S}_{\rm ext}$, takes the form*

$$\mathbb{R}\times G_s \ni (\tau, g): \quad (t,p)\mapsto (t+\tau, g\cdot p)\ .$$

We assume moreover that the generator of the $\mathbb{R}$ factor of G approaches the unit timelike normal to $\mathscr{S}_{\rm ext}$ as one recedes to infinity. If $\mathscr{S}_{\rm ext}/G_s$ is simply connected, then so is $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/G_s$.

At the heart of Theorem 3.1 lies Proposition 3.2 below. Before describing the result, some definitions are in order. Consider a spacelike manifold $S \subset \mathscr{M}$ of co-dimension two, and assume that there exists a smooth unit spacelike vector field n normal to S such that the vector fields $\pm n$ lie in distinct components of the bundle of spacelike vectors normal to S; we shall call *outwards* the component met by n, and the other one *inwards.* At every point $p \in S$ there exists then a unique future directed null vector field n^+ normal to S such that $\mathfrak{g}(n, n^+) = 1$, which we shall call the *outwards future null normal* to S. The *inwards future null normal* n^- is defined by the requirement that n^- is null, future directed, with $\mathfrak{g}(n, n^-) = -1$. In an asymptotically flat, or KK–asymptotically flat region the inwards direction at $\{r = R\}$ is defined to by $dr(n) < 0$.

We define the *null future inwards and outwards mean curvatures $\theta^\pm$ of S* as

$$\theta^\pm := \mathrm{tr}_\gamma(\nabla n^\pm)\ , \tag{3.2}$$

where γ is the metric induced on S. In (3.2) the symbol $n^\pm$ should be understood as representing any extension of the null normals $n^\pm$ to a neighborhood of S, and the definition is independent of the extension chosen.

We shall say that S is *weakly outer future trapped* if $\theta^+ \leq 0$. The notion of *weakly inner future trapped* is defined by requiring $\theta^- \leq 0$. A similar notion of *weakly outer* *or inner past trapped* is defined by considering the divergence of past pointing null normals. We will say *outer future trapped* if $\theta^+ < 0$, etc.

Let t be a time function on $\mathscr{M}$, and let $\gamma : [a,b] \to \mathscr{M}$ be a causal curve. The *time of flight* t_γ *of* γ is defined as

$$t_\gamma = t(\gamma(b)) - t(\gamma(a)) \ .$$

In what follows we will need the following, also proved in [9]:

Proposition 3.2 **([9], Proposition 5.3)**. — *Let $(\mathscr{M}, \mathfrak{g})$ be a stationary, asymptotically flat, or KK–asymptotically flat globally hyperbolic space-time satisfying the null energy condition. Let $S \subset \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ be future inwards marginally trapped. There exists a constant R_1 such that for all $R_2 \geq R_1$ there are no future directed null geodesics starting inwardly at S, ending inwardly at $\{r = R_2\} \subset \mathscr{M}_{\rm ext}$, and locally minimising the time of flight,*

4. The structure of the domain of outer communications

We wish, here, to point out a set of hypotheses which allows one to establish the KK–asymptotically flat counterpart of the Structure Theorem of [7, Section 4.2], Theorem 4.2 below. This shows in particular that the action of the isometry group on $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is of the form assumed in Theorem 3.1.

In this section we assume the existence of a connected subgroup $G = \mathbb{R} \times G_s$ (here the subscript "s" stands for "spacelike") of the group of isometries of $(\mathscr{M}, \mathfrak{g})$, where $G_s = G_1 \times G_2$ is a compact group, with the following action in the asymptotic region: let us write $\mathscr{M}_{\rm ext}$ as $\mathbb{R} \times \mathscr{S}_{\rm ext}$, where the $\mathbb{R}$ factor of G acts by translations on the $\mathbb{R}$ factor of $\mathbb{R} \times \mathscr{S}_{\rm ext}$. Each of G_s, G_1 and G_2 is allowed to be trivial, and neither is assumed to be commutative. Recalling that

$$(\mathscr{S}_{\rm ext}, \mathring{g}) = \Big(\big(\mathbb{R}^n \setminus \overline{B}(R)\big) \times N, \delta \oplus \mathring{k} \Big) \ , \tag{4.1}$$

we assume that G_1 is a subgroup of $SO(n)$ acting by rotations of the flat metric δ on $\mathbb{R}^n \setminus \overline{B}(R)$ and trivially on N, and that G_2 acts on the N factor by isometries of $\mathring{k}$ and trivially on $\mathbb{R}^n \setminus \overline{B}(R)$. Finally, we suppose that the Killing vector tangent to the $\mathbb{R}$ factor of $\mathscr{M}_{\rm ext}$, and denoted by $K_{(0)}$, is timelike on $\mathscr{M}_{\rm ext}$. Note that all the remaining Killing vectors, denoted by $K_{(i)}$, if any, have spacelike or trivial orbits in $\mathscr{M}_{\rm ext}$.

In the asymptotically flat case, the existence of coordinates as in (4.1) can be derived from asymptotic flatness if a timelike ADM four-momentum of $\mathscr{S}_{\rm ext}$ is assumed [10]. It would be of interst to determine whether or not this remains true in the KK–asymptotically flat setup.

The following definition is a direct generalisation of the one in [7]:

Definition 4.1. — *Let $(\mathscr{M}, \mathfrak{g})$ be a space-time containing a KK–asymptotically flat end $\mathscr{S}_{\rm ext}$, and let K be a stationary Killing vector field on $\mathscr{M}$. We will say that $(\mathscr{M}, \mathfrak{g}, K)$ is I^+–*regular *if K is complete, if the domain of outer communications $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ is globally hyperbolic, and if $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ contains a spacelike, connected, acausal hypersurface $\mathscr{S} \supset \mathscr{S}_{\rm ext}$, the closure $\overline{\mathscr{S}}$ of which is a topological manifold with boundary, consisting of the union of a compact set and of a finite number of asymptotic ends, such that the boundary $\partial\overline{\mathscr{S}} := \overline{\mathscr{S}} \setminus \mathscr{S}$ is a topological manifold satisfying*

$$\partial\overline{\mathscr{S}} \subset \mathscr{E}^+ := \partial\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \cap I^+(\mathscr{M}_{\rm ext}) \ , \tag{4.2}$$

with $\partial\overline{\mathscr{S}}$ *meeting every generator of* $\mathscr{E}^+$ *precisely once. See Figure 4.1.*

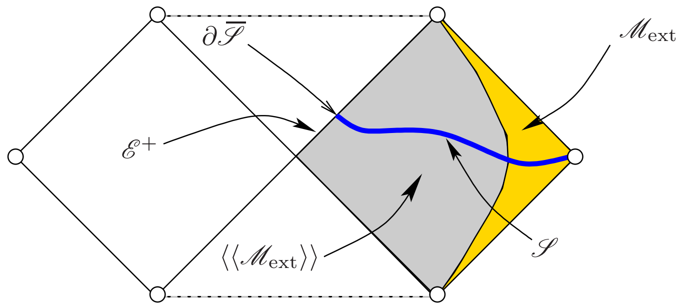

FIGURE 4.1. The hypersurface $\mathscr{S}$ from the definition of I^+–regularity.

The proof of the Structure Theorem [7, Theorem 4.5] carries over with only trivial modifications to the current setting:

THEOREM 4.2 **(Structure theorem)**. — *Suppose that* $(\mathscr{M},\mathfrak{g})$ *is an* I^+*–regular space-time invariant under an action of* $G=\mathbb{R}\times G_s$ *as above. There exists on* $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ *a smooth time function* t*, invariant under* G_s*, which, together with the flow of the Killing vector* $K_{(0)}$ *tangent to the orbits of the* $\mathbb{R}$ *factor of* G*, induces the diffeomorphisms*

$$\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\approx\mathbb{R}\times\mathring{\mathscr{S}}\ ,\qquad \overline{\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle}\cap I^+(\mathscr{M}_{\rm ext})\approx\mathbb{R}\times\overline{\mathring{\mathscr{S}}}\ , \tag{4.3}$$

where $\mathring{\mathscr{S}}:=t^{-1}(0)$ *is* KK*–asymptotically flat, (invariant under* G_s*), with the boundary* $\partial\overline{\mathring{\mathscr{S}}}$ *being a compact cross-section of* $\mathscr{E}^+$*. The smooth hypersurface with boundary* $\overline{\mathring{\mathscr{S}}}$ *is acausal, spacelike up-to-boundary, and the flow of* $K_{(0)}$ *is a translation along the* $\mathbb{R}$ *factor in* (4.3)*.*

5. The area function away from the axis

In this section we prove a generalization of [7, Theorem 5.4 and 5.6]. The main issue is, that an essential ingredient of the proof in [7] is simple connectedness of $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$, which is not expected for KK–asymptotically flat space-times with internal space $N=\mathbb{T}^k$. (We emphasize that we *do not* assume this form of N in this section, but this is the model which seems to be of main interest for applications of this work.) Here we use instead the closely related Proposition 3.2, obtaining:

THEOREM 5.1. — *Under the hypotheses of Theorem 3.1, suppose further that* $G=\mathbb{R}\times\mathbb{T}^s$ *with* $s+1$*–dimensional principal orbits,* $0\le s\le n-2$*. Assume moreover that* <u>*either*</u> $(\mathscr{M},\mathfrak{g})$ *is analytic,* <u>*or*</u> *that* $s=n-2$ *and* $(\mathscr{M},\mathfrak{g})$ *is* I^+*–regular. If the orthogonal integrability condition* (1.1) *holds, then the function*

$$W:=-\det\Big(\mathfrak{g}(K_{(\mu)},K_{(\nu)})\Big)_{\mu,\nu=0,\dots,s} \tag{5.1}$$

is strictly positive on $\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\setminus\mathscr{A}$*, and vanishes on* $\partial\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle\cup\mathscr{A}$ *.*

Proof. — We wish to adapt the proof of [7, Theorems 5.4 and 5.6] to the current setting. Let us show, first, that the existence of a non-empty, closed, embedded, null hypersurface $S^+ \subset \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, invariant under $\mathbb{T}^s$, is incompatible with what we know about the topology of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$:

If $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$ is a smooth manifold, and *if* $S^+/\mathbb{T}^s$ is a non-empty, closed, embedded hypersurface in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$, one can proceed as follows. Let γ be a closed path with strictly positive S^+–intersection number, as constructed in the last step of the proof of [7, Theorem 5.4]. Then $\pi(\gamma)$, where $\pi : \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle \to \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$ is the projection map, has strictly positive $S^+/\mathbb{T}^s$–intersection number, which contradicts simple connectedness of $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle/\mathbb{T}^s$, and proves the result.

However, it is not clear that both assumptions of the previous paragraph will hold in general, in which case the following argument applies: Suppose that S^+ is non-empty, let $\mathscr{S} \supset \mathscr{S}_{\rm ext}$ be any KK–asymptotically flat level set of a Cauchy time function in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$. Choose R_0 large enough so that the spacelike manifold $S_0 := \{r = R_0\} \cap \mathscr{S}$ is both past and future inwards trapped, and that any causal curve from S_0 to $\partial\mathscr{M}_{\rm ext}$ takes at least a coordinate time one before reaching $\partial\mathscr{M}_{\rm ext}$. Let $R_2 \geq \max(R_0.R)$, where R is as in Proposition 3.2. There exists a future directed causal curve from S_0 to $\{r = R_2\}$ which starts in the inwards direction at S_0, leaves $\mathscr{M}_{\rm ext}$, meets S^+, and returns to $\{r = R_2\}$. This shows that the set

$$\Omega := \{\gamma \mid \gamma \text{ is a causal curve from } S_0 \text{ to } \{r = R_2\} \text{ meeting } S^+\}$$

is non-empty. Let t_γ denote the coordinate arrival time of $\gamma \in \Omega$ to $\{r = R_2\}$ then $t_\gamma \geq t|_{S_0} + 2$. Let $\gamma_i \in \Omega$ be any sequence such that

$$t_{\gamma_i} \to \inf_{\gamma\in\Omega} t_\gamma \geq t|_{S_0} + 2\ .$$

Let γ_* be an accumulation curve of the γ_i's, global hyperbolicity implies that $\gamma_* : [a, b] \to \mathscr{M}$ is a non-trivial null geodesic from S_0 to $\{r = R_2\}$ without, in the terminology of [19], null S_0–focal points on $[a, b)$, inwards directed at S_0, and providing a local minimum of time of flight between S_0 and $\{r = R_2\}$. This, however, contradicts Proposition 3.2, hence S^+ is empty.

In the analytic case, the arguments of the proof of [7, Theorem 5.4] show that the existence of zeros of W in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ leads to the existence of an embedded hypersurface S^+ as above, contradicting what has just been said.

If $s = n - 2$ and $n = 4$, the proof of [7, Theorem 5.6] applies.

In what follows we assume that the reader is familiar with the notation, and arguments, of the proof of [7, Theorem 5.6].

Now, if $s = n - 2$ and $n \geq 5$, one needs to exclude the possibility that the leaves C_q pass through points on $\mathscr{A}$ which are intersection points of two or more axes of rotation. Suppose, for contradiction, that there exists such a point. Let $p_0 \in \langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$ be a corresponding point where the axes of rotation meet, then there exists a null $\mathbb{T}^s$–invariant (not necessarily embedded) hypersurface $\hat{S}_p$, totally geodesic in $\langle\langle \mathscr{M}_{\rm ext}\rangle\rangle$, passing through p_0. Choose a basis $\{K_{(1)}, \ldots, K_{(s)}\}$ of the Lie algebra of $\mathbb{T}^s$ such that the Killing vectors $\{K_{(r+1)}, \ldots, K_{(s)}\}$ form a largest linearly independent subfamily at p_0, while $\{K_{(1)}, \ldots, K_{(r)}\}$ vanish at p_0. Let $\mathbb{T}^{s-r}$ denote the group of isometries generated by $\{K_{(r+1)}, \ldots, K_{(s)}\}$, then $\mathscr{M}/\mathbb{T}^{s-r}$ is a smooth manifold near p_0, and $\hat{S}_p/\mathbb{T}^{s-r}$ is a smooth hypersurface there. We can equip $\mathscr{M}/\mathbb{T}^{s-r}$ with the quotient space-metric: for $Z, W \in T\mathscr{M}/\mathbb{T}^{s-r}$,

$$\gamma(Z, W) := \mathfrak{g}(\hat{Z}, \hat{W}) - h^{(i)(j)}\mathfrak{g}(\hat{Z}, K_{(i)})\mathfrak{g}(\hat{W}, K_{(j)})\ ,$$

where $h^{(i)(j)}$ is the matrix inverse to the matrix $g(K_{(i)}, K_{(j)})$, with $i,j = k+1,\dots,s$, and $(\hat W, \hat Z)$ are any vectors in $T\mathscr{M}$ which project on (W,Z). Then γ is Lorentzian. Furthermore, the null normal ℓ to $\hat S_p$ projects to a null vector in the quotient, as all Killing vectors are tangent to $\hat S_p$. So $\hat S_p/\mathbb{T}^{s-r}$ is a smooth null hypersurface through the projection q_0 of p_0 by the quotient map. We continue to denote by $K_{(i)}$, $i = 1,\dots,r$, the Killing vectors of $(\mathscr{M}/\mathbb{T}^{s-r},\gamma)$ generating the remaining $\mathbb{T}^r$ action. Then the $K_{(i)}$'s, $i = 1,\dots r$ are commuting Killing vectors vanishing at q_0. In normal coordinates, after perhaps redefining the $K_{(i)}$'s if necessary, the matrices $\nabla_\mu (K_{(i)})_\nu|_{q_0}$ can be represented by consecutive two-by-two blocks on the diagonal, with the associated non-trivial invariant spaces being spacelike. It follows that

$$n+1-(s-r) = \dim \mathscr{M}/\mathbb{T}^{s-r} \geq 2r+1 ,$$

where the "+1" at the right-hand-side accounts for at least one timelike direction. Since $n+1 = s+3$ by hypothesis, we obtain

$$3+r \geq 2r+1 ,$$

hence $r = 1$ or 2. Since we are assuming that we are at an intersection point of axes, r equals to two. Then $s-r = n-2-2 = n-4$, and $\dim \mathscr{M}/\mathbb{T}^{s-r} = n+1-(s-r) = 5$. This shows that the subspace of $T_{q_0}\mathscr{M}/\mathbb{T}^{s-r}$ invariant under $\mathbb{T}^r$ is one-dimensional timelike. But the normal vector at q_0 to $\hat S_p/\mathbb{T}^{s-r}$ is a null vector invariant under the action of $\mathbb{T}^r$. We conclude that $\hat S_p$ cannot pass through an intersection point of the axes. The remaining arguments of the proof of [7, Theorem 5.6] apply now without modification. □

6. The area function near the axis

In this section we prove:

THEOREM 6.1. — *Under the hypotheses of Theorem 3.1, suppose moreover that* $G = \mathbb{R}\times\mathbb{T}^s$ *with* $s+1$*–dimensional principal orbits,* $0 \leq s \leq n-2$*. If the orthogonal integrability condition* (1.1) *holds, then* $\mathrm{Span}\{K_{(0)},\dots,K_{(s)}\}$ *is timelike throughout the domain of outer communications.*

Note that the dimension of $\mathrm{Span}\{K_{(0)},\dots,K_{(s)}\}$ is *not* assumed to be constant.

Theorem 6.1 generalises to higher dimensions the *Ergoset theorem* of [7].

Proof. — Positivity of W on $\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle \setminus \mathscr{A}$, has already been established in Theorem 5.1. Consider thus a point $p \in \mathscr{A} \cap \langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle$. It follows from [7, Corollary 3.8] that $K_{(0)}$ is transverse to $\mathrm{Span}\{K_{(1)},\dots,K_{(s)}\}|_p$, so Theorem 6.2 below, and the calculations there, apply. If $\mathrm{Span}\{K_{(0)},\dots,K_{(s)}\}|_p$ is null, Theorem 6.2 shows that

$$\{q \in \mathscr{M} \mid W(q) = 0\} \cap (\langle\langle \mathscr{M}_{\mathrm{ext}}\rangle\rangle \setminus \mathscr{A}) \neq \emptyset ,$$

which is not possible by Theorem 5.1. On the other hand, (6.35) below shows that a spacelike $\mathrm{Span}\{K_{(0)},\dots,K_{(s)}\}|_p$ would lead to a negative function W at nearby points lying on geodesics orthogonal to $\mathrm{Span}\{K_{(0)},\dots,K_{(s)}\}|_p$, which is again not possible by Theorem 5.1. □

It remains to prove:

THEOREM 6.2. — *Let* $n \geq 3$*, and let* $(\mathscr{M},\mathfrak{g})$ *be an* $(n+1)$*–dimensional Lorentzian manifold with an effective action of* $\mathbb{R}\times\mathbb{T}^s$ *by isometries satisfying the orthogonal integrability condition* (1.1)*. Assume that the orbits of* $\mathbb{T}^s$ *are spacelike, that*

$Z(p)=0$ for some $p\in\mathscr{A}$, and that $K_{(0)}$ is transverse to $\mathrm{Span}\{K_{(1)},\ldots,K_{(s)}\}|_p$. *If* $\mathrm{Span}\{K_{(0)},\ldots,K_{(s)}\}|_p$ *is a null subspace of $T_p\mathscr{M}$, then W vanishes on*

$$\exp_p(\mathrm{Span}\{K_{(0)},\ldots,K_{(s)}\}|_p^\perp)$$

which, for any neighborhood $\mathscr{U}$ of p, has a non-empty intersection with $\mathscr{U}\setminus\mathscr{A}$.

Proof. — Throughout this proof we shall interchangeably think of the $K_{(\mu)}$'s as elements of the Lie algebra of the group of isometries of $(\mathscr{M},\mathfrak{g})$, or as vector fields on $\mathscr{M}$.

Without loss of generality we can assume that the linearly independent Killing vectors $K_{(i)}$, $i=1,\ldots,s$, have 2π–periodic orbits. By hypothesis we have

$$\mathfrak{g}(K_{(i)},K_{(i)})\geq 0\ ,\quad \text{with}\quad \mathfrak{g}(K_{(i)},K_{(i)})|_q=0 \quad\Longleftrightarrow\quad K_{(i)}|_q=0\ . \tag{6.1}$$

(Note that periodicity of orbits implies (6.1) in causal space-times. In view of (6.1), $Z(p)=0$ is only possible if some linear combination of the $K_{(i)}$'s vanishes at p, and then $W(p)=0$ as well. Thus

$$\mathscr{A}\subset\{p\in\mathscr{M}\mid W(p)=0\}\ .)$$

Let $G_p\subset\mathbb{T}^s$ denote the connected component of the identity of the set of $g\in\mathbb{T}^s$ which leave p fixed; since $Z(p)=0$ this is a closed non-trivial Lie subgroup of $\mathbb{T}^s$. Hence $G_p=\mathbb{T}^r$ for some $0<r\leq s$, and we can choose a new basis of $\mathrm{Span}\{K_{(1)},\ldots,K_{(s)}\}$, still denoted by $K_{(i)}$, so that all $K_{(i)}$'s remain 2π–periodic, and $K_{(1)},\ldots,K_{(r)}$ generate $\mathbb{T}^r$.

Since the isotropy group of p has dimension r, the spacelike subspace

$$\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p$$

of the tangent space at p has dimension $s-r$, hence its orthogonal

$$\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p^\perp$$

is a timelike subspace of dimension $n+1+r-s$. The first space is invariant under $\mathbb{T}^r$, and so must be the second.

Let $\hat T_p\in\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p^\perp$ be any timelike vector at p, set

$$T_p:=\int_{\mathbb{T}^r} g_*\hat T_p\,dg\in\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p^\perp\ , \tag{6.2}$$

where dg is the translation-invariant measure on $\mathbb{T}^r$ normalised to unit volume. Then T_p is invariant under $\mathbb{T}^r$. Hence the space $T_p^\perp$ of vectors orthogonal to T_p is also invariant under $\mathbb{T}^r$. Multiplying T_p by a suitable real, we can without loss of generality assume that T_p is unit, future directed.

A standard argument (see, e.g., [1, Appendix C]) shows that for $i=1,\ldots,r$ each $K_{(i)}$ vanishes on

$$\mathscr{A}_{p,(i)}:=\exp_p(\mathrm{Ker}\,\nabla K_{(i)})\ ,$$

and that $\mathscr{A}_{p,(i)}$ is totally geodesic. Note that $T_p\in\mathrm{Ker}\,\nabla K_{(i)}$ for $i=1,\ldots,r$, which implies that those $\mathscr{A}_{p,(i)}$'s are timelike, and that

$$\mathscr{A}_p:=\cap_{i=1}^r\mathscr{A}_{p,(i)}$$

is a non-empty totally geodesic timelike submanifold of $\mathscr{M}$ containing p.

Since $[K_{(\mu)},K_{(i)}]=0$ we have at p, for $i=1,\ldots,r$ and for all μ,

$$K^\alpha_{(\mu)}\nabla_\alpha K_{(i)}=K^\alpha_{(i)}\nabla_\alpha K_{(\mu)}=0\ ,$$

so $K_{(\mu)}\in\mathrm{Ker}\,\nabla K_{(i)}$ for $i=1,\ldots,r$, and the arguments of [1, Proposition C.1] show that each $K_{(\mu)}$ is tangent to all $\mathscr{A}_{p,(i)}$'s as well.

Alternatively, since the Killing vectors commute, $\mathfrak{g}(K_{(i)},K_{(i)})$ is invariant under the flow of $K_{(\mu)}$. So if $\mathfrak{g}(K_{(i)},K_{(i)})$ vanishes at p, then it vanishes at $\phi_t[K_{(0)}](p)$, where (as before) $\phi_t[K]$ denotes the flow of a Killing vector K; the vanishing of $K_{(i)}$ at $\phi_t[K_{(0)}](p)$ follows then from (6.1).

If $s=r$, we let $\mathscr{S}_{\mathscr{O}}=\exp_p|_{\mathscr{O}}(T_p^\perp)$, where $\mathscr{O}$ is any open neighborhood of p lying within the injectivity radius of $\exp_p|_{\mathscr{O}}$ sufficiently small so that $\mathscr{S}_{\mathscr{O}}$ is spacelike, while $\exp_p|_{\mathscr{O}}$ denotes the exponential map centred at p in the spacetime $(\mathscr{O},g|_{\mathscr{O}})$.

Otherwise we consider the intersection

$$\sigma_p := \mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p^\perp \cap T_p^\perp\ ,$$

since $T_p\in \mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p^\perp$, σ_p is a spacelike, $(n+r-s)$–dimensional, subspace of the tangent space at p, invariant under $\mathbb{T}^r$. Let $\mathscr{O}$ be a sufficiently small open neighborhood of p lying within the injectivity radius of p, then

$$\Sigma := \exp_p(\sigma_p)\cap\mathscr{O} \tag{6.3}$$

is a smooth $(n+r-s)$–dimensional spacelike submanifold of $\mathscr{O}$ invariant under $\mathbb{T}^r$.

Let $\hat G$ be the group generated by $\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}$, and let $\mathscr{S}_{\mathscr{O}}$ denote the union of the orbits of $\hat G$, within $\mathscr{O}$, passing through Σ. (Note that this reduces to the previous definition when $r=s$.) Passing to a subset of $\mathscr{O}$ if necessary, $\mathscr{S}_{\mathscr{O}}$ is then a smooth spacelike hypersurface in $\mathscr{O}$ to which all Killing vector field $K_{(i)}$, $i=1,\ldots,s$, are tangent. Indeed, this is already so by construction for $i=r+1,\ldots,s$. For the remaining i's, let T denote the field of future directed unit vectors normal to $\mathscr{S}_{\mathscr{O}}$. Again by construction we have

$$\mathscr{L}_{K_{(i)}}T=0\ ,\quad i=r+1,\ldots,s\ ,$$

where $\mathscr{L}$ denotes Lie derivation. This implies

$$\mathscr{L}_{K_{(i)}}\Big(\mathfrak{g}(K_{(\mu)},T)\Big)=0\ ,\quad i=r+1,\ldots,s\ ,$$

and since $\mathfrak{g}(K_{(j)},T)=0$ at Σ for $j=1,\ldots,s$, we obtain, along $\mathscr{S}_{\mathscr{O}}$,

$$\mathfrak{g}(K_{(i)},T)=0\ ,\quad i=1,\ldots,s\ . \tag{6.4}$$

Now, $K_{(0)}$ is transverse to $\mathscr{S}_{\mathscr{O}}$ by hypothesis (passing again to a subset of $\mathscr{O}$ if necessary). Moving $\mathscr{S}_{\mathscr{O}}$ with the flow of $K_{(0)}$ we obtain a function t, near p, defined by setting

$$t(p)=s\ \text{ iff }\ \phi_{-s}(p)\in\mathscr{S}_{\mathscr{O}}\ . \tag{6.5}$$

The function t is a time-function, as notation suggests: indeed, the level sets of t are spacelike, which implies that ∇t is timelike. Clearly $\mathscr{S}_{\mathscr{O}}=\{t=0\}$.Similarly to the proof of (6.4) along $\mathscr{S}_{\mathscr{O}}$, commutativity of $K_{(0)}$ with the $K_{(i)}$'s shows that the $K_{(i)}$'s are tangent to the level sets of t. Letting, away from $\mathscr{S}_{\mathscr{O}}$, T be the field of future directed unit vectors normal to the level sets of t, (6.4) holds now in a neighborhood of p.

We set

$$w := K^\flat_{(0)}\wedge\ldots\wedge K^\flat_{(s)}\ ,\qquad \hat w := K^\flat_{(1)}\wedge\ldots\wedge K^\flat_{(s)}\ , \tag{6.6}$$

where for any vector field Y we set $Y^b := \mathfrak{g}(Y,\cdot)$. By definition we have

$$w(K_{(0)},\ldots,K_{(s)})=-W\ ,\qquad \hat w(K_{(1)},\ldots,K_{(s)})=Z\ . \tag{6.7}$$

We need an equation of Carter [4]:

$$dW\wedge w = W dw\ . \tag{6.8}$$

To prove (6.8), let $F = \{W = 0\}$; note that the result is trivial on the interior $\mathring{F}$ of F, if non-empty. By continuity, it then suffices to prove (6.8) on $\mathscr{M} \setminus F$. So let $\mathscr{U}$ be the set of points in $\mathscr{M} \setminus F$ at which the Killing vectors are linearly independent. Consider any point $p \in \mathscr{U}$, and let (x^a, x^A), $a = 0, \ldots, s$, be local coordinates near p chosen so that $K_{(a)} = \partial_a$ and $\mathrm{Span}\{\partial_a\} \perp \mathrm{Span}\{\partial_A\}$; this is possible by (1.1). Then

$$w = -W dx^0 \wedge \ldots \wedge dx^s \ , \tag{6.9}$$

and (6.8) follows near p, hence on $\overline{\mathscr{U}} = \overline{\mathscr{M} \setminus F}$, and hence everywhere.

Recall that $K_{(0)}$ is causal at p, hence transverse to $\mathscr{S}_{\mathscr{O}}$. Passing to a subset of $\mathscr{O}$ if necessary, we redefine T to be the field of vectors normal to the level sets of the time function t, as defined (6.5), normalised so that $\mathfrak{g}(T, K_{(0)}) = 1$; the new T is thus a smooth non-zero multiple of the previous one. Since $\mathfrak{g}(K_{(i)}, T) = 0$,

$$w(T, K_{(1)}, \ldots K_{(s)}) = K^\flat_{(0)}(T)(K^\flat_{(1)} \wedge \ldots \wedge K^\flat_{(s)})(K_{(1)}, \ldots, K_{(s)}) = Z \ . \tag{6.10}$$

Let γ be any affinely parameterised geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) \perp K_{(\mu)}$ for all $\mu = 0, \ldots, s$; it is well known that then

$$\mathfrak{g}(K_{(\mu)}, \dot\gamma) = 0 \tag{6.11}$$

along γ. We then have by (6.8), (6.10) and (6.11),

$$\underbrace{Z\frac{dW}{ds}}_{(dW\wedge w)(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)})} = W dw(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)}) \ . \tag{6.12}$$

Let $\alpha^{(\mu)}$ denote the s–form obtained by omitting the $K_{(\mu)}$ factor in w, and multiplied by $(-1)^\mu$. Similarly let $\beta^{(i)}$ denote the $(s-1)$–form obtained by omitting the $K_{(i)}$ factor in $(-1)^i \hat w$. Using the summation convention on the index (μ) we have

$$dw(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)}) = (dK^\flat_{(\mu)} \wedge \alpha^{(\mu)})(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)}) \ , \tag{6.13}$$

Now,

$$\begin{aligned} (dK^\flat_{(0)} \wedge \alpha^{(0)})(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)}) &= dK^\flat_{(0)}(\dot\gamma, T)\alpha^{(0)}(K_{(1)}, \ldots, K_{(s)}) \\ &= Z dK^\flat_{(0)}(\dot\gamma, T) \ , \end{aligned} \tag{6.14}$$

while, again summing over (i),

(6.15)
$$\begin{aligned} (dK^\flat_{(i)} \wedge \alpha^{(i)})(\dot\gamma, T, K_{(1)}, \ldots, K_{(s)}) &= dK^\flat_{(i)}(\dot\gamma, T)\alpha^{(i)}(K_{(1)}, \ldots, K_{(s)}) \\ &\quad + \sum_j (-1)^j dK^\flat_{(i)}(\dot\gamma, K_{(j)})\alpha^{(i)}(\underbrace{T, K_{(1)}, \ldots, K_{(s)}}_{\text{no } K_{(j)}}) \\ &= dK^\flat_{(i)}(\dot\gamma, T)\alpha^{(i)}(K_{(1)}, \ldots, K_{(s)}) \\ &\quad + \sum_j (-1)^j dK^\flat_{(i)}(\dot\gamma, K_{(j)})\beta^{(i)}(\underbrace{K_{(1)}, \ldots, K_{(s)}}_{\text{no } K_{(j)}}) \ . \end{aligned}$$

Using $i_{K_{(j)}}d\hat w = \mathscr{L}_{K_{(j)}}\hat w - d(i_{K_{(j)}}\hat w) = -d(i_{K_{(j)}}\hat w)$, as well as further similar equations that follow from $\mathscr{L}_{K_{(i)}}K_{(j)}=0$, one has

$$\begin{aligned} d\hat w(\dot\gamma, K_{(1)},\ldots,K_{(s)}) &= (-1)^s d\hat w(K_{(1)},\ldots,K_{(s)},\dot\gamma) \\ &= (-1)^s i_{\dot\gamma} i_{K_{(s)}}\ldots i_{K_{(1)}} d\hat w \\ &= -(-1)^s i_{\dot\gamma} i_{K_{(s)}}\ldots i_{K_{(2)}} d(i_{K_{(1)}}\hat w) = \ldots \\ &= -i_{\dot\gamma} d(i_{K_{(s)}}\ldots i_{K_{(2)}} i_{K_{(1)}}\hat w) \\ &= -\frac{dZ}{ds}\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} d\hat w(\dot\gamma, K_{(1)},\ldots,K_{(s)}) &= (dK^\flat_{(i)}\wedge\beta^{(i)})(\dot\gamma, K_{(1)},\ldots,K_{(s)}) \\ &= \sum_j (-1)^j dK^\flat_{(i)}(\dot\gamma,K_{(j)})\beta^{(i)}(\underbrace{K_{(1)},\ldots,K_{(s)}}_{\text{no } K_{(j)}})\ . \end{aligned}$$

Comparing with (6.15), we conclude

$$(dK^\flat_{(i)}\wedge\alpha^{(i)})(\dot\gamma,T,K_{(1)},\ldots,K_{(s)}) \;=\; dK^\flat_{(i)}(\dot\gamma,T)\alpha^{(i)}(K_{(1)},\ldots,K_{(s)}) - \frac{dZ}{ds}\ .$$

Collecting all this, we obtain our key equation:

$$\frac{d}{ds}\left(\frac{W}{Z}\right) = \underbrace{\Big(Z^{-1}\alpha^{(i)}(K_{(1)},\ldots,K_{(s)})dK^\flat_{(i)} + dK^\flat_{(0)}\Big)(\dot\gamma,T)}_{=:f}\times\frac{W}{Z}\ . \tag{6.16}$$

We shall need the following:

Lemma 6.3. — *Let $(\mathscr{M},\mathfrak{g})$ be an $(n+1)$–dimensional Lorentzian manifold with an effective action of $\mathbb{R}\times\mathbb{T}^s$ by isometries. Suppose that $K_{(0)}$ is causal at p while* $\mathrm{Span}\{K_{(i)}\}_{i=1}^s|_p$ *is spacelike, and that the isotropy group of p is $\mathbb{T}^r$. Then*

$$0\le r\le n-s\ , \tag{6.17}$$

and if $r>0$ there exist coordinates (x^i,y^i,z^a), $i=1,\ldots,r$, and a basis $\{K_{(i)}\}_{i=1}^s$, consisting of 2π periodic Killing vectors, of the Lie algebra of $\mathbb{T}^s$ such that

$$K_{(i)} = (x^i\partial_{y^i} - y^i\partial_{x^i})\ ,\ i=1,\ldots,r\quad (\textit{no summation over } i)\ . \tag{6.18}$$

Furthermore, setting

$$\rho_{(i)} := \sqrt{(x^i)^2+(y^i)^2}\ ,\qquad \rho := \sqrt{\rho_{(1)}^2+\ldots+\rho_{(r)}^2}\ , \tag{6.19}$$

there exists a constant C such that we have, for all sufficiently small $\rho_{(i)}$ and $\rho_{(j)}$,

$$\forall\ i=1,\ldots,r\quad C^{-1}\rho_{(i)}^2\le\mathfrak{g}(K_{(i)},K_{(i)})\le C\rho_{(i)}^2\ , \tag{6.20}$$

$$\forall\ i=1,\ldots,r\ ,\ \forall\ j=r+1,\ldots,s\quad \mathfrak{g}(K_{(i)},K_{(j)})\le C\rho_{(i)}\rho_{(j)}\rho\ , \tag{6.21}$$

$$\forall\ i=1,\ldots,r\ ,\ \forall\ \mu\in\{0,r+1,\ldots,s\}\quad \mathfrak{g}(K_{(i)},K_{(\mu)})\le C\rho_{(i)}\rho\ . \tag{6.22}$$

Proof. — Let $\{\tilde K_{(i)}\}_{i=1,\ldots,s}$ denote any basis of the Lie algebra of $\mathbb{T}^s$, formed by 2π–periodic Killing vector fields. Let $\{\hat K_{(i)}\}_{i=1,\ldots,r}$ be any basis of the Lie algebra of $\mathbb{T}^r$, again formed by 2π–periodic Killing vector fields. We can complete $\hat K_{(i)}$ to a basis $\{\hat K_{(i)}\}_{i=1}^s$ of the Lie algebra of $\mathbb{T}^s$ using the $\tilde K_{(i)}$'s, and we set $\hat K_{(0)}=K_{(0)}$.

By construction, the manifold Σ defined by (6.3), is a smooth $(n-s+r)$–dimensional spacelike submanifold of $\mathscr{M}$ transverse at p to the $\hat K_{(i)}$'s, $i\in\{r+1,\ldots,s\}$ *and* to the vector T_p of (6.2). Let

$$\mathscr{U}\subset\Sigma$$

be a sufficiently small coordinate ball around p. Let, as before, $\hat G$ be obtained by exponentiating $\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}$, and let $\mathscr V$ be the union of the orbits of $\hat G$ passing through $\mathscr U$. Passing to a subset of $\mathscr U$ if necessary, we can without loss of generality assume that the action of $\mathbb R\times\mathbb T^{s-r}$ generated by the $\hat K_{(\mu)}$'s, with $\mu\in\{0,r+1,\ldots,s\}$, on $\mathscr V$ is free, and by elementary considerations one obtains

$$\mathscr V=\mathscr U\times\mathbb R\times\mathbb T^{s-r}\ .$$

We note that the function t of (6.5) defines a unique $\mathbb T^s$–invariant time function on $\mathscr V$, so that we have proved:

Proposition 6.4. — *Under the hypotheses of Theorem 6.1, there exists an $\mathbb R\times\mathbb T^s$–invariant stably causal neighborhood of p.* □

(We note that some considerations so far could have been considerably simplified if the conclusions of Proposition 6.4 have been known a priori, by averaging any time function as in the proposition over $\mathbb T^s$.)

Returning to the proof of Lemma 6.3, let h be the metric induced on $\mathscr S_{\mathscr O}$ by $\mathfrak g$. Then h is a Riemannian metric invariant under $\mathbb T^r$. Let γ denote the orbit-space metric on $\mathscr U$,

$$\forall\ X,Y\in T\mathscr U\qquad \gamma(X,Y)=h(X,Y)-h^{(i)(j)}h(X,K_{(i)})h(Y,K_{(j)})\ , \tag{6.23}$$

where $h^{(i)(j)}$ denotes the matrix inverse to $h(K_{(i)},K_{(j)})$, $i,j=r+1,\ldots,s$, and in (6.23) one sums over i,j in the last range. It is simple to check, using the Cauchy-Schwarz inequality, that γ is Riemannian, so that the group $\mathbb T^r$ acts locally on the Riemannian manifold $(\mathscr U,\gamma)$ by isometries, with complete orbits near p. We infer that near p the original orbit space $\mathscr M/(\mathbb R\times\mathbb T^s)$ is diffeomorphic to $\mathscr U/\mathbb T^r$.

Now, $\mathbb T^r$ acts effectively on $(T_p\mathscr U,\gamma|_p)$ by isometries, so we can view $\mathbb T^r$ as a closed abelian subgroup of $SO(n-s+r)$, such that the principal orbits of the action of $\mathbb T^r$ on $\mathbb R^{n-s+r}$ are r-dimensional.

Let $G\subset SO(n-s+r)$ denote any maximal torus containing $\mathbb T^r$, by [3, Theorem 16.2] G is conjugated to a standard maximal torus as in [16, Example 6.21], hence

$$r\le\dim G=\lfloor\frac{n-s+r}{2}\rfloor\le\frac{n-s+r}{2}\quad\Longrightarrow\quad 0\le r\le n-s\ . \tag{6.24}$$

Consider the simplest case $r=1$, then $\dim\mathrm{Ker}\nabla K_{(1)}=n+1-2=n-1$. Let $(x^1,y^1,z^a)\equiv(x^A,z^a)$ be the coordinates of [1, Proposition C.1] (denoted by (x^A,x^a) there, and constructed there under the assumption that the metric is Riemannian, but the result holds for a Lorentzian $\mathfrak g$ whenever $\mathrm{Ker}\nabla X$ contains a timelike vector), with n there replaced by $n+1$, X there equal to $K_{(1)}$, and ℓ there equal to one. The lemma follows now from [1, Equation (C.8)]:

$$\begin{aligned}\mathfrak g\ &=\ \sum_{i=1}^{\ell}\big((dx^i)^2+(dy^i)^2\big)+\sum_{A,B}O(\rho^2)dx^Adx^B+\sum_{A,a}O(\rho)dx^Adz^a\\&\quad+\mathfrak g_{ab}|_{\rho=0}dz^adz^b+\sum_{a,b}O(\rho^2)dz^adz^b\ ,\end{aligned}\tag{6.25}$$

where $\rho^2=\rho^2_{(1)}+\ldots+\rho^2_{(\ell)}$.

In general, by the already mentioned [3, Theorem 16.2] and [16, Example 6.21], there exists an orthonormal basis of T_pM so that the flows of $\hat K_{(i)}$ on T_pM, $i=1,\ldots,r$, are generated by linear combinations of vector fields $K_{(1)}$ and $K_{(2)}$ as in

(6.18). Equivalently, the $K_{(i)}$'s, $i = 1, \dots, r$, take the form (6.18) near p in the associated normal coordinates centred at p. Applying [1, Proposition C.1] to

$$X = K_{(1)} + \ldots + K_{(r)} \ ,$$

with ℓ there equal to r, our claims follow again from (6.25). □

For further reference we note the following variation of Lemma 6.3, with essentially identical, but somewhat simpler, proof:

LEMMA 6.5. — *Let (M, h) be an n–dimensional Riemannian manifold with an effective action of $\mathbb{T}^s$ by isometries.. If $\mathbb{T}^r$ is the isotropy group of p, then (6.17) holds, and for $r > 0$ there exist coordinates (x^i, y^i, z^a), $i = 1, \dots, r$, and a basis $\{K_{(i)}\}_{i=1}^s$, consisting of 2π periodic Killing vectors, of the Lie algebra of $\mathbb{T}^s$ such that (6.18) holds. Furthermore, letting ρ and $\rho_{(i)}$ be as in (6.19), there exists a constant C such that we have, for all sufficiently small $\rho_{(i)}$ and $\rho_{(j)}$,*

$$\forall\ i = 1, \dots, r \quad C^{-1}\rho_{(i)}^2 \le h(K_{(i)}, K_{(i)}) \le C\rho_{(i)}^2 \ , \tag{6.26}$$

$$\forall\ i \neq j \in \{1, \dots, r\} \quad h(K_{(i)}, K_{(j)}) \le C\rho_{(i)}\rho_{(j)}\rho \ , \tag{6.27}$$

$$\forall\ i = 1, \dots, r \ , \ \forall\ j = r+1, \dots, s, \qquad h(K_{(i)}, K_{(j)}) \le C\rho_{(i)}\rho \ . \tag{6.28}$$

□

We return now to the analysis of (6.16). The case $s = 1$ has already been covered in [7], so we assume $s \ge 2$. Set

$$\check{Z}_{(r)} := \det\left(\mathfrak{g}(K_{(\mu)}, K_{(\nu)})_{\mu,\nu=0,r+1,\dots,s}\right) \ , \quad Z_{(r)} := \det\left(\mathfrak{g}(K_{(i)}, K_{(j)})_{i,j=r+1,\dots,s}\right) \ .$$

Suppose, first, that $k = 1$. Then, after exchanging the zeroth and first row, and then the zeroth and first column, W is minus the determinant of a matrix of the form

$$\begin{pmatrix} \mathfrak{g}(K_{(1)}, K_{(1)}) & O(\rho^2) & \dots & O(\rho^2) \\ O(\rho^2) & \star & \dots & \star \\ \vdots & \vdots & \ddots & \vdots \\ O(\rho^2) & \star & \dots & \star \end{pmatrix} . \tag{6.29}$$

Equation (6.29), and a similar equation for Z, leads to

$$W = -\mathfrak{g}(K_{(1)}, K_{(1)})\check{Z}_{(1)} + O(\rho^4) \ , \quad Z = \mathfrak{g}(K_{(1)}, K_{(1)})Z_{(1)}(1 + O(\rho^2)) \ , \tag{6.30}$$

(recall that $Z_{(1)}(p)$ does not vanish by hypothesis). Using (6.20) we conclude that W/Z approaches $-\check{Z}_{(1)}/Z_{(1)}$ as one approaches p along γ.

Next, we wish to show that the function f defined in (6.16) is bounded; this requires an analysis of the term

(6.31)

$$\begin{aligned} Z^{-1}\alpha^{(i)}(K_{(1)}, \dots, K_{(s)})\, dK^\flat_{(i)}(\dot\gamma, T) \quad &= \quad Z^{-1}\alpha^{(1)}(K_{(1)}, \dots, K_{(s)})\, dK^\flat_{(1)}(\dot\gamma, T) + \\ &\sum_{i>1} Z^{-1}\alpha^{(i)}(K_{(1)}, \dots, K_{(s)})\, dK^\flat_{(i)}(\dot\gamma, T) \ . \end{aligned}$$

Writing $h_{\mu\nu}$ for $\mathfrak{g}(K_{(\mu)}, K_{(\nu)})$, by definition we have

$$\alpha^{(1)}(K_{(1)}, \dots, K_{(s)}) = \epsilon^{i_0 i_2 \dots i_s} h_{0i_0} h_{2i_1} \dots h_{si_s} = O(\rho^2) \ ,$$

$$i > 1: \quad \alpha^{(i)}(K_{(1)}, \dots, K_{(s)}) = (-1)^i \epsilon^{i_0 i_1 \dots i_s} h_{0i_0} \underbrace{h_{1i_1} \dots h_{si_s}}_{\text{no } h_{ij_i} \text{ factor}} = O(\rho^2) \ ,$$

and boundedness of f readily follows.

When $r = 2$ we set $\rho^2 = \sqrt{\rho_{(1)}^2 + \rho_{(2)}^2}$; then, after moving the zeroth row past the next two ones, similarly for the zeroth column, W is the minus the determinant of

$$\left(\begin{array}{ccccc} \mathfrak{g}(K_{(1)},K_{(1)}) & O(\rho_{(1)}\rho_{(2)}\rho) & O(\rho_{(1)}\rho) & \dots & O(\rho_{(1)}\rho) \\ O(\rho_{(1)}\rho_{(2)}\rho) & \mathfrak{g}(K_{(2)},K_{(2)}) & O(\rho_{(2)}\rho) & \dots & O(\rho_{(2)}\rho) \\ O(\rho_{(1)}\rho) & O(\rho_{(2)}\rho) & \star & \dots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O(\rho_{(1)}\rho) & O(\rho_{(2)}\rho) & \star & \dots & \star \end{array}\right) . \tag{6.32}$$

One finds

$$W = -\mathfrak{g}(K_{(1)},K_{(1)})\mathfrak{g}(K_{(2)},K_{(2)})\check{Z}_{(2)} + O(\rho_{(1)}^2\rho_{(2)}^2\rho^2) , \tag{6.33}$$

$$Z = \mathfrak{g}(K_{(1)},K_{(1)})\mathfrak{g}(K_{(2)},K_{(2)})Z_{(2)}(1+O(\rho^2)) , \tag{6.34}$$

The form of the error terms plays a key role when taking the quotient W/Z below, so it deserves a more careful justification. We start with the determinant Z:

$$Z = \epsilon^{i_1\dots i_s} h_{1i_1}\dots h_{si_s} .$$

Let us write

$$\epsilon^{i_1\dots i_s}_{i_1\neq 1} h_{1i_1}\dots h_{si_s}$$

for a sum where i_1 is not allowed to take the value one, and

$$\epsilon^{i_1\dots i_s}_{i_1\neq 1, i_2\neq 2} h_{1i_1}\dots h_{si_s}$$

for a sum where i_1 is not allowed to take the value one and i_2 is not allowed to take the value two, etc. Then

$$\begin{aligned} Z &= h_{11}\epsilon^{1i_2\dots i_s}h_{2i_2}\dots h_{si_s} + \epsilon^{i_1\dots i_s}_{i_1\neq 1}h_{1i_1}\dots h_{si_s} \\ &= \underbrace{h_{11}h_{22}\epsilon^{12i_3\dots i_s}h_{3i_3}\dots h_{si_s}}_{I} + \underbrace{h_{11}\epsilon^{1i_2\dots i_s}_{i_2\neq 2}h_{2i_2}\dots h_{si_s}}_{II} \\ &\quad + \underbrace{h_{22}\epsilon^{i_1 2 i_3\dots i_s}_{i_1\neq 1}h_{1i_1}h_{3i_3}\dots h_{si_s}}_{III} + \underbrace{\epsilon^{i_1\dots i_s}_{i_1\neq 1, i_2\neq 2}h_{1i_1}\dots h_{si_s}}_{IV} . \end{aligned}$$

The term I is the main term $\mathfrak{g}(K_{(1)},K_{(1)})\mathfrak{g}(K_{(2)},K_{(2)})Z_{(2)}$ in (6.34). In each term of the sum II one of the indices $i_k \neq i_2$ has to be a two, so each term in that sum contains a factor $h_{2i_2}h_{i_k2} = O(\rho_{(2)}^3\rho) = O(\rho_{(2)}^2\rho^2)$. Taking into account $|h_{11}| \leq C\rho_{(1)}^2$ we obtain $|II| \leq C\rho_{(1)}^2\rho_{(2)}^2\rho^2$, which can be factored as $h_{11}h_{22}Z_{(2)}O(\rho^2)$. The estimate on III follows by symmetry, the analysis of IV proceeds along the same lines.

It should be clear from (6.32) that the calculation for W is identical, after grouping $K_{(0)}$ with the $K_{(i)}$'s, $i = 3,\dots,s$. The only difference is in the last step, where we cannot factor out $\check{Z}_2$, as we are allowing it to vanish at p.

Without much further effort, the reader should be able to conclude that for all r

$$W = -\mathfrak{g}(K_{(1)},K_{(1)})\cdots\mathfrak{g}(K_{(r)},K_{(r)})\check{Z}_{(r)} + O(\rho_{(1)}^2\dots\rho_{(r)}^2\rho^2) , \tag{6.35}$$

$$Z = \mathfrak{g}(K_{(1)},K_{(1)})\cdots\mathfrak{g}(K_{(r)},K_{(r)})Z_{(r)}(1+O(\rho^2)) , \tag{6.36}$$

so that

$$\lim_{s\searrow 0}\frac{W}{Z}(\gamma(s)) = -\frac{\check{Z}_{(r)}}{Z_{(r)}}\Big|_p , \quad \text{with } Z_{(r)}(p)\neq 0 . \tag{6.37}$$

It follows that the quotient W/Z has a vanishing limit at $p \in \mathscr{A}$ either if $K_{(0)} \in \mathrm{Span}\{K_{(r+1)},\dots,K_{(s)}\}|_p$, or if $\mathrm{Span}\{K_{(0)},K_{(r+1)},\dots,K_{(s)}\}|_p$ is a null subspace of $T_p\mathscr{M}$. The former possibility does not occur since $K_{(0)}$ is transverse to $\mathrm{Span}\{K_{(r+1)},\ldots,K_{(s)}\}|_p$ by hypothesis.

Again for $r=2$, consider the function f of (6.16), the not-obviously-bounded part of which we write now as

(6.38)
$$\begin{aligned} Z^{-1}\alpha^{(i)}(K_{(1)},\ldots,K_{(s)})\,dK^\flat_{(i)}(\dot\gamma,T) &= Z^{-1}\alpha^{(1)}(K_{(1)},\ldots,K_{(s)})\,dK^\flat_{(1)}(\dot\gamma,T)+\\ &\ Z^{-1}\alpha^{(2)}(K_{(1)},\ldots,K_{(s)})\,dK^\flat_{(2)}(\dot\gamma,T)+\\ &\ \sum_{i>1} Z^{-1}\alpha^{(i)}(K_{(1)},\ldots,K_{(s)})\,dK^\flat_{(i)}(\dot\gamma,T)\,. \end{aligned}$$

By definition we have, for $s\ge 3$ (the calculation for $s=2$ is typographically different, but otherwise identical),

$$\begin{aligned} \alpha^{(1)}(K_{(1)},\ldots,K_{(s)}) &= \epsilon^{i_0i_2\ldots i_s}h_{0i_0}h_{2i_1}\ldots h_{si_s}\\ &= \underbrace{h_{22}}_{O(\rho^2_{(2)})}\epsilon^{i_02i_3\ldots i_s}h_{0i_0}h_{3i_3}\ldots h_{si_s}+\epsilon^{i_0i_2\ldots i_s}_{i_2\ne 2}h_{0i_0}\underbrace{h_{2i_2}}_{O(\rho_{(2)}\rho)}\ldots h_{si_s}\\ &= O(\rho_{(1)}\rho^2_{(2)}\rho)\,, \end{aligned}$$

because each term in each of the sums above contains a factor h_{r1}, $r\neq 1$, which is $O(\rho_{(1)}\rho)$; furthermore, one of the indices in the second sum has to be equal to two, which gives a further factor $O(\rho_{(2)}\rho)$ in the second sum. One similarly obtains

$$\alpha^{(2)}(K_{(1)},\ldots,K_{(s)}) = \epsilon^{i_0i_1i_3\ldots i_s}h_{0i_0}h_{1i_1}h_{3i_3}\ldots h_{si_s} = O(\rho^2_{(1)}\rho_{(2)}\rho)\,,$$
$$j>2:\quad \alpha^{(j)}(K_{(1)},\ldots,K_{(s)}) = (-1)^j\underbrace{\epsilon^{i_0\ldots i_s}}_{\text{no } i_j \text{ index}}h_{0i_1}\ldots h_{si_s} = O(\rho^2_{(1)}\rho^2_{(2)})$$

Now, this does not suffice for estimating a quotient by $Z\approx\rho^2_{(1)}\rho^2_{(2)}$ for the first two terms in (6.38). However, the missing powers of $\rho_{(1)}$ and of $\rho_{(2)}$ are provided by $dK^\flat_{(1)}(\dot\gamma,T)$ and $dK^\flat_{(1)}(\dot\gamma,T)$:

$$dK^\flat_{(i)}(\dot\gamma,T)=O(\rho_{(i)})\,.$$

Indeed, $dK^\flat_{(i)}(\dot\gamma,T)=0$ at $\{\rho_{(i)}=0\}$ for $i=1,\ldots,r$, and a Taylor expansion of order zero near $\{\rho_{(i)}=0\}$ gives the estimate.

Summarising, both for $r=1$ and $r=2$, we have shown that the function f defined in (6.16) is bounded along γ near p. A very similar analysis applies for higher r.

Now, if $\check Z_{(k)}=0$ at p, then the limit at p of W/Z along γ vanishes by (6.37). Using uniqueness of solutions of ODE's, it follows from (6.16) that W vanishes along γ. To finish the proof it suffices to notice that any γ with, e.g., $\dot\gamma(0)$ lying in the (x^1,y^1) plane of the coordinates of Lemma 6.3 immediately leaves $\mathscr{A}$. □

7. Uniqueness of static solutions and zeros of Killing vectors

As pointed out in [7], the proof of uniqueness of higher dimensional globally hyperbolic, static, vacuum black holes containing an asymptotically flat hypersurface, of positive energy type, with boundary contained away from the domain of outer communications, requires excluding zeros of the Killing vector on degenerate components of the event horizon. Our aim in this section is to prove that such zeros cannot occur, as needed for the argument in [7].

We start with the following result, pointed out to us by Abdelghani Zeghib, which is apparently well known among researchers acquainted with hyperbolic geometry. For completeness we provide the proof, as explained to us by Zeghib:

Proposition 7.1. — *Let X be a non-trivial Killing vector, and suppose that X vanishes at $p \in \mathscr{M}$. Then there exists a normal coordinate system (x^μ) near p such that:*

1. *either there exist constants $\beta_\mu \in \mathbb{R}$, $\mu = 0, \ldots, m \le n/2$, not all zero, such that*

$$X = \beta_0(x^0\partial_1 + x^1\partial_0) + \sum_{i=1}^{m} \beta_i(x^{2i+1}\partial_{2i} - x^{2i}\partial_{2i+1}) \; , \tag{7.1}$$

2. *or there exists constants $a \in \mathbb{R}^*$ and $\beta_i \in \mathbb{R}$, $i = 0, \ldots, m \le (n-1)/2$ such that*

$$X = a\left((x^0 - x^2)\partial_1 + x^1(\partial_0 + \partial_2)\right) + \sum_{i=1}^{m} \beta_i(x^{2i+1}\partial_{2i+2} - x^{2i+2}\partial_{2i+1}) \; . \tag{7.2}$$

Remarks 7.2. — 1. Recall that every orthochronous, orientation preserving Lorentz matrix is the exponential of a matrix $\lambda^\mu{}_\nu = \partial_\nu X^\mu$, where X^μ is a Minkowski space-time Killing vector vanishing at the origin. So (7.1)-(7.2) can also be used to obtain a canonical representation for Lorentz matrices.

2. The coordinates of (7.1) are unique, but those of (7.2) are not.

Proof. — Let $\lambda = \nabla X|_p$; in other words, $\lambda^\mu{}_\nu := \nabla_\nu X^\mu|_p$. Let e_a be any ON frame at p with e_0–timelike. Let (x^μ) denote the associated normal coordinates centered at p. It is well known, and in any case note very difficult to show using the fact that isometries map geodesics to geodesics, that $X = \lambda^\mu{}_\nu x^\nu \partial_\nu$. So to prove the result we need to classify the possible matrices λ, up to choice of ON-basis.

Suppose that $\sigma \in \mathbb{C} \setminus \mathbb{R}$ is a root of the characteristic polynomial of λ, let $u + \mathrm{i}v \in T_p\mathscr{M} \oplus \mathrm{i}T_p\mathscr{M}$ be the corresponding eigenvector. Keeping in mind that one-dimensional eigenspaces lead to real eigenvalues, the space $\mathrm{Span}\{u, v\}$ is a two-dimensional space invariant under λ. We claim that $\mathrm{Span}\{u, v\}$ is not null: otherwise it would contain a unique null direction, which would have to be mapped into itself by all the isometries $\exp(t\lambda)$. This would imply that $\mathrm{Span}\{u, v\}$ contains an eigenvector of λ with real eigenvalue, contradicting $\sigma \in \mathbb{C} \setminus \mathbb{R}$. Thus, $\mathrm{Span}\{u, v\}$ is either a) timelike or b) spacelike.

In the latter case b) we choose e_0 and e_1 so that $\mathrm{Span}\{u, v\} = \mathrm{Span}\{e_0, e_1\}$. Then the space $\mathrm{Span}\{u, v\}^\perp$ is a complementing spacelike subspace of $T_p\mathscr{M}$, invariant under λ, and we have reduced the problem to a Riemannian one, in dimension smaller by two.

In the former case a) we pass to an ON basis of $T_p\mathscr{M}$ so that $\mathrm{Span}\{u, v\} = \mathrm{Span}\{e_{n-1}, e_n\}$. Then the space $\mathrm{Span}\{u, v\}^\perp$ is a complementing timelike subspace of $T_p\mathscr{M}$ invariant under λ, and we have reduced the dimension by two.

Note that if at any stage of this dimension-reduction process the metric becomes Riemannian, then the iteration of the argument in the last paragraph provides a finite number of two-dimensional orthogonal invariant spaces plus a Riemannian space, say E, invariant under λ, with all E–eigenvalues of λ real.

Now, generally, since $\lambda_{\mu\nu}$ is anti-symmetric we have

$$0 = \lambda_{\mu\nu} u^\mu u^\nu = \sigma u_\mu u^\mu \; ,$$

which shows that u is null unless $\sigma = 0$. So on each timelike or spacelike one-dimensional eigenspace the action of the flow of X is trivial. Hence, if a Riemannian metric is obtained after any of the dimension-reduction steps described in this proof, after a finite number of further steps we obtain a basis where X takes the form (7.1).

Iterating, we can decompose $T_p\mathscr{M}$ as an orthogonal sum of invariant two-dimensional spaces plus an invariant remainder, say again E. If λ vanishes on E, then X takes the form (7.1), and we are done.

Otherwise λ maps a Lorentzian E to E, we shall still denote by λ the resulting map. By construction all roots of the characteristic polynomial of $\lambda|_E$ are real. Let σ be such a root, and let u be the corresponding vector. If u is timelike or spacelike, then $\mathrm{Span}\{u\}^\perp$ is a complementing invariant space, and we can further reduce the dimension by splitting off $\mathrm{Span}\{u\}$ from E, and renaming the new space E. We continue in this way until there are, in E, no eigenvectors which are timelike or null. In particular E has no proper Riemannian eigenspaces.

Again, if at some stage one of u's is timelike, we are in the case (7.1).

So, there eventually remains a space E invariant under λ, with λ having only real eigenvalues, and only null eigenvectors. Suppose that there exist two such eigenvectors, u and v, then $\mathrm{Span}\{u, v\}$ is timelike, invariant under the flow of X, with the complementing space Riemannian, or trivial. We avoid a contradiction with the fact that λ, restricted to E, has no proper Riemannian eigenspaces only if $\dim E = 2$, leading to (7.1), and the proof is complete in this case.

Otherwise E contains only one null eigenvector u, and no invariant subspaces which are timelike or spacelike. The space $\mathrm{Span}\{u\}^\perp$ is a null subspace of E invariant under λ. Let $\{u, e_i\}$ be a basis of $\mathrm{Span}\{u\}^\perp$, then the e_i's are necessarily spacelike. There exists a matrix $\alpha_i{}^j$ and numbers α_i such that

$$\lambda e_i = \alpha_i u + \alpha_i{}^j e_j \ .$$

The numbers (α_i) behave as a vector under rotations of $\mathrm{Span}\{e_1, \ldots\}$, so we can choose a rotation matrix $\omega_i{}^j$ so that in the new basis $\hat e_i = \omega_i{}^j e_j$ we have

$$\lambda \hat e_i = \hat\alpha_i u + \hat\alpha_i{}^j \hat e_j \ ,$$

with $(\hat\alpha_i) = (\hat\alpha_1, 0, \ldots, 0)$. But then the space $\mathrm{Span}\{\hat e_2, \ldots\}$ is a Riemannian subspace of E invariant under λ, which leads to a contradiction unless $\{e_i\}$ contains only one element, and then $\dim E = 3$. This shows that $\{e_1\}^\perp$ is a two-dimensional Lorentzian space containing u. We can choose an ON basis $\{e_0, e_2\}$ of $\{e_1\}^\perp$, with e_0 timelike, so that $u = e_0 + e_2$. The equation $\lambda u = \sigma u$, where $\sigma \in \mathbb{R}$ is the eigenvalue, gives

$$\lambda u = (\lambda^\mu{}_0 + \lambda^\mu{}_2) e_\mu = \sigma(e_0 + e_2) \ .$$

Equivalently, keeping in mind $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$,

$$\lambda^0{}_2 = \sigma = \lambda^2{}_0 \ , \quad \underbrace{\lambda^1{}_0}_{=:a} + \lambda^1{}_2 = 0 \ .$$

So

$$\lambda^\mu{}_\nu = \begin{pmatrix} 0 & a & \sigma \\ a & 0 & -a \\ \sigma & a & 0 \end{pmatrix} \iff \lambda_{\mu\nu} = \begin{pmatrix} 0 & -a & -\sigma \\ a & 0 & -a \\ \sigma & a & 0 \end{pmatrix} \ . \tag{7.3}$$

Calculating $\det(\lambda - \sigma\,\mathrm{id})$, one finds that λ has both σ and $-\sigma$ as eigenvalues, which at this stage is consistent only if σ vanishes. If $a = 0$ we obtain a contradiction with the fact that $\lambda|_E$ is non-trivial, so (7.2) holds with $a \neq 0$, and the result is established. □

We wish, now to show that [6, Theorem 1.1] remains valid in higher dimensions, under the following proviso: For consistency of notation with the remainder of this work, let us denote by $\mathscr{S}$ the manifold Σ there. One then needs to assume that the doubling of $\mathscr{S}$ across all non-degenerate components of its boundary, and compactification of all asymptotically flat regions except one, leads to a manifold of positive energy type, as defined in [7, Section 1.1]. Under this condition, the arguments of the proof of [6, Theorem 1.1] go through without modifications except for the proof that there are no zeros of X on degenerate components of $\partial\overline{\mathscr{S}} = \overline{\mathscr{S}}\setminus\mathscr{S}$. In [6] such zeros were "excluded" by the incorrect Theorem 3.1 there. To take care of this, recall that it is assumed in [6, Theorem 1.1] that a vacuum space-time $(\mathscr{M}, \mathfrak{g})$ has a hypersurface-orthogonal Killing vector X which is timelike along a spacelike hypersurface $\mathscr{S}$. Further, it is assumed that X vanishes on the boundary $\partial\overline{\mathscr{S}}$, which is supposed there to be a compact two-dimensional topological manifold, and which we allow in this work to be any compact topological manifold of co-dimension two in $\mathscr{M}$. It is shown in [7, Section 5.2] that the set, say $\mathscr{E}$, where $\mathfrak{g}(X, X)$ vanishes, is foliated by locally totally geodesic null hypersurfaces, away from the points where X vanishes. Hence each leaf of $\mathscr{E}$ is smooth on an open dense set, so $\partial\mathscr{S}$ is smooth on the open dense subset of $\partial\mathscr{S}$ consisting of points at which X does not vanish. Note that $\mathscr{E}$ might fail to be embedded in general, but this is irrelevant for the proof here because $\partial\mathscr{S}$ is a compact embedded topological manifold by hypothesis. In vacuum, on every smooth leaf of $\mathscr{E}$, and hence on every smooth component of $\partial\mathscr{S}$, the surface gravity κ is constant (see, e.g., [26, Theorem 2.1]). It follows that the problem with the incorrect [6, Theorem 3.1] is solved by the following:

PROPOSITION 7.3. — *Let $(\mathscr{M}, \mathfrak{g})$ be an $(n+1)$–dimensional Lorentzian manifold with Killing vector field X, and suppose that*

$$\Omega := \partial\{p \in \mathscr{M} \mid \mathfrak{g}(X, X) < 0\}\ . \tag{7.4}$$

is a topological hypersurface. Assume that

1. *either X is hypersurface-orthogonal and Ω has vanishing surface gravity wherever defined,*
2. *or Ω is differentiable.*

Then X has no zeros on Ω.

Proof. — Suppose, first, that X is of the form (7.2) in a geodesically convex neighborhood $\mathscr{U}$ of p globally coordinatised by normal coordinates. This, together with elementary properties of normal coordinates, implies

$$\mathfrak{g}(X, X) = a^2(x^0 - x^2)^2 + \sum_{i=1}^{m} \beta_i^2 \left((x^{2i+1})^2 + (x^{2i+2})^2\right) + O(|x|^4)\ , \tag{7.5}$$

where $|x|^2 = (x^0)^2 + \ldots + (x^n)^2$. It follows from (7.2) that X is tangent to the two hypersurfaces

$$\mathscr{N}^{\pm} = \{x^0 = x^2\ ,\ \pm x^2 > 0\}\ ,$$

non-vanishing there.

Consider any point $q \in \Omega$ at which X does not vanish. As shown in [7], the hypersurface Ω is smooth near q, and any geodesic γ initially normal to X_q stays on Ω, except perhaps when it reaches a point at which X vanishes.

So, suppose that γ is such a geodesic from $q \in \Omega$ to p, with p being the first point on γ at which X vanishes. If $\dot x^0 \neq \dot x^2$ at p, (7.5) shows that X is spacelike along γ near and away from p, contradicting the fact that X is null on Ω. We conclude that $\dot\gamma$ is tangent at p to the hypersurface $\{x^0 = x^2\}$, but then $\gamma \cap \mathscr{U}$ is included in $\{x^0 = x^2\}$. Consequently

$$\Omega \cap \mathscr{U} \subset \{x^0 = x^2\} \ . \tag{7.6}$$

Since Ω is a topological hypersurface by hypothesis, we obtain that

$$\Omega \cap \mathscr{U} = \{x^0 = x^2\} \ . \tag{7.7}$$

(In particular Ω is smooth near p.)

In the case where X is not necessarily hypersurface orthogonal, but we assume a priori that Ω is differentiable, the argument is somewhat similar, with a weaker conclusion: Let $\gamma \subset \Omega$ be any differentiable curve, then we must have $\dot x^0 = \dot x^2$ at p. Since Ω is a hypersurface, this implies that

$$T_p\Omega = T_p\{x^0 = x^2\} \ . \tag{7.8}$$

So, while (7.7) does not necessarily hold, the tangent spaces coincide at p in both cases.

Consider, now any differentiable curve σ through p on which $\dot x^0 \neq \dot x^2 \neq 0$ at p. Equation (7.5) shows that on σ the Killing vector X is spacelike near and away from p. By (7.8) such curves are transverse to Ω, which shows that there exist points arbitrarily close to Ω at which X is *spacelike* on both sides of Ω. This contradicts (7.4), and shows that (7.2) cannot arise under our hypotheses.

It remains to analyze Killing vectors of the form (7.1). In this case

$$\mathfrak{g}(X,X) = \beta_0^2\left(-(x^0)^2 + (x^1)^2\right) + \sum_{i=1}^m \beta_i^2\left((x^{2i})^2 + (x^{2i+1})^2\right) + O(|x|^4) \ . \tag{7.9}$$

Suppose, first, that $\beta_0 = 0$. Then $\mathrm{Ker}\lambda = \mathrm{Span}\{\partial_0, \partial_1\}|_p$. Now, because the flow of a Killing vector maps geodesics to geodesics, X vanishes on every geodesic γ with $\gamma(0) = p$ such that $\dot\gamma(0) \in \mathrm{Ker}\lambda$. So X vanishes throughout the timelike hypersurface $\{x^2 = \ldots = x^n = 0\}$. At every point q of this hypersurface, in adapted normal coordinates centered a q the tensor $\nabla_c X_d|_q$ takes the form (**??**) with $\beta_0 = 0$. This implies that X is spacelike or vanishing throughout a neighborhood of p, so $\beta_0 = 0$ cannot occur.

Now, if Ω is differentiable at p, an argument very similar to the one above shows that

$$T_p\Omega \subset E_+ \cup E_- \ , \quad \text{where } E_\pm := \{\dot x^0 = \pm \dot x^1\} \ .$$

So either $T_p\Omega = E_+$ or $T_p\Omega = E_-$. But, the curves with $\dot x^0 = \dot x^1/2$ at p are transverse both to E_- and to E_+, with X spacelike on those curves near and away from p on both sides of $\mathscr{E}_\pm$, contradicting the definition of Ω. So, under the assumption of differentiability of Ω the proof is complete.

Assuming, next, that X is hypersurface-orthogonal, we claim that $\beta_i = 0$. Indeed, let $X^\flat$ be the field of one-forms defined as $X^\flat = \mathfrak{g}(X, \cdot)$. Then

$$\begin{aligned} X^\flat &= \beta_0(x^0dx^1 - x^1dx^0) + \sum_{i=1}^m \beta_i(x^{2i}dx^{2i+1} - x^{2i+1}dx^{2i}) + O(|x|^{3/2}) \ , \\ dX^\flat &= 2\beta_0\, dx^0 \wedge dx^1 + \sum_{i=1}^m 2\beta_i\, dx^{2i} \wedge dx^{2i+1} + O(|x|^{3/2}) \ , \end{aligned}$$

and the staticity condition $X^\flat \wedge dX^\flat = 0$ gives $\beta_i = 0$, $i = 1, \ldots, m$.

Arguments similar to the ones already given show now that

$$\Omega \cap \mathscr{U} \cap \{x^2 = \ldots = x^n = 0\;,\; x^0 = \pm x^1\} \neq \emptyset\;.$$

Next, from (7.9) we have

$$d\left(\mathfrak{g}(X,X)\right) = 2\beta_0^2\left(-x^0dx^0 + x^1dx^1\right) + 2\sum_{i=1}^{m}\beta_i^2\left(x^{2i}dx^{2i} + x^{2i+1}dx^{2i+1}\right) + O(|x|^3)\;,$$

and recall that this vanishes on Ω wherever Ω is differentiable, by definition of degeneracy. But on $S := \{x^2 = \ldots = x^n = 0\;,\; x^0 = \pm x^1\}$, with $|x|$ sufficiently small, we clearly have $d\left(\mathfrak{g}(X,X)\right) \neq 0$. If points on S are differentiability points of Ω we are done; otherwise, notice that $d\left(\mathfrak{g}(X,X)\right) \neq 0$ on a space-time neighborhood of $S \cap \{0 < |x| < \epsilon\}$ for some $\epsilon > 0$, and since differentiability points are dense on Ω the horizon cannot be degenerate. □

Proposition 7.1 allows us also to solve a question concerning the codimension of zero-sets of Killing vectors within null hypersurfaces, that arose in [7, Section 5]:

Proposition 7.4. — *Let X be a Killing vector. Suppose that X vanishes at p. Then the intersection of the zero-set of X with a null hypersurface $\mathscr{N}$ is, near p, a smooth submanifold of $\mathscr{N}$ with $\mathscr{N}$–codimension at least two, unless $T\mathscr{N}$ contains a null generator on which X vanishes, or is tangent to it.*

Proof. — Suppose, first, that near p the Killing vector X takes the form (7.1). If $\beta_0 \neq 0$ and if at least one $\beta_i = 0$, with $i \geq 1$, is non-zero, then X vanishes on a smooth submanifold through p of codimension larger than or equal to four, and the result is straightforward. If $\beta_0 = 0$, the result follows from the fact that the codimension the zero set of X in $\mathscr{M}$ equals that in $\mathscr{N}$. Otherwise only β_0 is different from zero, and the zero-set of X through p is a smooth spacelike submanifold S of co-dimension two. A straightforward examination of the tangent planes at p shows that the intersection with any null-hypersurface $\mathscr{N}$ is a set of co-dimension at least two unless the null tangent plane of $\mathscr{N}$ at p contains one of the null normals to S. But then the corresponding generator of $\mathscr{N}$ through p will contain, at least near p, a null orbit of X accumulating at p. The analysis of (7.2) is similar. □

8. Concluding remarks

As discussed in more detail in [7], event horizons in well behaved stationary asymptotically flat space-times are smooth hypersurfaces. The key to the proof of this fact is [8, Theorem 6.18], with a purely local proof except for the requirement that the conclusions of the area theorem hold. So any set of global conditions ensuring the validity of the area theorem imply the result. Now, smoothness of the event horizon is needed to prove the existence of a supplementary isometry in the space-time, via the so-called *rigidity theorem* [20, 21, 23]. While it is clear that some version of this statement remains correct for KK–asymptotically flat space-times, we have not investigated this issue any further since our main results here assume more Killing vectors than provided by the rigidity theorem. Under the hypotheses of Theorem 6.1, smoothness of the event horizon follows from the locally totally geodesic character of leaves of the zero-level set of the area function W, see [7, Corollary 5.13].

We note that the key elements of the uniqueness argument for non-degenerate Kerr black holes, derived from I^+–regularity and asymptotic flatness, are: a) simple connectedness; b) smoothness of the event horizon; c) product structure of the domain of outer communications; d) the reduction of the problem to a singular harmonic map with well understood uniqueness properties, with e) well understood boundary conditions.

In this paper, assuming KK–asymptotic flatness, we noted that b) holds but is less essential given the number of Killing vectors assumed; we proved c); we pointed out a version of a) sufficient to define the twist potentials, and to prove positivity of the area function. All this establishes d). Theorem 6.1, perhaps the most involved result here, provides an essential step towards e). However a complete proof, that the resulting reduced equations satisfy the right boundary conditions at $\mathscr{A}\cup\partial\langle\langle\mathscr{M}_{\rm ext}\rangle\rangle$ for uniqueness, has to be carried out yet, both for non-degenerate and degenerate horizons. (Recall that the question of boundary conditions at degenerate horizons is open even with $n = 3$). We are hoping to return to at least some of those issues in a near future.

ACKNOWLEDGEMENTS I am grateful to A. Zeghib for pointing out Proposition 7.1 and its proof, and to J. Lopes Costa for comments about a previous version of this paper.

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