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A new proof of the Bianchi type IX attractor theorem

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Abstract

We consider the dynamics towards the initial singularity of Bianchi type IX vacuum and orthogonal perfect fluid models with a linear equation of state. The ‘Bianchi type IX attractor theorem’ states that the past asymptotic behavior of generic type IX solutions is governed by Bianchi type I and II vacuum states (Mixmaster attractor). We give a comparatively short and self-contained new proof of this theorem. The proof we give is interesting in itself, but more importantly it illustrates and emphasizes that type IX is special, and to some extent misleading when one considers the broader context of generic models without symmetries.

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1 Introduction

Based on work reviewed and developed in [1] and by Rendall [2], Ringström succeeded, some eight, nine years ago, to produce the first major theorem about asymptotic dynamics of type IX vacuum and orthogonal perfect fluid models [3, 4]: The Bianchi type IX attractor theorem, which states that the past asymptotic behavior of generic type IX solutions is governed by Bianchi type I and II vacuum states (which constitute the Mixmaster attractor); see Theorem 5.1. In this paper we provide a comparatively short and simple new proof of this theorem. Our proof rests on three cornerstones:

- (i) We introduce new bounded variables that yield a relatively compact state space, which eliminates several of the complications associated with the unbounded variables used by Ringström.
- (ii) We make systematic use of the Lie contraction hierarchy of invariant subsets admitted by Bianchi type IX where monotone functions restrict the asymptotic dynamics to boundaries of boundaries.
- (iii) We systematically employ methods, arguments, and results from the theory of dynamical systems, see e.g. [1, 5, 6, 7] and references therein.

However, to find a more succinct argument is not our primary motivation to re-investigate Bianchi type IX. Our proof demonstrates that Bianchi type IX is special in comparison with the other oscillatory Bianchi models, i.e., type $VI_{-1/9}$ and type VIII. This special nature of Bianchi type IX is associated with the geometric condition that the three structure constants have the same sign, which results in an extraordinary simplification of the problem (which is easily overlooked) and thus makes the treatment of Bianchi type IX models relatively straightforward. The fact that Bianchi type IX is rather special has broad ramifications for our understanding of generic spacelike singularities [8, 9]: Bianchi type IX is probably not quite as good a role model for the asymptotic behavior of generic inhomogeneous models as is commonly asserted.

Ringström’s Bianchi type IX attractor theorem is a remarkable theorem. However, it is imperative to point out that this theorem has limited implications, e.g., nothing is rigorously known about dynamical chaotic properties (although there are good grounds for beliefs). All claims about chaos in Einstein’s equations rest on the (plausible) belief that the asymptotic dynamics of Einstein’s equations is reflected by a discrete map (the Kasner map), but this is far from evident and has not been proved so far. These and related issues are discussed in the paper ‘Mixmaster: Fact and Belief’ [10]. The present paper, however, concentrates on rigorous results.

This paper is essentially self-contained. In Section 2 we begin with a brief discussion of the dynamical systems approach, where we establish the connection with the metric approach. We briefly introduce Hubble-normalized variables, but we concentrate on a set of new bounded variables that yield a relatively compact state space—this is the first cornerstone of our analysis. In Section 3 we discuss the levels of the Bianchi type IX Lie contraction hierarchy: Apart from reviewing results on the Bianchi type I and type II subsets, we present a thorough analysis of Bianchi type VII_0 . The proofs we give are novel and, in particular, independent of results on Bianchi type IX—this is the second cornerstone of our analysis. In Section 4 we present the results of the local analysis of the fixed points of the dynamical system and discuss non-generic asymptotically self-similar behavior. Section 5 is the core of this paper: We present a new and succinct proof of the Bianchi type IX attractor theorem (which is stated as Theorem 5.1). The proof is considerably shorter than the proof given by Ringström [4] (which is in turn based on results in [1, 2, 3]). In the proof we systematically employ methods from the theory of dynamical systems—the third cornerstone of our analysis. In Section 6 we state and prove a number of consequences of Theorem 5.1. These results follow relatively easily from Theorem 5.1 when combined with the knowledge of the flow on the Mixmaster attractor (which is the union of the type I and II subsets). We conclude in Section 7 with a discussion of the main themes of this paper and we put Bianchi type IX in a broader context; in particular we emphasize that the present Bianchi type IX models are too special in some respects

Bianchi type	n_α	n_β	n_γ
I	0	0	0
II	0	0	+
VI ₀	0	-	+
VII ₀	0	+	+
VIII	-	+	+
IX	+	+	+

Table 1: The class A Bianchi types are characterized by different signs of the variables $(n_\alpha, n_\beta, n_\gamma)$, where $(\alpha\beta\gamma)$ is any permutation of (123) . In addition to the above representations there exist equivalent representations associated with an overall change of sign of the variables; e.g., another type IX representation is $(- - -)$.

to serve as good role models for generic spacelike singularities. Throughout this paper we use units so that $c = 1$ and $8\pi G = 1$, where c is the speed of light and G the gravitational constant.

2 Basic equations

Consider a vacuum or orthogonal perfect fluid spatially homogeneous (SH) Bianchi type IX model (i.e., the fluid 4-velocity is assumed to be orthogonal to the SH symmetry surfaces). It is well known that there exists a symmetry-adapted (co-)frame $\{\hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3\}$ such that the metric for these models takes the form

$${}^4g = -dt \otimes dt + g_{11}(t) \hat{\omega}^1 \otimes \hat{\omega}^1 + g_{22}(t) \hat{\omega}^2 \otimes \hat{\omega}^2 + g_{33}(t) \hat{\omega}^3 \otimes \hat{\omega}^3, \quad (1a)$$

$$\text{where } d\hat{\omega}^1 = -\hat{n}_1 \hat{\omega}^2 \wedge \hat{\omega}^3, \quad d\hat{\omega}^2 = -\hat{n}_2 \hat{\omega}^3 \wedge \hat{\omega}^1, \quad d\hat{\omega}^3 = -\hat{n}_3 \hat{\omega}^1 \wedge \hat{\omega}^2, \quad (1b)$$

and where $\hat{n}_\alpha = +1 \forall \alpha$; see e.g. [1, 4] and references therein.

Remark. Class A Bianchi models of different Bianchi types are characterized by different structure constants \hat{n}_α ; the different cases are listed in Table 1.

Let

$$n_1(t) := \hat{n}_1 \frac{g_{11}}{\sqrt{\det g}}, \quad n_2(t) := \hat{n}_2 \frac{g_{22}}{\sqrt{\det g}}, \quad n_3(t) := \hat{n}_3 \frac{g_{33}}{\sqrt{\det g}}, \quad (2)$$

where $\det g = g_{11}g_{22}g_{33}$. Furthermore, define

$$\theta = -\text{tr } k \quad \text{and} \quad \sigma_\beta^\alpha = -k_\beta^\alpha + \frac{1}{3} \text{tr } k \delta_\beta^\alpha = \text{diag}(\sigma_1, \sigma_2, \sigma_3) \quad \left(\Rightarrow \sum_\alpha \sigma_\alpha = 0 \right), \quad (3)$$

where $k_{\alpha\beta}$ denotes the second fundamental form associated with (1) of the SH hypersurfaces $t = \text{const}$. The quantities θ and $\sigma_{\alpha\beta}$ can be interpreted as the expansion and the shear, respectively, of the normal congruence of the SH hypersurfaces. The spatial volume density changes according to $d\sqrt{\det g}/dt = \theta\sqrt{\det g}$. In a cosmological context it is customary to replace θ by the Hubble variable $H = \theta/3 = -\text{tr } k/3$. Evidently, for the present Bianchi type IX models there is a one-to-one correspondence between the ‘orthonormal frame variables’ $(H, \sigma_\alpha, n_\alpha)$ (with $\sum_\alpha \sigma_\alpha = 0$) and $(g_{\alpha\beta}, k_{\alpha\beta})$.

Remark. The orthonormal frame variables $(H, \sigma_\alpha, n_\alpha)$ can be used to describe any model of the family of Bianchi class A models; see Table 1. In Bianchi type VIII and IX, the metric is obtained from (n_1, n_2, n_3) by (2); for the lower Bianchi types, the other frame variables, i.e., (H, σ_α) , are needed as well to reconstruct the metric; see [11] for a group theoretical approach.

In the perfect fluid case we assume an orthogonal perfect fluid with density ρ and pressure p that satisfy a linear equation of state $p = w\rho$. We require the energy conditions (weak/strong/dominant)

to hold, hence $\rho > 0$ and¹

$$-\frac{1}{3} < w < 1; \quad (4)$$

we exclude the special cases $w = -\frac{1}{3}$ and $w = 1$, where the energy conditions are only marginally satisfied.²

2.1 Hubble-normalized variables and equations

In the Hubble-normalized dynamical systems approach we define dimensionless orthonormal frame variables according to

$$(\Sigma_\alpha, N_\alpha) = (\sigma_\alpha, n_\alpha)/H, \quad \Omega = \rho/(3H^2). \quad (5)$$

Remark. For all class A models except type IX the Gauss constraint guarantees that H remains positive if it is positive initially. In Bianchi type IX, however, it is known from a theorem by Lin and Wald [14] that all type IX vacuum and orthogonal perfect fluid models with $w \geq 0$ first expand ($H > 0$), reach a point of maximum expansion ($H = 0$), and then recollapse ($H < 0$).³ Therefore, although the variable transformation (5) breaks down at the point of maximum expansion, the variables $(\Sigma_\alpha, N_\alpha)$ correctly describe the dynamics in the expanding phase.

Since the past singularity is of particular interest in our considerations, it is convenient to choose the *time direction towards the past singularity*. We introduce a new dimensionless time variable τ_- according to

$$d\tau_-/dt = -H. \quad (6)$$

The Einstein fields equations can be reformulated in terms of the Hubble variable H and the Hubble-normalized variables [1]. Since H is the only variable that carries dimension, the equation for H ,

$$dH/d\tau_- = (1+q)H, \quad (7)$$

decouples from the equations for Σ_α and N_α , which are given by the system (as follows from [1])

$$d\Sigma_\alpha/d\tau_- = (2-q)\Sigma_\alpha + {}^3S_\alpha, \quad (8a)$$

$$dN_\alpha/d\tau_- = -(q+2\Sigma_\alpha)N_\alpha \quad (\text{no sum over } \alpha), \quad (8b)$$

where

$$q = 2\Sigma^2 + \frac{1}{2}(1+3w)\Omega, \quad \Sigma^2 = \frac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2), \quad (9a)$$

$$\text{and } {}^3S_\alpha = \frac{1}{3}[N_\alpha(2N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2], \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}. \quad (9b)$$

It is straightforward to show that $\rho \propto \exp(3[1+w]\tau_-)$ and $\tau_- \rightarrow \infty$ towards the past singularity, so that $\rho \rightarrow \infty$ in this limit; see [10].

Apart from the trivial constraint $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$,⁴ there exists the Gauss constraint

$$\Sigma^2 + \frac{1}{12}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)] = 1 - \Omega \leq 1. \quad (10)$$

Since this constraint can be used to globally solve for Ω , the reduced state space is given as the space of all $(\Sigma_1, \Sigma_2, \Sigma_3)$ and (N_1, N_2, N_3) satisfying (10) and $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$. The Gauss constraint (10) reveals a serious disadvantage of the Hubble-normalized variables: The range of the variables is unbounded, i.e., the state space is not relatively compact.

¹Note, however, that the well-posedness of the Einstein equations (for solutions without symmetry) has been questioned in the case $-1/3 < w < 0$, see [12].

²The case $w = 1$ is known as the stiff fluid case, for which the speed of sound is equal to the speed of light. The asymptotic dynamics of stiff fluid solutions is simpler than the oscillatory behavior characterizing the models with range $-\frac{1}{3} < w < 1$, and well understood [4, 13]. (In the terminology introduced below, the stiff fluid models are asymptotically self-similar.) We will therefore refrain from discussing the stiff fluid case in this paper.

³In the locally rotationally symmetric case it has been proved that the range of w can be extended to $w > -\frac{1}{3}$, see [15]. There are good reasons to believe that the assumption of local rotational symmetry is superfluous, but this has not been established yet.

⁴It is common to globally solve $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ by introducing new variables according to $\Sigma_1 = -2\Sigma_+$, $\Sigma_2 = \Sigma_+ - \sqrt{3}\Sigma_-$, $\Sigma_3 = \Sigma_+ + \sqrt{3}\Sigma_-$, which yields $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2$. However, since this breaks the permutation symmetry of the three spatial axes (exhibited by type IX models), we choose to retain the variables $\Sigma_1, \Sigma_2, \Sigma_3$.

2.2 Bounded variables

The preparatory step in our approach to Ringström's Bianchi type IX attractor theorem is to reformulate the Einstein equations for class A Bianchi models in terms of variables that span a relatively compact state space. (The aim is to avoid the problems that are caused by the unboundedness of the Hubble-normalized variables in the dynamical system (8); note, however, that the key arguments in our proof, e.g., Lemma 5.3, are independent of the choice of variables.) Define

$$D := \sqrt{H^2 + \frac{1}{6}(n_1 n_2 + n_1 n_3 + n_2 n_3)}. \quad (11)$$

The Gauss constraint reads $3D^2 = \rho + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{4}(n_1^2 + n_2^2 + n_3^2)$, hence $D > 0$. This makes it possible to introduce variables that are normalized w.r.t. the 'dominant' variable D instead of H , i.e.,

$$(\bar{\Sigma}_\alpha, \bar{N}_\alpha, \bar{H}) = (\sigma_\alpha, n_\alpha, H)/D, \quad \bar{\Omega} = \rho/(3D^2). \quad (12)$$

By construction, $\bar{\Sigma}_1 + \bar{\Sigma}_2 + \bar{\Sigma}_3 = 0$.

Remark. Contrary to the Hubble-normalized variables, the new variables (12) are globally well-defined.

We choose a dimensionless time variable $\bar{\tau}_-$, which is defined by

$$\frac{d\bar{\tau}_-}{dt} = -D; \quad (13)$$

accordingly, $\bar{\tau}_-$ is directed towards the past singularity.

The quantity D decouples from the other equations for dimensional reasons,

$$\frac{dD}{d\bar{\tau}_-} = [(1 + \bar{q})\bar{H} + \bar{F}] D. \quad (14)$$

The remaining dimensionless system of equations reads

$$\frac{d\bar{H}}{d\bar{\tau}_-} = \bar{q}(1 - \bar{H}^2) - \bar{F}\bar{H}, \quad (15a)$$

$$\frac{d\bar{\Sigma}_\alpha}{d\bar{\tau}_-} = \bar{\Sigma}_\alpha [(2 - \bar{q})\bar{H} - \bar{F}] + {}^3\bar{S}_\alpha, \quad (15b)$$

$$\frac{d\bar{N}_\alpha}{d\bar{\tau}_-} = -\bar{N}_\alpha [\bar{q}\bar{H} + 2\bar{\Sigma}_\alpha + \bar{F}], \quad (\text{no sum over } \alpha) \quad (15c)$$

where

$$\bar{q} = 2\bar{\Sigma}^2 + \frac{1}{2}(1 + 3w)\bar{\Omega}, \quad \bar{\Sigma}^2 = \frac{1}{6}(\bar{\Sigma}_1^2 + \bar{\Sigma}_2^2 + \bar{\Sigma}_3^2), \quad (16a)$$

$$\bar{F} = \frac{1}{6}(\bar{N}_1\bar{N}_2\bar{\Sigma}_3 + \bar{N}_1\bar{\Sigma}_2\bar{N}_3 + \bar{\Sigma}_1\bar{N}_2\bar{N}_3), \quad (16b)$$

$${}^3\bar{S}_\alpha = \frac{1}{3}[\bar{N}_\alpha(2\bar{N}_\alpha - \bar{N}_\beta - \bar{N}_\gamma) - (\bar{N}_\beta - \bar{N}_\gamma)^2], \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}. \quad (16c)$$

The vacuum case is characterized by $\bar{\Omega} = 0$ while $\bar{\Omega} > 0$ in the fluid case. Note that the 'deceleration parameter' \bar{q} is non-negative.

In contrast to the system for the Hubble-normalized variables, there exist two non-trivial constraints, the Gauss constraint and a constraint resulting directly from (11),

$$\bar{\Sigma}^2 + \frac{1}{12}(\bar{N}_1^2 + \bar{N}_2^2 + \bar{N}_3^2) + \bar{\Omega} = 1, \quad (17a)$$

$$\bar{H}^2 + \frac{1}{6}(\bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3) = 1. \quad (17b)$$

Evidently, in Bianchi type IX, the range of the new variables (12) is *bounded*. This is true for the entire class A: We minimize the expression $\bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3$ appearing in (17b) under the condition $\bar{N}_1^2 + \bar{N}_2^2 + \bar{N}_3^2 \leq 12$ resulting from (17a); this leads to $\bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3 \geq -6$ and

hence $\bar{H}^2 \leq 2$ for all class A models; restriction to Bianchi type IX yields $\bar{H}^2 < 1$, since $\bar{N}_\alpha > 0 \forall \alpha$. Therefore, the system (15) is a system on a relatively compact state space for the entire class A.

The dimensionless state space of the Bianchi type IX orthogonal perfect fluid models with a linear equation of state is 5-dimensional while the state space of the vacuum models is 4-dimensional. The same is true for Bianchi type VIII, while the state spaces of the remaining class A Bianchi models have less degrees of freedom; see Table 2. Once the dynamics in the dimensionless state space is understood, D is obtained from a quadrature by integrating (14), and subsequently the metric can be reconstructed. Note that we choose to use (17a) to solve for $\bar{\Omega}$ when $\bar{\Omega} > 0$; however, it is still useful to consider the evolution equation

$$d\bar{\Omega}/d\bar{\tau}_- = -\bar{\Omega}[2\bar{q}\bar{H} - (1 + 3w)\bar{H} + 2\bar{F}]. \quad (18)$$

Remark. Since \bar{q} is bounded as $\bar{\tau}_- \rightarrow \infty$, the asymptotics of \bar{H} can be bounded by exponential functions from above and below. It follows that the equation $dt/d\bar{\tau}_- = -D^{-1}$ can be integrated to yield t as a function of $\bar{\tau}_-$ such that $t \rightarrow 0$ as $\bar{\tau}_- \rightarrow +\infty$.

In all Bianchi types except type IX the constraints force \bar{H} to have a sign for all $\bar{\tau}_-$, e.g., $\bar{H} > 0$ (which entails that these models are forever expanding⁵). In Bianchi type IX the subset $\bar{H} > 0$ is a future invariant subset of the state space, which follows from the inequality $d\bar{H}/d\bar{\tau}_- \geq 0$ on $\bar{H} = 0$. (There is a close relationship between this fact and the results of [14].)

Let $(\alpha\beta\gamma)$ denote any permutation of (123). We define the Bianchi type IX state space \mathbf{B}_{IX} as the future-invariant set

$$\mathbf{B}_{\text{IX}} = \{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0, \bar{H} > 0, \bar{\Sigma}_\alpha, \bar{\Sigma}_\beta, \bar{\Sigma}_\gamma, \bar{\Omega} \geq 0\}, \quad (19)$$

where the variables are subject to the constraints (17) and $\bar{\Sigma}_1 + \bar{\Sigma}_2 + \bar{\Sigma}_3 = 0$; the set \mathbf{B}_{IX} comprises both the vacuum subset and the fluid subset, i.e., $\bar{\Omega} \geq 0$.

Setting one or more of these variables $(\bar{N}_\alpha, \bar{N}_\beta, \bar{N}_\gamma)$ to zero (which corresponds to Lie contractions [16]) yields invariant boundary subsets which represent more special Bianchi types. In this spirit, the boundary⁶ $\partial\mathbf{B}_{\text{IX}}$ of the Bianchi type IX state space is given by

$$\partial\mathbf{B}_{\text{IX}} = \bar{\mathbf{H}}_0 \cup \bar{\mathbf{B}}_{\text{VII}_0}, \quad (20)$$

where

$$\mathbf{H}_0 = \{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0, \bar{H} = 0, \bar{\Omega} \geq 0\}, \quad (21a)$$

$$\mathbf{B}_{\text{VII}_0} = \{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma = 0, \bar{H} > 0, \bar{\Omega} \geq 0\}, \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}. \quad (21b)$$

Note that $\mathbf{B}_{\text{VII}_0}$ denotes the collection of the three equivalent Bianchi type VII₀ subspaces; if we want to refer to one of the subsets in particular we use the notation $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}$ for the set $\{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma = 0, \bar{H} > 0\}$. The notation is such that the subscript denotes the non-zero variables among $\{\bar{N}_\alpha, \bar{N}_\beta, \bar{N}_\gamma\}$; accordingly, $\mathbf{B}_{\text{VII}_0} = \mathcal{B}_{\bar{N}_1 \bar{N}_2} \cup \mathcal{B}_{\bar{N}_2 \bar{N}_3} \cup \mathcal{B}_{\bar{N}_3 \bar{N}_1}$.

The constraints (17) imply that the boundary of the set \mathbf{H}_0 consists of the three points

$$\mathbf{Z}_\alpha : \bar{H} = 0, \bar{\Sigma}_\alpha = 0, \bar{N}_\alpha = 0, \bar{N}_\beta = \bar{N}_\gamma = \sqrt{6}, \bar{\Omega} = 0, \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}; \quad (22)$$

the boundary of $\mathbf{B}_{\text{VII}_0}$ is the union of the closures of the Bianchi type II subspaces, which are given by $\mathcal{B}_{\bar{N}_\alpha} = \{\bar{N}_\alpha > 0, \bar{N}_\beta = 0, \bar{N}_\gamma = 0, \bar{\Omega} \geq 0\}$, $\alpha = 1, 2, 3$, collectively denoted by \mathbf{B}_{II} , and the three points \mathbf{Z}_α , i.e.,

$$\partial\mathbf{B}_{\text{VII}_0} = \bar{\mathbf{B}}_{\text{II}} \cup \{\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3\}. \quad (23)$$

For completeness, we note that \mathbf{B}_{I} is the set $\{\bar{N}_\alpha = 0, \bar{N}_\beta = 0, \bar{N}_\gamma = 0, \bar{\Omega} \geq 0\}$; we have $\partial\mathbf{B}_{\text{II}} = \mathbf{B}_{\text{I}}$. A Bianchi subset contraction diagram for type IX is given in Figure 1.

⁵This excludes the Bianchi type I and type VII₀ representations of Minkowski spacetime.

⁶In our context, $\partial\mathbf{B}_*$ denotes $\bar{\mathbf{B}}_* \setminus \mathbf{B}_*$; the word ‘boundary’ is chosen in lack of better terminology. Accordingly, since $\mathbf{B}_{\text{IX}} = \{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0, \bar{H} > 0, \bar{\Omega} \geq 0\}$, $\partial\mathbf{B}_{\text{IX}}$ is the set where one of the variables $(\bar{N}_\alpha, \bar{N}_\beta, \bar{N}_\gamma)$ or \bar{H} is set to zero, while the vacuum subset $\{\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0, \bar{H} > 0, \bar{\Omega} = 0\}$ does not appear in $\partial\mathbf{B}_{\text{IX}}$. This convention is adapted to the formulation of Lemma 5.2.

Type	Symbol	Range of $(N_\alpha, N_\beta, N_\gamma)$	State space properties	D
I	\mathbf{B}_I	$\bar{N}_\alpha = 0, \bar{N}_\beta = 0, \bar{N}_\gamma = 0$	$\bar{H} \equiv 1, \bar{\Sigma}^2 \leq 1$	2
II	\mathbf{B}_{II}	$\bar{N}_\alpha = 0, \bar{N}_\beta = 0, \bar{N}_\gamma > 0$	$\bar{H} \equiv 1, \bar{\Sigma}^2 + \frac{1}{12}\bar{N}_\gamma^2 \leq 1$	3
VI ₀	\mathbf{B}_{VI_0}	$\bar{N}_\alpha = 0, \bar{N}_\beta < 0, \bar{N}_\gamma > 0$	$\bar{H} \in (0, 2), \bar{\Sigma}^2 + \frac{1}{12}[\bar{N}_\beta - \bar{N}_\gamma]^2 \leq \bar{H}^2$	4
VII ₀	\mathbf{B}_{VII_0}	$\bar{N}_\alpha = 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0$	$\bar{H} \in (0, 1), \bar{\Sigma}^2 + \frac{1}{12}[\bar{N}_\beta - \bar{N}_\gamma]^2 \leq \bar{H}^2$	4
VIII	\mathbf{B}_{VIII}	$\bar{N}_\alpha < 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0$	$\bar{H} \in (0, 2), \bar{\Sigma}^2 < \bar{H}^2$	5
IX	\mathbf{B}_{IX}^*	$\bar{N}_\alpha > 0, \bar{N}_\beta > 0, \bar{N}_\gamma > 0$	$\bar{H} \in (-1, 1), \bar{\Sigma}^2 \not\leq \bar{H}^2$	5

Table 2: The dimensionless state spaces associated with class A Bianchi models; here, $(\alpha\beta\gamma)$ is any permutation of (123). In addition to the above representations there exist equivalent representations associated with an overall change of sign of the variables $(\bar{N}_1, \bar{N}_2, \bar{N}_3)$ or \bar{H} . The quantity D denotes the dimension of the state space (in the fluid case); the dimensionality of the state space in the vacuum cases is given by $D - 1$. The Bianchi type IX state space decomposes into a future-invariant half and a past-invariant half. By \mathbf{B}_{IX} we denote the future-invariant half, i.e., $\bar{H} > 0$; cf. (19).

Remark. The constraints imply that $\bar{H} = 1$ on $\mathbf{B}_I \cup \mathbf{B}_{II}$; hence the dominant variables reduce to the standard Hubble-normalized variables for type I and II.

Remark. Each of the sets $\mathbf{B}_I, \mathbf{B}_{II}, \mathbf{B}_{VII_0}, \mathbf{B}_{IX}$ (and \mathbf{H}_0) decomposes into a vacuum subset ($\bar{\Omega} = 0$) and a fluid subset ($\bar{\Omega} > 0$). We refer to these subsets using the superscripts ^{vac.} and ^{fl.}.

Remark. Note that there does not appear a subset \mathbf{B}_{VI_0} as a boundary subset of \mathbf{B}_{IX} . In the concluding remarks, Section 7, we argue that this simple fact has far-reaching consequences.

3 The boundaries of the state space: $\mathbf{B}_I, \mathbf{B}_{II}, \mathbf{B}_{VII_0}$

In the analysis of a dynamical system that is defined on a (relatively) compact state space, the analysis of the boundary subsets is crucial. Often there exists a hierarchy of boundary subsets that governs the asymptotic behavior of solutions, see e.g. [17, 18] for examples how such hierarchies can be exploited. In the case of Bianchi type IX models, the ‘Lie contraction hierarchy’ of the boundaries is depicted in Figure 1.

3.1 The Bianchi type I subset

The Bianchi type I subset \mathbf{B}_I is given by $\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$, hence $\bar{H} = 1$ and $\bar{\Omega} = 1 - \bar{\Sigma}^2 \geq 0$. The *vacuum subset*, $\bar{\Omega} = 0$, consists of a circle of fixed points—the *Kasner circle* K° , which is characterized by $\bar{\Sigma}^2 = 1$. Each fixed point on K° represents a Kasner solution (Kasner metric). There exist six special points: Q_α are given by $(\bar{\Sigma}_\alpha, \bar{\Sigma}_\beta, \bar{\Sigma}_\gamma) = (-2, 1, 1)$; the Taub points T_α are given by $(\bar{\Sigma}_\alpha, \bar{\Sigma}_\beta, \bar{\Sigma}_\gamma) = (2, -1, -1)$. The former are associated with locally rotationally symmetric (LRS) solutions whose intrinsic geometry is non-flat; the latter correspond to the flat LRS solutions—the Taub representation of Minkowski spacetime.

The Bianchi type I perfect fluid subset is the set $1 - \bar{\Omega} = \bar{\Sigma}^2 < 1$. From (18) it is straightforward to deduce that there exists a central fixed point, the *Friedmann fixed point* F, given by $\bar{\Sigma}_1 = \bar{\Sigma}_2 = \bar{\Sigma}_3 = 0$, which corresponds to the isotropic Friedmann-Robertson-Walker (FRW) solution. Solutions with $0 < \bar{\Sigma}^2 < 1$ are given by radial straight lines originating from F and ending at K° . These results rely on the assumption $w < 1$, see (4).

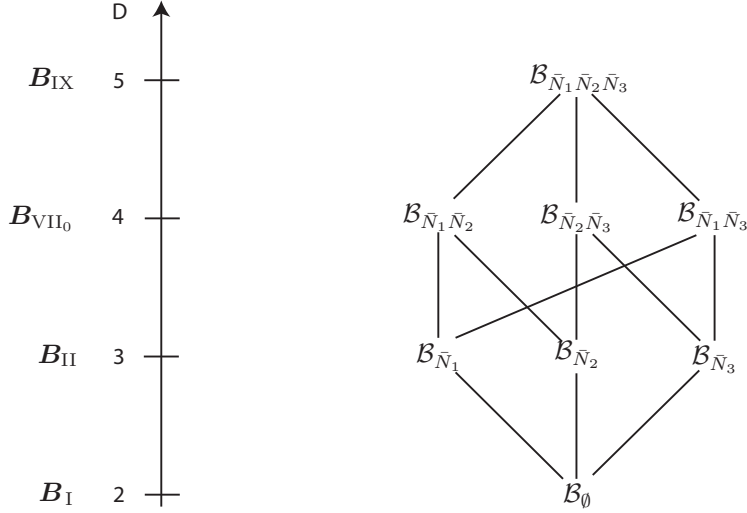


Figure 1: Subset contraction diagram for Bianchi type IX. D denotes the dimension of the dimensionless state space for the various models with an orthogonal perfect fluid with linear equation of state; the associated vacuum subsets have one dimension less; see also Table 2. The notation is such that the subscript of \mathcal{B}_* denotes the non-zero variables, e.g. $\mathcal{B}_{\bar{N}_1 \bar{N}_2}$ denotes the type VII₀ subset with $\bar{N}_1 > 0$, $\bar{N}_2 > 0$ and $\bar{N}_3 = 0$.

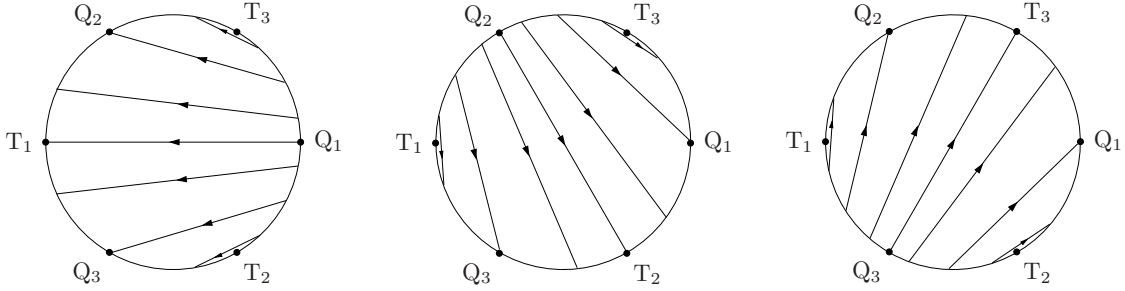


Figure 2: Projection of the flow on the type II subsets $\mathcal{B}_{\bar{N}_1}^{\text{vac}}$, $\mathcal{B}_{\bar{N}_2}^{\text{vac}}$, $\mathcal{B}_{\bar{N}_3}^{\text{vac}}$ onto $(\bar{\Sigma}_1, \bar{\Sigma}_2, \bar{\Sigma}_3)$ -space.

3.2 The Bianchi type II subset

The Bianchi type II subset \mathcal{B}_{II} has three equivalent representations: $\mathcal{B}_{\bar{N}_1}$, $\mathcal{B}_{\bar{N}_2}$, $\mathcal{B}_{\bar{N}_3}$. Let us consider $\mathcal{B}_{\bar{N}_\gamma}$, which is given by $\bar{N}_\alpha = \bar{N}_\beta = 0$, $\bar{N}_\gamma > 0$ (hence $\bar{H} = 1$). On $\mathcal{B}_{\bar{N}_\gamma}$, the Gauss constraint $\bar{\Sigma}^2 + \frac{1}{12}\bar{N}_\gamma^2 + \bar{\Omega} = 1$ can be used to replace \bar{N}_γ by $\bar{\Omega}$ as a dependent variable. The system (8) thus becomes

$$\frac{d\bar{\Sigma}_{\alpha/\beta}}{d\bar{\tau}_-} = (2-q)\bar{\Sigma}_{\alpha/\beta} + {}^3S_{\alpha/\beta}, \quad \frac{d\bar{\Sigma}_\gamma}{d\bar{\tau}_-} = (2-q)\bar{\Sigma}_\gamma + {}^3S_\gamma, \quad \frac{d\bar{\Omega}}{d\bar{\tau}_-} = -\bar{\Omega}[2q - (1+3w)], \quad (24)$$

where $q = 2\bar{\Sigma}^2 + \frac{1}{2}(1+3w)\bar{\Omega}$ and ${}^3S_{\alpha/\beta} = -4(1-\bar{\Sigma}^2-\bar{\Omega})$, ${}^3S_\gamma = 8(1-\bar{\Sigma}^2-\bar{\Omega})$; we have $\bar{\Sigma}^2 + \bar{\Omega} < 1$.

Let us first consider the vacuum subset $\mathcal{B}_{\bar{N}_\gamma}^{\text{vac}}$, i.e., $\bar{\Omega} = 0$. There do not exist any fixed points on $\mathcal{B}_{\bar{N}_\gamma}^{\text{vac}}$, but the boundary coincides with the Kasner circle K° . The orbits of (24) form a family of straight lines in $\mathcal{B}_{\bar{N}_\gamma}^{\text{vac}}$; each orbit is a heteroclinic orbit, since it connects two different fixed points. If the initial point is Q_γ , the final point is T_γ (LRS orbit); the points T_α and T_β are not connected with any other fixed point (they are ‘fixed points’ under the present ‘vacuum Bianchi type II map’), see Figure 2.

While there do not exist any fixed points in $\mathcal{B}_{\bar{N}_\gamma}^{\text{vac}}$, there exists one fixed point in $\mathcal{B}_{\bar{N}_\gamma}^{\text{fl}}$ with $\bar{\Omega} > 0$, the *Collins-Stewart fixed point* CS_γ , which corresponds to one representation of the LRS solutions found by Collins and Stewart [19]. CS_γ is given by $(\bar{\Sigma}_\alpha, \bar{\Sigma}_\beta, \bar{\Sigma}_\gamma) = \frac{1}{8}(1+3w)(1, 1, -2)$ and $\bar{\Omega} =$

$1 - \frac{1}{16}(1 + 3w)$ (which yields $\bar{N}_\gamma = \frac{3}{4}\sqrt{1-w}\sqrt{1+3w}$). The fixed point CS_γ is the source for all orbits in $\mathcal{B}_{\bar{N}_\gamma}^{\text{fl}}$. For a detailed discussion of these results see [1].

3.3 Bianchi type VII₀: A new analysis

A detailed analysis of the dynamics of Bianchi type VII₀ models is essential for an understanding of type IX asymptotic dynamics. To underline the importance of Bianchi type VII₀, we present a new analysis of the global dynamics of type VII₀ solutions; the proofs we give are novel and, in particular, independent of results on Bianchi type IX. (The proof given in [4, 20] relies on results on Bianchi type IX.)

The dynamical system on $\mathcal{B}_{\text{VII}_0}$ ($= \mathcal{B}_{\bar{N}_1\bar{N}_2} \cup \mathcal{B}_{\bar{N}_2\bar{N}_3} \cup \mathcal{B}_{\bar{N}_3\bar{N}_1}$) does not admit fixed points with $\bar{\Omega} > 0$ (while, of course, there exist fixed points on $\partial\mathcal{B}_{\text{VII}_0}$). However, each of the three vacuum subsets $\mathcal{B}_{\bar{N}_\alpha\bar{N}_\beta}^{\text{vac}}$ contains a line of fixed points with $\bar{N}_\alpha = \bar{N}_\beta > 0$:

$$\text{TL}_\gamma = \left\{ 0 < \bar{H} < 1, \bar{\Sigma}_\alpha = \bar{\Sigma}_\beta = -\bar{H}, \bar{\Sigma}_\gamma = 2\bar{H}, \bar{N}_\alpha = \bar{N}_\beta = \sqrt{6}\sqrt{1-\bar{H}^2}, \bar{N}_\gamma = 0, \bar{\Omega} = 0 \right\}. \quad (25a)$$

These lines of fixed points (‘Taub lines’) connect the Taub points T_γ on the Kasner circle with the fixed points Z_γ . Each of the fixed points on TL_γ represents the Minkowski spacetime in a Bianchi type VII₀ LRS symmetry foliation; this is in analogy to the Taub points T_γ themselves. (Note that $\bar{\Sigma}_\alpha = \bar{\Sigma}_\beta = -\bar{H}, \bar{\Sigma}_\gamma = 2\bar{H}$ corresponds to $\Sigma_\alpha = \Sigma_\beta = -1, \Sigma_\gamma = 2$ in Hubble-normalized variables. $\bar{N}_\alpha = \bar{N}_\beta \in (0, \sqrt{6})$ corresponds to $N_\alpha = N_\beta \in (0, \infty)$.) The second family of LRS vacuum subsets are the three sets

$$\mathcal{QL}_\gamma = \left\{ 0 < \bar{H} < 1, \bar{\Sigma}_\alpha = \bar{\Sigma}_\beta = \bar{H}, \bar{\Sigma}_\gamma = -2\bar{H}, \bar{N}_\alpha = \bar{N}_\beta = \sqrt{6}\sqrt{1-\bar{H}^2}, \bar{N}_\gamma = 0, \bar{\Omega} = 0 \right\}, \quad (25b)$$

which connect the exceptional points Z_γ with the non-flat LRS fixed points Q_γ . These lines are not lines of fixed points, since $d\bar{H}/d\bar{\tau}_- = 4\bar{H}^2(1-\bar{H}^2)$, but heteroclinic orbits $\text{Z}_\gamma \rightarrow \text{Q}_\gamma$. The corresponding solutions represent the non-flat LRS Kasner solutions in a Bianchi type VII₀ symmetry foliation.

The vacuum case $\bar{\Omega} = 0$. On the representative $\mathcal{B}_{\bar{N}_\alpha\bar{N}_\beta}^{\text{vac}}$ of $\mathcal{B}_{\text{VII}_0}^{\text{vac}}$, consider the function $(2\bar{H} - \bar{\Sigma}_\gamma)$. This function is non-negative, since $\bar{\Sigma}_\gamma \in [-2\bar{H}, 2\bar{H}]$, which follows from the constraints when we use the relation $12\bar{\Sigma}^2 = 3\bar{\Sigma}_\gamma^2 + (\bar{\Sigma}_\alpha - \bar{\Sigma}_\beta)^2$; furthermore, $(2\bar{H} - \bar{\Sigma}_\gamma)$ is increasing on $\bar{\mathcal{B}}_{\bar{N}_\alpha\bar{N}_\beta}^{\text{vac}} \setminus (\overline{\text{TL}_\gamma} \cup \text{K}^\circ)$, since

$$\frac{d}{d\bar{\tau}_-} (2\bar{H} - \bar{\Sigma}_\gamma) = \frac{1}{24} (2\bar{H} - \bar{\Sigma}_\gamma) [(\bar{N}_\alpha + \bar{N}_\beta)^2 (2\bar{H} - \bar{\Sigma}_\gamma) + (\bar{N}_\alpha - \bar{N}_\beta)^2 (2\bar{H} + \bar{\Sigma}_\gamma)]. \quad (26)$$

The monotonicity principle⁷ implies that the α -limit is a fixed point on $\overline{\text{TL}_\gamma} = \{\text{T}_\gamma\} \cup \text{TL}_\gamma \cup \{\text{Z}_\gamma\}$, since the fixed points on the Kasner circle are excluded as possible α -limit points because they are (transversally hyperbolic) saddles and sinks.

Remark. In addition, the monotonicity principle implies that the ω -limit of every orbit in $\mathcal{B}_{\bar{N}_\alpha\bar{N}_\beta}^{\text{vac}}$ (except for the fixed points on TL_γ themselves) is a fixed point on the Kasner circle (where T_γ is excluded); clearly, only the transversally hyperbolic sinks come into question.

The results concerning the α -limits of orbits that are obtained from the monotonicity principle can be strengthened considerably:

Lemma 3.1. *The only type VII₀ vacuum orbit that converges to Z_γ as $\bar{\tau}_- \rightarrow -\infty$ is the orbit \mathcal{QL}_γ ; any other orbit converges to one of the points on TL_γ as $\bar{\tau}_- \rightarrow -\infty$; conversely, each point on TL_γ is the α -limit set for a one-parameter set of orbits.*

⁷The monotonicity principle [1] gives information about the global asymptotic behavior of solutions of a dynamical system. If M is a C^1 function on the state space X that is strictly decreasing along orbits, then

$$\omega(x) \subseteq \{\xi \in \bar{X} \setminus X \mid \lim_{\zeta \rightarrow \xi} M(\zeta) \neq \sup_X M\}, \quad \alpha(x) \subseteq \{\xi \in \bar{X} \setminus X \mid \lim_{\zeta \rightarrow \xi} M(\zeta) \neq \inf_X M\}$$

for all $x \in X$.

The proof of these statements requires a more detailed analysis of the dynamical system (in the neighborhood of TL_γ , in particular). The reader who is not interested in these details might prefer to continue with the discussion of the Bianchi type VII₀ fluid case below; note, however, that the proof we give here is independent of the proof given in [4, 20] (and, in particular, it is completely independent of results on Bianchi type IX).

Proof. Let us introduce a set of variables that are adapted to the special features of the Bianchi type VII₀ state space, using the Hubble-normalized formulation as a starting point.⁸ Consider $\mathcal{B}_{N_\alpha N_\beta}^{\text{vac.}}$ and let

$$\Sigma_\alpha - \Sigma_\beta = 2\sqrt{3} \sin \vartheta \cos 2\psi, \quad \Sigma_\gamma = 2 \cos \vartheta, \quad N_\alpha - N_\beta = \sqrt{12} \sin \vartheta \sin 2\psi, \quad N_\alpha N_\beta = 3\zeta^2, \quad (27)$$

where $(\vartheta, \psi, \zeta) \in [0, \pi] \times [0, \pi] \times (0, \infty)$. The condition $\vartheta = \pi$ yields \mathcal{QL}_γ , while $\vartheta = 0$ yields TL_γ . The constraint is automatically satisfied by this choice of variables; the dynamical system takes the form

$$\frac{d\vartheta}{d\tau_-} = (1 - \cos \psi)(1 - \cos \vartheta) \sin \vartheta, \quad (28a)$$

$$\frac{d\psi}{d\tau_-} = -4 \sin \vartheta \sqrt{3 \left(\frac{\zeta^2}{\sin^2 \vartheta} + \frac{1 - \cos \psi}{2} \right)} - 2 \sin \psi (1 - \cos \vartheta), \quad (28b)$$

$$\frac{d\zeta}{d\tau_-} = -\zeta [2(1 - \cos \vartheta) - \sin^2 \vartheta (1 - \cos \psi)]. \quad (28c)$$

Consider a non-LRS orbit in $\mathcal{B}_{N_\alpha N_\beta}^{\text{vac.}}$, i.e., assume $0 < \vartheta < \pi$. The function $\zeta / \sin \vartheta$ appearing in (28b) is a monotone function,

$$\frac{d}{d\tau_-} \frac{\zeta}{\sin \vartheta} = -\frac{\zeta}{\sin \vartheta} (1 + \cos \psi)(1 - \cos \vartheta); \quad (29)$$

in particular, for sufficiently small τ_- , $\zeta / \sin \vartheta$ is bounded from below by a positive constant. The function ϑ is monotonically decreasing in the reversed direction of time; most importantly, $\vartheta \rightarrow 0$ as $\tau_- \rightarrow -\infty$. (Proof: Assume the contrary, i.e., $\vartheta \rightarrow \vartheta_\infty = \text{const} > 0$ as $\tau_- \rightarrow -\infty$.⁹ The expression

$$\frac{d}{d\tau_-} \left(\log \frac{\zeta}{\sin \vartheta} - \log \tan \frac{\vartheta}{2} \right) = -2(1 - \cos \vartheta) \quad (30)$$

converges to the limit $c = -2(1 - \cos \vartheta_\infty)$ as $\tau_- \rightarrow -\infty$. We thus obtain the asymptotic behavior $\zeta \sim e^{-c\tau_-}$ as $\tau_- \rightarrow -\infty$; accordingly, from (28b), $d\psi/d\tau_- \sim -e^{-c\tau_-}$ and hence $\psi \sim e^{-c\tau_-}$ as $\tau_- \rightarrow -\infty$. Therefore, the integral of $\cos \psi$ behaves asymptotically like the cosine integral $\text{Ci}(e^{-c\tau_-})$; in particular, the limit exists as $\tau_- \rightarrow -\infty$. This, however, contradicts the assumption $\vartheta \rightarrow \vartheta_\infty > 0$ because of (28a).) Likewise, for sufficiently small τ_- , the function ψ is strictly monotonically decreasing, i.e., $d\psi/d\tau_- < 0$. This is a consequence of the property $\vartheta \rightarrow 0$ as $\tau_- \rightarrow -\infty$ (and $\zeta / \sin \vartheta$ being bounded away from zero).

In the limit $\vartheta \rightarrow 0$, the system (28) takes the form

$$\frac{d\vartheta}{d\tau_-} = \frac{1}{2}(1 - \cos \psi)\vartheta^3 (1 + O(\vartheta^2)), \quad \frac{d\zeta}{d\tau_-} = -\zeta [\cos \psi \vartheta^2 + O(\vartheta^4)], \quad (31a)$$

$$\frac{d\psi}{d\tau_-} = -4\vartheta \sqrt{3 \left(\frac{\zeta^2}{\sin^2 \vartheta} + \frac{1 - \cos \psi}{2} \right)} + O(\vartheta^2). \quad (31b)$$

We introduce an alternative time variable, σ_- , by requiring $d\sigma_- = \vartheta^2 d\tau_-$. Evidently, σ_- is a monotone function of τ_- ; most importantly, $\sigma_- \rightarrow -\infty$ as $\tau_- \rightarrow -\infty$. (Proof: Assume the

⁸It follows from Section 2 that $\Sigma_\delta = \bar{\Sigma}_\delta / \bar{H}$ and $N_\delta = \bar{N}_\delta / \bar{H}$, $\delta = 1, 2, 3$; in particular, $\bar{\Sigma}_\gamma = 2\bar{H} \Leftrightarrow \Sigma_\gamma = 2$; the time variables satisfy $d\tau_- = \bar{H} d\bar{\tau}_-$.

⁹This assumption does not suffice to conclude that $d\vartheta/d\tau_- \rightarrow 0$ as $\tau_- \rightarrow -\infty$, since the second derivatives are in general not bounded, nevertheless, the heuristic reasoning is correct: $d\vartheta/d\tau_-$ approaches zero as $\tau_- \rightarrow -\infty$ and thus ψ approaches a multiple of 2π in this limit, cf. (28a), which is a contradiction to (28b).

contrary, i.e., $\sigma_- \rightarrow \sigma_\infty > -\infty$ as $\tau_- \rightarrow -\infty$. Expressed in σ_- , Eq. (31a) reads $d \log \vartheta / d\sigma_- = (1 - \cos \psi)(1 + O(\vartheta^2))$. However, this contradicts the fact that $\vartheta \rightarrow 0$ as $\sigma_- \rightarrow \sigma_\infty$. Using the variable σ_- (which can be identified as a logarithmic time variable), the system (31) becomes

$$\frac{d \log \vartheta}{d\sigma_-} = \frac{1}{2}(1 - \cos \psi)(1 + O(\vartheta^2)), \quad \frac{d \log \zeta}{d\sigma_-} = -\cos \psi + O(\vartheta^2), \quad (32a)$$

$$\frac{d\psi}{d\sigma_-} = -4\vartheta^{-1} \sqrt{3 \left(\frac{\zeta^2}{\sin^2 \vartheta} + \frac{1 - \cos \psi}{2} \right)} + O(\vartheta); \quad (32b)$$

in addition,

$$\frac{d}{d\sigma_-} \left(\log \frac{\zeta}{\sin \vartheta} \right) = -\frac{1}{2}(1 + \cos \psi)(1 + O(\vartheta^2)). \quad (33)$$

The function $\zeta / \sin \vartheta$ is monotone, cf. (29); furthermore, $\zeta / \sin \vartheta \rightarrow \infty$ as $\sigma_- \rightarrow -\infty$. (Proof: Assume the contrary, i.e., $\zeta / \sin \vartheta \rightarrow a_\infty = \text{const} > 0$ as $\sigma_- \rightarrow -\infty$. Expressing (30) in terms of σ_- and integrating this equation yields $\vartheta \sim e^{\sigma_-}$ as $\sigma_- \rightarrow -\infty$. Consistency with the differential equation for ϑ , cf. (32a), requires the integral of $\cos \psi$ as $\sigma_- \rightarrow -\infty$ to converge to -1 . This, however, contradicts the differential equation for ψ , cf. (32b).)

Finally, we obtain

$$\frac{d\psi}{d\sigma_-} = -4\sqrt{3}\vartheta^{-1} \frac{\zeta}{\sin \vartheta} (1 + o(1)), \quad (34)$$

where the function on the r.h.s. goes to $-\infty$ monotonically as $\sigma_- \rightarrow -\infty$ (since ϑ^{-1} and $\zeta / \sin \vartheta$ are monotone and grow beyond all bounds as $\sigma_- \rightarrow -\infty$). We conclude that the integral

$$\int_{-\infty}^{\sigma_-} d\sigma'_- \cos \psi(\sigma'_-) \quad (35)$$

exists. (A statement like this can be regarded as a continuous version of the Leibniz criterion for alternating series.) Therefore, by integrating the differential equation for ζ it follows that ζ converges to a positive constant as $\sigma_- \rightarrow -\infty$. (The solution of the equation for ϑ implies that the lower order terms cannot contribute.)

We therefore obtain that the Hubble-normalized variables N_α and N_β converge to (one and the same) constant as $\tau_- \rightarrow -\infty$; consequently, the α -limit set of a non-LRS orbit is a point with $\Sigma_\alpha = \Sigma_\beta = -1$, $\Sigma_\gamma = 2$, $N_\alpha = N_\beta > 0$ (and $N_\gamma = 0$). Expressed in terms of the dominant variables, the α -limit set of a non-LRS orbit in $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$ is a point on TL_γ . \square

Remark. Based on this information it is straightforward to consider and analyze the asymptotic system of differential equations, which arises by inserting the asymptotic behavior of solutions into (28). The asymptotic oscillations of solutions can be read off directly.

The fluid case $\bar{\Omega} > 0$. On the representative $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{fl.}}$ of $\mathcal{B}_{\text{VII}_0}^{\text{fl.}}$ we adapt a monotone function found by Uggla, given in [1],

$$\zeta_0 = (2\bar{H} - v\bar{\Sigma}_\gamma)^{-2(1+v)} \bar{\Omega} (\bar{N}_\alpha \bar{N}_\beta)^v \quad \text{with} \quad v = \frac{1}{4}(1 + 3w). \quad (36)$$

Note that $0 < v < 1$ since $-\frac{1}{3} < w < 1$; hence $2\bar{H} - v\bar{\Sigma}_\gamma > 0$ if $\bar{H} > 0$ (since $\bar{\Sigma}_\gamma \in (-2\bar{H}, 2\bar{H})$ because of the constraints) and thus $0 \leq \zeta_0 < \infty$ on $\bar{\mathcal{B}}_{\bar{N}_\alpha \bar{N}_\beta} \setminus \{Z_\gamma\}$. On this subset we find

$$\begin{aligned} \frac{d}{d\bar{\tau}_-} \zeta_0 &= -(2\bar{H} - v\bar{\Sigma}_\gamma)^{-1} \zeta_0 \left[\frac{2}{3}(1 - v^2)(\bar{\Sigma}_\alpha - \bar{\Sigma}_\beta)^2 + 2(2v\bar{H} - \bar{\Sigma}_\gamma)^2 \right], \\ \frac{d^3 \zeta_0}{d\bar{\tau}_-^3} \Big|_{d\zeta_0/d\bar{\tau}_- = 0} &= -(2\bar{H} - v\bar{\Sigma}_\gamma)^{-1} \zeta_0 \left[\frac{4}{3}(1 - v^2)(\bar{N}_\alpha^2 - \bar{N}_\beta^2)^2 + (1 - v)^2 (8v\bar{\Omega} - \frac{2}{3}(\bar{N}_\alpha - \bar{N}_\beta)^2) \right], \end{aligned}$$

hence ζ_0 is monotonically decreasing on $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{fl.}}$.

This allows us to apply the monotonicity principle: Using that $\zeta_0 = 0$ on $\bar{\mathcal{B}}_{\text{II}}$ and on the vacuum subset $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$ we conclude that the α -limit of every orbit in $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{fl.}}$ is one of the fixed points Z_γ .

The asymptotic approach to the fixed points Z_γ provides an example for asymptotic self-similarity breaking, see [21] for details and explicit decay rates. (Note that the points Z_γ do not appear as fixed points in the Hubble-normalized approach; instead, they are associated with ‘infinity’ in the Hubble-normalized state space and thus do not correspond to self-similar solutions).

The results on the ω -limit sets of type VII₀ orbits are not needed in the proof of the Bianchi type IX attractor theorem below. However, for completeness we now analyze the possible ω -limit sets of orbits in $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{fl}}$. The monotonicity principle, applied to the function (36), implies that the ω -limit sets of orbits must be contained in $\bar{\mathcal{B}}_{\bar{N}_\alpha}$ or $\bar{\mathcal{B}}_{\bar{N}_\beta}$ (which are part of $\bar{\mathcal{B}}_{\text{II}}$) or/and in the vacuum subset $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$. However, on $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$ only TL_α is admissible. To prove this assume that there exists an orbit γ that has an ω -limit point P on $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$ and that this point does not lie on TL_γ . Then the orbit through P and the ω -limit set of that orbit must be contained in $\omega(\gamma)$. As proved above, this ω -limit set consists of one single fixed point K_P on the Kasner circle (which acts as a sink, when viewed as a fixed point on the [closure of the] vacuum subset). Since $\bar{\Omega}^{-1}d\bar{\Omega}/d\bar{\tau}_- = -3(1-w)$ on the Kasner circle, K_P is a (transversally hyperbolic) sink also when viewed as a fixed point on $\bar{\mathcal{B}}_{\bar{N}_\alpha \bar{N}_\beta}$. The conclusion that the sink K_P is in $\omega(\gamma)$ contradicts the assumption $\text{P} \in \omega(\gamma)$. This leaves only the fixed points on TL_γ as possible ω -limits on the vacuum subset $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{vac.}}$. In fact, for each fixed point $\text{L}_\gamma \in \text{TL}_\gamma$ there exists exactly one orbit in $\mathcal{B}_{\bar{N}_\alpha \bar{N}_\beta}^{\text{fl}}$ converging to L_γ as $\bar{\tau}_- \rightarrow \infty$. This follows from the center manifold theorem by noting that the fixed point L_γ acts as a center saddle (which is because $\bar{\Omega}^{-1}d\bar{\Omega}/d\bar{\tau}_- = -3(1-w)\bar{H}$ along TL_γ). One easily identifies the orbit that converges to L_γ as being an LRS orbit (the LRS case is exactly solvable). We have thus shown that convergence to TL_γ occurs for a non-generic set of orbits; the ω -limit set of every other orbit lies on the sets $\bar{\mathcal{B}}_{\bar{N}_\alpha}$ and $\bar{\mathcal{B}}_{\bar{N}_\beta}$ (and is thus of type I or II).

The details are as follows: The remaining LRS solutions either converge to Q_γ (a one-parameter family) or converge to F as $\bar{\tau}_- \rightarrow \infty$ (one solution). The local analysis of these fixed points, see Section 4 for the general case, implies that the former are embedded into a two-parameter families of orbits converging to Q_γ , while the latter are embedded into a one-parameter families of orbits converging to F. Furthermore, there exist two (equivalent) orbits converging to each of the fixed points $\text{CS}_\alpha, \text{CS}_\beta$.¹⁰ The generic scenario, however, is convergence to one of the transversally hyperbolic sinks on the Kasner circle. To prove this statement suppose that there exists an orbit γ that possesses an ω -limit point P on $\mathcal{B}_{\bar{N}_\alpha} (\mathcal{B}_{\bar{N}_\beta})$ with $\text{P} \neq \text{CS}_\alpha$ ($\text{P} \neq \text{CS}_\beta$). Then the orbit through P and the α -limit set of that orbit must be contained in $\omega(\gamma)$. As stated in Section 3.2, this α -limit set coincides with CS_α (CS_β) in $\mathcal{B}_{\bar{N}_\alpha} (\mathcal{B}_{\bar{N}_\beta})$. As this point is a hyperbolic saddle in $\bar{\mathcal{B}}_{\bar{N}_\alpha \bar{N}_\beta}$, the orbit converging to CS_α (CS_β) as $\bar{\tau}_- \rightarrow \infty$ must be contained in $\omega(\gamma)$ as well. However, this is a contradiction to the fact that $\omega(\gamma)$ is a subset of $\bar{\mathcal{B}}_{\bar{N}_\alpha} \cup \bar{\mathcal{B}}_{\bar{N}_\beta}$. Analogously, one can prove that $\omega(\gamma)$ and the interior of $\mathcal{B}_{\text{I}}^{\text{fl}}$ are disjoint, which leads to the statement. Summing up, the ω -limit of a generic orbit in $\mathcal{B}_{\text{VII}_0}^{\text{fl}}$ is one of the transversally hyperbolic sinks on the Kasner circle.

4 Non-generic solutions: Asymptotic self-similarity

In the previous section we have identified the fixed points associated with the system (15) on the Bianchi boundary subsets of $\bar{\mathcal{B}}_{\text{IX}}$. A local dynamical systems analysis of the fixed points shows whether or not these points attract type IX orbits in the limit $\bar{\tau}_- \rightarrow \infty$.

K° Evaluated on the Kasner circle, Eq. (15) implies $\bar{N}_\alpha^{-1}d\bar{N}_\alpha/d\bar{\tau}_-|_{\text{K}^\circ} = -2(1 + \bar{\Sigma}_\alpha)$ ($\alpha = 1, 2, 3$) and $\bar{\Omega}^{-1}d\bar{\Omega}/d\bar{\tau}_-|_{\text{K}^\circ} = -3(1-w)$. Each fixed point K on $\text{K}^\circ \setminus \{\text{T}_1, \text{T}_2, \text{T}_3\}$ is a transversally hyperbolic saddle that has one unstable mode and three stable modes. The unstable manifold of K coincides with a vacuum type II orbit, see Figure 2; the three-dimensional stable manifold is contained in $\bar{\mathcal{B}}_{\text{VII}_0}$. (The one-dimensional center manifold is K° itself.) Therefore, there do not exist any type IX solutions that converge to K as $\bar{\tau}_- \rightarrow \infty$. The *Taub points* $\{\text{T}_1, \text{T}_2, \text{T}_3\}$ are not transversally hyperbolic. Each Taub point

¹⁰Equivalence of orbits refers to the discrete symmetries of the problem that are associated with interchanging the axes.

T_α possesses a two-dimensional stable manifold and a three-dimensional center manifold. The (closure of the) two-dimensional stable manifold of T_α coincides with the LRS subset of $\overline{\mathcal{B}}_{\bar{N}_\alpha}$ (which contains the two special orbits $Q_\alpha \rightarrow T_\alpha$ and $F \rightarrow T_\alpha$ on the vacuum subset of $\mathcal{B}_{\bar{N}_\alpha}$ and on the Bianchi type I fluid subset, respectively.) The three-dimensional center manifold coincides with the vacuum subset of $\overline{\mathcal{B}}_{\bar{N}_\beta \bar{N}_\gamma}$ (which is a vacuum $\overline{\mathcal{B}}_{\text{VII}_0}$ set). Therefore, the center manifold reduction theorem [6] reduces the problem to analyzing Bianchi type VII₀ vacuum dynamics. In Section 3.3 we have shown that T_α is excluded as an ω -limit point for orbits in $\overline{\mathcal{B}}_{\bar{N}_\beta \bar{N}_\gamma}^{\text{vac.}}$; the existence of the monotone function (26) implies that T_α is a center saddle in $\overline{\mathcal{B}}_{\text{IX}}$. Consequently, there do not exist any type IX solutions that converge to any of the Taub points as $\bar{\tau}_- \rightarrow \infty$.

F Eq. (8) implies that $\bar{\Sigma}_\alpha^{-1} d\bar{\Sigma}_\alpha / d\bar{\tau}_- |_{\text{F}} = \frac{3}{2}(1-w)$ and $\bar{N}_\alpha^{-1} d\bar{N}_\alpha / d\bar{\tau}_- |_{\text{F}} = -\frac{1}{2}(1+3w)$. Therefore, F is a hyperbolic saddle that possesses a two-dimensional unstable manifold, which coincides with the Bianchi type I subset, and a three-dimensional stable manifold. Accordingly, F attracts a two-parametric family of type IX orbits as $\bar{\tau}_- \rightarrow \infty$. These solutions have a so-called isotropic singularity.

CS _{α} The fixed points CS _{α} ($\alpha = 1, 2, 3$) are hyperbolic with a three-dimensional unstable and a two-dimensional stable manifold; the former coincides with $\mathcal{B}_{\bar{N}_\alpha}$, the latter is associated with the equations $\bar{N}_\beta^{-1} d\bar{N}_\beta / d\bar{\tau}_- |_{\text{CS}_\alpha} = \frac{3}{4}(1+3w)$ (for $\beta \neq \alpha$). Therefore, each of the fixed points CS _{α} attracts an (equivalent) one-parameter set of type IX orbits in the limit $\bar{\tau}_- \rightarrow \infty$.

TL _{α} Each fixed point of the line TL _{α} has a three-dimensional center manifold and a two-dimensional stable manifold. The center manifold coincides with the vacuum subset $\overline{\mathcal{B}}_{\bar{N}_\beta \bar{N}_\gamma}^{\text{vac.}}$; in Section 3.3 we have proved that the points of TL _{α} take the role of sources. We thus conclude that the fixed points on TL _{α} are center saddles. The two-dimensional stable manifold of each point of TL _{α} is contained in the LRS subset \mathcal{LRS}_α of \mathcal{B}_{IX} (which is the hyperplane given by the conditions $\bar{\Sigma}_\beta = \bar{\Sigma}_\gamma$ and $\bar{N}_\beta = \bar{N}_\gamma$); more specifically, the closure of the union of the unstable manifolds coincides with the closure of \mathcal{LRS}_α . Therefore, for each fixed point on TL _{α} there exists a one-parameter family of type IX orbits that converges to this point as $\bar{\tau}_- \rightarrow \infty$; these orbits correspond to LRS solutions. (Conversely, generic LRS type IX solutions converge to TL _{α} , see e.g. [1].)

The solutions whose ω -limit is one of the fixed points form a subfamily of measure zero of the (four-parameter) family of Bianchi type IX solutions. Following the nomenclature of [4] we thus refer to these solutions as *non-generic* solutions of Bianchi type IX. Alternatively, to capture the asymptotic behavior of these solution, we use the term *past asymptotically self-similar* solutions. (Since a fixed point in the Hubble-normalized dynamical systems formulation corresponds to a self-similar solution, see e.g. [1], solutions that converge to one of the above fixed points are asymptotically self-similar.)

Apart from the invariant Bianchi contraction subsets there exists other invariant subsets of the full state space. The most important are the three equivalent LRS subsets \mathcal{LRS}_γ defined by $\bar{\Sigma}_\alpha = \bar{\Sigma}_\beta$ and $\bar{N}_\alpha = \bar{N}_\beta$ for $(\alpha\beta\gamma) \in \{(123), (231), (312)\}$. The past asymptotically self-similar solutions comprise the LRS Bianchi type IX solutions. As seen above, generic LRS solutions converge to TL _{α} towards the past (and each solution that converges to TL _{α} is LRS), but there exist exceptional LRS solutions that converge to F or CS _{α} . The remaining orbits whose limit point is either F or CS _{α} correspond to past asymptotically self-similar solutions that are non-LRS. Clearly, every solution that converges to F or CS _{α} is a non-vacuum solution, since $\bar{\Omega} \neq 0$ at F and CS _{α} .

It is natural to ask how the non-generic orbits are embedded in the state space $\overline{\mathcal{B}}_{\text{IX}}$. The LRS orbits form the three LRS subsets \mathcal{LRS}_α , which are the hyperplanes given by the conditions $\bar{\Sigma}_\beta = \bar{\Sigma}_\gamma$, $\bar{N}_\beta = \bar{N}_\gamma$, where $(\alpha\beta\gamma) \in \{(123), (231), (312)\}$. The orbits whose ω -limit set is the fixed point CS _{α} (for some α) form the set \mathcal{CS}_α in \mathcal{B}_{IX} ; we call \mathcal{CS}_α the Collins-Stewart manifold. The local analysis of the fixed point CS _{α} and the regularity of the dynamical system (8) imply that the Collins-Stewart manifold \mathcal{CS}_α is a two-dimensional surface; it can be viewed as a two-dimensional manifold with boundary embedded in $\overline{\mathcal{B}}_{\text{IX}}$ (where this boundary lies in $\overline{\mathcal{B}}_{\text{VII}_0}$). Analogously, the

orbits whose α -limit set is the fixed point F form the set \mathcal{F} in \mathbf{B}_{IX} , which we call the isotropic singularity manifold, since solutions converging to F are those with an isotropic singularity. The isotropic singularity manifold \mathcal{F} is a three-dimensional hypersurface; it can be viewed as a three-dimensional manifold with boundary.

In the subsequent section we will state and prove the Bianchi type IX attractor theorem, which concerns the behavior of generic Bianchi type IX models (i.e., those that are not asymptotic self-similar, which provides an example of asymptotic self-similarity breaking; for other such examples, see [22]).

5 A new proof of the Bianchi type IX attractor theorem

Definition. Consider a solution of Bianchi type IX that is either vacuum or associated with a perfect fluid satisfying $-\frac{1}{3} < w < 1$. Such a solution is called generic if it is not past asymptotically self-similar.

Remark. Accordingly, a solution is generic if its ω -limit set is neither the point F , nor any of the points CS_α , nor a point on TL_α ; in other words, a generic solution corresponds to an orbit in \mathbf{B}_{IX} that is neither contained in the submanifold \mathcal{F} , nor in \mathcal{CS}_α , nor in the hyperplane \mathcal{LRS}_α . Therefore, the set of generic Bianchi type IX states is an open set in \mathbf{B}_{IX} .

Definition. The Mixmaster attractor \mathcal{A}_{IX} (alternatively referred to as the Bianchi type IX attractor) is defined to be the subset of \mathbf{B}_{IX} given by union of the Bianchi type I and II vacuum subsets, i.e., $\mathcal{A}_{\text{IX}} = \mathbf{B}_{\text{I}}^{\text{vac.}} \cup \mathbf{B}_{\text{II}}^{\text{vac.}}$. Accordingly, \mathcal{A}_{IX} consists of the three representations of the Bianchi type II vacuum subset and the Kasner circle (the Bianchi type I vacuum subset), i.e.,

$$\mathcal{A}_{\text{IX}} = \text{K}^\circ \cup \mathcal{B}_{N_1}^{\text{vac.}} \cup \mathcal{B}_{N_2}^{\text{vac.}} \cup \mathcal{B}_{N_3}^{\text{vac.}}. \quad (37)$$

The main result concerning generic Bianchi type IX models is the Bianchi type IX attractor theorem, which is due to Ringström [4]; this result rests on earlier work that is reviewed and derived in [1], and on [2] and [3]. In the following we state the Bianchi type IX attractor theorem in a version adapted to our purposes.

Theorem 5.1 ([4]). A generic orbit γ in \mathbf{B}_{IX} has an ω -limit set that is a subset of the Mixmaster attractor, i.e., $\omega(\gamma) \subseteq \mathcal{A}_{\text{IX}} = \mathbf{B}_{\text{I}}^{\text{vac.}} \cup \mathbf{B}_{\text{II}}^{\text{vac.}}$.

Remark. Note that we have chosen the time direction towards the past singularity, see (6) and (13). When we use the standard future directed time, ‘ ω -limit set’ is replaced by ‘ α -limit set’.

Remark. For an equivalent formulation of Theorem 5.1 let $X(\bar{\tau}_-) = (\bar{\Sigma}_1, \bar{\Sigma}_2, \bar{\Sigma}_3, \bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{H})(\bar{\tau}_-)$ be a generic solution of Bianchi type IX. Then

$$\|X(\bar{\tau}_-) - \mathcal{A}_{\text{IX}}\| \rightarrow 0 \quad (\bar{\tau}_- \rightarrow \infty), \quad (38)$$

where the distance $\|X - \mathcal{A}_{\text{IX}}\|$ is given as $\min_{Y \in \mathcal{A}_{\text{IX}}} \|X - Y\|$.

Remark. Alternatively, the statement of the theorem can be expressed as follows: Along every generic orbit we have

$$\bar{N}_1 \bar{N}_2 + \bar{N}_1 \bar{N}_3 + \bar{N}_2 \bar{N}_3 \rightarrow 0 \quad \text{and} \quad \bar{\Omega} \rightarrow 0 \quad \text{as} \quad \bar{\tau}_- \rightarrow \infty. \quad (39)$$

In combination with the constraint (17b) we further obtain $\bar{H} \rightarrow 1$ as $\bar{\tau}_- \rightarrow \infty$. Since, by definition, $\bar{H} = [1 + \frac{1}{6}(N_1 N_2 + N_1 N_3 + N_2 N_3)]^{-1/2}$, see (10) and (11), it follows that $(N_1 N_2 + N_1 N_3 + N_2 N_3) \rightarrow 0$ when $\bar{\tau}_- \rightarrow \infty$. An immediate consequence, cf. (12), is that $\Omega/\bar{\Omega} \rightarrow 1$ and $d\tau_-/d\bar{\tau}_- \rightarrow 1$ as $\bar{\tau}_- \rightarrow \infty$, so that $\tau_- \rightarrow \infty$ as $\bar{\tau}_- \rightarrow \infty$. Using these results, Theorem 5.1 instantly yields the original formulation of this theorem in [4].

In the following we give a new proof of Ringström’s Bianchi type IX attractor theorem. The proof we present is subdivided into a number of lemmas, which culminate in Theorem 5.1. The additional remarks in this subsection are not needed directly for the proofs, but give further insights about the main ideas and the lines of argument.

Lemma 5.2. *Every orbit in \mathbf{B}_{IX} possesses a non-empty ω -limit set that is contained in the subset $\overline{\mathbf{B}}_{\text{VII}_0}$ of the boundary $\partial\mathbf{B}_{\text{IX}}$.*

Proof. Since the state space \mathbf{B}_{IX} is relatively compact, every orbit has an ω -limit point in $\overline{\mathbf{B}}_{\text{IX}} = \overline{\mathbf{H}}_0 \cup \overline{\mathbf{B}}_{\text{VII}_0}$, cf. (20). The function

$$\bar{\Delta} := \bar{H}^{-3} \bar{N}_1 \bar{N}_2 \bar{N}_3 \quad (40)$$

is positive and strictly monotonically decreasing on \mathbf{B}_{IX} , since

$$\frac{d\bar{\Delta}}{d\bar{\tau}_-} = -3 \frac{\bar{q}}{\bar{H}} \bar{\Delta}, \quad \left. \frac{d^2\bar{\Delta}}{d\bar{\tau}_-^2} \right|_{\bar{\Sigma}^2=0} = 0, \quad \left. \frac{d^3\bar{\Delta}}{d\bar{\tau}_-^3} \right|_{\bar{\Sigma}^2=0} = -2\bar{H}^{-1} ({}^3\bar{S}_1^2 + {}^3\bar{S}_2^2 + {}^3\bar{S}_3^2) \bar{\Delta}; \quad (41)$$

in the perfect fluid case the first derivative is negative since $\bar{q} = 2\bar{\Sigma}^2 + \frac{1}{2}(1+3w)\bar{\Omega} > 0$; in the vacuum case, i.e., $\bar{\Omega} = 0$, $d\bar{\Delta}/d\bar{\tau}_- = 0$ is possible since $\bar{q} = 0$ when $\bar{\Sigma}^2 = 0$, but then the third derivative is negative, since ${}^3\bar{S}_1^2 + {}^3\bar{S}_2^2 + {}^3\bar{S}_3^2 > 0$ when $\bar{\Sigma}^2 = 0$ because of the constraints. Application of the monotonicity principle to the function $\bar{\Delta}$ yields that the ω -limit set of every orbit is contained in $\partial\mathbf{B}_{\text{IX}}$, where the set \mathbf{H}_0 is excluded because $\bar{\Delta} = \infty$ on \mathbf{H}_0 . \square

The set $\overline{\mathbf{B}}_{\text{VII}_0}$ can be decomposed according to

$$\overline{\mathbf{B}}_{\text{VII}_0} = \mathbf{B}_{\text{VII}_0} \cup \mathbf{B}_{\text{II}} \cup \mathbf{B}_{\text{I}} \cup \{Z_1, Z_2, Z_3\}. \quad (42)$$

In the subsequent lemmas we exclude the possibility that generic orbits have ω -limits in $\mathbf{B}_{\text{VII}_0}$ or $\{Z_\alpha\}$; for pedagogical reasons some of the lemmas refer to the (simpler) vacuum case and the fluid case separately. Furthermore, we prove that not only in the vacuum case but also in the fluid case only the vacuum subsets of \mathbf{B}_{I} and \mathbf{B}_{II} come into question. Taken together these lemmas then directly lead to Theorem 5.1.

Lemma 5.3 (Vacuum case). *A generic orbit in \mathbf{B}_{IX} cannot have an ω -limit point on the subset $\mathbf{B}_{\text{VII}_0}$.*

Proof. (Vacuum case, i.e., $\bar{\Omega} = 0$. In this case ‘generic’ reduces to non-LRS, cf. Section 4.) In preparation for the proof we begin by considering a fixed point L_α of the line TL_α , see (25a); in particular, $\bar{N}_\alpha = 0$, and $\bar{N}_\beta = \bar{N}_\gamma > 0$ at L_α . The point L_α has the following properties: L_α is not hyperbolic, but on its center manifold, which is the set $\mathbf{B}_{\text{VII}_0}$, there exists a one-parameter set of orbits converging to L_α as $\bar{\tau}_- \rightarrow -\infty$; in contrast, no orbit converges to L_α as $\bar{\tau}_- \rightarrow \infty$; see the previous discussion of Bianchi type VII₀ models. In this sense, L_α resembles a (transversal) source for $\mathbf{B}_{\text{VII}_0}$. Furthermore, from (15c) we obtain that $\bar{N}_\alpha^{-1} d\bar{N}_\alpha/d\bar{\tau}_- = -6\bar{H}$ at L_α , hence L_α possesses a one-dimensional stable subspace that lies in \mathbf{B}_{IX} . The center manifold reduction theorem applies [6]: There exists a neighborhood of L_α such that the flow of the full nonlinear system is equivalent to the flow of the decoupled system

$$\frac{d}{d\bar{\tau}_-} \bar{N}_\alpha = -6\bar{H} \bar{N}_\alpha \quad (43a)$$

$$\frac{d}{d\bar{\tau}_-} B_{\text{VII}_0} = F(B_{\text{VII}_0}), \quad (43b)$$

where B_{VII_0} denotes the collection of the variables of $\mathbf{B}_{\text{VII}_0}$. Since L_α is a (transversal) source for the second subsystem, L_α is a center saddle and there exists exactly one orbit whose ω -limit is L_α ; this orbit coincides with the unstable manifold of L_α (for which $\bar{N}_\alpha > 0$, $\bar{N}_\beta > 0$, $\bar{N}_\gamma > 0$) and can straightforwardly be identified as a Bianchi type IX LRS orbit (because the vacuum type IX LRS subset is exactly solvable).

Now consider a non-LRS orbit γ in the vacuum subset of \mathbf{B}_{IX} and assume that the ω -limit set $\omega(\gamma)$ contains a point on $\mathbf{B}_{\text{VII}_0}$, i.e., a point P such that $\bar{N}_\alpha = 0$, $\bar{N}_\beta > 0$, $\bar{N}_\gamma > 0$. We distinguish three possible cases: (i) $P \in \text{TL}_\alpha$; (ii) $P \in \mathbf{B}_{\text{VII}_0}$, $P \notin \text{TL}_\alpha$, $P \notin \mathcal{QL}_\alpha$; (iii) $P \in \mathcal{QL}_\alpha$.

Consider case (i), i.e., assume $P = L_\alpha \in \omega(\gamma)$. There are two possibilities: Either $\omega(\gamma) = \{L_\alpha\}$, then γ is the orbit that coincides with the stable manifold of L_α ; this is impossible since the stable manifold is an LRS orbit; or $\omega(\gamma) \supsetneq \{L_\alpha\}$. The saddle structure of L_α allows us to draw the following conclusion: Since $\{L_\alpha\} \subsetneq \omega(\gamma)$, it follows that also the stable manifold of L_α must be a subset of $\omega(\gamma)$. However, since the stable manifold lies in \mathbf{B}_{IX} , this is a contradiction to Lemma 5.2. Hence, the existence of a point $P \in \text{TL}_\alpha$ in $\omega(\gamma)$ is excluded for the non-LRS orbit γ .

Case (ii) is analogous. Since P is contained in $\omega(\gamma)$, so is the entire Bianchi type VII₀ orbit through P and therefore the α - and ω -limit sets of that orbit. The α -limit set is a fixed point L_α on the line TL_α , cf. the analysis of Bianchi type VII₀ models; accordingly, $L_\alpha \supsetneq \omega(\gamma)$ which leads to a contradiction to Lemma 5.2 in analogy with case (i).

Finally, consider case (iii). Since $P \in \omega(\gamma)$, the entire orbit through P (and the α - and ω -limit points of that orbit) must be contained in $\omega(\gamma)$ as well; hence, $\overline{\mathcal{QL}_\alpha} \subseteq \omega(\gamma)$. Because we have already excluded cases (i) and (ii), we know that $\omega(\gamma) \subseteq \overline{\mathbf{B}_{\text{II}}} \cup (\bigcup_\delta \overline{\mathcal{QL}_\delta})$. However, the heteroclinic orbits \mathcal{QL}_δ , $\delta = 1, 2, 3$, are not connected, but ‘isolated branches’ of the set $\overline{\mathbf{B}_{\text{II}}} \cup (\bigcup_\delta \overline{\mathcal{QL}_\delta})$. Since such structures can never be part of a limit set, cf. the remark below, the assumption $\mathcal{QL}_\alpha \subseteq \omega(\gamma)$ and therefore (iii) result in a contradiction. This finishes the proof of the lemma. \square

Remark. Let us elaborate on case (iii) in the proof of the theorem and show explicitly why the isolated heteroclinic orbits \mathcal{QL}_δ , $\delta = 1, 2, 3$, are excluded from the possible ω -limit set. In the proof of Lemma 5.3, case (iii), we assume that there exists an orbit γ such that $P \in \mathcal{QL}_\alpha$ is an element of $\omega(\gamma)$; hence there exists a diverging sequence of times, $(\varsigma_n)_{n \in \mathbb{N}}$, such that $\gamma(\varsigma_n) \rightarrow P$ ($n \rightarrow \infty$). Let \mathcal{V} be a sufficiently small neighborhood of Z_α , preferably generated by an open ball, so that $P \notin \mathcal{V}$. For sufficiently large n , there exists times σ_n , $\varsigma_{n-1} < \sigma_n < \varsigma_n$, such that $\gamma(\sigma_n) \in \mathcal{V}$. (This is a simple consequence of the continuous dependence of the flow on initial data; recall that \mathcal{QL}_α is a heteroclinic orbit connecting the fixed point Z_α with the fixed point Q_α .) Consequently, there exist times κ_n , $\varsigma_{n-1} < \kappa_n < \varsigma_n$, such the orbit γ enters \mathcal{V} at κ_n (i.e., $\gamma(\kappa_n) \in \partial\mathcal{V}$, $\gamma(\kappa_n + \epsilon) \in \mathcal{V}$ for sufficiently small $\epsilon > 0$). For all n , $\gamma(\kappa_n)$ is contained in the complement of a sufficiently small neighborhood \mathcal{U} of the point $\partial\mathcal{V} \cap \mathcal{QL}_\alpha$ (because the flow of the dynamical system points out of \mathcal{V} in $\partial\mathcal{V} \cap \mathcal{U}$). By going over to a subsequence $\gamma(\kappa_n)$, this implies that γ has a ω -limit point on $\partial\mathcal{V} \setminus \mathcal{U}$, i.e., an ω -limit point in $\mathbf{B}_{\text{VII}_0}$ (or \mathbf{H}_0) that is not contained on \mathcal{QL}_δ , $\delta = 1, 2, 3$. This is a contradiction to the assumption $\omega(\gamma) \subseteq \overline{\mathbf{B}_{\text{II}}} \cup (\bigcup_\delta \overline{\mathcal{QL}_\delta})$.

Remark. The concept of ‘isolated branches’—used in the proof of Lemma 5.3 and discussed in the previous remark—is a very useful picture to have in mind also for the general situation. The state space $\overline{\mathbf{B}_{\text{IX}}}$ can be depicted roughly as the space between three branches (the three $\mathbf{B}_{\text{VII}_0}$ subsets $\mathcal{B}_{\overline{N}_1 \overline{N}_2}$, $\mathcal{B}_{\overline{N}_2 \overline{N}_3}$, $\mathcal{B}_{\overline{N}_3 \overline{N}_1}$) sticking out from a common basis (‘trunk’) which is the set $\overline{\mathbf{B}_{\text{II}}}$; the end points of the branches are the points Z_1, Z_2, Z_3 ; see Figure 3. The main observation is that the flow on the branches is (more or less) unidirectional, i.e., directed away from the points Z_α (although there are the lines TL_α of fixed points on $\mathbf{B}_{\text{VII}_0}$ which make the situation much more subtle). A generic orbit γ cannot have ω -limit points in the interior of the state space, see Lemma 5.2, however, the picture strongly suggests that ω -limit points on the branches are excluded as well: This is simply because an orbit γ cannot ‘climb up’ to a point on $\mathbf{B}_{\text{VII}_0}$ along the $\mathbf{B}_{\text{VII}_0}$ branches themselves (which is due to the continuous dependence of the flow on initial data). Lemma 5.4 is intimately connected with this idea.

Remark. Lemma 5.3 implies that the ω -limit set of a non-generic orbit γ (in the vacuum subset of \mathbf{B}_{IX}) is a subset of $\overline{\mathbf{B}_{\text{II}}} \cup \{Z_\alpha\}$. It follows trivially that $\{Z_\alpha\} \subsetneq \omega(\gamma)$ is impossible, which is simply because these points are isolated from the set $\overline{\mathbf{B}_{\text{II}}}$ (while limit sets are necessarily connected). (By studying the LRS subset it is easy to show that $\{Z_\alpha\} \subseteq \omega(\gamma)$ is excluded for LRS orbits.) We will give an independent proof of this statement (and its extension to the fluid case) in Lemma 5.4.

Lemma 5.4. *Let γ be an orbit in \mathbf{B}_{IX} . Then $\omega(\gamma) \supsetneq \{Z_1\}$ (or $\{Z_2\}$, $\{Z_3\}$) is impossible.*

Proof. Since the statement of the lemma is trivial for non-generic orbits (past asymptotically self-similar orbits, cf. Section 4), we restrict ourselves to generic orbits γ . Assume that there exists a (generic) orbit γ such that the ω -limit set of γ contains Z_1 , i.e., $\omega(\gamma) \supsetneq \{Z_1\}$. Accordingly, there

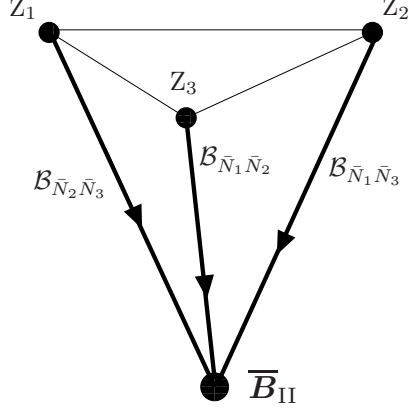


Figure 3: A simplistic picture of the state space $\overline{\mathcal{B}}_{\text{IX}}$. $\mathcal{B}_{\text{VII}_0}$ consists of three ‘branches’ ($\mathcal{B}_{\overline{N}_1\overline{N}_2}$, $\mathcal{B}_{\overline{N}_1\overline{N}_3}$, $\mathcal{B}_{\overline{N}_2\overline{N}_3}$) that intersect in a common subset, $\overline{\mathcal{B}}_{\text{II}}$. The flow of the $\mathcal{B}_{\text{VII}_0}$ subsets is directed away from the points Z_α towards $\overline{\mathcal{B}}_{\text{II}}$.

exists a diverging sequence of times $(\varsigma_n)_{n \in \mathbb{N}}$ such that $\gamma(\varsigma_n) \rightarrow Z_1$ as $n \rightarrow \infty$. There exists a neighborhood \mathcal{V} of Z_1 such that γ intersects the complement of \mathcal{V} infinitely many times (otherwise γ would converge to Z_1). Therefore we can construct a sequence of times $(\sigma_n)_{n \in \mathbb{N}}$, $\sigma_n < \varsigma_n \forall n$, such that $\gamma(\sigma_n) \in \partial\mathcal{V}$ and $\gamma|_{(\sigma_n, \varsigma_n]} \in \mathcal{V}$ for all n . (Clearly, $|\varsigma_n - \sigma_n|$ diverges as $n \rightarrow \infty$, because $\gamma(\varsigma_n)$ converges to the fixed point Z_1 .) The sequence $(\gamma(\sigma_n))_{n \in \mathbb{N}}$ possesses a converging subsequence, i.e., there exists a point P such that $\gamma(\sigma_n) \rightarrow P$ as $n \rightarrow \infty$ (where the index n now runs over the index set of the considered subsequence). By construction, $P \in \partial\mathcal{V}$; furthermore, by definition, P is an ω -limit point of the orbit γ ; Lemma 5.2 implies that $P \in \mathcal{B}_{\text{VII}_0}$ (since \mathcal{B}_{I} and \mathcal{B}_{II} are disjoint from \mathcal{V}). As discussed in the subsection ‘‘Bianchi type $\mathcal{B}_{\text{VII}_0}$: A new analysis,’’ the ω -limit of the orbit through the point P , which we call γ_P , is a fixed point on TL_1 or $\overline{\mathcal{B}}_{\text{II}}$ (generically, $\omega(\gamma_P)$ is a fixed point on the Kasner circle); in particular, $\omega(\gamma_P)$ does not contain Z_1 . Assume that $\omega(\gamma_P) \in \overline{\mathcal{B}}_{\text{II}}$; then, by continuous dependence on initial data, for sufficiently large n , $\gamma|_{(\sigma_n, \cdot)}$ shadows the orbit γ_P and reaches $\overline{\mathcal{B}}_{\text{II}}$; but this is a contradiction to the fact that $\gamma|_{(\sigma_n, \cdot)} \subset \mathcal{V}$. Assume that $\omega(\gamma_P)$ is one of the fixed points of TL_1 ; if this fixed point is not contained in \mathcal{V} , we immediately obtain a contradiction; otherwise, we proceed in close analogy to the proof of Lemma 5.3; we exploit the (center) saddle property of $\omega(\gamma_P)$ and we obtain that $\gamma|_{(\sigma_n, \cdot)}$, for sufficiently large n , shadows first γ_P , then follows some orbit emanating from $\omega(\gamma_P)$ and eventually approaches a fixed point on the Kasner circle; this is again the desired contradiction. \square

Lemma 5.5 (Fluid case). *A generic orbit in \mathcal{B}_{IX} cannot have an ω -limit point on the subset $\mathcal{B}_{\text{VII}_0}$.*

Proof. (Fluid case, i.e., $\overline{\Omega} > 0$.) Consider a generic orbit γ in \mathcal{B}_{IX} and assume that the ω -limit set $\omega(\gamma)$ contains a point P on $\mathcal{B}_{\text{VII}_0}$. We distinguish three possible cases: (i) P is an element of the fluid subset of $\mathcal{B}_{\text{VII}_0}$, i.e., $\overline{\Omega}|_P > 0$; (ii) $P \in \mathcal{QL}_\alpha$; (iii) $P \in \text{TL}_\alpha$; (iv) P is an element of the vacuum subset of $\mathcal{B}_{\text{VII}_0}$, but $P \notin \text{TL}_\alpha$, $P \notin \mathcal{QL}_\alpha$.

Consider case (i). Since $P \in \omega(\gamma)$, the orbit γ_P through P and its α -limit $\alpha(\gamma_P)$ must also be contained in $\omega(\gamma)$. Our analysis of the $\mathcal{B}_{\text{VII}_0}$ subset shows that $\alpha(\gamma_P) = \{Z_\alpha\}$. Consequently, $\{P, Z_\alpha\} \subset \omega(\gamma)$; but this is a contradiction to Lemma 5.4. Case (ii) is completely analogous.

Consider case (iii), i.e., $\omega(\gamma)$ contains a fixed point L_α of the line TL_α . The scenario $\omega(\gamma) = \{L_\alpha\}$ is impossible, since γ is a generic orbit and the stable manifold of L_α is a subset of the LRS subset. In analogy to the proof of Lemma 5.3 we can exploit the (center) saddle structure of L_α , which is reflected in (43) and the additional ‘fluid’ equation $\overline{\Omega}^{-1} d\overline{\Omega}/d\tau_- = -3(1-w)\overline{H}$. Hence, since $\{L_\alpha\} \subsetneq \omega(\gamma)$, it follows that $\omega(\gamma)$ contains a point Q on the stable manifold of L_α ; since $Q \notin \mathcal{B}_{\text{IX}}$ by Lemma 5.2, we have $Q \in \mathcal{B}_{\text{VII}_0} \cap \mathcal{LRS}_\alpha$, i.e., Q lies on the Bianchi type VII_0 LRS orbit that converges to L_α . Because $\overline{\Omega} > 0$ for this orbit (and thus for Q), this brings us back to case (i); a

contradiction ensues.

Consider case (iv). Since the α -limit of the orbit through P is a fixed point L_α on TL_α , case (iv) can be reduced to case (iii). \square

Lemma 5.6. *A generic orbit in \mathbf{B}_{IX} cannot have an ω -limit point on the fluid subset of $\overline{\mathbf{B}}_{\text{II}}$.*

Proof. The lemma is obviously true in the vacuum case. In the fluid case, assume that there exists a generic orbit γ such that $\omega(\gamma)$ contains a point $P \in \mathbf{B}_{\text{I}} \cup \mathbf{B}_{\text{II}}$ with $\bar{\Omega}|_P > 0$. First, assume that P is an element of \mathbf{B}_{II} , e.g., in the $\mathcal{B}_{\bar{N}_\alpha}$ subset. The orbit through P has the fixed point CS_α as its α -limit set, see Section 3; hence $\text{CS}_\alpha \in \omega(\gamma)$. Using the saddle structure of CS_α , see Section 4, we conclude that $\omega(\gamma)$ also contains a point Q of the stable manifold of CS_α . However, since this stable manifold is a subset of $\mathbf{B}_{\text{VII}_0} \cup \mathbf{B}_{\text{IX}}$, we have $Q \in \mathbf{B}_{\text{VII}_0} \cup \mathbf{B}_{\text{IX}}$ and thus a contradiction to Lemma 5.2 or Lemma 5.5. Second, assume $P \in \mathbf{B}_{\text{I}}$. Using the same line of arguments (where F takes the role of CS_α) we obtain that $Q \in \mathbf{B}_{\text{II}} \cup \mathbf{B}_{\text{VII}_0} \cup \mathbf{B}_{\text{IX}}$ and thus a contradiction to what has already been proved. \square

Lemmas 5.3 and 5.5 imply that the ω -limit set of a generic orbit in \mathbf{B}_{IX} must be either contained in the vacuum subset of $\overline{\mathbf{B}}_{\text{II}}$ (where we recall that $\overline{\mathbf{B}}_{\text{II}} = \mathbf{B}_{\text{II}} \cup \mathbf{B}_{\text{I}}$) or it is a fixed point of the set $\{Z_1, Z_2, Z_3\}$. It remains to prove that the latter scenario is impossible.

Lemma 5.7. *There does not exist any orbit in \mathbf{B}_{IX} that converges to Z_1 (Z_2, Z_3) as $\bar{\tau}_- \rightarrow \infty$.*

Proof. The non-trivial case (which is at the same time the case that is relevant for our purposes) is the non-LRS case. We perform a proof by contradiction. The main idea is to consider a non-LRS orbit that is assumed to converge to the fixed point Z_1 ; the convergence to Z_1 then implies that \bar{N}_1 decays rapidly so that the orbit is forced to shadow a $\mathbf{B}_{\text{VII}_0}$ orbit as Z_1 is approached. However, there does not exist any orbit on $\mathbf{B}_{\text{VII}_0}$ that converges to Z_1 as $\bar{\tau}_- \rightarrow \infty$; a contradiction must ensue.

This idea is formalized by using a non-negative function ζ_1 on $\overline{\mathbf{B}}_{\text{IX}}$ which is zero at Z_1 and whose restriction to $\mathbf{B}_{\text{VII}_0}$ is monotonically increasing along the flow of $\mathbf{B}_{\text{VII}_0}$. Since the orbit γ in \mathbf{B}_{IX} that is assumed to converge to Z_1 must shadow a type VII_0 orbit, we expect the function ζ_1 to increase along γ as well, or at least to decrease at a rate small enough so that the integral still exists. This leads directly to a contradiction, because by assumption ζ_1 must go to zero as Z_1 is approached. In essence this is the main idea employed by Ringström [4] in a similar context, who, however, used H -normalized variables, and a function introduced by Wainwright and Hsu [23]; to facilitate comparison we choose basically the same function. Consider

$$\zeta_1 = \frac{(\bar{\Sigma}_2 - \bar{\Sigma}_3)^2 + (\bar{N}_2 - \bar{N}_3)^2}{\bar{N}_2 \bar{N}_3}, \quad (44a)$$

$$\frac{d\zeta_1}{d\bar{\tau}_-} = -\frac{1}{\bar{N}_2 \bar{N}_3} [2(\bar{\Sigma}_2 - \bar{\Sigma}_3)^2(\bar{\Sigma}_1 - 2\bar{H}) + 2(\bar{\Sigma}_2 - \bar{\Sigma}_3)(\bar{N}_2 - \bar{N}_3)\bar{N}_1]. \quad (44b)$$

The second term in the brackets can be simply estimated by using

$$2(\bar{\Sigma}_2 - \bar{\Sigma}_3)(\bar{N}_2 - \bar{N}_3) \leq (\bar{\Sigma}_2 - \bar{\Sigma}_3)^2 + (\bar{N}_2 - \bar{N}_3)^2 = \zeta_1 \bar{N}_2 \bar{N}_3. \quad (45)$$

We combine the constraints (17a) and (17b) to find

$$\bar{\Sigma}^2 - \bar{H}^2 + \frac{1}{12} [\bar{N}_1^2 + (\bar{N}_2 - \bar{N}_3)^2] + \bar{\Omega} = \frac{1}{6} \bar{N}_1 (\bar{N}_2 + \bar{N}_3).$$

Since $\bar{\Sigma}^2 = \frac{1}{4} \bar{\Sigma}_1^2 + \frac{1}{12} (\bar{\Sigma}_2 - \bar{\Sigma}_3)^2$ we obtain

$$\frac{1}{4} (\bar{\Sigma}_1 - 2\bar{H})(\bar{\Sigma}_1 + 2\bar{H}) + \frac{1}{12} [\bar{N}_1^2 + (\bar{\Sigma}_2 - \bar{\Sigma}_3)^2 + (\bar{N}_2 - \bar{N}_3)^2] + \bar{\Omega} = \frac{1}{6} \bar{N}_1 (\bar{N}_2 + \bar{N}_3).$$

If $\bar{\Sigma}_1 - 2\bar{H} > 0$, which is the ‘worst case scenario’ for our considerations, then

$$\bar{\Sigma}_1 - 2\bar{H} \leq \frac{1}{6} \bar{N}_1 (\bar{N}_2 + \bar{N}_3) \frac{4}{\bar{\Sigma}_1 + 2\bar{H}} < \frac{1}{6} (\bar{N}_2 + \bar{N}_3) \frac{\bar{N}_1}{\bar{H}} < \lambda \frac{\bar{N}_1}{\bar{H}} \quad (46)$$

for some sufficiently large positive constant λ (since \bar{N}_α are bounded); if $\bar{\Sigma}_1 - 2\bar{H} \leq 0$, this inequality holds trivially.

Assume that there exists a non-LRS orbit γ in \mathbf{B}_{IX} that converges to Z_1 as $\bar{\tau}_- \rightarrow \infty$. Inserting the inequalities (45) and (46) into (44b) yields

$$\frac{d\zeta_1}{d\bar{\tau}_-} \geq -2\lambda \frac{\bar{N}_1}{H} \zeta_1 - \bar{N}_1 \zeta_1 \geq -\frac{\bar{N}_1}{H^2} \zeta_1, \quad (47)$$

where the latter inequality is true for sufficiently large $\bar{\tau}_-$, since $\bar{H} \rightarrow 0$ along γ . At a reference time $\bar{\tau}_- = \hat{\tau}$, which we choose to be sufficiently large (in order for (47) to hold), the function ζ_1 takes a value $\hat{\zeta}_1 > 0$. Integration of the differential inequality (47) yields

$$\log \zeta_1(\bar{\tau}_-) \geq \log \hat{\zeta}_1 - \int_{\hat{\tau}}^{\bar{\tau}_-} \frac{\bar{N}_1}{\bar{H}^2}(\bar{\tau}) d\bar{\tau} \quad (48)$$

along the orbit γ .

To estimate the r.h. side in equation (48) we exploit the convergence of γ to Z_1 . This convergence entails that $\bar{N}_1 \rightarrow 0$ and $\bar{N}_2 \rightarrow \sqrt{6}$, $\bar{N}_3 \rightarrow \sqrt{6}$; the constraints then automatically imply $\bar{H} \rightarrow 0$, $\bar{\Sigma}^2 \rightarrow 0$, and $\bar{\Omega} \rightarrow 0$. Let $\delta = 2\sqrt{6} - \bar{N}_2 - \bar{N}_3$. Convergence to Z_1 implies that $\delta \rightarrow 0$. Using the constraint (17b) and the fact that $\bar{N}_1 = O(\bar{H}^3)$ as $\bar{H} \rightarrow 0$ (which follows from the monotonicity of $\bar{\Delta}$) we obtain

$$\bar{H}^2 + \frac{2}{\sqrt{6}}\bar{N}_1 = \frac{1}{\sqrt{6}}\delta + o(\delta) \quad \Rightarrow \quad \bar{H}^2 = \frac{1}{\sqrt{6}}\delta + o(\delta) \quad \text{for } \delta \rightarrow 0. \quad (49)$$

The constraint (17a) gives

$$\bar{\Sigma}^2 + \bar{\Omega} = \frac{1}{\sqrt{6}}\delta + o(\delta) \quad \Rightarrow \quad \frac{\bar{\Sigma}^2 + \bar{\Omega}}{\bar{H}^2} = 1 + o(1) \quad \text{for } \delta \rightarrow 0. \quad (50)$$

For the quantity $\bar{q} = 2(\bar{\Sigma}^2 + \bar{\Omega}) - \frac{3}{2}(1-w)\bar{\Omega}$ we therefore derive the estimate $\bar{q}/\bar{H}^2 = 2 + o(1) - \frac{3}{2}(1-w)\bar{\Omega}/\bar{H}^2 \geq \frac{1}{2}(1+3w) + o(1)$, while $\bar{q}/\bar{H}^2 = 2 + o(1)$ in the vacuum case. We conclude that there exists a constant $\alpha > 0$ such that

$$\frac{\bar{q}}{\bar{H}^2} \geq \frac{\alpha}{3} \quad (51)$$

for sufficiently small values of δ , i.e., for sufficiently large values of $\bar{\tau}_-$ along γ . Using this inequality in the integrated version of equation (41), i.e.,

$$\bar{\Delta} = \frac{\bar{N}_1 \bar{N}_2 \bar{N}_3}{\bar{H}^3} \propto \exp \left[-3 \int_{\hat{\tau}}^{\bar{\tau}_-} \frac{\bar{q}}{\bar{H}} d\bar{\tau}_- \right], \quad (52)$$

results in the estimate

$$\frac{\bar{N}_1}{\bar{H}^3} \leq C \exp \left[-\alpha \int_{\hat{\tau}}^{\bar{\tau}_-} \bar{H} d\bar{\tau}'_- \right], \quad (53)$$

where C is some positive constant and where $\bar{\tau}_-$ is assumed to be sufficiently large.

Therefore, \bar{N}_1 goes to zero at a fast rate which ensure finiteness of the integral in (48) as $\bar{\tau}_- \rightarrow \infty$:

$$\int_{\hat{\tau}}^{\infty} \frac{\bar{N}_1}{\bar{H}^2} d\bar{\tau}_- = \int_{\hat{\tau}}^{\infty} \frac{\bar{N}_1}{\bar{H}^3} \bar{H} d\bar{\tau}_- \leq C\alpha^{-1} \left[1 - \exp \left(-\alpha \int_{\hat{\tau}}^{\infty} \bar{H} d\bar{\tau}_- \right) \right] \leq C\alpha^{-1}. \quad (54)$$

We thus conclude from (48) that ζ_1 remains bounded away from zero as $\bar{\tau}_- \rightarrow \infty$; but this contradicts the assumption that the orbit γ converges to the fixed point Z_1 , since then $\zeta_1 \rightarrow 0$ along γ . \square

Remark. The function ζ_1 can be used to give an independent proof of the statement $Z_1 \notin \omega(\gamma) \forall \gamma$ (which is closely related to Lemma 5.4); we briefly sketch this proof. Since the case $\omega(\gamma) = \{Z_1\}$ is treated in Lemma 5.7, consider an orbit γ such that $\omega(\gamma) \subsetneq \{Z_1\}$. If $\gamma(\zeta_n) \rightarrow Z_1$ as $n \rightarrow \infty$, then

$\zeta_1(\zeta_n) \rightarrow 0$ as $n \rightarrow \infty$. Relations (47) and (51) hold whenever an orbit is in a sufficiently small neighborhood of Z_1 . Therefore, in analogy to the considerations in the proof of Lemma 5.7, it is impossible to achieve $\zeta_1(\zeta_n) \rightarrow 0$, if γ has an ω -limit point that does not lie on the LRS subset. (In there were such a point it would be impossible for ζ_1 to decrease sufficiently much between that point and $\gamma(\zeta_n)$, n sufficiently large.) A priori it is possible that γ has an ω -limit point on the LRS subset (note that $\zeta_1 = 0$ on the LRS subset). However, a study of the LRS dynamics shows that orbits emanate from Z_1 , but there do not exist orbits that converge to Z_1 as $\bar{\tau}_- \rightarrow \infty$. Using the same reasoning as above (e.g., in the remark following Lemma 5.3 or in the proof of Lemma 5.4) we therefore exclude the possibility that γ has an ω -limit point on the LRS subset as well. Consequently, $\omega(\gamma)$ cannot have an ω -limit point except Z_1 ; but this is a contradiction to the assumption.

The collection of the lemmas finally yields Theorem 5.1 and thus completes our argument.

Theorem 5.1. *A generic orbit γ in \mathcal{B}_{IX} has an ω -limit set that is a subset of the Mixmaster attractor, i.e., $\omega(\gamma) \subseteq \mathcal{A}_{\text{IX}} = \mathcal{B}_{\text{I}}^{\text{vac.}} \cup \mathcal{B}_{\text{II}}^{\text{vac.}}$.*

Proof. Lemma 5.2 implies that the ω -limit set of a generic orbit in \mathcal{B}_{IX} is a subset of $\overline{\mathcal{B}}_{\text{VII}_0}$, which is given by (42). Lemmas 5.3 and 5.5 exclude $\mathcal{B}_{\text{VII}_0}$, Lemma 5.6 excludes the fluid subsets of \mathcal{B}_{I} and \mathcal{B}_{II} . Lemma 5.7 (in combination with Lemma 5.4) excludes $\{Z_1, Z_2, Z_3\}$. This leaves the vacuum subset of $\mathcal{B}_{\text{I}} \cup \mathcal{B}_{\text{II}}$ as the only possible superset of the ω -limit set of a generic orbit. \square

Remark. To complete the statement of Theorem 5.1 it is important to note that both $\omega(\gamma) \subseteq \mathcal{B}_{\text{I}}^{\text{vac.}}$ and $\omega(\gamma) \subseteq \mathcal{B}_{\text{II}}^{\text{vac.}}$ is impossible. The latter is obvious, since $\mathcal{B}_{\text{II}}^{\text{vac.}}$ consists of a collection of heteroclinic orbits (transitions) with end points on $\mathcal{B}_{\text{I}}^{\text{vac.}}$ ($= K^\circ$), see Section 3. The proof that $\omega(\gamma) \subseteq \mathcal{B}_{\text{I}}^{\text{vac.}}$ ($= K^\circ$) is impossible is contained in the local analysis of Section 4. (For an alternative proof see [3].)

Remark. It is important to emphasize that the Theorem states that $\omega(\gamma) \subseteq \mathcal{A}_{\text{IX}}$. Whether $\omega(\gamma)$ actually coincides with \mathcal{A}_{IX} (at least generically) or whether it is a proper subset of \mathcal{A}_{IX} is open. This question and related issues are discussed in detail in [10].

6 Consequences

The Bianchi type IX attractor theorem in conjunction with our understanding of the flow on the attractor subset implies a number of consequences that we formulate as corollaries. Some of these correspond to results presented in [4]; our approach, however, is rather different.

On the Mixmaster attractor \mathcal{A}_{IX} , the dynamical system (15) generates an intricate network of structures that are invariant under the flow: heteroclinic cycles and finite and infinite heteroclinic chains. (These heteroclinic structures arise by concatenating the vacuum type II orbits on $\mathcal{B}_{\bar{N}_1}$, $\mathcal{B}_{\bar{N}_2}$, and $\mathcal{B}_{\bar{N}_3}$, see Figure 2.) The ‘simplest’ structure (i.e., the structure that contains the smallest number of fixed points) is a heteroclinic cycle with three fixed points on K° , see Figure 4. We refer to [10] and references therein for a comprehensive discussion.

Lemma 6.1. *If $P \in \mathcal{A}_{\text{IX}}$ is an ω -limit point of a type IX orbit, then the entire heteroclinic cycle/chain through P must be contained in the ω -limit set.*

Proof. The lemma follows from basic facts of the theory of dynamical systems [5]. \square

Lemma 6.1 entails that the heteroclinic cycles and chains are potential limit set candidates for generic type IX orbits.

Corollary 6.2. *The asymptotic behavior of a generic Bianchi type IX solution is oscillatory with oscillations between at least three fixed points on K° .*

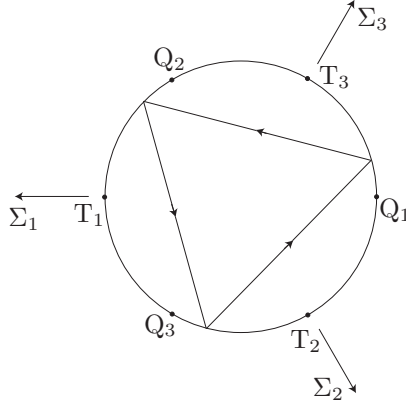


Figure 4: The ‘simplest’ structure on the Mixmaster attractor: a heteroclinic cycle with three fixed points. In general, concatenating of the vacuum type II orbits of Figure 2 yields infinite heteroclinic chains.

Proof. The simplest structure on \mathcal{A}_{IX} that is a potential ω -limit set for a generic type IX orbit is a heteroclinic cycle with three fixed points on K° , see Figure 4. A type IX orbit converging to such a heteroclinic cycle (if such an orbit exists) exhibits oscillations between three Kasner points. \square

Remark. The oscillatory behavior implies that generic asymptotic type IX dynamics constitute an example of asymptotic self-similarity breaking [22].

Corollary 6.3. *If one of the Taub points $\{T_1, T_2, T_3\}$ is an ω -limit point of a type IX orbit, then the ω -limit set contains Kasner fixed points arbitrarily close to the Taub points.*

Proof. Assume the contrary, i.e., suppose that there exists a type IX orbit γ such that T_α (for some α) is an element of $\omega(\gamma)$, while at the same time there exists a neighborhood \mathcal{U} of T_α such that $\omega(\gamma) \cap \mathcal{U} \cap \text{K}^\circ = \{T_\alpha\}$. However, $\omega(\gamma) \cap \mathcal{U} \supsetneq \{T_\alpha\}$, since $\omega(\gamma)$ is connected and strictly larger than T_α . (There do not exist type IX orbits that converge to a Taub point, see Section 4.) Taking into account the structure of orbits on \mathcal{A}_{IX} we conclude that $\omega(\gamma) \cap \mathcal{U} = \bar{\zeta} \cap \mathcal{U}$, where $\bar{\zeta}$ denotes the closure of the type II orbit $Q_\alpha \rightarrow T_\alpha$. However, since ω -limit sets cannot contain ‘isolated branches’ of this type, we obtain a contradiction. \square

Remark. Corollary 6.3 implies that the ω -limit set contains an infinite set of Kasner fixed points in a neighborhood of the Taub point(s), but this set is not necessarily a continuum of fixed points.

Lemma 6.4. *Let \mathcal{W} be a neighborhood of the Mixmaster attractor \mathcal{A}_{IX} . Then there exists a smaller neighborhood \mathcal{V} of \mathcal{A}_{IX} , $\mathcal{V} \subset \mathcal{W}$, such that each solution with initial data in \mathcal{V} (at $\bar{\tau}_- = 0$) remains in \mathcal{W} for all $\bar{\tau}_- > 0$.*

Proof. Henceforth, we denote by $X(\hat{x}; \bar{\tau}_-)$ the type IX solution generated by initial data $\hat{x} \in \mathcal{B}_{\text{IX}}$, i.e., $X(\hat{x}, \bar{\tau}_-)$ is a type IX solution and $X(\hat{x}, 0) = \hat{x}$. We employ the two functions $\bar{\Omega}$ and $\bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3$, cf. (39), as a measure of the distance from the Mixmaster attractor. The choice of a small neighborhood of the Mixmaster attractor then corresponds to both $\bar{\Omega}$ and $\bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3$ being bounded by a small constant ϵ . Choose a small neighborhood \mathcal{W} of \mathcal{A}_{IX} (corresponding to a choice $\epsilon_{\mathcal{W}}$) and consider a neighborhood \mathcal{V} of \mathcal{A}_{IX} with $\mathcal{V} \subset \mathcal{W}$, $\epsilon_{\mathcal{V}} \ll \epsilon_{\mathcal{W}}$. A solution $X(\hat{x}, \bar{\tau}_-)$ with initial data $\hat{x} \in \mathcal{V}$ remains in \mathcal{W} at least for $0 \leq \bar{\tau}_- < \bar{T}_-$. The constant \bar{T}_- depends on $\epsilon_{\mathcal{V}}$ and, a priori, on the initial data; we intend to show that $\bar{T}_- = \infty$ irrespective of the choice of initial data if $\epsilon_{\mathcal{V}}$ is sufficiently small.

First, consider Eq. (18) for $\bar{\Omega}$. At the Kasner circle we obtain $\bar{\Omega}^{-1} d\bar{\Omega}/d\bar{\tau}_-|_{\text{K}^\circ} = -3(1-w)$; hence there exists a neighborhood \mathcal{U}_I of K° such that $\bar{\Omega}$ is (exponentially) decreasing as long as the solution stays \mathcal{U}_I . In general, we obtain $\bar{\Omega}^{-1} d\bar{\Omega}/d\bar{\tau}_- < (1+3w)$ in \mathcal{W} , i.e., a limit on the possible increase. Decompose \mathcal{W} into $\mathcal{W}_I = \mathcal{W} \cap \mathcal{U}_I$ and $\mathcal{W}_{\text{II}} = \mathcal{W} \setminus \mathcal{U}_I$. Using the flow on \mathcal{A}_{IX}

(and elementary results from the theory of dynamical systems) we see that the solution $X(\hat{x}, \bar{\tau}_-)$ oscillates between \mathcal{W}_I and \mathcal{W}_{II} , where the sojourn times in \mathcal{W}_I are large compared to the sojourn times in \mathcal{W}_{II} . (The ratio of the respective sojourn times diverges with $\epsilon_V \rightarrow 0$.) Using the estimates for $d\bar{\Omega}/d\bar{\tau}_-$ it follows that $\bar{\Omega}$ exhibits a (rapid) overall decrease for $\bar{\tau}_- \in [0, \bar{T}_-)$ (i.e., as long as $X(\hat{x}, \bar{\tau}_-) \in \mathcal{W}$). Consequently, at $\bar{\tau}_- = \bar{T}_-$, the quantity $\bar{\Omega}$ is smaller than initially, $\bar{\Omega}|_{\bar{T}_-} < \epsilon_V$.

Second, consider $\bar{N} := \bar{N}_1\bar{N}_2 + \bar{N}_1\bar{N}_3 + \bar{N}_2\bar{N}_3$. For $(\alpha\beta\gamma) \in \{(123), (231), (312)\}$ we obtain $(\bar{N}_\alpha\bar{N}_\beta)^{-1} d(\bar{N}_\alpha\bar{N}_\beta)/d\bar{\tau}_- = -2(\bar{q}\bar{H} - \bar{\Sigma}_\gamma + \bar{F})$ from (15c). Let \mathcal{W}_T be a (small) neighborhood of the Taub points. Then $\bar{N}_\alpha\bar{N}_\beta$ and thus \bar{N} is (exponentially) decreasing in $\mathcal{W}_I \setminus \mathcal{W}_T$. In \mathcal{W}_{II} we obtain $\bar{N}^{-1} d\bar{N}/d\bar{\tau}_- < \text{const}$, i.e., a limit on the possible increase. To obtain information about \bar{N} in \mathcal{W}_T we use the center manifold reduction theorem: The set \mathcal{W}_T contains a piece of the Taub line TL_γ (up to some value of $\bar{N}_\alpha = \bar{N}_\beta$). The analysis of $\bar{\mathcal{B}}_{VII_0}^{\text{vac}}$ in Section 3.3 implies that each fixed point L_γ on TL_γ is the (non-hyperbolic) source for a one-parameter family of orbits in $\bar{\mathcal{B}}_{VII_0}^{\text{vac}}$. The family of orbits emerging from L_γ forms a two-dimensional invariant surface $\mathcal{S}[L_\gamma]$ in $\bar{\mathcal{B}}_{VII_0}^{\text{vac}} \cap \mathcal{W}_T$. The proof of Lemma 3.1 entails that there exists a universal constant c_T such that

$$\sup_{\mathcal{S}[L_\gamma]}(\bar{N}_\alpha\bar{N}_\beta) \leq c_T \inf_{\mathcal{S}[L_\gamma]}(\bar{N}_\alpha\bar{N}_\beta) \quad (55)$$

for each surface $\mathcal{S}[L_\gamma]$ (in fact, c_T is only marginally larger than 1). We thus obtain, in $\bar{\mathcal{B}}_{VII_0}^{\text{vac}} \cap \mathcal{W}_T$, a foliation of $\bar{\mathcal{B}}_{VII_0}^{\text{vac}}$ into two-dimensional invariant surfaces across which the relative variation of $(\bar{N}_\alpha\bar{N}_\beta)$ is small. Since each point L_γ is a center saddle with a two-dimensional stable subspace/manifold, the foliation of $\bar{\mathcal{B}}_{VII_0}^{\text{vac}} \cap \mathcal{W}_T$ carries over, by the center manifold theorem, to a foliation of \mathcal{W}_T into four-dimensional invariant hypersurfaces $\mathcal{H}[L_\gamma]$. (Each hypersurface is associated with the direct sum of the stable subspace and the tangent space of $\mathcal{S}[L_\gamma]$ at L_γ .) Furthermore, for sufficiently small \mathcal{W}_T , Eq. (55) carries over from $\mathcal{S}[L_\gamma]$ to $\mathcal{H}[L_\gamma]$. Therefore, since every solution in \mathcal{W}_T is contained in one of the hypersurfaces $\mathcal{H}[L_\gamma]$, we obtain $(\bar{N}_\alpha\bar{N}_\beta)(\bar{\tau}_-) \leq c_T(\bar{N}_\alpha\bar{N}_\beta)(\bar{\sigma}_-)$ for all $\bar{\tau}_-$ such that the solution is contained in \mathcal{W}_T , where $\bar{\sigma}_-$ is the time the solution enters \mathcal{W}_T . (Note in particular that this result is independent of the actual time the solution spends in the neighborhood \mathcal{W}_T .) Summing up: For the solution $X(\hat{x}, \bar{\tau}_-)$ we find a decrease of \bar{N} in $\mathcal{W}_I \setminus \mathcal{W}_T$ (comparatively long sojourn times), an increase in \mathcal{W}_{II} (comparatively short sojourn times) and a bound on the increase in \mathcal{W}_T (irrespective of the sojourn times). From these facts we infer an overall decrease of \bar{N} for $\bar{\tau}_- \in [0, \bar{T}_-)$ (i.e., as long as $X(\hat{x}, \bar{\tau}_-) \in \mathcal{W}$). Consequently, at $\bar{\tau}_- = \bar{T}_-$, the quantity \bar{N} is smaller than initially, $\bar{N}|_{\bar{T}_-} < \epsilon_V$.

Since both $\bar{\Omega}|_{\bar{T}_-} < \epsilon_V < \epsilon_W$ and $\bar{N}|_{\bar{T}_-} < \epsilon_V < \epsilon_W$, the solution $X(\hat{x}, \bar{\tau}_-)$ does not actually leave \mathcal{W} at \bar{T}_- but is still in $\mathcal{V} (\subset \mathcal{W})$ for some time beyond \bar{T}_- . It is immediate that this leads to $\bar{T}_- = \infty$, and hence the lemma is established. \square

Remark. Note that not all solutions in \mathcal{V} or \mathcal{W} are generic. In these neighborhoods of \mathcal{A}_{IX} , there also exist LRS solutions, which converge to TL_α , $\alpha = 1, 2, 3$, instead of to \mathcal{A}_{IX} .

Corollary 6.5. *Convergence to the Mixmaster attractor is uniform on compact sets of generic initial data: Let \mathcal{X} be a compact set in \mathcal{B}_{IX} that does not intersect any of the manifolds \mathcal{F} , \mathcal{CS}_α , \mathcal{LRS}_α , so that each initial data $\hat{x} \in \mathcal{X}$ generates a generic type IX solution. Let $X(\hat{x}; \bar{\tau}_-)$ denote the type IX solution with $X(\hat{x}, 0) = \hat{x}$. Then*

$$\|X(\hat{x}; \bar{\tau}_-) - \mathcal{A}_{IX}\| \rightarrow 0 \quad (\bar{\tau}_- \rightarrow \infty) \quad (56)$$

uniformly in $\hat{x} \in \mathcal{X}$.

Proof. Let \mathcal{W} be a neighborhood of the Mixmaster attractor \mathcal{A}_{IX} . We need to show that there exists \bar{T}_- such that $X(\hat{x}; \bar{\tau}_-) \in \mathcal{W}$ for all $\bar{\tau}_- \geq \bar{T}_-$, for all $\hat{x} \in \mathcal{X}$. Let \mathcal{V} be as in Lemma 6.4. Theorem 5.1 implies that for each $\hat{x} \in \mathcal{X}$ there exists $\bar{\sigma}_-(\hat{x})$ such that $X(\hat{x}; \bar{\tau}_-) \in \mathcal{V}$. Since \mathcal{X} is a compact set of generic initial data, $\bar{T}_- := \sup_{\hat{x} \in \mathcal{X}} \bar{\sigma}_-(\hat{x})$ exists. (If this supremum did not exist, we could consider a sequence of initial data along which $\bar{\sigma}_-$ diverges to construct generic initial data violating Theorem 5.1.) Applying Lemma 6.4 we conclude that $X(\hat{x}; \bar{\tau}_-) \in \mathcal{W}$ for all $\bar{\tau}_- \geq \bar{T}_-$, for all $\hat{x} \in \mathcal{X}$. \square

Corollary 6.6. *For generic solutions of Bianchi type IX the Weyl curvature scalar $C_{abcd}C^{abcd}$ (and therefore also the Kretschmann scalar) becomes unbounded towards the past.*

Proof. At a fixed point on K° (which represents a Kasner solution) the Hubble-normalized Weyl curvature scalar $C_{abcd}C^{abcd}/(48H^4)$ is given by $27(2 - \Sigma_1\Sigma_2\Sigma_3)$. Therefore, $C_{abcd}C^{abcd}/(48H^4) \in [0, 4]$ on the Kasner circle K° , where $C_{abcd}C^{abcd}/(48H^4) = 0$ holds only at the Taub points $\{T_1, T_2, T_3\}$. Since the ω -limit set of a generic type IX solution must necessarily contain a fixed point on K° different from the Taub points by Corollary 6.3, we conclude that $C_{abcd}C^{abcd}$ becomes unbounded towards the past; we simply use that $H \rightarrow \infty$ as $\bar{t}_- \rightarrow \infty$ ($t \rightarrow 0$). \square

Corollary 6.7. *Taking into account both the expanding and contracting phases of Bianchi type IX solutions, generic Bianchi type IX initial data generate an inextendible maximally globally hyperbolic development associated with past and future singularities where the curvature becomes unbounded.*

Remark. This is a direct consequence of the previous corollary. It follows straightforwardly that the analogous statement holds for the asymptotically self-similar solutions as well, the only exceptions being the type IX vacuum LRS solutions.

7 Discussion

In this paper we give new and comparatively short proofs of the main rigorous results on Bianchi type IX asymptotic dynamics: Ringström's Bianchi type IX attractor theorem [4], Theorem 5.1, and its consequences. To find more succinct arguments is not our primary motivation to re-investigate the problem. By emphasizing the importance of the Lie contraction hierarchy our proof demonstrates that Bianchi type IX is special in comparison with the other oscillatory Bianchi types: type VIII and type $VI_{-1/9}$. Let us elaborate.

Among the class A Bianchi models Bianchi type IX is characterized by the condition that the three structure constants possess the same sign, see Table 1; in terms of the dynamical systems variables of Section 2 positivity of the structure constants is expressed by the conditions $\bar{N}_1 > 0$, $\bar{N}_2 > 0$, $\bar{N}_3 > 0$. The pivotal feature of type IX dynamics is the following fact: The set of asymptotic states that are accessible to Bianchi type IX models in the past asymptotic limit is represented by the 'Bianchi type IX Lie contraction hierarchy' of Figure 1. Since this hierarchy is obtained by successively setting the type IX structure constants to zero (Lie contractions), the first level is taken by the three equivalent representations of the Bianchi type VII_0 state space, the second level consists of the three representation of Bianchi type II, and the third level coincides with Bianchi type I, see Figure 1. Accordingly, admissible past asymptotic states of type IX models are of Bianchi type I, where $\bar{N}_1 = 0$, $\bar{N}_2 = 0$, $\bar{N}_3 = 0$, of Bianchi type II, where $\bar{N}_\alpha > 0$, $\bar{N}_\beta = \bar{N}_\gamma = 0$, or of Bianchi type VII_0 , where $\bar{N}_\alpha > 0$, $\bar{N}_\beta > 0$, $\bar{N}_\gamma = 0$; cf. Figure 1. (As usual, $(\alpha\beta\gamma)$ runs over the set $\{(123), (231), (312)\}$.) Models of Bianchi type VI_0 are not among the admissible past asymptotic states, since type VI_0 does not appear in the Bianchi type IX Lie contraction hierarchy; this is because one of the structure constants is necessarily negative in type VI_0 . (Using dynamical systems terminology to summarize: The ω -limit set of every type IX orbit lies on the boundary of the type IX state space, which is the union of the type I, type II, and type VII_0 state spaces; in contrast, the type VI_0 state space is not part of the boundary.)

For simplicity and clarity, let us restrict our discussion to the vacuum case. Vacuum Bianchi type I solutions (Kasner solutions) are represented by fixed points on the Kasner circle, K° , while vacuum Bianchi type II solutions are represented by heteroclinic orbits that connect one fixed point on K° with another; see Figure 2. The Mixmaster attractor (which is simply the union of the Kasner circle and the three equivalent representations of the type II vacuum subset) is covered by an intricate network of these heteroclinic orbits, which can be concatenated to form heteroclinic cycles and chains. While Bianchi type I and type II thus fit together seamlessly to form the fabric of type IX asymptotics, Bianchi type VII_0 is the odd one out. Solutions of type VII_0 do not connect one fixed point on the Kasner circle with another, but connect a fixed point on K° with a fixed point on the

Taub line, for which $\bar{N}_\alpha = \bar{N}_\beta > 0$. As a consequence, Bianchi type VII₀ orbits are incompatible with the network of heteroclinic cycles/chains; in particular, Bianchi type VII₀ orbits cannot be concatenated with Bianchi type II orbits to form heteroclinic chains. This incompatibility is crucial: It can be regarded as the underlying reason that excludes Bianchi type VII₀ from the set of possible past asymptotic states. Indeed, the new proof of Theorem 5.1 given in this paper reflects this idea accurately.

The exclusion of Bianchi type VII₀ from the set of admissible past asymptotic states is the cornerstone of the analysis of type IX asymptotics. Once established, the exclusion of type VII₀ automatically leaves the Mixmaster attractor (or a subset thereof) as the past attractor and thus yields Theorem 5.1. So what about Bianchi type VIII then? There is but one difference between type VIII and type IX that is relevant in the context of past asymptotic dynamics: The Lie contraction hierarchies of Bianchi type VIII and Bianchi type IX differ on the first level, compare Figure 1 with Figure 5. The boundary of the type VIII state space encompasses a Bianchi type VII₀ state space and two equivalent representations of the Bianchi type VI₀ state space instead of three type VII₀ representations; hence the Bianchi type VIII Lie contraction hierarchy contains a representation of each class A model, whereas Bianchi type IX does not. (This is merely one aspect of Bianchi type VIII being more general than type IX. Another aspect is the violation of the permutation symmetry in type VIII.)

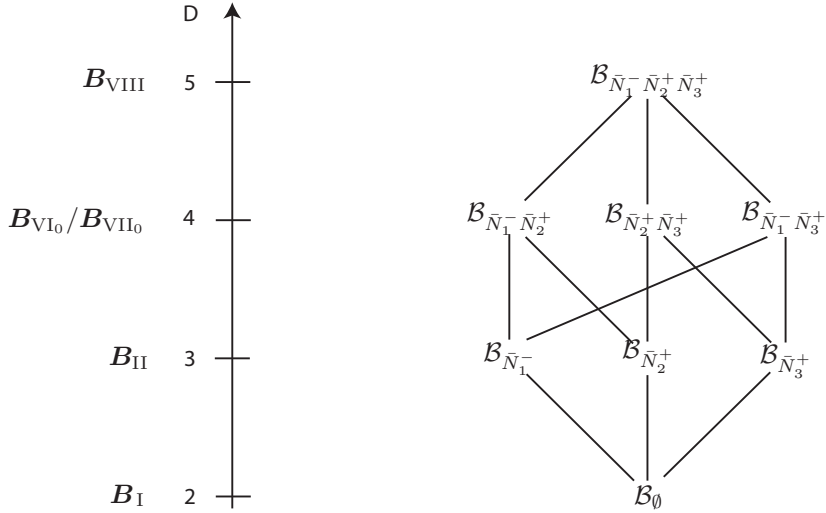


Figure 5: This subset contraction diagram for Bianchi type VIII is to be compared with the one of type IX, see Figure 1. The subscript of \mathcal{B}_* denotes the non-zero variables; variables with the superscript $+$ are positive, variables with the superscript $-$ are negative; e.g. $\mathcal{B}_{\bar{N}_1^- \bar{N}_2^+}$ denotes the type VI₀ subset with $\bar{N}_1 < 0$, $\bar{N}_2 > 0$ and $\bar{N}_3 = 0$. Note that $\mathcal{B}_{\bar{N}_1^-}$ is a completely equivalent representation of $\mathcal{B}_{\bar{N}_1^+}$.

Bianchi type VI₀ and Bianchi type VII₀ possess the same scale-automorphism group which generates corresponding monotone functions [1]. However, the state spaces on which these monotone functions act are rather different: While the type VII₀ state space is unbounded and contains a line of fixed points, the type VI₀ state space is bounded and does not contain this line of fixed points. The reason for these differences between type VI₀ and VII₀ is that type VI₀ does not admit an LRS subset, while VII₀ does. In type VII₀ the vacuum LRS subset contains a one-parameter set of representations of Minkowski spacetime which are different, but equivalent, from the Taub representation in Bianchi type I. In the dynamical systems picture this corresponds to a line of fixed points, TL_α , which is necessarily absent in type VI₀. These features indicate that type VI₀ dynamics is simpler than that of type VII₀, which is indeed the case, see, e.g., [9, 1, 21]. Paradoxically, the very simplicity of the type VI₀ flow is the reason why the analysis of Bianchi type VIII asymptotics is more intricate and complex than that of type IX. Let us elaborate. In contrast to Bianchi type VII₀ solutions, orbits of Bianchi type VI₀ are compatible with the network of hete-

roclinic chains on the Mixmaster attractor. This is a crucial fact, because it implies that Bianchi type VI_0 orbits can be concatenated with heteroclinic chains. Therefore, Bianchi type VI_0 states cannot be ruled out a priori as possible asymptotic states of Bianchi type VIII solutions. It is clear that this causes the (analog of the) proofs of Theorem 5.1 to fail in type VIII. However, the failure of the methods of proof does not imply the failure of the statement: It is expected that (a generic version of) Theorem 5.1 holds and that the Mixmaster attractor is the past attractor for type VIII models. However, it is conceivable that arguments of a completely different kind, like stochastic arguments, are necessary to prove that Bianchi type VI_0 is excluded (generically) from being involved in the asymptotic dynamics of solutions. A statement like this would then be the core of a (generic) version of Theorem 5.1 for Bianchi type VIII.

A closely related issue concerns the validity of Lemma 6.4 and the related Corollary 6.5. Lemma 6.4 fails in Bianchi type VIII. There exists (generic) initial data that is arbitrarily close to the Mixmaster attractor but transported beyond a given neighborhood of the attractor by the flow of the dynamical system. The reason for this difference between type VIII and IX is the by now familiar one: The contraction hierarchy of type VIII contains type VI_0 while that of type IX does not. However, lack of uniform convergence does not imply that there is no convergence at all in type VIII: The Mixmaster attractor might still be the past attractor.

Our arguments support the thesis that Bianchi type IX is special. The existence of discrete symmetries associated with axes permutations, which follow from the positivity of the structure constants, leads to a simplification of the problem and makes the treatment of type IX dynamics relatively straightforward. Type VIII on the other hand is less symmetric and is thus harder to grasp. The same is true for the other remaining oscillatory Bianchi models—the Bianchi type $VI_{-1/9}$ models. These types might therefore be more relevant for our understanding of generic spacelike singularities.

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