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On analytic aymptotically-flat vacuum and electrovac metrics, periodic in time*

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Abstract

By an argument similar to that of Gibbons and Stewart [5], we show that there are no weakly-asymptotically-simple, analytic vacuum or electrovac solutions of the Einstein equations which are periodic in time, in a way made precise.

1 Introduction

There are plausible physical reasons for thinking that, while solutions of the Einstein vacuum or electrovac equations which are genuinely periodic in a suitable time-coordinate, without being static or stationary, may exist, such solutions could not be asymptotically flat. This is because in any context in which the Bondi mass-loss formula can be established, for example in an appropriately asymptotically-flat space-time, periodicity would imply zero mass-loss, which would imply zero radiation, and then one would hope to obtain stationarity. There have been various attempts to prove this result for vacuum, starting with [9] and [10] (summarised in English in [11]) and more recently [5] (see also [13]).

In [9], the author considers vacuum metrics which are everywhere nonsingular, weak and asymptotically-flat and which can be expanded in a series in some parameter, with the flat metric as the first term in the series. Each term in the series is assumed to be periodic in a fixed Minkowski time coordinate and to satisfy the de Donder gauge condition. The second and third terms, call them v_{ab} and w_{ab} respectively, are expanded as Fourier series in the background time-coordinate and the Einstein equations then imply that v_{ab} satisfies the source-free wave equation, and w_{ab} satisfies a wave equation whose source is a quadratic expression in v_{ab}. Assuming that the solution for v_{ab} is everywhere regular, the author shows that there cannot be an asymptotically-flat solution for w_{ab} unless v_{ab} vanishes. Therefore the space-time is flat. In [10], a similar calculation when v_{ab} is regular only outside a certain radius leads to the conclusion that v_{ab} must be time-independent in order to have asymptotically-flat w_{ab}, and the space-time is stationary.

In [5] the authors use the spin-coefficient formalism as set up in, for example [12], to study the system of conformal Einstein equations given in [2]. They introduce a coordinate system based on two families of null hypersurfaces, incoming from past null infinity $\mathcal{I}^-$ and labeled by constant v and outgoing near $\mathcal{I}^-$ and labeled by constant u. They have a definition of periodicity which enables them to prove that, at $\mathcal{I}^-$, the u-derivatives of all orders of all components of the metric are independent of v. They conclude that if the metric is analytic in these coordinates, then it necessarily has a Killing vector, which in these coordinates is $\partial/\partial v$, at least in a neighbourhood of $\mathcal{I}^-$. Thus any analytic metric, periodic in their sense, has such a Killing vector. This leads

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to an unexpected difficulty in that, by construction, their Killing vector is null wherever it is defined, while reducing at $\mathcal{I}^-$ to a constant translation along the generators. These are strong conditions and in fact no Killing vector in flat space has these properties. Thus flat space is not periodic according to their definition and nor is any of the familiar stationary, asymptotically flat solutions, for example the Schwarzschild solution.

This raises the question of whether a different, weaker definition of periodic would permit metrics stationary near $\mathcal{I}$ and indeed allow only these for analytic, asymptotically-flat vacuum metrics. This is what we shall find. In this article I shall follow the method of [5] for both the vacuum and electrovac field equations, but in a different coordinate and tetrad system, similar to the one used at $\mathcal{I}$ in [8], and to prove the existence of a symmetry at the event horizon in [6] and at a compact Cauchy horizon in [7]. The method is essentially the same as in [5], though a little more complicated, and one arrives at the same conclusion, but now with a Killing vector which is time-like in the interior, at least near to $\mathcal{I}^-$.

Clearly, one would like either to drop the assumption of analyticity, for example following the lead of [3] or [4] with a similar problem, or to prove that it follows from the assumptions of periodicity and asymptotic-flatness. It remains to be seen whether this can be done.

The plan of this article is as follows. We begin, in the next section, by introducing a (fairly standard) coordinate and tetrad system for use in weakly-asymptotically-simple space-times, and obtain some familiar conditions for quantities at $\mathcal{I}^-$. Then in section 3 we recall the system of conformal Einstein equations, impose periodicity, and state and prove the main result. The proof requires an induction using some of the conformal Einstein and Einstein-Maxwell equations written out in Newman-Penrose formalism. The relevant equations are collected in an Appendix.

2 Coordinates and tetrad

We are concerned with space-times which are weakly-asymptotically-simple and analytic. Thus we assume that there is a conformal rescaling of the physical metric $\tilde{g}_{ab}$ (following the convention of [12] that the physical metric is tilded, rather than the unphysical one):

$$g_{ab} = \Omega^2 \tilde{g}_{ab}, \tag{1}$$

where $\Omega \to 0$ at $\mathcal{I}^-$, which is a smooth (in fact analytic) null hypersurface in the conformally-extended space-time.

Still following [12], we modify the original choice of Ω to a new $\hat{\Omega} = \Theta\Omega$ where Θ is chosen to satisfy

$$\Box\Theta = \frac{1}{6} R\Theta,$$

where R is the Ricci scalar of the metric g_{ab}. This has the effect of setting to zero the Ricci scalar of the metric $\hat{g}_{ab} = \Theta^2 g_{ab}$. We choose data for a characteristic initial value problem for Θ as follows: choose an incoming null hypersurface N_0 which meets $\mathcal{I}^-$ in a spherical cut S_0; choose Θ on S_0 so that the new metric of S_0 is that of a unit sphere; choose Θ on $\mathcal{I}^-$ so that the spin-coefficient $\hat{\rho}$ vanishes there; and choose Θ on N_0 so that the spin-coefficient $\hat{\mu}$ vanishes there. These choices, which constrain the conformal gauge, are standard and simplify what follows. Having rescaled to the hatted metric, we henceforth drop the hats.

We work with Bondi coordinates (v, θ, ϕ) at $\mathcal{I}^-$, which is to say that v is an affine parameter along the generators, vanishing at S_0, and (θ, ϕ) are standard coordinates at the metric sphere S_0 and are constant along the generators. We define null hypersurfaces N_v in the interior as null surfaces meeting $\mathcal{I}^-$ orthogonally at the 2-spheres S_v of constant v, then define v to be constant on N_v, and label null geodesic generators of the surfaces N_v by the values of $x^A = (\theta, \phi)$ which they meet at $\mathcal{I}^-$. We introduce the first leg of a (Newman-Penrose) null tetrad as $n_a dx^a = dv$ and introduce r as the affine parameter along geodesics tangent to n^a, with $r = 0$ at $\mathcal{I}^-$. Now take ℓ^a as the other null normal to the two-surfaces $S_{v,r}$ of constant v and r, and take m^a as a choice of complex null tangent to $S_{v,r}$, oriented and normalised as is conventional. There is residual freedom to rotate m^a, which we exploit below. Then in the chosen coordinates, the tetrad vectors can be expanded as

$$\ell_a dx^a = dr + H dv,\ n_a dx^a = dv,\ m_a dx^a = \omega dv + R_A dx^A, \tag{2}$$

and the corresponding NP operators as

$$D = \partial_v - H\partial_r + C^A\partial_A,\ \Delta = \partial_r,\ \delta = P^A\partial_A, \tag{3}$$

in terms of real functions H and C^A and complex functions ω, R_A and P^A satisfying

$$\omega + C^A R_A = 0 = P^A R_A,\ \overline{P}^A R_A = -1.$$

We introduce the notation $\doteq$ for equality at $\mathcal{I}^-$, then

$$C^A \doteq 0 \doteq H.$$

From the commutator (28a) in the Appendix, we find $\gamma + \overline{\gamma} = 0$. One of the spin-coefficient equations (see e.g. p219 of [12]) is

$$\Delta m^a = -\tau n^a + (\gamma - \overline{\gamma}) m^a).$$

so that we may rotate m^a to $e^{i\chi}m^a$ and choose the function $\chi(v, r, \theta, \phi)$ along the generators of N_v so as to remove the imaginary part of γ. This sets γ to zero, and we still have the freedom to rotate m^a with a function χ independent of r.

Now the whole set of commutators (28a)-(28d) reduces to

$$\gamma = \nu = \rho - \overline{\rho} = \mu - \overline{\mu} = \tau - \overline{\alpha} - \beta = \tau - \overline{\pi} = 0, \tag{4}$$

and

$$\begin{aligned} \Delta H &= -(\epsilon + \overline{\epsilon}) \\ \Delta C^A &= -2\pi P^A - 2\overline{\pi}\overline{P}^A \\ \Delta P^A &= -\mu P^A - \overline{\lambda}\,\overline{P}^A \end{aligned} \tag{5}$$

and

$$\begin{aligned} \delta H &= -\kappa \\ \delta C^A - D P^A &= -(\rho + \epsilon - \overline{\epsilon}) P^A - \sigma \overline{P}^A \\ \overline{\delta} P^A - \delta \overline{P}^A &= (\alpha - \overline{\beta}) P^A - (\overline{\alpha} - \beta)\overline{P}^A \end{aligned} \tag{6}$$

where the differential equations have been simplified by use of (4). By an earlier choice, v is affine on the generators of $\mathcal{I}^-$ which are geodesics. The spin-coefficient equation

$$D\ell^a = (\epsilon + \overline{\epsilon})\ell^a - \overline{\kappa} m^a - \kappa \overline{m}^a,$$

(again from p219 of [12]) now implies

$$\epsilon + \overline{\epsilon} \doteq 0 \doteq \kappa$$

and we can rotate m^a on $\mathcal{I}^-$ to set $\epsilon \doteq 0$, since

$$D m^a = \overline{\pi}\ell^a + (\epsilon - \overline{\epsilon}) m^a.$$

We shall see shortly that $\sigma \doteq 0$ and then, by (6), $DP^A \doteq 0$, so that P^A takes the value everywhere on $\mathcal{I}^-$ that it takes at S_0. Then we have

$$\delta \doteq \frac{1}{\sqrt{2}}(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi),\ \alpha - \overline{\beta} \doteq -\frac{1}{\sqrt{2}}\cot\theta. \tag{7}$$

3 The conformal Einstein and Einstein-Maxwell equations

The conformal Einstein equations [2] are a system of equations in the unphysical metric g_{ab} which are equivalent to the vacuum field equations on the physical metric $\tilde{g}_{ab}$. One may analogously define a set of conformal Einstein-Maxwell equations and these are what we shall consider. They can be taken to consist of six sets of first-order equations: the derivatives of the tetrad components in terms of the spin-coefficients, (5) and (6); the spin-coefficient equations (29a)-(29r); the Bianchi identities (30a)-(30h), but with $\Omega\phi_i$ substituted for Ψ_i, and contracted Bianchi identities (31a)-(31c), given here subject to the tetrad conditions (4) and the conformal gauge condition $\Lambda = 0$; the zero-rest-mass equation (32a)-(32h) for the field

$$\varphi_{ABCD} = \Omega^{-1}\Psi_{ABCD},$$

where $\Psi_{ABCD} = \tilde{\Psi}_{ABCD}$ is the physical Weyl spinor; the Maxwell equations (34a)-(34d) for the rescaled Maxwell spinor $\phi_{AB} = \Omega^{-1}\tilde{\phi}_{AB}$, where $\tilde{\phi}_{AB}$ is the physical Maxwell spinor; and a set of equations for the conformal factor written in terms of the variables $s_a = \partial_a\Omega$ and $F = 4\Box\Omega$,

$$\begin{aligned} \nabla_a s_b - g_{ab}F &= -\Omega\Phi_{ab} + 2\Omega^3\phi_{AB}\overline{\phi}_{A'B'} \\ \partial_a F &= -\Phi_{ab}s^b + 2\Omega^2 s^{BB'}\phi_{AB}\overline{\phi}_{A'B'}. \end{aligned} \tag{8}$$

With the aid of the conformal Einstein equations, Friedrich has proved a range of existence and uniqueness theorems for asymptotically-flat solutions of the vacuum equations, including existence for analytic solutions from analytic data [2]. We shall use the corresponding electrovac system first to draw conclusions about quantities at $\mathcal{I}^-$.

One may write out the system (8) in the NP formalism first defining

$$s_0 = \ell^a s_a,\ s_1 = m^a s_a,\ s_2 = n^a s_a.$$

We shan't need all the resulting equations, but the system includes the following:

$$Ds_1 - \overline{\pi}s_0 + \kappa s_2 - (\epsilon - \overline{\epsilon})s_1 = -\Omega\Phi_{01} + 2\Omega^3\phi_0\overline{\phi}_1 \tag{9}$$
$$Ds_2 + (\epsilon + \overline{\epsilon})s_2 - \pi s_1 - \overline{\pi}\,\overline{s}_1 - F = -\Omega\Phi_{11} + 2\Omega^3\phi_1\overline{\phi}_1 \tag{10}$$
$$\delta s_2 + (\overline{\alpha} + \beta)s_2 - \mu s_1 - \overline{\lambda}\,\overline{s}_1 = -\Omega\Phi_{12} + 2\Omega^3\phi_1\overline{\phi}_2 \tag{11}$$
$$\overline{\delta}s_1 - (\alpha - \overline{\beta})s_1 - \mu s_0 + \rho s_1 + F = -\Omega\Phi_{11} + 2\Omega^3\phi_1\overline{\phi}_1 \tag{12}$$
$$\delta s_1 - (\beta - \overline{\alpha})s_1 - \overline{\lambda}s_0 + \sigma s_2 = -\Omega\Phi_{02} + 2\Omega^3\phi_0\overline{\phi}_2 \tag{13}$$
$$\Delta s_0 + \pi s_1 + \overline{\pi}_1\overline{s}_1 - F = -\Omega\Phi_{11} + 2\Omega^3\phi_1\overline{\phi}_1 \tag{14}$$
$$\Delta s_1 + \overline{\pi}s_1 = -\Omega\Phi_{12} + 2\Omega^3\phi_1\overline{\phi}_2 \tag{15}$$
$$\Delta s_2 = -\Omega\Phi_{22} + 2\Omega^3\phi_2\overline{\phi}_2 \tag{16}$$

and

$$\begin{aligned} \delta F = & -s_0(\Phi_{12} + 2\Omega^2\phi_1\overline{\phi}_2) - s_2(\Phi_{01} + 2\Omega^2\phi_0\overline{\phi}_1) \\ & +\overline{s}_1(\Phi_{02} + 2\Omega^2\phi_0\overline{\phi}_2) + s_1(\Phi_{11} + 2\Omega^2\phi_1\overline{\phi}_1) \end{aligned} \tag{17}$$
$$\begin{aligned} \Delta F = & -s_0(\Phi_{22} + 2\Omega^2\phi_2\overline{\phi}_2) - s_2(\Phi_{11} + 2\Omega^2\phi_1\overline{\phi}_1) \\ & +\overline{s}_1(\Phi_{12} + 2\Omega^2\phi_1\overline{\phi}_2) + s_1(\Phi_{21} + 2\Omega^2\phi_2\overline{\phi}_1) \end{aligned} \tag{18}$$

At $\mathcal{I}^-$, $\Omega = 0 = r$ but $\Omega_{,r} \neq 0$, so that

$$s_0 \doteq 0,\ s_1 \doteq 0,\ s_2 \mathrel{\dot{\neq}} 0. \tag{19}$$

Then (9), (12), (13) give

$$\kappa \doteq F \doteq \sigma \doteq 0 \tag{20}$$

where the vanishing of σ justifies the last comment in the previous section, and (10), (11) give

$$Ds_2 \doteq 0 \doteq \delta s_2 + (\overline{\alpha} + \beta)s_2,$$

which will require s_2 to be constant on $\mathcal{I}^-$ once we have made $\overline{\alpha}+\beta \doteq 0$. From the spin-coefficient equations given in the Appendix, we find that (29a) and (17) give

$$\Phi_{00} \doteq 0 \doteq \Phi_{01}, \tag{21}$$

while (29d), (29e) give

$$D\alpha \doteq 0 \doteq D\beta \doteq D\pi. \tag{22}$$

Thus far the conclusions are general for asymptotically-flat space-times. Now we impose the condition of time-like periodicity. By this, we mean that there is a discrete isometry taking any point of the physical space-time into its chronological future. To define time-like periodicity for an asymptotically-flat space-time, we require this isometry to extend to an isometry of a neighbourhood of $\mathcal{I}^-$ which preserves the generators of $\mathcal{I}^-$. This therefore has to be a supertranslation:

$$v \to v + a(\theta, \phi),$$

on $\mathcal{I}^-$.

Our aim is the following theorem:

Theorem 3.1 *A weakly-asymptotically simple, vacuum or electrovac, time-periodic space-time which is analytic in a neighbourhood of $\mathcal{I}^-$ in the coordinates introduced above necessarily has a Killing vector which is time-like in the interior and extends to a translation on $\mathcal{I}^-$.*

The proof, as in [5], is by an induction showing that all r-derivatives at $\mathcal{I}^-$ of all metric components are independent of v. Then by analyticity the metric components are all independent of v, and $K = \partial/\partial v$ is a Killing vector. Where we differ from [5] is that K is time-like in the interior, near to $\mathcal{I}^-$, but the method is much the same. To begin the induction we shall establish a lemma:

Lemma 3.2 *The following are zero on $\mathcal{I}^-$:*

$$\rho, \sigma, \pi, \kappa, \mu, \lambda, \epsilon, s_0, s_1, F, \varphi_0, \varphi_1, \Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{12}, \Phi_{22}, \phi_0$$

$$D\alpha, D\beta, D\Phi_{11}, D\varphi_2, D\varphi_3, D\varphi_4, D\phi_1, D\phi_2.$$

Proof:

We have some of these already from (19)-(22). Since mass-loss is in general non-negative, periodicity will also imply zero-mass-loss. From the mass-loss formula, for example equation (35) in [1] but converted to $\mathcal{I}^-$, this means zero 'news-function', which implies the following conditions on the components φ_i of the field φ_{ABCD}:

$$\varphi_0 \doteq 0 \doteq \varphi_1,$$

as well as zero electromagnetic radiation, which implies the following condition on the components ϕ_i of the Maxwell field ϕ_{AB}:

$$\phi_0 \doteq 0. \tag{23}$$

Next we introduce the main method of argument of [5] which we shall use repeatedly: note from (29h) that

$$D\mu \doteq (\delta + 2\beta)\pi, \tag{24}$$

while from (28a) $[\delta, D] \doteq 0$. Therefore

$$D^2\mu \doteq 0,$$

which integrates at once to give

$$\mu \doteq \mu_0 + v\mu_1,$$

in terms of v-independent functions μ_0, μ_1. This is incompatible with periodicity unless $\mu_1 \doteq 0$, that is unless

$$D\mu \doteq 0. \tag{25}$$

This is a powerful technique for deducing $DQ \doteq 0$ from $D^2Q \doteq 0$ for quantities Q, but it needs care. We claim that the tetrad, underlying spinor dyad, spin-coefficients, curvature components and their derivatives all are periodic on $\mathcal{I}^-$. Where we differ from [5] is that they use a second null coordinate u instead of the affine parameter r. Now $\partial/\partial v$ which for them is $(\partial/\partial v)_u$ i.e. differentiation at fixed u, is a null vector, while for us it is $(\partial/\partial v)_r$ i.e. differentiation at fixed r, which is time-like in the interior.

Returning to (24), we have

$$(\delta + 2\beta)\pi \doteq \frac{1}{\sqrt{2}}(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi + \cot\theta - \overline{\pi})\pi \doteq 0,$$

or equivalently

$$(\partial_\theta + \frac{i}{\sin\theta}\partial_\phi)(\sin\theta\pi) \doteq \sin\theta\overline{\pi}\pi,$$

which can be integrated over a sphere of constant v on $\mathcal{I}^-$ to imply $\pi \doteq 0$. From (4), (7) this gives

$$\alpha \doteq -\beta \doteq -\frac{1}{2\sqrt{2}}\cot\theta,$$

when (29l) leads to

$$\Phi_{11} \doteq \frac{1}{2}, \tag{26}$$

so that, in particular, $D\Phi_{11} \doteq 0$. Also from (25), since $\mu = 0$ on N_0, we deduce

$$\mu \doteq 0.$$

This argument works in a similar way for λ: from (29h) and (30e)

$$D\lambda \doteq \Phi_{20},\ D\Phi_{20} \doteq 0,$$

so $D^2\lambda \doteq 0$ and, by periodicity,

$$D\lambda \doteq \Phi_{20} \doteq 0.$$

From (30g), (30c)

$$D\Phi_{22} \doteq (\delta + 2\beta)\Phi_{21},\ D\Phi_{21} \doteq 0,$$

so $D^2\Phi_{22} = 0$ and by periodicity

$$D\Phi_{22} \doteq (\delta + 2\beta)\Phi_{21} \doteq 0.$$

The angular operator on Φ_{21} above is an instance of the Newman-Penrose edth, and has trivial kernel in this case, so by globality on the spherical cross-sections of $\mathcal{I}^-$, $\Phi_{21} \doteq 0$. From (29m)

$$(\delta + 4\beta)\lambda \doteq \Phi_{21} \doteq 0,$$

when the angular operator is another instance of edth, and again by globality $\lambda \doteq 0$. Now from (29n), $\Phi_{22} = 0$ at S_0, where N_0 meets $\mathcal{I}^-$, and $D\Phi_{22} \doteq 0$, so we have $\Phi_{22} \doteq 0$.

Turning to φ_i, by (32b) $D\varphi_2 \doteq 0$. Then by (32c) $D^2\varphi_3 \doteq 0$ so by periodicity $D\varphi_3 \doteq 0$, and the same argument from (32d) gives $D\varphi_4 \doteq 0$. (From these equations, we also have $\overline{\delta}\varphi_2 \doteq 0$ so that $\varphi_2 \doteq$ constant, and $(\overline{\delta} + 2\alpha)\varphi_3 \doteq 0$, which has a finite dimensional set of solutions.)

Turning to the Maxwell spinor, by (34a) and (23), $D\phi_1 \doteq 0$. Then by differentiating (34b) $D^2\phi_2 \doteq 0$ so by periodicity $D\phi_2 \doteq 0 \doteq \overline{\delta}\phi_1$.

□

That completes the proof of the Lemma, now we turn to the proof of the Theorem. We set up an induction:

Suppose inductively that $\partial_v \Delta^j Q \doteq 0$ *for* $0 \le j \le k$ *with* Q *one of*

$$H, C^A, P^A, \epsilon, \pi, \lambda, \beta, \alpha, F, \varphi_i, \Phi_{mn}, \phi_i \tag{27}$$

and for $0 \le j \le k+1$ *with* $Q = s_i$.

This is easily seen to hold for $k = 0$, so we need to deduce it for $k+1$ from its truth for k. (Note that $\partial_v \doteq D$.) Under the inductive hypothesis, this follows for H, C^A, P^A from (5); for $\epsilon, \pi, \lambda, \beta, \alpha$, respectively, from (29i), (29r), (29o), (29j) and (29f), subject to the conditions (4); for F from (18); for $\varphi_i, i = 0, 1, 2, 3$ from (32e)-(32h) subject to (4), leaving φ_4 to be done; for $\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{12}$ from (33b), (33f), (33d) and (33h) respectively, leaving Φ_{22} and Φ_{11} to be done; for ϕ_0 and ϕ_1 from (34c) and (34d) respectively, leaving ϕ_2 to be done.

For φ_4, under the inductive hypothesis, we deduce

$$D^2\Delta^{k+1}\varphi_4 \doteq 0$$

from (32d) and then periodicity implies

$$D\Delta^{k+1}\varphi_4 \doteq 0;$$

then for Φ_{22}, from (33g) we deduce

$$D^2\Delta^{k+1}\Phi_{22} \doteq 0$$

and then periodicity implies

$$D\Delta^{k+1}\Phi_{22} \doteq 0,$$

while for Φ_{11} we now use (31c).

For ϕ_2, from (34b) we deduce

$$D^2\Delta^{k+1}\phi_2 \doteq 0,$$

and then periodicity implies

$$D\Delta^{k+1}\phi_2 \doteq 0.$$

This completes the inductive step for the first set of quantities Q. For $Q = s_i$ we use (14)-(16).

Thus r-derivatives of all orders of the quantities in (27), which includes the metric functions H, C^A and P^A, are independent of v. Now analyticity forces these functions to be independent of v, and therefore, by (2), the metric components are all independent of v and $\partial/\partial v$ is a Killing vector. The norm of this Killing vector is H and from (5), (29f) and (26) we see

$$H \doteq 0, \ \Delta H \doteq 0, \ \Delta^2 H \doteq 1,$$

so that H is positive at least close to $\mathcal{I}^-$ and the Killing vector is time-like in the interior.

□

Appendix

Here we collect the spin-coefficient equations that we shall need. A general reference for this is [12]. First, given the NP operators D, Δ, δ, their commutators in general are:

$$[\Delta, D] = (\gamma + \overline{\gamma})D + (\epsilon + \overline{\epsilon})\Delta - (\tau + \overline{\pi})\overline{\delta} - (\overline{\tau} + \pi)\delta \tag{28a}$$
$$[\delta, D] = (\overline{\alpha} + \beta - \overline{\pi})D + \kappa\Delta - \sigma\overline{\delta} - (\overline{\rho} + \epsilon - \overline{\epsilon})\delta \tag{28b}$$
$$[\delta, \Delta] = -\overline{\nu}D + (\tau - \overline{\alpha} - \beta)\Delta + \overline{\lambda}\,\overline{\delta} + (\mu - \gamma + \overline{\gamma})\delta \tag{28c}$$
$$\left[\overline{\delta}, \delta\right] = (\overline{\mu} - \mu)D + (\overline{\rho} - \rho)\Delta - (\overline{\alpha} - \beta)\overline{\delta} - (\overline{\beta} - \alpha)\delta. \tag{28d}$$

These equations can be viewed as defining the spin-coefficients from the tetrad. Next, what are usually called just the spin-coefficient equations but which can be understood as defining the curvature components from the spin-coefficients:

$$D\rho - \overline{\delta}\kappa = (\rho^2 + \sigma\overline{\sigma}) + (\epsilon + \overline{\epsilon})\rho - \overline{\kappa}\tau - \kappa(3\alpha + \overline{\beta} - \pi) + \Phi_{00} \tag{29a}$$
$$D\sigma - \delta\kappa = (\rho + \overline{\rho})\sigma + (3\epsilon - \overline{\epsilon})\sigma - (\tau - \overline{\pi} + \overline{\alpha} + 3\beta)\kappa + \Psi_0 \tag{29b}$$
$$D\tau - \Delta\kappa = (\tau + \overline{\pi})\rho + (\overline{\tau} + \pi)\sigma + (\epsilon - \overline{\epsilon})\tau - (3\gamma + \overline{\gamma})\kappa + \Psi_1 + \Phi_{01} \tag{29c}$$
$$D\alpha - \overline{\delta}\epsilon = (\rho + \overline{\epsilon} - 2\epsilon)\alpha + \beta\overline{\sigma} - \overline{\beta}\epsilon - \kappa\lambda - \overline{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10} \tag{29d}$$
$$D\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\overline{\rho} - \overline{\epsilon})\beta - (\mu + \gamma)\kappa - (\overline{\alpha} - \overline{\pi})\epsilon + \Psi_1 \tag{29e}$$
$$D\gamma - \Delta\epsilon = (\tau + \overline{\pi})\alpha + (\overline{\tau} + \pi)\beta - (\overline{\epsilon} + \epsilon)\gamma - (\gamma + \overline{\gamma})\epsilon + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda \tag{29f}$$
$$D\lambda - \overline{\delta}\pi = (\rho\lambda + \overline{\sigma}\mu) + \pi^2 + (\alpha - \overline{\beta})\pi - \overline{\kappa}\nu - (3\epsilon - \overline{\epsilon})\lambda + \Phi_{20} \tag{29g}$$
$$D\mu - \delta\pi = (\overline{\rho}\mu + \sigma\lambda) + \pi\overline{\pi} - (\epsilon + \overline{\epsilon})\mu - \pi(\overline{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda \tag{29h}$$
$$D\nu - \Delta\pi = (\pi + \overline{\tau})\mu + (\overline{\pi} + \tau)\lambda + (\gamma - \overline{\gamma})\pi - (3\epsilon + \overline{\epsilon})\nu + \Psi_3 + \Phi_{21} \tag{29i}$$
$$\Delta\lambda - \overline{\delta}\nu = -(\mu + \overline{\mu})\lambda - (3\gamma - \overline{\gamma})\lambda + (3\alpha + \overline{\beta} + \pi - \overline{\tau})\nu - \Psi_4 \tag{29j}$$
$$\delta\rho - \overline{\delta}\sigma = \rho(\overline{\alpha} + \beta) - \sigma(3\alpha - \overline{\beta}) + (\rho - \overline{\rho})\tau + (\mu - \overline{\mu})\kappa - \Psi_1 + \Phi_{01} \tag{29k}$$
$$\delta\alpha - \overline{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta + \gamma(\rho - \overline{\rho}) + \epsilon(\mu - \overline{\mu}) - \Psi_2 + \Phi_{11} + \Lambda \tag{29l}$$
$$\delta\lambda - \overline{\delta}\mu = (\rho - \overline{\rho})\nu + (\mu - \overline{\mu})\pi + \mu(\alpha + \overline{\beta}) + \lambda(\overline{\alpha} - 3\beta) - \Psi_3 + \Phi_{21} \tag{29m}$$
$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\overline{\lambda}) + (\gamma + \overline{\gamma})\mu - \overline{\nu}\pi + (\tau - 3\beta - \overline{\alpha})\nu + \Phi_{22} \tag{29n}$$
$$\delta\gamma - \Delta\beta = (\tau - \overline{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\overline{\nu} - \beta(\gamma - \overline{\gamma} - \mu) + \alpha\overline{\lambda} + \Phi_{12} \tag{29o}$$
$$\delta\tau - \Delta\sigma = (\mu\sigma + \overline{\lambda}\rho) + (\tau + \beta - \overline{\alpha})\tau - (3\gamma - \overline{\gamma})\sigma - \kappa\overline{\nu} + \Phi_{02} \tag{29p}$$
$$\Delta\rho - \overline{\delta}\tau = -(\rho\overline{\mu} + \sigma\lambda) + (\overline{\beta} - \alpha - \overline{\tau})\tau + (\gamma + \overline{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda \tag{29q}$$
$$\Delta\alpha - \overline{\delta}\gamma = (\rho + \epsilon)\nu - (\tau + \beta)\lambda + (\overline{\gamma} - \overline{\mu})\alpha + (\overline{\beta} - \overline{\tau})\gamma - \Psi_3. \tag{29r}$$

Now the Bianchi identities, namely the system:

$$\nabla_{A'}^{\ D}\Psi_{ABCD} = \nabla_{(A}^{\ B'}\Phi_{BC)A'B'},$$

simplified by the conditions we are imposing namely $\Lambda = 0$ and conditions (4). Thus we omit

γ, ν and eliminate $\tau, \alpha, \overline{\rho}, \overline{\mu}$ in favour of π, β, ρ, μ:

$$
\begin{aligned}
D\Psi_1 - \overline{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} &= -3\kappa\Psi_2 + (2\epsilon + 4\rho)\Psi_1 + (-3\pi + 4\beta)\Psi_0 \\
&\quad + \overline{\pi}\Phi_{00} - 2(\rho + \epsilon)\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \overline{\kappa}\Phi_{02} && \text{(30a)} \\
D\Psi_2 - \overline{\delta}\Psi_1 + \Delta\Phi_{00} - \overline{\delta}\Phi_{01} &= -2\kappa\Psi_3 + 3\rho\Psi_2 + 2\overline{\beta}\Psi_1 - \lambda\Psi_0 \\
&\quad - \mu\Phi_{00} - 2(2\pi - \overline{\beta})\Phi_{01} - 2\overline{\pi}\Phi_{10} + 2\rho\Phi_{11} + \overline{\sigma}\Phi_{02} && \text{(30b)} \\
D\Psi_3 - \overline{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} &= -\kappa\Psi_4 - (2\epsilon - 2\rho)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1 \\
&\quad + 2\mu\Phi_{10} - 2\pi\Phi_{11} - (4\beta - \overline{\pi})\Phi_{20} - 2(\rho - \epsilon)\Phi_{21} + \overline{\kappa}\Phi_{22} && \text{(30c)} \\
D\Psi_4 - \overline{\delta}\Psi_3 + \Delta\Phi_{20} - \overline{\delta}\Phi_{21} &= -(4\epsilon - \rho)\Psi_4 + (6\pi - 2\overline{\beta})\Psi_3 - 3\lambda\Psi_2 \\
&\quad - 2\lambda\Phi_{11} - 2\mu\Phi_{20} - 2\overline{\beta}\Phi_{21} + \overline{\sigma}\Phi_{22} && \text{(30d)} \\
\Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} &= -\mu\Psi_0 - (4\overline{\pi} + 2\beta)\Psi_1 + 3\sigma\Psi_2 \\
&\quad - \overline{\lambda}\Phi_{00} + 2(\overline{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} + (\rho + 2\epsilon - 2\overline{\epsilon})\Phi_{02} - 2\kappa\Phi_{12} && \text{(30e)} \\
\Delta\Psi_1 - \delta\Psi_2 - \Delta\Phi_{01} + \overline{\delta}\Phi_{02} &= -2\mu\Psi_1 - 3\overline{\pi}\Psi_2 + 2\sigma\Psi_3 \\
&\quad + 2\mu\Phi_{01} + (3\pi - 4\overline{\beta})\Phi_{02} + 2\overline{\pi}\Phi_{11} - 2\rho\Phi_{12} && \text{(30f)} \\
\Delta\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} &= -3\mu\Psi_2 + (-2\overline{\pi} + 2\beta)\Psi_3 + \sigma\Psi_4 \\
&\quad - 2\mu\Phi_{11} - \overline{\lambda}\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \overline{\pi})\Phi_{21} + (\rho - \epsilon - \overline{\epsilon})\Phi_{22} && \text{(30g)} \\
\Delta\Psi_3 - \delta\Psi_4 - \Delta\Phi_{21} + \overline{\delta}\Phi_{22} &= -4\mu\Psi_3 + (-\overline{\pi} + 4\beta)\Psi_4 + 2\lambda\Phi_{12} + 2\mu\Phi_{21} - \pi\Phi_{22}. && \text{(30h)}
\end{aligned}
$$

The contracted Bianchi identities with $\Lambda = 0$, corresponding to

$$\nabla^{AA'}\Phi_{ABA'B'} = 0,$$

become the system

$$
\begin{aligned}
D\Phi_{11} - \delta\Phi_{10} + \Delta\Phi_{00} - \overline{\delta}\Phi_{01} &= -2\mu\Phi_{00} - (\pi + 2\alpha)\Phi_{01} - (\overline{\pi} + 2\overline{\alpha})\Phi_{10} + 4\rho\Phi_{11} \\
&\quad + \overline{\sigma}\Phi_{02} + \sigma\Phi_{20} - \overline{\kappa}\Phi_{12} - \kappa\Phi_{21} && \text{(31a)} \\
D\Phi_{12} - \delta\Phi_{11} + \Delta\Phi_{01} - \overline{\delta}\Phi_{02} &= -3\mu\Phi_{01} - \overline{\lambda}\Phi_{10} + 2(\overline{\beta} - \alpha)\Phi_{02} \\
&\quad + (3\rho - 2\overline{\epsilon})\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22} && \text{(31b)} \\
D\Phi_{22} - \delta\Phi_{21} + \Delta\Phi_{11} - \overline{\delta}\Phi_{12} &= -4\mu\Phi_{11} - \lambda\Phi_{02} - \overline{\lambda}\Phi_{20} + (\pi + 2\overline{\beta})\Phi_{12} \\
&\quad + (\overline{\pi} + 2\beta)\Phi_{21} + 2(\rho - \epsilon - \overline{\epsilon})\Phi_{22}. && \text{(31c)}
\end{aligned}
$$

Next the zero-rest-mass equations for the spin-2 field $\varphi_{ABCD} = \Omega^{-1}\Psi_{ABCD}$:

$$
\begin{aligned}
D\varphi_1 - \overline{\delta}\varphi_0 &= -3\kappa\varphi_2 + (2\epsilon + 4\rho)\varphi_1 - (-\pi + 4\alpha)\varphi_0 && \text{(32a)} \\
D\varphi_2 - \overline{\delta}\varphi_1 &= -2\kappa\varphi_3 + 3\rho\varphi_2 - (-2\pi + 2\alpha)\varphi_1 - \lambda\varphi_0 && \text{(32b)} \\
D\varphi_3 - \overline{\delta}\varphi_2 &= -\kappa\varphi_4 - (2\epsilon - 2\rho)\varphi_3 + 3\pi\varphi_2 - 2\lambda\varphi_1 && \text{(32c)} \\
D\varphi_4 - \overline{\delta}\varphi_3 &= -(4\epsilon - \rho)\varphi_4 + (4\pi + 2\alpha)\varphi_3 - 3\lambda\varphi_2 && \text{(32d)} \\
\Delta\varphi_0 - \delta\varphi_1 &= (4\gamma - \mu)\varphi_0 - (4\tau + 2\beta)\varphi_1 + 3\sigma\varphi_2 && \text{(32e)} \\
\Delta\varphi_1 - \delta\varphi_2 &= \nu\varphi_0 + (2\gamma - 2\mu)\varphi_1 - 3\tau\varphi_2 + 2\sigma\varphi_3 && \text{(32f)} \\
\Delta\varphi_2 - \delta\varphi_3 &= 2\nu\varphi_1 - 3\mu\varphi_2 + (-2\tau + 2\beta)\varphi_3 + \sigma\varphi_4 && \text{(32g)} \\
\Delta\varphi_3 - \delta\varphi_4 &= 3\nu\varphi_2 - (2\gamma + 4\mu)\varphi_3 + (-\tau + 4\beta)\varphi_4. && \text{(32h)}
\end{aligned}
$$

The Bianchi identities simplified by the zero-rest-mass equations (32a)-(32h) and with $\Psi_i = \Omega\varphi_i$:

$$-D\Phi_{01} + \delta\Phi_{00} = -s_0\varphi_1 + \bar{s}_1\varphi_0 + \bar{\pi}\Phi_{00} - 2(\rho+\epsilon)\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \bar{\kappa}\Phi_{02} \tag{33a}$$

$$\Delta\Phi_{00} - \bar{\delta}\Phi_{01} = -s_0\varphi_2 + \bar{s}_1\varphi_1 - \mu\Phi_{00} - 2(2\pi - \bar{\beta})\Phi_{01} - 2\bar{\pi}\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} \tag{33b}$$

$$-D\Phi_{21} + \delta\Phi_{20} = -s_0\varphi_3 + \bar{s}_1\varphi_2 + 2\mu\Phi_{10} - 2\pi\Phi_{11} - (4\beta - \bar{\pi})\Phi_{20} - 2(\rho-\epsilon)\Phi_{21} + \bar{\kappa}\Phi_{22} \tag{33c}$$

$$\Delta\Phi_{20} - \bar{\delta}\Phi_{21} = -s_0\varphi_4 + \bar{s}_1\varphi_3 - 2\lambda\Phi_{11} - 2\mu\Phi_{20} - 2\bar{\beta}\Phi_{21} + \bar{\sigma}\Phi_{22} \tag{33d}$$

$$\begin{aligned} D\Phi_{02} - \delta\Phi_{01} = {} & -s_2\varphi_0 + s_1\varphi_1 - \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} \\ & + (\rho + 2\epsilon - 2\bar{\epsilon})\Phi_{02} - 2\kappa\Phi_{12} \end{aligned} \tag{33e}$$

$$-\Delta\Phi_{01} + \bar{\delta}\Phi_{02} = -s_2\varphi_1 + s_1\varphi_2 + 2\mu\Phi_{01} + (3\pi - 4\bar{\beta})\Phi_{02} + 2\bar{\pi}\Phi_{11} - 2\rho\Phi_{12} \tag{33f}$$

$$\begin{aligned} D\Phi_{22} - \delta\Phi_{21} = {} & -s_2\varphi_2 + s_1\varphi_3 - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + 2(\beta + \bar{\pi})\Phi_{21} \\ & + (\rho - \epsilon - \bar{\epsilon})\Phi_{22} \end{aligned} \tag{33g}$$

$$-\Delta\Phi_{21} + \bar{\delta}\Phi_{22} = -s_2\varphi_3 + s_1\varphi_4 + 2\lambda\Phi_{12} + 2\mu\Phi_{21} - \pi\Phi_{22}. \tag{33h}$$

Finally the Maxwell equations with (4) become:

$$D\phi_1 - \bar{\delta}\phi_0 = (2\bar{\beta} - \pi)\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \tag{34a}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2 \tag{34b}$$

$$\delta\phi_1 - \Delta\phi_0 = \mu\phi_0 + 2\bar{\pi}\phi_1 - \sigma\phi_2 \tag{34c}$$

$$\delta\phi_2 - \Delta\phi_1 = 2\mu\phi_1 + (\bar{\pi} - 2\beta)\phi_2. \tag{34d}$$

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