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REPORT No. 23, 2008/2009, fall

ISSN 1103-467X

ISRN IML-R- -23-08/09- -SE+fall

The peeling in the “very external region” of non linear perturbations of the Kerr spacetime

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Abstract

Let an initial data metric \bar{g} be, outside a ball B_{R_0} centered in the origin, the induced metric on Σ_0 of a Kerr spacetime (with a mass M and angular momentum J whose ratio, J/M , depends on the size of R_0) plus small corrections which decay at spacelike infinity faster than r^{-3} ; let, in the same region, a symmetric tensor \bar{k} be the second fundamental form of the Kerr spacetime plus small corrections which decay at spacelike infinity faster than r^{-4} , let \bar{g} and \bar{k} satisfy the constraint equations. Then, using the previous results of [Ch-Kl] and [Kl-Ni1], the global existence of the external region of a global spacetime, outside the region of influence of B_{R_0} follows.

In this external region the various components of the Riemann tensor decay along the outgoing null directions in agreement with the suggestion of the “Peeling Theorem”.

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1 Introduction and results

The problem of the global stability for the Kerr spacetime is a very difficult and open problem. The more difficult issue is that of proving the existence of solutions of the vacuum Einstein equations with initial data “near to Kerr” in the whole external region up to the event horizon, which is also an unknown of the problem.¹ If we consider the existence problem in a part of the external region sufficiently far from the Kerr event horizon, we call hereafter the “very external region”, the result is included, in the version of Minkowski stability result proved by S.Klainerman and the present author with initial data near the flat ones (see [Kl-Ni1] and also [Ch-Kl]). How far the “very external region” has to be from the event horizon² will depend on the mass, M , of the Kerr spacetime, see later for a detailed discussion.

What is known up to now relative to the whole outer region are some relevant uniform boundedness results for solutions to the wave equation in the Kerr spacetime used as a background spacetime, see Dafermos-Rodnianski, [Da-Ro] and references therein.³

If we are interested to the asymptotic behaviour of the Riemann components toward (a portion of)⁴ the null infinity, more precisely to the question if the non linear perturbation of the “very external region” of the Kerr spacetime satisfy the peeling properties⁵ one has to observe that also this much easier, but not easy, problem is an open problem.

In [Ch-Kl] and in [Kl-Ni1], the null asymptotic behaviour of some of the null components of the Riemann tensor, specifically the α and the β components, see later for their definitions, is different from the one expected from the “Peeling Theorem”, [W], as the decay proved there is slower. Nevertheless in a subsequent paper S.Klainerman and the present author, [Kl-Ni2], proved for these components an asymptotic behaviour consistent with the “Peeling Theorem” under stronger asymptotic conditions for the initial data. Unfortunately these conditions do not include initial data “near to Kerr”, due to the fact that, as the angular momentum J of the Kerr spacetime is different from zero, in the Kerr initial data there are metric components decaying as r^{-2} , a decay not sufficient to guarantee the peeling in [Kl-Ni2].

Therefore the new result we expect to be true and that we would like to prove is the following one:

Expected result: *Assuming the initial data on the hypersurface Σ_0/B_{R_0} near to the Kerr initial data, where B_{R_0} is a ball centered at the origin sufficiently*

¹Of course for small perturbations the horizon is expected to stay “near” to the Kerr horizon for $r_+ = m + \sqrt{m^2 - a^2}$.

²More precisely, the intersection of this region with Σ_0 from the ball centered in the origin with radius $r_+ = m + \sqrt{m^2 - a^2}$.

³See also for the $J = 0$ case, [Bl] and references therein.

⁴As we are concerned to a spacetime region whose initial data are given on Σ_0 outside a ball centered in the origin of radius $R_0 \gg r_+$ this implies that we cannot obtain the asymptotic behaviour for the Riemann tensor on the whole null infinity, but only on a portion of it.

⁵The decay of the Riemann components satisfying or not the “Peeling Theorem” have been a problem studied by many authors, see [Kr1], [Kr2] and references therein.

large, it is possible to prove a global existence result for the “very external region” such that the null Riemann components decay in agreement with the “Peeling Theorem”.

In fact we prove a weaker result where the size of the Kerr mass, M , depends on how large is R_0 , namely:

Proved result: *Assuming the initial data on the initial hypersurface Σ_0/B_{R_0} , where B_{R_0} is a ball centered at the origin sufficiently large, near to the initial data of a Kerr spacetime whose mass M is (upper) bounded by a quantity depending on the size R_0 , it is possible to prove a global existence result for the “very external region” such that the null Riemann components decay in agreement with the “Peeling Theorem”.*

The detailed version of this theorem is given at the end of this section. Its proof can be divided in five steps which we summarize here and will describe in detail in the following sections:

a) From the explicit expression of the Kerr metric, in the Boyer-Lindquist coordinates, $\{t, r, \theta, \phi\}$,⁶

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4Mar \sin^2 \theta}{\Sigma} d\phi dt + R^2 \sin^2 \theta d\phi^2$$

we can write its restriction to the $t = 0$ hypersurface, Σ_0 , as

$$\begin{aligned} \mathbf{g} &= \mathbf{g}_{(Sch)} + \left(\frac{a^2 \sin^2 \theta}{r^2} + O\left(\frac{a^2 m}{r^3}\right) + O\left(\frac{a^4}{r^4}\right) \right) dr^2 + \left(\frac{a^2 \cos^2 \theta}{r^2} \right) r^2 d\theta^2 \\ &+ \left(\frac{a^2}{r^2} + O\left(\frac{a^2 m}{r^3}\right) + O\left(\frac{a^4}{r^4}\right) \right) r^2 \sin^2 \theta d\phi^2 . \end{aligned} \quad (1.1)$$

where $\mathbf{g}_{(Sch)}$ denotes the restriction of the Schwarzschild metric on the initial hypersurface:

$$\mathbf{g}_{(Sch)} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) .$$

It follows that the components of the correction to the $\mathbf{g}_{(Sch)}$ metric, $(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)})$, have terms of order $O(a^2/r^2)$ or higher. Therefore if we consider in the external region initial data which are a “small modification” of those of a Kerr spacetime with a small parameter $a = J/m \leq \varepsilon_0$ we can write, for the metric on Σ_0 ,

$$\mathbf{g} = \mathbf{g}_{(Kerr)} + \delta\mathbf{g} = \mathbf{g}_{(Sch)} + [(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)}) + \delta\mathbf{g}] \quad (1.2)$$

and if $\delta\mathbf{g}$ is a small correction to the Kerr initial data with a better decay toward spacelike infinity, then it follows that the whole correction to $\mathbf{g}_{(Sch)}$ satisfies the conditions of the theorem proved in [KI-Ni1].

Observe that the initial decay conditions in the present case are even stronger than those required in [KI-Ni1] where the components of the metric restriction

⁶See the appendix eq.4.146 for the definitions of the various coefficients.

on Σ_0 have to decay faster than $r^{-\frac{3}{2}}$. This will turn out to be crucial in the sequel.

These considerations allow us to conclude that, outside the dependance region of the sphere B_{R_0} , we have a global spacetime satisfying all the properties proved in [Kl-Ni1] which can be interpreted as a “small” non linear perturbation of the very external region of a Kerr spacetime with a small angular momentum. Moreover expression 1.1 and the fact that in [Kl-Ni1] the decay required for \mathbf{g} is $r^{-\frac{3}{2}}$ allows to use part of the extra decay of the term $O(a^2/r^2)$ to avoid the condition $a = J/M$ small. In fact we can choose $\gamma < 1/2$ and require

$$\frac{a^2}{R_0^\gamma} \leq \epsilon_0 \quad \text{equivalent to} \quad \frac{J}{M} \leq \sqrt{R_0^\gamma \epsilon_0} . \quad (1.3)$$

If we consider the Kerr spacetimes satisfying the inequality $a = J/M \leq M^\gamma$ then condition 1.3 is satisfied when

$$M \leq \sqrt{R_0^\gamma \epsilon_0} \quad (1.4)$$

which tells us that provided the Kerr mass satisfies inequality 1.4 no conditions on J is imposed.⁸ In the sequel we denote $B_{R_0} \equiv B$.

b) From the results in [Kl-Ni1], it follows that the decay in the null directions of the various components of the Riemann tensor⁹ do not follow the “Peeling theorem”. More precisely the components α and β decay as $r^{-\frac{7}{2}}$ while we “expect” r^{-5} and r^{-4} respectively.¹⁰ In a subsequent paper, [Kl-Ni2], S.Klainerman and the author proved that the decay suggested from the “Peeling theorem” could be obtained assuming a stronger spacelike decay for the initial data. In particular we proved the following result:

Theorem 1.1. *Let assume that on Σ_0/B the metric and the second fundamental form have the following asymptotic behaviour*¹¹

$$\begin{aligned} g_{ij} &= g_S{}_{ij} + O_{q+1}(r^{-(3+\epsilon)}) \\ k_{ij} &= O_q(r^{-(4+\epsilon)}) \end{aligned} \quad (1.5)$$

where g_S denotes the restriction of the Schwarzschild metric on the initial hypersurface:

$$g_S = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$

Let us assume that a smallness condition for the initial data is satisfied.¹² Then

⁷Those Kerr spacetimes which do not have a naked singularity and can therefore be interpreted as representing the spacetime around a black hole with angular momentum J .

⁸Remark that, unfortunately, this does not allow to choose $R_0 \cong 2M$ as to prove the global existence in the external region, see [Kl-Ni1], Chapter 3, the following inequality $M/R_0 \leq \sqrt{\epsilon_0}$ has to be satisfied.

⁹Components relative to a null frame adapted to the null outgoing and incoming cones which foliate the “very external region” we will denote \mathcal{K} .

¹⁰In principle some log powers can be present, see J.A.V.Kroon,[Kr2].

¹¹Here $f = O_q(r^{-a})$ means that f asymptotically behaves as $O(r^{-a})$ and its partial derivatives $\partial^k f$, up to order q behave as $O(r^{-a-k})$. Here with g_{ij} we mean the components written in Cartesian coordinates.

¹²The details of the smallness condition are in [Kl-Ni2].

along the outgoing null hypersurfaces $C(u)$ (of the external region \mathcal{K}) the following limits hold, with $\epsilon' < \epsilon$ and \underline{u} and u the generalization of the Finkelstein variables $u = t - r_*$ $u = t + r_*$ in the Schwarzschild spacetime:

$$\begin{aligned}
\lim_{C(u); \underline{u} \rightarrow \infty} r(1 + |u|)^{(4+\epsilon)} \underline{\alpha} &= C_0 \\
\lim_{C(u); \underline{u} \rightarrow \infty} r^2(1 + |u|)^{(3+\epsilon)} \underline{\beta} &= C_0 \\
\lim_{C(u); \underline{u} \rightarrow \infty} r^3 \rho &= C_0 \\
\lim_{C(u); \underline{u} \rightarrow \infty} r^3 \sigma &= C_0 \\
\lim_{C(u); \underline{u} \rightarrow \infty} r^4(1 + |u|)^{(1+\epsilon)} \beta &= C_0 \\
\sup_{(u, \underline{u}) \in \mathcal{K}} r^5(1 + |u|)^{\epsilon'} |\alpha| &\leq C_0 .
\end{aligned} \tag{1.6}$$

$\underline{\alpha}, \underline{\beta}, \dots$ are the null components of the Riemann tensor defined with respect to a null frame adapted to the double null foliation, see [Kl-Ni1] Chapter 2, we recall here their definition for completeness,

$$\begin{aligned}
\alpha(R)(e_a, e_b) &= R(e_a, e_4, e_b, e_4) , \quad \underline{\alpha}(R)(e_a, e_b) = R(e_a, e_3, e_b, e_3) \\
\beta(R)(e_a) &= \frac{1}{2} R(e_a, e_4, e_3, e_4) , \quad \underline{\beta}(R)(e_a) = \frac{1}{2} R(e_a, e_3, e_3, e_4) \\
\rho(R) &= \frac{1}{4} R(e_3, e_4, e_3, e_4) , \quad \sigma(R) = \frac{1}{4} \rho({}^*R) = \frac{1}{4} {}^*R(e_3, e_4, e_3, e_4)
\end{aligned} \tag{1.7}$$

where $\{e_a, e_b, e_3, e_4\}$ is a null frame adapted to the foliation.

This result was obtained, basically, in two steps. The first one consisted in proving that a family of energy-type norms, \tilde{Q} , made with the Bel-Robinson tensor associated to the Riemann tensor R , of the same type as those used to prove the global existence of the spacetime in the external region \mathcal{K} see [Kl-Ni1], but with a different weight in the integrand, were bounded in terms of the same norms relative to the initial data. This allowed to prove that the various null components of the Riemann tensor, besides the decay in r (or \underline{u}) already proved, have a decay factor in the $|u|$ variable. In the second step it was proved, integrating along the incoming cones, that the extra decay in the u variable can be transformed in an extra decay in the r variable proving the final result.

Unfortunately Theorem 1.1 cannot be applied to the present case as the initial conditions of the previous theorem do not include the initial data 1.2, due to the part $(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)})$ which decays as $O(r^{-2})$.

This can be understood observing that the extra weight in the \tilde{Q} norms, required in [Kl-Ni2], is a factor $|u|^{(3+\epsilon)}$ and, as on the initial hypersurface $|u|$ is equivalent to r , it follows that the decay $O(r^{-2})$ of $(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)})$ is not sufficient to make the initial data norms $\tilde{Q}|_{\Sigma_0}$ bounded. To be more explicit let us look at the second term of the \tilde{Q} norm in [Kl-Ni2], eq.(2.6), on the initial hypersurface,

$$\int_{\Sigma_0 \cap \mathcal{K}} |u|^{3+\epsilon} Q(\hat{\mathcal{L}}_O R)(\bar{K}, \bar{K}, T, T) .$$

Recalling the expression of $Q(\hat{\mathcal{L}}_O R)(\bar{K}, \bar{K}, T, T)$, see [Kl-Nil], Chapter 4,

$$\begin{aligned} Q(\hat{\mathcal{L}}_O R)(\bar{K}, \bar{K}, T, T) &= \frac{1}{8} \underline{u}^4 |\alpha|^2 + \frac{1}{8} u^4 |\underline{\alpha}|^2 + \frac{1}{2} (\underline{u}^4 + \frac{1}{2} u^2 \underline{u}^2) |\beta|^2 \\ &+ \frac{1}{2} (u^4 + 4 \underline{u}^2 u^2 + \underline{u}^4) (\rho^2 + \sigma^2) + \frac{1}{2} (u^4 + \frac{1}{2} u^2 \underline{u}^2) |\underline{\beta}|^2 \end{aligned}$$

and observing that on $\Sigma_0 \cap \mathcal{K}$ u, \underline{u} behave like r , we have

$$\begin{aligned} \int_{\Sigma_0 \cap \mathcal{K}} |u|^{3+\epsilon} Q(\hat{\mathcal{L}}_O R)(\bar{K}, \bar{K}, T, T) &\cong \\ \int_{R_0}^{\infty} dr r^{2+(3+\epsilon)+4} [|\alpha|^2 + |\underline{\alpha}|^2 + |\beta|^2 + (\rho^2 + \sigma^2) + |\underline{\beta}|^2] &. \end{aligned}$$

This integral is convergent if all the null components of $\hat{\mathcal{L}}_O R$ decay faster than r^{-5} which implies that the various components of \mathbf{g} have to decay faster than r^{-3} , while $(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)})$ decays as r^{-2} and in particular $\rho(\hat{\mathcal{L}}_O R)$ decays as r^{-4} .¹³

The idea to overcome this difficulty is based, intuitively, on the fact that the Kerr spacetime is static and $\frac{\partial}{\partial t}$ is a Killing vector field. Therefore if, instead of considering the Riemann components, we consider their time derivatives, they do not depend anymore on the Kerr part of the Riemann tensor, Their initial data can have a good decay and then, reapplying Theorem 1.1, a good decay along the null directions. Once we have this decay by a simple “time integration” we can obtain the decay of the Riemann tensor.

This argument as described here is not rigorous, but could be directly implemented if we consider a “linear perturbation” of the (very external) Kerr spacetime.¹⁴ In the non linear case the proof of its validity has to be redone from scratch, but it turns out that the basic idea is still valid in the following sense:

Let us consider instead of the time derivative of the Riemann tensor its (modified) Lie derivative $\hat{\mathcal{L}}_T R$,¹⁵ where T is not anymore a Killing vector, but only “nearly Killing”,¹⁶ then we can define some \tilde{Q} norms relative to $\hat{\mathcal{L}}_T R$ with appropriate weights and prove that they are bounded in terms of the corresponding initial data norms.

To prove that these norms are bounded in terms of the initial data ones, is not obvious even if we are in a given spacetime \mathcal{K} . The reason is that the “Error”¹⁷ which has to be controlled depends on the ρ component of the $\hat{\mathcal{L}}_T$ derivative

¹³One could expect a decay r^{-3} , but the result is better as the part of the ρ component which decays as r^{-3} is spherically symmetric and therefore cancelled by $\hat{\mathcal{L}}_O$. Here the fact that $\hat{\mathcal{L}}_O$ is a “modified” Lie derivative, see later on for its definition, is not relevant.

¹⁴This is, basically, part of a paper by G.Caciotta and T.Raparelli, to appear, where it is also discussed in which sense we can define a linear perturbation of the Kerr spacetime.

¹⁵See later for its precise definition.

¹⁶With “nearly Killing” we mean that its deformation tensor is small with respect to some Sobolev norms.

¹⁷with Error we mean the difference between the norms on the final boundary of a region of \mathcal{K} and those defined on (the portion of) the initial hypersurface.

of the Riemann tensor, $\hat{\mathcal{L}}_T R$, which cannot be estimated by the $\tilde{\mathcal{Q}}$ norms (they can only provide an estimate of $\rho - \bar{\rho}$, see [Kl-Ni1], Chapter 5); on the other side ρ is estimated by an integration along the incoming directions which, to be done, requires the control of the other Riemann null components which are at their turn estimated in terms of the $\tilde{\mathcal{Q}}$ norms. This clearly suggests the need of a bootstrap mechanism that we show in detail in the following sections.

c) The initial data correction, $\delta\mathbf{g}$ have to be chosen with a decay in r such that all the components of $\hat{\mathcal{L}}_T R$ on Σ_0 have a decay which makes the $\tilde{\mathcal{Q}}$ norms on Σ_0 finite. Once this is achieved to prove that the same norms in the whole region \mathcal{K} are bounded in terms of the initial data ones amounts to prove that the “Error”, which is a spacetime integral describing the difference between the final and initial norms, can be bounded by the same $\tilde{\mathcal{Q}}$ norms multiplied by a factor proportional to $\varepsilon_0 R_0^{-1}$, where ε_0 is a parameter defining the smallness of the initial data corrections. Formally we have

$$\tilde{\mathcal{Q}} \leq \tilde{\mathcal{Q}}|_{\Sigma_0} + c \frac{\varepsilon_0}{R_0} \tilde{\mathcal{Q}} \quad (1.8)$$

implying

$$\tilde{\mathcal{Q}} \leq \frac{1}{1 - c \frac{\varepsilon_0}{R_0}} \tilde{\mathcal{Q}}|_{\Sigma_0} .$$

The control of these norms allows to prove, with the same techniques used in [Kl-Ni1] and in the first part of [Kl-Ni2], the following decay along the outgoing null directions for the null components of $\hat{\mathcal{L}}_T R$, $\alpha(\hat{\mathcal{L}}_T R)$, $\beta(\hat{\mathcal{L}}_T R)$,

$$\begin{aligned} \sup_{\mathcal{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2} + \frac{\varepsilon}{2}} |\alpha(\hat{\mathcal{L}}_T R)| &\leq C_0 , & \sup_{\mathcal{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2} + \frac{\varepsilon}{2}} |\beta(\hat{\mathcal{L}}_T R)| &\leq C_0 \\ \sup_{\mathcal{K}} r^3 |u|^{3 + \varepsilon'} |\rho(\hat{\mathcal{L}}_T R) - \overline{\rho(\hat{\mathcal{L}}_T R)}| &\leq C_0 , & \sup_{\mathcal{K}} r^3 |u|^{3 + \frac{\varepsilon}{2}} |\sigma(\hat{\mathcal{L}}_T R) - \overline{\sigma(\hat{\mathcal{L}}_T R)}| &\leq C_0 \\ \sup_{\mathcal{K}} r^2 |u|^{4 + \frac{\varepsilon}{2}} \underline{\beta}(\hat{\mathcal{L}}_T R) &\leq C_0 , & \sup_{\mathcal{K}} r |u|^{5 + \frac{\varepsilon}{2}} |\underline{\alpha}(\hat{\mathcal{L}}_T R)| &\leq C_0 . \end{aligned} \quad (1.9)$$

As said before while discussing point **b)**, this is only the first step for getting the right decay. The next one, which is basically a repetition of what has been done in [Kl-Ni2], consists in transforming the $|u|$ decay in a r decay, integrating along the (portion) of the incoming cones. As proved in detail later on, Theorem 2.3, this provides the following estimates

$$\begin{aligned} \sup_{\mathcal{K}} r^5 |u|^{1 + \frac{\varepsilon''}{2}} |\alpha(\hat{\mathcal{L}}_T R)| &\leq C_1 , & \sup_{\mathcal{K}} r^4 |u|^{2 + \frac{\varepsilon''}{2}} |\beta(\hat{\mathcal{L}}_T R)| &\leq C_1 \\ \sup_{\mathcal{K}} r^3 |u|^{3 + \frac{\varepsilon'}{2}} |\rho(\hat{\mathcal{L}}_T R) - \overline{\rho(\hat{\mathcal{L}}_T R)}| &\leq C_1 , & \sup_{\mathcal{K}} r^3 |u|^{3 + \frac{\varepsilon'}{2}} |\sigma(\hat{\mathcal{L}}_T R) - \overline{\sigma(\hat{\mathcal{L}}_T R)}| &\leq C_1 \\ \sup_{\mathcal{K}} r^2 |u|^{4 + \frac{\varepsilon}{2}} \underline{\beta}(\hat{\mathcal{L}}_T R) &\leq C_1 , & \sup_{\mathcal{K}} r |u|^{5 + \frac{\varepsilon}{2}} |\underline{\alpha}(\hat{\mathcal{L}}_T R)| &\leq C_1 . \end{aligned} \quad (1.10)$$

where

$$\varepsilon'' < \varepsilon' < \varepsilon .$$

d) Differently from [Kl-Ni2] our goal is not yet obtained. In fact we have a decay in agreement with the peeling only for the null components of $\hat{\mathcal{L}}_T R$. To get the decay for the null components of R we have to integrate the previous components along the timelike curves $r = \text{const}$. To do it we have first to connect the null components of $\hat{\mathcal{L}}_T R$ to the partial t -derivatives of the null components of R .

Observe that, as we will show in detail later on, the decay coming from inequalities 1.10 seems sufficiently good for integrating in the t variable and getting again uniform decay estimates, but as we will integrate the time derivatives of the various components, $\partial_t \alpha(e_a, \epsilon_b), \partial_t \beta(e_a), \dots$ we have first to prove that these time derivatives satisfy the same bounds as those in 1.10.

To connect, for instance, $\alpha(\hat{\mathcal{L}}_T R)$ to $\partial_t \alpha(e_a, \epsilon_b)$ one has first observe that $\hat{\mathcal{L}}_T$ is the modified Lie derivative introduced in [Ch-Kl], see its definition in [Kl-Ni1], page 75, def.(3.2.2); therefore one has first to estimate, starting from the proved decay of $\alpha(\hat{\mathcal{L}}_T R)$, the decay satisfied by $\alpha(\mathcal{L}_T R)$ and then from it the one satisfied by $\partial_t \alpha(e_a, \epsilon_b)$. This can be done using the equations which connect these various expressions. To connect, for instance, $\alpha(\hat{\mathcal{L}}_T R)$ with $\alpha(\mathcal{L}_T R)$ we use the definition,

$$\mathcal{L}_T R = \hat{\mathcal{L}}_T R + \frac{1}{2} {}^{(T)}[R] - \frac{3}{8} (\text{tr}^{(T)} \pi) R \quad (1.11)$$

where ${}^{(T)}\pi$ is the deformation tensor of the vector field T .

From 1.11 and the explicit expression of ${}^{(T)}\pi$, eq.2.86, one realizes that to prove that $\alpha(\mathcal{L}_T R)$ and $\partial_t \alpha(e_a, \epsilon_b)$ satisfy the same bounds of $\alpha(\hat{\mathcal{L}}_T R)$ requires to control the decay of the connection coefficients of the spacetime \mathcal{K} and of the components of the non derived Riemann tensor R which we have, up to now, at our disposal, therefore those provided in [Kl-Ni1].

e) This is a slightly delicate point. In fact if we assume for the connection coefficients of the spacetime \mathcal{K} and of the components of the Riemann tensor R exactly the decay proved in [Kl-Ni1] it is easy to realize that this will not allow to obtain a decay for $\partial_t \alpha(e_a, \epsilon_b)$ sufficient to integrate and to obtain the expected r^{-5} decay for $\alpha(R)$, but at most we can prove a decay $O((\log r)r^{-5})$ (Viceversa this would be enough for $\beta(R)$, while the other components already satisfy the appropriate ‘‘Peeling decay’’).

It is to face this problem that we take advantage from the fact, observed discussing point **a)**, that the corrections to the Schwarzschild metric on Σ_0/B decay faster than the minimum (spacelike) decay required in [Kl-Ni1]. This fact will allow, as proved in the first part of [Kl-Ni2], to obtain for the various null components of R (with the exception of ρ) and for those connection coefficients or their combinations which are identically zero in Kerr, an extra $|u|^{\frac{1}{2}-\delta}$ decay factor with $\delta > 0$ arbitrarily small.¹⁸

This result allows us to prove that $\alpha(\mathcal{L}_T R)$ and $\partial_t \alpha(e_a, \epsilon_b)$ satisfy the same bounds as $\alpha(\hat{\mathcal{L}}_T R)$ and that, therefore, we can perform the integration along

¹⁸It is important to observe that the combination of connection coefficients which appear in the ${}^{(T)}\pi$ components are, in fact, identically zero in Kerr.

the T integral curves, obtaining the expected decay. Therefore we have proved the following theorem:

Theorem 1.2. *Assume that the initial data on the initial hypersurface Σ_0/B (where B is a ball centered at the origin of linear size R_0 sufficiently large) are near to the initial data of a Kerr spacetime with angular momentum J satisfying the condition $J/M \leq M$ and the mass M satisfying, with $\gamma > 0$ sufficiently small,¹⁹*

$$M \leq \sqrt{R_0^\gamma \varepsilon_0} . \quad (1.12)$$

Denoted with \mathbf{g} the metric on Σ_0 we have

$$\mathbf{g} = \mathbf{g}_{(Kerr)} + \delta\mathbf{g} = \mathbf{g}_{(Sch)} + [(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)}) + \delta\mathbf{g}] \quad (1.13)$$

where

$$(\mathbf{g}_{(Kerr)} - \mathbf{g}_{(Sch)}) = O\left(\frac{a^2}{r^2}\right) . \quad (1.14)$$

Moreover let the remaining correction $\delta\mathbf{g}$ be such that,

$$\delta\mathbf{g} = o_q\left(\frac{\varepsilon_0}{r^{-3}}\right)$$

and let k be a symmetric tensor such that

$$k = k_{(Kerr)} + o_{q-1}\left(\frac{\varepsilon_0}{r^{-4}}\right) ,$$

with $q \leq 6$.²⁰

Under these assumptions it is possible to prove a global existence result for the “very external region”, \mathcal{K} , endowed with a double null canonical foliation with g and k the restrictions to Σ_0 of the Lorentzian metric \mathbf{g} and of the second fundamental form associated to the vector field

$$T = \Omega(e_3 + e_4)$$

where e_3, e_4 are the null vector fields adapted to the double canonical foliation and Ω is defined through the relation

$$\mathbf{g}(L, \underline{L}) = -(2\Omega)^{-2} ,$$

where L, \underline{L} are the null geodesic vector field associated to the double canonical foliation. Finally all the null Riemann components decay in agreement with the “Peeling Theorem” .

¹⁹ $\gamma \leq \frac{1}{4}$ would be enough.

²⁰A function f is $o_q(r^{-k})$ if $\partial_l f = o(r^{-k-l})$ for $l = 0, 1, \dots, q$.

The result proved in this theorem tell us, in broad terms, that perturbing the (external) initial data of a slow rotating Kerr spacetime the “peeling behaviour” of the various Riemann components is preserved. One can argue that the allowed “perturbation” is very mild, see the assumptions on $\delta\mathbf{g}$ in the theorem, but the previous results in [Kl-Ni2] suggest, although do not prove, that a lower decaying correction $\delta\mathbf{g}$ to Kerr would destroy the peeling, at least for $\alpha(R)$. If for instance we assume an r^{-2} decay all the strategy based on the fact that the r^{-2} decaying part of the metric does not depend on t would fail. Even with a decay $r^{-\gamma}$ with $\gamma \in [2, 3]$ the present proof could not be completed and the [Kl-Ni2] results would again suggest a peeling violation.

Remark: *As done for J the smallness of the correction δg can be relaxed if we increase (not in a linear way) the value of R_0 .*

In the sequel, just for notational simplification we will consider the situation with R_0 given and sufficiently large and $a = J/M \leq \sqrt{\varepsilon_0}$.

Acknowledgments: *Part of this work has been done during a visit of the author to the Institut Mittag-Leffler (Djursholm, Sweden), where he was invited for the General Relativity semester and where he enjoyed many scientific discussions in a very stimulating scientific atmosphere. Moreover the author is deeply indebted to Sergiu Klainerman for pointing to him the importance of considering the Lie derivative, with respect to the “time” vector field T , of the Riemann tensor to obtain more detailed estimates for the various components of the Riemann tensor. Besides he is also indebted for many illuminating discussions he had with him about this subject and many related ones. He also thanks G.Caciotta for pointing to him some delicate aspects of the proof to be clarified. He also wants to state that the present result is deeply based on the previous works [Kl-Ni1], [Kl-Ni2] and on the original seminal work by D.Christodoulou and S.Klainerman, [Ch-Kl].”*

2 Proof of the result

2.1 The very external region \mathcal{K}

The main difference between the result we obtain and the “Expected result” is that in the present proof we assume the ratio of the Kerr angular momentum with the mass, J/M , small.²¹ This allows us to treat the initial data difference between Kerr and Schwarzschild as a perturbation of the initial data. Even in this case to get the global existence in the external region is not at all easy, but it is a result already at our disposal, see [Kl-Ni1]. If we do not require J/M small,²² together with proving the asymptotic decay we have to prove the global existence of spacetimes which in the “very external region” are non linear perturbations around a (J, M) -Kerr spacetime and we cannot fully use the results of [Kl-Ni1]. The main difficulty is that the rotation generators vector fields cannot be considered anymore approximately “Killing” and this makes difficult to prove the boundeness of some of the \mathcal{Q} norms.²³

Assuming, therefore, J/M small,

$$J = Ma \leq M\varepsilon_0^{\frac{1}{2}},$$

we can apply the results in [Kl-Ni1]. Moreover, as it will turn out to be necessary later on, we can use the fact that the decay of the correction to the Schwarzschild metric on Σ_0 is faster than the decay required there and that, therefore, the Riemann tensor has an extra factor “decay” in the $|u|$ variable. This implies that also the decay behaviour of some of the connection coefficients is improved by the same factor. Even this result has not to be proved here as it is already contained in the first part of [Kl-Ni2]. Therefore we only recall this result:

Theorem 2.1. *Assume the initial data $\{\bar{g}, \bar{k}\}$ on the initial hypersurface Σ_0/B have the following asymptotic behaviour*

$$\begin{aligned} \bar{g} - g_{Kerr} &= o_4(r^{-3}) \\ \bar{k} - k_{Kerr} &= o_3(r^{-4}) \end{aligned} \tag{2.15}$$

and a smallness condition $\mathcal{J} \leq \varepsilon_0$ is given as in [Kl-Ni1], then there exists a complete spacetime \mathcal{K}^{24} with a double null canonical foliations such that its Riemann components behave in the following way, with $C_1 = O(\varepsilon_0)$,

$$\begin{aligned} \sup_{\mathcal{K}_2} r^{\frac{7}{2}} |u|^{\frac{1}{2}-\delta} |\alpha(R)| &\leq C_1, \quad \sup_{\mathcal{K}_2} r^{\frac{7}{2}} |u|^{\frac{1}{2}-\delta} |\beta(R)| \leq C_1 \\ \sup_{\mathcal{K}_2} r^3 |\rho(R)| &\leq C_1, \quad \sup_{\mathcal{K}_2} r^3 |u|^{1-\delta} |\sigma(R)| \leq C_1 \\ \sup_{\mathcal{K}_2} r^2 |u|^{2-\delta} |\underline{\beta}(R)| &\leq C_1, \quad \sup_{\mathcal{K}_2} r |u|^{3-\delta} |\underline{\alpha}(R)| \leq C_1. \end{aligned} \tag{2.16}$$

²¹As we said before all the proof can be performed under the more general condition 1.12.

²²More precisely if we do not require J/M depending on R_0 .

²³More specifically the $\mathcal{Q}(\hat{\mathcal{L}}_O R)$ norms, see [Kl-Ni1] Chapter 3 for their definitions or later on.

²⁴It is the very external region outside the influence region of the initial ball B .

with $\delta > 0$ and arbitrarily small. Moreover all the connection coefficients (and their combinations) which are identically zero in the Kerr spacetime acquire the extra decay factor $|u|^{\frac{1}{2}-\delta}$. In particular the following inequalities hold

$$\begin{aligned} \sup_{\mathcal{K}} r^2 |u|^{1-\delta} |tr\chi + tr\underline{\chi}| &\leq C_1 \\ \sup_{\mathcal{K}} r^2 |u|^{1-\delta} |\omega + \underline{\omega}| &\leq C_1 . \end{aligned} \quad (2.17)$$

Moreover all the norms of the connection coefficients, $\tilde{\mathcal{O}}$, are proportional to ε_0 .

Proof: With the exception of 2.17 whose proof is in the appendix, the proof of this result is the content of part I of Theorem 1.2, of Theorem 2.2 and of the definition given there of the $\tilde{\mathcal{O}}$ norms, see Appendix A.2, of [Kl-Ni2]. Nevertheless in the appendix we give a simplified proof of this result which does not use the results proved in [Kl-Ni2].

Remarks:

a) It is important to remark, see subsection 4.1, that the estimates 2.17 for $\omega + \underline{\omega}$ and $tr\chi + tr\underline{\chi}$ are valid in the whole spacetime, but the estimates on Σ_0 of the same quantities are stronger due to the decay assumptions for the metric and the second fundamental form k . These stronger initial data assumptions are needed to have bounded $\tilde{\mathcal{Q}}$ norms on Σ_0 .

b) As it will be clear in the following sections and specifically in Subsection 2.7, Theorem 2.4, the result we prove requires all the estimates which are valid in spacetime \mathcal{K} ; nevertheless a weaker result can be obtained using only the estimates proved in [Kl-Ni1]. In this case, see Theorem 2.5, we obtain for the α component a decay with an extra logarithmic term in agreement with some results discussed in J.A.V.Kroon works, see [Kr1], [Kr2] and references therein.

2.2 The boundness of the $\tilde{\mathcal{Q}}$ norms

Once the \mathcal{K} spacetimes are defined together with the asymptotic behaviour of the various Riemann components and connection coefficients, we can start proving the better asymptotic behaviour for the α and β components of the Riemann tensor. As explained at point **b**) of the introduction we cannot apply the result of [Kl-Ni2] in a direct way and we have to look first for a better asymptotic behaviour of the $\hat{\mathcal{L}}_T R$ null components. This is done defining some Bel-Robinson norms similar to those introduced in [Kl-Ni2], see equations (2.6),..., (2.11), but relative to $\hat{\mathcal{L}}_T R$ and with a different weight factor, $|u|^{5+\epsilon}$ instead of $|u|^{3+\epsilon}$. We define, therefore, the following $\tilde{\mathcal{Q}}$ norms:

$$\begin{aligned} \tilde{\mathcal{Q}}(u, \underline{u}) &= \tilde{\mathcal{Q}}_1(u, \underline{u}) + \tilde{\mathcal{Q}}_2(u, \underline{u}) + \sum_{i=2}^{q-2} \tilde{\mathcal{Q}}_{(q)}(u, \underline{u}) \\ \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) &= \underline{\tilde{\mathcal{Q}}}_1(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}_2(u, \underline{u}) + \sum_{i=2}^{q-2} \underline{\tilde{\mathcal{Q}}}_{(q)}(u, \underline{u}) , \end{aligned} \quad (2.18)$$

where, denoting \mathcal{K} the “very external region” whose existence has already been proved and the regions $V(u, \underline{u}) \subset \mathcal{K}$, with $S(u, \underline{u}) = C(u) \cap \underline{C}(\underline{u})$,²⁵

$$V(u, \underline{u}) = J^{(-)}(S(u, \underline{u})) \cap \mathcal{K} ,$$

$$\begin{aligned} \tilde{\mathcal{Q}}_1(u, \underline{u}) &\equiv \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &\quad + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, e_4) \\ \tilde{\mathcal{Q}}_2(u, \underline{u}) &\equiv \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_4) \\ &\quad + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2 \tilde{R})(\bar{K}, \bar{K}, T, e_4) \quad (2.19) \\ &\quad + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_4) \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{Q}}_{(q)}(u, \underline{u}) &\equiv \sum_{i=2}^{q-2} \left\{ \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \hat{\mathcal{L}}_O^i \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_4) \right. \\ &\quad + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^{i+1} \tilde{R})(\bar{K}, \bar{K}, T, e_4) \quad (2.20) \\ &\quad \left. + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \hat{\mathcal{L}}_O^{i-1} \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_4) \right\} \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{Q}}_1(u, \underline{u}) &\equiv \int_{\underline{C}(\underline{u}) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &\quad + \int_{\underline{C}(\underline{u}) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, e_3) , \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{Q}}_2(u, \underline{u}) &\equiv \int_{\underline{C}(\underline{u}) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_3) \\ &\quad + \int_{\underline{C}(\underline{u}) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2 \tilde{R})(\bar{K}, \bar{K}, T, e_3) \quad (2.21) \\ &\quad + \int_{\underline{C}(\underline{u}) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_3) \end{aligned}$$

²⁵Recall that in \mathcal{K} a double null canonical foliation exists made by outgoing and incoming “cones” $\{C(u)\}, \{\underline{C}(\underline{u})\}$.

$$\begin{aligned}
\tilde{\mathcal{Q}}_{(q)}(u, \underline{u}) &\equiv \sum_{i=2}^{q-2} \left\{ \int_{C(\lambda) \cap V(\lambda, \nu)} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \hat{\mathcal{L}}_O^i \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_3) \right. \\
&\quad + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^{i+1} \tilde{R})(\bar{K}, \bar{K}, T_0, e_3) \\
&\quad \left. + \int_{C(u) \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_{T_0} \hat{\mathcal{L}}_O^{i-1} \tilde{R})(\bar{K}, \bar{K}, \bar{K}, e_3) \right\}. \tag{2.22}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{Q}}_1(\Sigma_0) &\equiv \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, T) \\
&\quad + \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, T) \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{Q}}_2(\Sigma_0) &\equiv \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, T) \\
&\quad + \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^2 \tilde{R})(\bar{K}, \bar{K}, T, T) \\
&\quad + \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{R})(\bar{K}, \bar{K}, \bar{K}, T) . \tag{2.24}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=2}^{q-2} \tilde{\mathcal{Q}}_{(q)}(\Sigma_0) &\equiv \sum_{i=2}^{q-2} \left\{ \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_T \hat{\mathcal{L}}_O^i \tilde{R})(\bar{K}, \bar{K}, \bar{K}, T) \right. \\
&\quad + \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O^{i+1} \tilde{R})(\bar{K}, \bar{K}, T, T) \\
&\quad \left. + \int_{\Sigma_0 \cap V(u, \underline{u})} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \hat{\mathcal{L}}_O^{i-1} \tilde{R})(\bar{K}, \bar{K}, \bar{K}, T) \right\} \tag{2.25}
\end{aligned}$$

We also introduce the following quantity:

$$\tilde{\mathcal{Q}}_{\mathcal{K}} \equiv \sup_{\{u, \underline{u} | S(u, \underline{u}) \subseteq \mathcal{K}\}} \{ \tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\mathcal{Q}}(u, \underline{u}) \}. \tag{2.26}$$

The initial conditions which define the spacetime \mathcal{K} have been chosen in such a way that the $\tilde{\mathcal{Q}}, \underline{\mathcal{Q}}$ norms are bounded on Σ_0 (due to the fact that we are considering $\tilde{R} = \hat{\mathcal{L}}_T R$ instead of R). This is discussed in greater detail in subsection 4.1 of the appendix.

As discussed in the introduction, see **b**), to fulfill the expectation that, everywhere in \mathcal{K} , the ‘‘Kerr part is ‘‘basically’’ subtracted in $\hat{\mathcal{L}}_T R$, we have to prove that these norms are bounded everywhere, that is that $\tilde{\mathcal{Q}}_{\mathcal{K}}$ is bounded. To prove it, proceeding in a similar way as in [Kl-Nil], we define a spacetime integral $\hat{\mathcal{E}}(u, \underline{u})$, we call the ‘‘Error’’,

$$\hat{\mathcal{E}}(u, \underline{u}) := \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\mathcal{Q}}(u, \underline{u}) \right) - \left(\tilde{\mathcal{Q}}(\Sigma_0 \cap V(u, \underline{u})) + \underline{\mathcal{Q}}(\Sigma_0 \cap V(u, \underline{u})) \right) \tag{2.27}$$

and prove that the “Error” can be bounded by

$$\widehat{\mathcal{E}}(u, \underline{u}) \leq \varepsilon_0 f(R_0) \left(\widetilde{\mathcal{Q}}(u, \underline{u}) + \underline{\widetilde{\mathcal{Q}}}(u, \underline{u}) \right) + \sup_{\Sigma_0 \cap V(u, \underline{u})} \left(|r^3|u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3|u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \quad (2.28)$$

with f a function decaying in R_0 .

The boundedness of $\widetilde{\mathcal{Q}}_{\mathcal{K}}$, as we said before, allows us to control the $L^2(C)$ and $L^2(\underline{C})$ weighted norms for the null components of $\tilde{R} = \hat{\mathcal{L}}_T R$ and their derivatives up to fourth order, with the exception of $\rho(\tilde{R})$ and $\sigma(\tilde{R})$, and, using them, to obtain the sup norms and the decay of the same quantities.²⁶ The terms $\rho(\tilde{R})$ and $\sigma(\tilde{R})$ have to be treated differently. In fact only $\rho - \bar{\rho}, \sigma - \bar{\sigma}$ can be estimated by the $\underline{\mathcal{Q}}$ norm associated to $\hat{\mathcal{L}}_O \tilde{R}$, via Poincarè Lemma.

The norms $\sup_{\mathcal{K}} |r^3|u|^{3-\delta} (\rho(\tilde{R}), \sigma(\tilde{R}))|$ are estimated directly, using the evolution equation for ρ and σ along the incoming cones, in terms of the corresponding initial data norm.²⁷

Here the need of a bootstrap mechanism shows up: $\sup_{\mathcal{K}} |r^3|u|^{3-\delta} (\rho(\tilde{R}), \sigma(\tilde{R}))|$ are estimated, using the transport equations along the incoming cones, in terms of the initial data $\sup_{\Sigma_0 \cap \mathcal{K}} |r^3|u|^{3-\delta} (\rho(\tilde{R}), \sigma(\tilde{R}))|$ and of an integral part which depends on the remaining null Riemann components. On the other side these remaining components are estimated in terms of the $\underline{\mathcal{Q}}(\tilde{R})$ norms which at their turn, to prove they are bounded, require a control of the $(\rho(\tilde{R}), \sigma(\tilde{R}))$ components.

2.3 The bootstrap assumption and the “Error” estimate.

The strategy to estimate the “Error terms” is, basically, a much simpler version of the one used in [Kl-Ni1]. The reason of the simplification is that in [Kl-Ni1] the estimate of the error terms required some bootstrap assumptions on the \mathcal{R} norms of the Riemann components and on the \mathcal{O} norms of the connection coefficients, see the discussion in Chapter 4. This was due to the fact that the control of the $\underline{\mathcal{Q}}$ norm was an intermediate step to prove global existence. Here the global existence of the spacetime \mathcal{K} , is already proved, and we need a simpler bootstrap assumption only to estimate $(\rho(\hat{\mathcal{L}}_T R), \sigma(\hat{\mathcal{L}}_T R))$. The various steps are the following:

a) We know that due to the initial conditions on Σ_0 the following estimate holds, with $\tilde{R} = \hat{\mathcal{L}}_T R$:

Lemma 2.1. *Under the initial data assumptions on \bar{g} and \bar{k} given in Theorem 2.1 we have the following estimates,*

$$\sup_{\Sigma_0 \cap V(u, \underline{u})} |r^{6-\delta} \overline{\rho(\tilde{R})}| \leq c\varepsilon_0 \quad , \quad \sup_{\Sigma_0 \cap V(u, \underline{u})} |r^{7-\delta} \overline{\sigma(\tilde{R})}| \leq c\varepsilon_0 \quad . \quad (2.29)$$

²⁶This is done in any detail in Chapter 5 of [Kl-Ni1] for the derivatives of R up to second order, the higher derivatives are easier and are estimated exactly in the same way.

²⁷Of course we have also to prove that the initial data is finite and small, we discuss it later on.

Proof: The proof of Lemma 2.1 is in the appendix.

Therefore we can define a spacetime region $\bar{V} \equiv V(\bar{u} = -R_0, \bar{u} = R_0 + \delta u) \subset \mathcal{K}$ where, by continuity,

$$\sup_{(u, \underline{u}) \in \bar{V}} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 \leq \varepsilon_1^2 . \quad (2.30)$$

with $\varepsilon_1 = c\varepsilon_0 > \varepsilon_0$, but sufficiently small. In this region it is easy to prove, using also 2.30, that, with an appropriate constant c ,²⁸

$$\sup_{(u, \underline{u}) \in \bar{V}} \{ \tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \} \leq c\varepsilon_0(\varepsilon_0 + \varepsilon_1) . \quad (2.31)$$

b) This “local” result allows to state a bootstrap mechanism:

i) Define $J^{(-)}(S(u', \underline{u}')) \equiv \mathcal{K}' \subset \mathcal{K}$ as the largest possible region where

$$\sup_{(u, \underline{u}) \in \mathcal{K}'} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 \leq \varepsilon_1^2 \quad (2.32)$$

ii) Prove that using 2.32 the error in the region \mathcal{K}' satisfies the following bound

$$\begin{aligned} & \sup_{(u, \underline{u}) \in \mathcal{K}'} \hat{\mathcal{E}}(u, \underline{u}) \quad (2.33) \\ & \leq \varepsilon_0 f(R_0) \left(\sup_{(u, \underline{u}) \in \mathcal{K}'} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right) + \sup_{(u, \underline{u}) \in \mathcal{K}'} \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right) \end{aligned}$$

where $f(R_0)$ goes to zero as $R_0 \rightarrow \infty$.

iii) This implies that, for an appropriate constant c

$$\begin{aligned} & \sup_{(u, \underline{u}) \in \mathcal{K}'} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right) \quad (2.34) \\ & \leq \frac{1}{1 - c\varepsilon_0} \left[\left(\tilde{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') + \underline{\tilde{\mathcal{Q}}}(\Sigma_0 \cap \mathcal{K}') \right) + c\varepsilon_0 \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right] \end{aligned}$$

iv) The estimate of $\left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right)$ in 2.34 allows to prove, using the transport equation for $|r^{3-\frac{2}{p}} |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|_{p,S}$ (and the analogous one for $\nabla \rho$), the following estimate

$$\begin{aligned} & \sup_{(u, \underline{u}) \in \mathcal{K}'} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 \leq \sup_{\Sigma_0 \cap \mathcal{K}'} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 \quad (2.35) \\ & + c \left[\sup_{(u, \underline{u}) \in \mathcal{K}'} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right) + \varepsilon_0^2 \sup_{(u, \underline{u}) \in \mathcal{K}'} \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right] \\ & \leq \sup_{\Sigma_0 \cap \mathcal{K}'} |r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 \\ & + c \left[\left(\tilde{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') + \underline{\tilde{\mathcal{Q}}}(\Sigma_0 \cap \mathcal{K}') \right) + (\varepsilon_0 + \varepsilon_0^2) \sup_{(u, \underline{u}) \in \mathcal{K}'} \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right] \end{aligned}$$

²⁸We do not report the proof here as it is just a particular case of the proof of point iii).

which implies, choosing ε_0 sufficiently small

$$\begin{aligned}
& \sup_{(u, \underline{u}) \in \mathcal{K}'} |r^3 |u|^{3-\delta} \overline{(\rho(\tilde{R}), \sigma(\tilde{R}))}|^2 & (2.36) \\
& \leq \frac{c}{1 - c\varepsilon_0} \left(\sup_{\Sigma_0 \cap \mathcal{K}'} |r^3 |u|^{3-\delta} \overline{(\rho(\tilde{R}), \sigma(\tilde{R}))}|^2 + \left(\tilde{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') + \underline{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') \right) \right) \\
& \leq \frac{c}{1 - c\varepsilon_0} \left(\varepsilon_0^2 + \left(\tilde{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') + \underline{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}') \right) \right) \leq \frac{c\varepsilon_0^2}{1 - c\varepsilon_0} .
\end{aligned}$$

v) Choosing ε_0 sufficiently small the previous inequality implies that

$$\sup_{(u, \underline{u}) \in \mathcal{K}'} |r^3 |u|^{3-\delta} \overline{(\rho(\tilde{R}), \sigma(\tilde{R}))}|^2 \leq \frac{\varepsilon_1^2}{2} \quad (2.37)$$

which contradicts the definition of \mathcal{K}' unless it coincides with the whole space-time \mathcal{K} . We have, therefore proved the estimate

$$\sup_{(u, \underline{u}) \in \mathcal{K}} |r^3 |u|^{3-\delta} \overline{(\rho(\tilde{R}), \sigma(\tilde{R}))}|^2 \leq \varepsilon_1^2 . \quad (2.38)$$

vi) Using these estimates and the assumption made on ε_1 we can compute the error in the whole region \mathcal{K} proving that

$$\begin{aligned}
& \sup_{(u, \underline{u}) \in \mathcal{K}} \hat{\mathcal{E}}(u, \underline{u}) & (2.39) \\
& \leq c\varepsilon_0 \left(\sup_{(u, \underline{u}) \in \mathcal{K}} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\mathcal{Q}}(u, \underline{u}) \right) + \sup_{(u, \underline{u}) \in \mathcal{K}} \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right) .
\end{aligned}$$

which, at its turn, implies that, for an appropriate constant c , repeating the result at point iv),

$$\begin{aligned}
& \sup_{(u, \underline{u}) \in \mathcal{K}} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\mathcal{Q}}(u, \underline{u}) \right) & (2.40) \\
& \leq \frac{1}{1 - c\varepsilon_0} \left(\tilde{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}) + \underline{\mathcal{Q}}(\Sigma_0 \cap \mathcal{K}) + \varepsilon_0 \sup_{(u, \underline{u}) \in \mathcal{K}} \left(|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2 \right) \right) \\
& \leq c\varepsilon_0 (\varepsilon_0 + \varepsilon_1^2)
\end{aligned}$$

vii) The control of these $\tilde{\mathcal{Q}}$ norms and of $|r^3 |u|^{3-\delta} \overline{(\rho(\tilde{R}), \sigma(\tilde{R}))}|^2$ allows to control the sup norms of all the components of $\hat{\mathcal{L}}_T R$ with some weights (not yet the final ones).

If we had used only the estimates proved in [Kl-Ni1] then in all the previous steps the exponent $3-\delta$ would be substituted by $\frac{5}{2}$.

Remarks:

1) The estimates 2.37 which are proved in the sequel could be improved, with some extra work, for the σ component due to the fact that its estimate on Σ_0 is a

better one. The result we prove here is obtained looking at the transport equation for ρ and σ instead of $\bar{\rho}$ and $\bar{\sigma}$ and does not allow to distinguish between ρ and σ .

2) In the previous argument that ε_1 be small is not really needed. In fact, as said before, we do not need it as the spacetime \mathcal{K} is already given.

3) Here and in the whole work we denote with c many different constants, which, assumed ε_0 small, do not depend on ε_0 or ε_1 .

4) In this argument and more in general in the whole work the fact the R_0 has to be large is used, as in [Kl-Ni1], to prove that the spacetime \mathcal{K} does exist and follows from the requirement that M/R_0 has to be small. As discussed in the introduction, see eq.1.3, the requirement of the smallness of the ratio J/M angular momentum can be weakened increasing R_0 .

To prove the results described in i).....vii) we have to prove inequality 2.33 and the first inequality of 2.35. This is the content of the following

Theorem 2.2. *In the spacetime \mathcal{K} , see Theorem 2.1, the following inequalities hold*

$$\begin{aligned} \sup_{(u, \underline{u}) \in \mathcal{K}} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 &\leq \sup_{\Sigma_0 \cap \mathcal{K}} |r^3 |u|^{3-\delta} (\overline{\rho(\tilde{R})}, \overline{\sigma(\tilde{R})})|^2 \\ + c \left[\sup_{(u, \underline{u}) \in \mathcal{K}} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right) + \varepsilon_0^2 \left(1 + \sup_{(u, \underline{u}) \in \mathcal{K}} (|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2) \right) \right] \end{aligned} \quad (2.41)$$

$$\begin{aligned} \sup_{(u, \underline{u}) \in \mathcal{K}} \hat{\mathcal{E}}(u, \underline{u}) \\ \leq \varepsilon_0 f(R_0) \left(\sup_{(u, \underline{u}) \in \mathcal{K}} \left(\tilde{\mathcal{Q}}(u, \underline{u}) + \underline{\tilde{\mathcal{Q}}}(u, \underline{u}) \right) + \sup_{(u, \underline{u}) \in \mathcal{K}} (|r^3 |u|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |u|^{3-\delta} \overline{\sigma(\tilde{R})}|^2) \right). \end{aligned} \quad (2.42)$$

Proof of 2.41: We prove the estimate for the ρ component, the one for σ goes exactly in the same way.

The evolution equation of ρ along the incoming cones is:

$$\frac{d}{du} \rho + \frac{3}{2} \Omega \text{tr} \chi \rho = \Omega \left(-\text{div} \underline{\beta} - 2\eta \cdot \underline{\beta} - \frac{1}{2} \hat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} \right). \quad (2.43)$$

Applying Lemma 4.1.5 of [Kl-Ni1] it follows:

$$|r^{(3-\frac{2}{p})} \rho(\tilde{R})|_{p,S}(u, \underline{u}) \leq c_0 \left(|r^{(3-\frac{2}{p})} \rho(\tilde{R})|_{p,S}(u_0, \underline{u}) + \int_{u_0}^u |r^{(3-\frac{2}{p})} \tilde{F}|_{p,S}(\lambda, \underline{u}) d\lambda \right) \quad (2.44)$$

where

$$\tilde{F} := \Omega \left(-\text{div} \underline{\beta}(\tilde{R}) - 2\eta \cdot \underline{\beta}(\tilde{R}) - \frac{1}{2} \hat{\chi} \cdot \underline{\alpha}(\tilde{R}) + \zeta \cdot \underline{\beta}(\tilde{R}) \right) \quad (2.45)$$

and

$$S(u_0, \underline{u}) = \underline{C}(\underline{u}) \cap \Sigma_0 . \quad (2.46)$$

As in \mathcal{K} , $|u| \leq |u_0|$, multiplying the left hand side of 2.44 with $|u|^{3-\delta}$ we obtain the inequality

$$\begin{aligned} & |r^{(3-\frac{2}{p})}|u|^{3-\delta} \rho(\tilde{R})|_{p,S}(u, \underline{u}) \\ & \leq c_0 \left(|r^{(3-\frac{2}{p})}|u_0|^{3-\delta} \rho(\tilde{R})|_{p,S}(u_0, \underline{u}) + \int_{u_0}^u |r^{(3-\frac{2}{p})}| \lambda^{3-\delta} \tilde{F}|_{p,S}(\lambda, \underline{u}) d\lambda \right) . \end{aligned} \quad (2.47)$$

We consider this inequality for $p = 4$ as this is what we need to get sup estimates. We have to prove that the estimate of $|r^{(3-\frac{2}{p})}| \lambda^{3-\delta} \tilde{F}|_{p=4,S}(\lambda, \underline{u})$ in terms of the $\tilde{\mathcal{Q}}$ norms makes the integral in 2.47 uniformly bounded. In the inequality

$$\begin{aligned} |r^{(3-\frac{2}{p})}| \lambda^{3-\delta} \tilde{F}|_{p=4,S} & \leq c \left(|r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \nabla \underline{\beta}|_{p=4,S} + 2|r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \eta \underline{\beta}|_{p=4,S} \right. \\ & \left. + \frac{1}{2}|r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \hat{\chi} \underline{\alpha}|_{p=4,S} + |r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \zeta \underline{\beta}|_{p=4,S} \right) \end{aligned} \quad (2.48)$$

Next Lemma estimates the term $|r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \nabla \underline{\beta}|_{p=4,S}(\lambda, \underline{u})$ and the term $|r^{(3-\frac{2}{4})}| \lambda^{(3-\delta)} \eta \underline{\beta}|_{p=4,S}$, the other ones have a better decay and can be treated in a similar way.

Lemma 2.2. *The following estimate holds*

$$\begin{aligned} & |r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \nabla \underline{\beta}|_{p=4,S}(\lambda, \underline{u}) \\ & \leq \frac{c}{|\lambda|^{1+\frac{(\epsilon+2\delta)}{2}}} \left(\tilde{\mathcal{Q}}^{\frac{1}{2}}(\lambda, \underline{u}) + \varepsilon_0 \left(1 + \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3| \lambda^{3-\delta} \overline{\rho(\tilde{R})}) + |r^3| \lambda^{3-\delta} \overline{\sigma(\tilde{R})}) \right) \right) \end{aligned} \quad (2.49)$$

$$\begin{aligned} & |r^{(3-\frac{2}{4})}| \lambda^{(3-\delta)} \eta \cdot \underline{\beta}|_{p=4,S} \\ & \leq \frac{1}{|\lambda|^{2-\frac{\delta}{2}}} \left(\sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} \tilde{\mathcal{Q}}(\lambda, \underline{u})^{\frac{1}{2}} + \varepsilon_0 \left(1 + \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3| \lambda^{3-\delta} \overline{\rho(\tilde{R})}) + |r^3| \lambda^{3-\delta} \overline{\sigma(\tilde{R})}) \right) \right) \end{aligned} \quad (2.50)$$

Proof: See the appendix. This implies that

$$\begin{aligned} \int_{u_0}^u |r^{(3-\frac{2}{4})}| \lambda^{3-\delta} \nabla \underline{\beta}(\tilde{R})|_{p=4,S} d\lambda & = O \left(\frac{1}{|u|^{\frac{(\epsilon+2\delta)}{2}}} \right) \left(\varepsilon_0 + \sup_{(u, \underline{u}) \in \mathcal{K}} \tilde{\mathcal{Q}}(u, \underline{u})^{\frac{1}{2}} \right. \\ & \left. + \frac{\varepsilon_0}{|\lambda|^{1-\frac{2\delta+\epsilon}{2}}} \sup_{(u, \underline{u}) \in \mathcal{K}} (|r^3| \lambda^{3-\delta} \overline{\rho(\tilde{R})}) + |r^3| \lambda^{3-\delta} \overline{\sigma(\tilde{R})}) \right) . \end{aligned}$$

and

$$\begin{aligned} & \int_{u_0}^u |r^{(3-\frac{2}{4})}| \lambda^{(3-\delta)} \eta \cdot \underline{\beta}|_{p=4,S} d\lambda \\ & \leq c \frac{1}{|u|^{1-\frac{\delta}{2}}} \left(\varepsilon_0 + \sup_{(u, \underline{u}) \in \mathcal{K}} \tilde{\mathcal{Q}}(u, \underline{u})^{\frac{1}{2}} + \varepsilon_0 \sup_{(u, \underline{u}) \in \mathcal{K}} (|r^3| \lambda^{3-\delta} \overline{\rho(\tilde{R})}) + |r^3| \lambda^{3-\delta} \overline{\sigma(\tilde{R})}) \right) . \end{aligned} \quad (2.51)$$

The other terms in 2.48 can be treated in the same way, therefore we obtain finally

$$\begin{aligned} & |r^{(3-\frac{2}{p})}|u|^{3-\delta}\rho(\tilde{R})|_{p,S}(u,\underline{u}) \leq c_0 \left[|r^{(3-\frac{2}{p})}|u|^{3-\delta}\rho(\tilde{R})|_{p,S}(u_0,\underline{u}) \right. \\ & \left. + c \frac{1}{|u|^{1-\frac{\delta}{2}}} \left(\sup_{(u,\underline{u}) \in \mathcal{K}} (\tilde{\mathcal{Q}}(u,\underline{u})^{\frac{1}{2}} + \underline{\tilde{\mathcal{Q}}}(u,\underline{u})^{\frac{1}{2}}) + \varepsilon_0 \left(1 + \sup_{(u,\underline{u}) \in \mathcal{K}} (|r^3|\lambda|^{3-\delta}\overline{\rho(\tilde{R})}| + |r^3|\lambda|^{3-\delta}\overline{\sigma(\tilde{R})}|) \right) \right) \right]. \end{aligned}$$

Deriving tangentially the transport equation for ρ and proceeding as before, we obtain analogous estimates for $|r^{(4-\frac{2}{p})}|u|^{3-\delta}\nabla\rho(\tilde{R})|_{p,S}(\lambda,\underline{u})$, then applying Lemma 4.1.3 of [Kl-Ni1] and taking the sup norms we obtain the desired inequality 2.41 proving the first part of the theorem.

Remarks: *We do not prove here this estimate for $\nabla\rho(\tilde{R})$ as we will prove a better estimate in Lemma 2.3. This improved estimate will be needed in the sequel.*

Proof of 2.42:

In the $\tilde{\mathcal{Q}}$ norms, there are some differences with respect to those defined in [Kl-Ni1], the more important being, obviously, the presence in these norms of the extra weight factor $|u|^{5+\epsilon}$; the control of the norms with an extra factor $|u|^\sigma$ has been already examined in [Kl-Ni2].

The important property which has been proved there and which is one of the central features of the whole approach is that the ‘‘Error’’, $\hat{\mathcal{E}}(u,\underline{u})$, which has to be controlled to prove the boundedness of the $\tilde{\mathcal{Q}}$ norms, has exactly the same structure as the one relative to the \mathcal{Q} norms. This is because, in the external region (or more in general, in the region where u is negative), the extra terms in $\hat{\mathcal{E}}(u,\underline{u})$ produced by the weight factor $|u|^\sigma$ give a negative contribution and, therefore, can be neglected.²⁹

We start examining the general structure of $\hat{\mathcal{E}}(u,\underline{u})$. Proceeding exactly as in [Kl-Ni2], see equations (3.15), (3.16), (3.17), we know that the expression of $\hat{\mathcal{E}}(u,\underline{u})$ for a Weyl tensor W , (which in the present case is $\hat{\mathcal{L}}_O\tilde{R}$, $\hat{\mathcal{L}}_T\tilde{R}$ or terms with more (modified) Lie derivatives) is, neglecting the negative terms of the error,³⁰

$$\begin{aligned} \hat{\mathcal{E}}(W)(X, Y, Z) = & \\ & - \int_{V(u,\underline{u})} |u|^\sigma \left[Div Q(W)_{\beta\gamma\delta} X^\beta Y^\gamma Z^\delta + \frac{1}{2} Q^{\alpha\beta\gamma\delta}(W) \left({}^{(X)}\pi_{\alpha\beta} Y_\gamma Z_\delta \right. \right. \\ & \left. \left. + {}^{(Y)}\pi_{\alpha\beta} Z_\gamma X_\delta + {}^{(Z)}\pi_{\alpha\beta} X_\gamma Y_\delta \right) \right] \end{aligned} \quad (2.52)$$

where ${}^{(X)}\pi_{\alpha\beta}$ is the deformation tensor relative to the vector field X .

²⁹See eq. (3.15) of [Kl-Ni2].

³⁰The final expression of the error will be the sum of the $\hat{\mathcal{E}}(W)(u,\underline{u})$ with respect to all the possible W appearing in the $\tilde{\mathcal{Q}}$ norms.

Symbolically we can write, denoting with Π a generic term at the connection coefficients level:³¹

$$\widehat{\mathcal{E}}(W) = \sum \int_{\mathcal{K}} |u|^{5+\epsilon} \tau_-^a \tau_+^b (D^\alpha \tilde{R}) \cdot (D^\beta \tilde{R}) \cdot (D^\gamma \Pi) \quad (2.53)$$

where $\alpha + \beta + \gamma \geq 2$, the factor $\tau_-^a \tau_+^b$ represents, with appropriate a and b , the weights already present in the analogous terms of [Kl-Ni1] and W denote the term $\widehat{\mathcal{L}}_X^\delta \tilde{R}$ with $\delta \in \{1, 2, \dots, q-1\}$, see 2.20, ..., 2.22. The factor $|u|^{5+\epsilon}$ is the extra weight factor which we have to cope with.

Remark: *That all the terms of the “Error” have the structure 2.53 can be seen explicitly in Chapter 6 of [Kl-Ni1] where all the terms are explicitly written and analyzed.*

At first glance we might expect that each of the terms $D^\alpha \tilde{R}, D^\beta \tilde{R}, D^\gamma \Pi$ appearing on the right hand side of 2.53,³² can absorb a factor $|u|^{\frac{5+\epsilon}{2}}$ for the reason that the L^2 norms of the various Riemann components have these weight factors as these norms are estimated in terms of these $\widehat{\mathcal{Q}}$ norms (defined with these weight factors). This will more than compensate the presence of the weight $|u|^{5+\epsilon}$ in 2.53.³³

Unfortunately this is not quite true, indeed the symbolic expression $D^\alpha \tilde{R} \cdot D^\beta \tilde{R} \cdot D^\gamma \Pi$ hides the presence of terms such as $\rho(\tilde{R}), \sigma(\tilde{R})$ or $\text{tr}\chi, \text{tr}\underline{\chi}, \omega, \underline{\omega}$. To prove that the term $\rho(\tilde{R}), \sigma(\tilde{R})$ can absorb in its L^2 norms some additional weight requires more care and has been proved when we estimate separately, via a bootstrap mechanism, $\sup_{\mathcal{K}} |r^3 |u|^{3-\delta} \rho(\tilde{R})|$ and $\sup_{\mathcal{K}} |r^3 |u|^{3-\delta} \sigma(\tilde{R})|$.³⁴

Moreover the norms of the connection coefficients $\text{tr}\chi, \text{tr}\underline{\chi}, \omega, \underline{\omega}$ cannot absorb any additional weight factor as they are different from zero even in the Schwarzschild spacetime. Nevertheless, with the present definition of the vector field T ³⁵ and of the canonical foliation these connection coefficients only appear in the “Error”, when together with ρ , in the linear combinations $\text{tr}\chi + \text{tr}\underline{\chi}$ and $\omega + \underline{\omega}$ which are identically zero in Kerr spacetime and have a better and sufficient asymptotic behaviour as it has been show in Theorem 2.1.

Most of the error terms, nevertheless, do not have this problem and they can be estimated as in [Kl-Ni1]; the control of the extra weight can be done as said before and has been carefully proved in a kind of linearized version discussed in

³¹With this we mean terms proportional to connection coefficients or linear combinations of them.

³² $D^\gamma \Pi$ denotes the covariant derivative of an arbitrary deformation tensor. Although the term Π is at the level of the connection coefficient and therefore has “a derivative less” with respect to the R terms this fact is already present at the level of decay in [Kl-Ni1] and, therefore, does not change the possible presence of the extra decay factor.

³³To make this statement clearer recall that, apart from a ρ term the estimate we need to prove for the “Error” is $\mathcal{E} \leq c\epsilon_0 \widehat{\mathcal{Q}}$.

³⁴The additional weight which $\rho(\tilde{R}), \sigma(\tilde{R})$ can tolerate is smaller than the one for the remaining Riemann components, but still sufficient to prove the result. Observe, moreover, that the proof we present here of inequality 2.42 has to be used “before” to prove inequality 2.33 and to complete the bootstrap mechanism.

³⁵The same as in [Kl-Ni2], different from [Kl-Ni1].

a paper by G.Caciotta and T.Raparelli to appear. Here we do not repeat their proof and we examine only some of the more problematic error terms.

2.3.1 Some error terms

As we said the more delicate terms are those containing $\rho(\tilde{R})$ (and also $\sigma(\tilde{R})$, but the terms with σ have always an extra r^{-1} decay and therefore are easier). These terms are of the following kind:

$$\int_{\mathcal{K}} |u|^{5+\epsilon} \tau_-^a \tau_+^b \rho(\tilde{R}) (\hat{\mathcal{L}}_X^\alpha \tilde{R}) (D^\gamma \Pi) \quad (2.54)$$

where $(\hat{\mathcal{L}}_X^\alpha \tilde{R})$ is, symbolically, a null component of $\hat{\mathcal{L}}_X^\alpha \tilde{R}$, with $\alpha \geq 1$. This last term multiplied by its weight including the factor $|u|^{\frac{5+\epsilon}{2}}$ can be estimated by a \tilde{Q} norm. The factor $\rho(\tilde{R})$ has been estimated before, using the evolution equation for ρ along the incoming cones, in terms of the initial data term, $\sup_{\Sigma_0 \cap \mathcal{K}} |r^3 |u|^{3-\delta} \rho(\tilde{R})|$.

To show how to control the error term 2.54 we estimate it following [Kl-Ni1], Chapter 6. More precisely we estimate one of the terms with this structure, namely a term analogous to the one in the first line of eq.(6.2.1).

The term we are considering is one of the many terms which compose the part of the error \mathcal{E}_1 , see [Kl-Ni1] page 242, relative to the spacetime region $V(\bar{u}, \underline{u}) \subset \mathcal{K}$, the part over Σ_0 of $J^{(-)}(S(\bar{u}, \underline{u}))$. The boundaries of this region are

$$\Sigma_0 \cap J^{(-)}(S(\bar{u}, \underline{u})) \quad , \quad C(\bar{u}, [\underline{u}_0(\bar{u}), \underline{u}]) \quad , \quad \underline{C}(\underline{u}, [u_0(\underline{u}), \bar{u}]) \quad ,$$

where

$$\underline{u}_0(\bar{u}) = |\bar{u}| \quad , \quad u_0(\underline{u}) = -\underline{u} \quad .$$

Therefore we consider

$$\int_{V(\bar{u}, \underline{u})} |u|^{5+\epsilon} \tau_+^6 \alpha(\hat{\mathcal{L}}_T \tilde{R}) \cdot \Theta(T, \tilde{R}) \quad (2.55)$$

a term of $\int_{V(\bar{u}, \underline{u})} |u|^{5+\epsilon} Div Q(\hat{\mathcal{L}}_T \tilde{R})_{\beta\gamma\delta} \bar{K}^\beta \bar{K}^\gamma \bar{K}^\delta$, and, more specifically, between the various terms which compose $\Theta(T, \tilde{R})$ we choose, see equation (6.1.14) of [Kl-Ni1], the term proportional to ${}^{(T)}p_4(\rho(\tilde{R}), \sigma(\tilde{R}))$. In conclusion the term we want to control is

$$\int_{V(\bar{u}, \underline{u})} |u|^{5+\epsilon} \tau_+^6 \alpha(\hat{\mathcal{L}}_T \tilde{R}) \cdot \left({}^{(T)}p_4(\rho(\tilde{R}), \sigma(\tilde{R})) \right) \quad . \quad (2.56)$$

Proceeding as in [Kl-Ni1] Chapter 6 page 260, we write³⁶

$$\begin{aligned}
& \left| \int_{V(\bar{u}, \underline{u})} |u|^{5+\epsilon} \tau_+^6 \alpha(\hat{\mathcal{L}}_T \tilde{R}) \cdot \left({}^{(T)}p_4(\rho(\tilde{R}), \sigma(\tilde{R})) \right) \right| \tag{2.57} \\
& \leq c \int_{u_0(\underline{u})}^{\bar{u}} du' \left(\int_{C(u') \cap V(\bar{u}, \underline{u})} |u'|^{5+\epsilon} \tau_+^6 |\alpha(\hat{\mathcal{L}}_T \tilde{R})|^2 \right)^{\frac{1}{2}} \left(\int_{C(u') \cap V(\bar{u}, \underline{u})} |{}^{(T)}p_4|^2 |u'|^{5+\epsilon} \tau_+^6 |\rho(\tilde{R})|^2 \right)^{\frac{1}{2}} \\
& \leq c \tilde{Q}_1^{\frac{1}{2}} \int_{u_0(\underline{u})}^{\bar{u}} du' \left(\int_{C(u') \cap V(\bar{u}, \underline{u})} |{}^{(T)}p_4|^2 |u'|^{\frac{5+\epsilon}{2}} \tau_+^3 \rho(\tilde{R})^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The estimate of the last integral is:

$$\begin{aligned}
& \left[\left(\int_{C(u') \cap V(\bar{u}, \underline{u})} |{}^{(T)}p_4|^2 |u'|^{\frac{5+\epsilon}{2}} \tau_+^3 \rho(\tilde{R})^2 \right)^{\frac{1}{2}} \right] \\
& \leq \left(\sup_{V(\bar{u}, \underline{u})} |u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \right) \frac{1}{|u|^{\frac{1}{2} - (\frac{\epsilon}{2} + \delta)}} \left(\int_{C(u') \cap V(\bar{u}, \underline{u})} |{}^{(T)}p_4|^2 \right)^{\frac{1}{2}} \\
& \leq \left(\sup_{V(\bar{u}, \underline{u})} |u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \right) \frac{1}{|u|^{\frac{3}{2} - (\frac{\epsilon}{2} + \delta)}} \left(\int_{u_0(u')}^{\infty} \frac{1}{\tau_+^2} |\tau_+ \tau_- {}^{(T)}p_4|_{p=2,S}^2 \right)^{\frac{1}{2}} \\
& \leq \left(\sup_{V(\bar{u}, \underline{u})} |u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \right) \frac{1}{|u'|^{2 - (\frac{\epsilon}{2} + \delta)}} \left(\sup_{V(\bar{u}, \underline{u})} |\tau_+ \tau_- {}^{(T)}p_4|_{p=2,S}^2 \right)^{\frac{1}{2}} \\
& \leq c \frac{\varepsilon_0}{|u'|^{2 - (\frac{\epsilon}{2} + \delta)}} \left(\sup_{V(\bar{u}, \underline{u})} |u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \right) \tag{2.58}
\end{aligned}$$

recalling that $u_0(u') = |u'|$. The last two inequalities use the estimate for p_4 proved in [Kl-Ni1], Proposition 6.1.4, which could be improved. Therefore as from (the first part of) Theorem 2.2 we have

$$|u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \leq (\varepsilon_0 + c \tilde{Q}_K^{\frac{1}{2}})$$

we conclude that

$$\begin{aligned}
& \int_{\mathcal{K}} |u|^{5+\epsilon} \tau_+^6 \alpha(\hat{\mathcal{L}}_T \tilde{R}) \cdot \Theta(T, \tilde{R}) \leq c \tilde{Q}_1^{\frac{1}{2}} \int_{u_0}^{\bar{u}} du' \left(\int_{C(u') \cap \mathcal{K}} |{}^{(T)}p_4|^2 |u'|^{\frac{5+\epsilon}{2}} \tau_+^3 \rho(\tilde{R})^2 \right)^{\frac{1}{2}} \\
& \leq c \tilde{Q}_1^{\frac{1}{2}} \left(\int_{u_0}^{\bar{u}} \frac{1}{|u'|^{\frac{5}{2} - (\frac{\epsilon}{2} + \delta)}} du' \right) \left(\sup_{\mathcal{K}} |u'|^{3-\delta} \tau_+^3 \rho(\tilde{R}) \right) \left(\sup_{\mathcal{K}} |\tau_+ \tau_- {}^{(T)}p_4|_{p=2,S}^2 \right)^{\frac{1}{2}} \\
& \leq c \frac{\varepsilon_0}{R_0^{\frac{3}{2} - (\frac{\epsilon}{2} + \delta)}} \tilde{Q}_K^{\frac{1}{2}} (\varepsilon_0 + c \tilde{Q}_K^{\frac{1}{2}}) \leq c \frac{\varepsilon_0}{R_0} (\varepsilon_0^2 + \tilde{Q}_K). \tag{2.59}
\end{aligned}$$

³⁶We consider only the ρ part as the easier σ part can be treated in a similar way.

in agreement with 2.42.

Conclusions: *The previous discussion and estimate, although done for a specific term of the “Error” should convince the reader that the $\tilde{\mathcal{Q}}$ norms are bounded. The remaining part of the paper is devoted to obtain decay informations from them.*

2.4 The decay estimates for the $\hat{\mathcal{L}}_T R$ null components

Form the boundedness of the $\tilde{\mathcal{Q}}$ norms we obtain, proceeding as in [Kl-Ni1] and in the first part of [Kl-Ni2], the following sup estimates:³⁷

$$\begin{aligned} \sup_{\mathcal{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2} + \frac{\epsilon}{2}} |\alpha(\hat{\mathcal{L}}_T R)| &\leq C_0, & \sup_{\mathcal{K}} r^{\frac{7}{2}} |u|^{\frac{5}{2} + \frac{\epsilon}{2}} |\beta(\hat{\mathcal{L}}_T R)| &\leq C_0 \\ \sup_{\mathcal{K}} r^3 |u|^{3 + \frac{\epsilon}{2}} |\rho(\hat{\mathcal{L}}_T R) - \overline{\rho(\hat{\mathcal{L}}_T R)}| &\leq C_0, & \sup_{\mathcal{K}} r^3 |u|^{3 + \frac{\epsilon}{2}} |\sigma(\hat{\mathcal{L}}_T R) - \overline{\sigma(\hat{\mathcal{L}}_T R)}| &\leq C_0 \\ \sup_{\mathcal{K}} r^2 |u|^{4 + \frac{\epsilon}{2}} \underline{\beta}(\hat{\mathcal{L}}_T R) &\leq C_0, & \sup_{\mathcal{K}} r |u|^{5 + \frac{\epsilon}{2}} |\underline{\alpha}(\hat{\mathcal{L}}_T R)| &\leq C_0. \end{aligned} \quad (2.60)$$

As in [Kl-Ni2] we use the transport equations for $\beta(\hat{\mathcal{L}}_T R)$ and $\alpha(\hat{\mathcal{L}}_T R)$ along the incoming cones to obtain a better decay in r for these components. More precisely we prove the following theorem:

Theorem 2.3. *In the spacetime \mathcal{K} , from the estimates 2.60 we derive the following ones*

$$\sup_{\mathcal{K}} r^4 |u|^{2 + \frac{\epsilon''}{2}} |\beta(\hat{\mathcal{L}}_T R)| \leq C_1 \quad (2.61)$$

$$\sup_{\mathcal{K}} r^5 |u|^{1 + \frac{\epsilon''}{2}} |\alpha(\hat{\mathcal{L}}_T R)| \leq C_1 \quad (2.62)$$

where $\epsilon'' < \epsilon$.

Proof of 2.61: From the Bianchi equations, see (3.2.8) of [Kl-Ni1], it follows that $\beta = \beta(\hat{\mathcal{L}}_T R)$ satisfies, along the incoming null hypersurface $\underline{C}(\nu)$, the evolution equation

$$\mathfrak{D}_3 \beta + \text{tr} \chi \beta = \nabla \rho + [2\underline{\omega} \beta + \star \nabla \sigma + 2\hat{\chi} \cdot \underline{\beta} + 3(\eta \rho + \star \eta \sigma)] \quad (2.63)$$

which can be rewritten as

$$\frac{\partial \beta_a}{\partial \lambda} + \Omega \text{tr} \chi \beta_a = 2\Omega \underline{\omega} \beta_a + \Omega [\nabla_a \rho + \star \nabla_a \sigma + 2(\hat{\chi} \cdot \underline{\beta})_a + 3(\eta \rho + \star \eta \sigma)_a] \quad (2.64)$$

where³⁸ $\beta_a = \beta(e_a)(\hat{\mathcal{L}}_T R)$. From this equation, see Chapter 4 of [Kl-Ni1], we

³⁷Recall that the decay of the initial data has an $\epsilon' > \epsilon$. Hereafter all the capital C constants, C, C_0, C_1, \dots are proportional to ϵ_0 .

³⁸All the notations used in this paper without an explicit definition are those already introduced in [Kl-Ni1]. The movig frame compatible with equation 2.64 is the Fermi transported, see detailed discussion in [Kl-Ni1].

obtain the following inequality,

$$\begin{aligned} \frac{d}{d\lambda} |r^{(2-\frac{2}{p})} \beta|_{p,S} &\leq \|2\Omega\underline{\omega} - (1-1/p)(\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}})\|_\infty |r^{(2-\frac{2}{p})} \beta|_{p,S} \quad (2.65) \\ &+ \|\Omega\|_\infty \left(|r^{(2-\frac{2}{p})} \nabla\rho|_{p,S} + 3|r^{(2-\frac{2}{p})} \eta\rho|_{p,S} + |r^{(2-\frac{2}{p})} \tilde{F}|_{p,S} \right), \end{aligned}$$

where $\tilde{F}(\cdot) = 2\hat{\chi} \cdot \underline{\beta} + (*\nabla\sigma + 3^*\eta\sigma)$.³⁹ Integrating along $\underline{C}(\nu)$, with $\lambda_1 = u|_{\underline{C}(\nu) \cap \Sigma_0}$, we obtain

$$\begin{aligned} |r^{(2-\frac{2}{p})} \beta|_{p,S}(\lambda, \nu) &\leq |r^{(2-\frac{2}{p})} \beta|_{p,S}(\lambda_1) \\ &+ \int_{\lambda_1}^\lambda \|2\Omega\underline{\omega} - (1-1/p)(\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}})\|_\infty |r^{(2-\frac{2}{p})} \beta|_{p,S}(\lambda', \nu) \\ &+ \|\Omega\|_\infty \left(\int_{\lambda_1}^\lambda |r^{(2-\frac{2}{p})} \nabla\rho|_{p,S} + 3 \int_{\lambda_1}^\lambda |r^{(2-\frac{2}{p})} \eta\rho|_{p,S} + \frac{1}{2} \int_{\lambda_1}^\lambda |r^{(2-\frac{2}{p})} \tilde{F}|_{p,S} \right). \end{aligned}$$

In \mathcal{K} we have at least the following behaviour⁴⁰

$$\|\Omega\underline{\omega}\|_\infty = O(r^{-1}|\lambda|^{-1}) \quad \text{and} \quad \|\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}\|_\infty = O(r^{-2}|\lambda|^{-\frac{1}{2}}).$$

Therefore we can apply the Gronwall's Lemma obtaining:

$$\begin{aligned} |r^{2-\frac{2}{p}} \beta|_{p,S}(\lambda, \nu) &\leq c \left[|r^{2-\frac{2}{p}} \beta|_{p,S}(\lambda_1) + \|\Omega\|_\infty \left(\int_{\lambda_1}^\lambda |r^{2-\frac{2}{p}} \nabla\rho|_{p,S} \right. \right. \\ &\quad \left. \left. + 3 \int_{\lambda_1}^\lambda |r^{2-\frac{2}{p}} \eta\rho|_{p,S} + \frac{1}{2} \int_{\lambda_1}^\lambda |r^{2-\frac{2}{p}} \tilde{F}|_{p,S} \right) \right] \quad (2.66) \end{aligned}$$

and recalling that $\|\Omega\|_\infty \leq C$, we will inglobe, hereafter, this factor in the constant c . Multiplying both sides by $r^2|\lambda|^{2+\frac{\epsilon''}{2}}$, with $\epsilon > \epsilon' > \epsilon'' > 0$, remembering that $r(\lambda, \nu) < r(\lambda_1, \nu)$ and $|\lambda| < |\lambda_1|$, we obtain

$$\begin{aligned} |r^{4-\frac{2}{p}} |\lambda|^{2+\frac{\epsilon''}{2}} \beta|_{p,S}(\lambda, \nu) &\leq c \left(|r^{4-\frac{2}{p}} |\lambda|^{2+\frac{\epsilon''}{2}} \beta|_{p,S}(\lambda_1) \right. \\ &\quad \left. + \int_{\lambda_1}^\lambda |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \nabla\rho|_{p,S} + 3 \int_{\lambda_1}^\lambda |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \eta\rho|_{p,S} + \frac{1}{2} \int_{\lambda_1}^\lambda |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \tilde{F}|_{p,S} \right). \quad (2.67) \end{aligned}$$

We examine the integrals in 2.67 and prove that they are uniformly bounded in ν .⁴¹

a) $\int_{\lambda_1}^\lambda |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \nabla\rho(\hat{\mathcal{L}}_T R)|_{p,S}$:

To prove the boundedness of this integral we use the following lemma:

³⁹The term $*\nabla\sigma + 3^*\eta\sigma$ behaves as the term $\nabla_a\rho + 3\eta_a\rho$ and, therefore, we will not consider it explicitly.

⁴⁰Consistent with the decay properties proved in [Kl-Nil]. In fact in \mathcal{K} the estimates for the second term are better with respect to the $|\lambda|$ decay.

⁴¹Recall that all the null components $\alpha, \beta, \rho, \sigma, \dots$ refer to $\hat{\mathcal{L}}_T R$.

Lemma 2.3. *In the spacetime \mathcal{K} from the boundedness of the $\tilde{\mathcal{Q}}$ norms the following estimate holds:*

$$\sup_{\mathcal{K}} |r^{4-\frac{2}{p}} |\lambda|^{\frac{6+\epsilon'}{2}} \nabla \rho(\tilde{R})|_{p,S} \leq C . \quad (2.68)$$

Proof: See the appendix.

b) $\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \eta \rho|_{p,S}$: Recalling that η satisfies in [Kl-Ni1],

$$|r^{2-2/p} |\lambda|^{\frac{1}{2}} \eta|_{p,S}(\lambda, \nu) \leq c , \quad p \in [2, \infty] . \quad (2.69)$$

Using the previous estimate for $\rho(\tilde{R})$ in \mathcal{K} ,

$$\sup_{\mathcal{K}} \left| |u|^{(3-\delta)} \tau_+^3 \rho(\tilde{R}) \right| \leq \frac{c}{R_0} (\varepsilon_0 + \tilde{\mathcal{Q}}_{\mathcal{K}}^{\frac{1}{2}}) \leq C ,$$

it follows immediately

$$\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \eta \rho|_{p,S} \leq c \int_{\lambda_1}^{\lambda} \frac{1}{r |\lambda'|^{\frac{3}{2} - \frac{2\delta+\epsilon''}{2}}} \leq c . \quad (2.70)$$

c) $\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \tilde{F}|_{S,p}$:

From the expression $\tilde{F}(\cdot) = {}^* \nabla \sigma + 3^* \eta \sigma + 2 \hat{\chi} \cdot \underline{\beta}$ and the previous remark concerning ${}^* \nabla \sigma + 3^* \eta \sigma$, we have only to prove that

$$\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \hat{\chi} \underline{\beta}|_{p,S} \leq c \quad (2.71)$$

This is easy, as, from the estimates 2.60, we have

$$\sup_{\mathcal{K}} |r^2 |\lambda|^{4+\frac{5}{2}} \underline{\beta}| \leq C_0 , \quad \sup_{\mathcal{K}} ||\lambda|^{\frac{1}{2}} r^{2-2/p} \hat{\chi}|_{p,S} \leq C_0 .$$

Therefore

$$\int_{\lambda_1}^{\lambda} |r^{4-\frac{2}{p}} |\lambda'|^{2+\frac{\epsilon''}{2}} \tilde{F}|_{p,S} \leq c \int_{\lambda_1}^{\lambda} \frac{1}{|\lambda'|^{\frac{5}{2} + \frac{\epsilon-\epsilon''}{2}}} \leq \int_{\lambda_1}^{\lambda_0} \frac{1}{|\lambda'|^{\frac{5}{2} + \frac{\epsilon-\epsilon''}{2}}} \leq \frac{c}{R_0^{\frac{3}{2} + \frac{\epsilon-\epsilon''}{2}}} , \quad (2.72)$$

where $\lambda_0 = R_0$. Collecting all these estimates for the integrals in 2.67 we infer that

$$|r^{4-\frac{2}{p}} |\lambda|^{2+\frac{\epsilon''}{2}} \beta|_{p,S}(\lambda, \nu) \leq c \left(|r^{4-\frac{2}{p}} |\lambda|^{2+\frac{\epsilon''}{2}} \beta|_{p,S}(\lambda_1) + 1 \right) \quad (2.73)$$

and using the initial data assumptions we conclude that,

$$|r^{4-\frac{2}{p}} |\lambda|^{2+\frac{\epsilon''}{2}} \beta|_{p,S}(\lambda, \nu) \leq c , \quad p \in [2, 4] . \quad (2.74)$$

To prove the sup estimate 2.61 we have to repeat for $\nabla\beta$ the previous estimate for β . This requires again the transport equation for $\nabla\beta$ along the \underline{C} “cones” which at its turn requires the control of an extra derivative for ρ and σ . This is the reason why we need a greater regularity in the initial data which translates in the introduction of \mathcal{Q} norms with more Lie derivatives than in [Kl-Ni1]. We do not write the proof here as it goes, with the obvious changes, exactly as for the β estimate.

Proof of 2.62: We look at the transport equation for $|r^{(1-\frac{2}{p})}\alpha(\hat{\mathcal{L}}_T R)|_{p,S}$. From the evolution equation satisfied by α , see [Kl-Ni1], Chapter 3, equation (3.2.8),

$$\frac{\partial\alpha}{\partial\lambda} + \frac{1}{2}\Omega\text{tr}\underline{\chi}\alpha = 4\Omega\underline{\omega}\alpha + \Omega[\nabla\hat{\otimes}\beta + (-3(\hat{\chi}\rho + \hat{\chi}\sigma) + (\zeta + 4\eta)\hat{\otimes}\beta)] ,$$

it follows that

$$\begin{aligned} \frac{\partial|\alpha|^p}{\partial\lambda} &= p|\alpha|^{p-1}\frac{\partial|\alpha|}{\partial\lambda} = p|\alpha|^{p-2}\alpha \cdot \frac{\partial\alpha}{\partial\lambda} \\ &= p|\alpha|^{p-2}\alpha \cdot \left(-\frac{1}{2}\Omega\text{tr}\underline{\chi} + 4\Omega\underline{\omega}\right)\alpha + \Omega\nabla\hat{\otimes}\beta + F(\cdot) \\ &= p\left(-\frac{1}{2}\Omega\text{tr}\underline{\chi} + 4\Omega\underline{\omega}\right)|\alpha|^p + p|\alpha|^{p-2}\Omega\alpha \cdot (\nabla\hat{\otimes}\beta) + p|\alpha|^{p-2}\alpha \cdot F(\cdot) \end{aligned} \quad (2.75)$$

where

$$F = \Omega(-3(\hat{\chi}\rho + \hat{\chi}\sigma) + (\zeta + 4\eta)\hat{\otimes}\beta) .$$

Immediately,

$$\frac{\partial|\alpha|^p}{\partial\lambda} + \frac{p}{2}\Omega\text{tr}\underline{\chi}|\alpha|^p = 4p\Omega\underline{\omega}|\alpha|^p + p|\alpha|^{p-2}\Omega\alpha \cdot (\nabla\hat{\otimes}\beta) + p|\alpha|^{p-2}\alpha \cdot F(\cdot) . \quad (2.76)$$

As, see [Kl-Ni1], Chapter 4,

$$\begin{aligned} \frac{d}{d\lambda} \left(\int_{S(\lambda,\nu)} r^\sigma |\alpha|^p d\mu_\gamma \right) &= \int_{S(\lambda,\nu)} r^\sigma \left(\frac{\partial|\alpha|^p}{\partial\lambda} + \left(1 + \frac{\sigma}{2}\right)\Omega\text{tr}\underline{\chi}|\alpha|^p \right) \\ &\quad - \frac{\sigma}{2} \int_{S(\lambda,\nu)} r^\sigma |\alpha|^p (\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}) , \end{aligned} \quad (2.77)$$

choosing $\sigma = p(1 - \frac{2}{p})$ we obtain

$$\begin{aligned} \frac{d}{d\lambda} \left(\int_{S(\lambda,\nu)} |r^{(1-\frac{2}{p})}\alpha|^p d\mu_\gamma \right) &= \int_{S(\lambda,\nu)} r^{p(1-\frac{2}{p})} \left(\frac{\partial|\alpha|^p}{\partial\lambda} + \frac{p}{2}\Omega\text{tr}\underline{\chi}|\alpha|^p \right) \\ &\quad - \frac{p(1-\frac{2}{p})}{2} \int_{S(\lambda,\nu)} |r^{(1-\frac{2}{p})}\alpha|^p (\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}) \\ &= \int_{S(\lambda,\nu)} r^{p(1-\frac{2}{p})} (4p\Omega\underline{\omega}|\alpha|^p + p|\alpha|^{p-2}\Omega\alpha \cdot (\nabla\hat{\otimes}\beta) + p|\alpha|^{p-2}\alpha \cdot F(\cdot)) \\ &\quad - \left(\frac{p}{2} - 1\right) \int_{S(\lambda,\nu)} |r^{(1-\frac{2}{p})}\alpha|^p (\Omega\text{tr}\underline{\chi} - \overline{\Omega\text{tr}\underline{\chi}}) . \end{aligned} \quad (2.78)$$

Therefore with $p \in [2, 4]$ we have

$$\begin{aligned} \frac{d}{d\lambda} |r^{(1-\frac{2}{p})} \alpha|_{p,S}^p &= \int_{S(\lambda,\nu)} \left(4p\Omega\underline{\omega} - \left(\frac{p}{2} - 1\right) (\Omega \text{tr}\underline{\chi} - \overline{\Omega \text{tr}\underline{\chi}}) \right) |r^{(1-\frac{2}{p})} \alpha|^p \\ &+ \int_{S(\lambda,\nu)} pr^{p(1-\frac{2}{p})} |\alpha|^{p-2} \Omega \alpha \cdot (\nabla \widehat{\otimes} \beta) + \int_{S(\lambda,\nu)} pr^{p(1-\frac{2}{p})} |\alpha|^{p-2} \alpha \cdot F(\cdot) \end{aligned}$$

and, therefore,

$$\begin{aligned} p|r^{(1-\frac{2}{p})} \alpha|_{p,S}^{p-1} \frac{d}{d\lambda} |r^{(1-\frac{2}{p})} \alpha|_{p,S} &\leq \|\Omega\|_\infty \left(4p|\underline{\omega}|_\infty + \left(\frac{p}{2} - 1\right) |\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}|_\infty \right) |r^{(1-\frac{2}{p})} \alpha|_{p,S}^p \\ &+ p\|\Omega\|_\infty \|r^5 |\lambda|^{2+\frac{\epsilon''}{2}} \nabla \beta\|_\infty \frac{1}{r^{(4+\frac{2}{p})} |\lambda|^{2+\frac{\epsilon''}{2}}} \int_{S(\lambda,\nu)} |r^{(1-\frac{2}{p})} \alpha|^{p-1} + p\|r^{1-\frac{2}{p}} F\|_\infty \int_{S(\lambda,\nu)} |r^{(1-\frac{2}{p})} \alpha|^{p-1} \\ &\leq \|\Omega\|_\infty \left(4p|\underline{\omega}|_\infty + \left(\frac{p}{2} - 1\right) |\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}|_\infty \right) |r^{(1-\frac{2}{p})} \alpha|_{p,S}^p \\ &+ p\|\Omega\|_\infty \|r^5 |\lambda|^{2+\frac{\epsilon''}{2}} \nabla \beta\|_\infty \frac{1}{r^4 |\lambda|^{2+\frac{\epsilon''}{2}}} |r^{(1-\frac{2}{p})} \alpha|_{p,S}^{p-1} + p\|rF\|_\infty |r^{(1-\frac{2}{p})} \alpha|_{p,S}^{p-1} \end{aligned}$$

where in the last inequality we applied the Holder inequality

$$\begin{aligned} \int_{S(\lambda,\nu)} r^{(p-1)(1-\frac{2}{p})} |\alpha|^{p-1} &\leq \left(\int_{S(\lambda,\nu)} 1^p \right)^{\frac{1}{p}} \left(\int_{S(\lambda,\nu)} [r^{(p-1)(1-\frac{2}{p})} |\alpha|^{p-1}]^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq cr^{\frac{2}{p}} \left(\int_{S(\lambda,\nu)} r^{p(1-\frac{2}{p})} |\alpha|^p \right)^{\frac{p-1}{p}} \leq cr^{\frac{2}{p}} |r^{(1-\frac{2}{p})} \alpha|_{p,S}^{p-1}. \end{aligned}$$

Therefore the following inequality holds:

$$\begin{aligned} \frac{d}{d\lambda} |r^{(1-\frac{2}{p})} \alpha|_{p,S} &\leq \|\Omega\|_\infty \left(4p|\underline{\omega}|_\infty + \left(\frac{p}{2} - 1\right) |\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}|_\infty \right) |r^{(1-\frac{2}{p})} \alpha|_{p,S} \\ &+ p\|\Omega\|_\infty \frac{1}{r^4 |\lambda|^{2+\frac{\epsilon''}{2}}} \|r^5 |\lambda|^{2+\frac{\epsilon''}{2}} \nabla \beta\|_\infty + p\|rF\|_\infty. \quad (2.79) \end{aligned}$$

$F = \Omega(-3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma) + (\zeta + 4\eta)\widehat{\otimes}\beta)$ satisfies the following bound:

$$\begin{aligned} \|F\|_\infty &\leq c\|\Omega\|_\infty \|\hat{\chi}\|_\infty (\|\rho\|_\infty + \|\sigma\|_\infty) + (\|\zeta\|_\infty + \|\eta\|_\infty) \|\beta\|_\infty \\ &\leq c \left(\frac{1}{r^2 |\lambda|^{\frac{1}{2}}} \|r^2 |\lambda|^{\frac{1}{2}} \hat{\chi}\|_\infty \frac{1}{r^3 |\lambda|^{3-\delta}} \|r^3 |\lambda|^{3-\delta} (\rho, \sigma)\| + \frac{1}{r^2 |\lambda|^{\frac{1}{2}}} \left(\|r^2 |\lambda|^{\frac{1}{2}} \zeta\|_\infty \right. \right. \\ &\quad \left. \left. + \|r^2 |\lambda|^{\frac{1}{2}} \eta\|_\infty \right) \frac{1}{r^4 |\lambda|^{2+\epsilon''}} \|r^4 |\lambda|^{2+\epsilon''} \beta\|_\infty \right) \\ &\leq c \frac{1}{r^5 |\lambda|^{\frac{7-2\delta}{2}}} \left(\|r^3 |\lambda|^{3-\delta} (\rho, \sigma)\|_\infty + \|r^4 |\lambda|^{2+\epsilon''} \beta\|_\infty \right) \leq cC_1 \frac{1}{r^5 |\lambda|^{\frac{7-2\delta}{2}}} \quad (2.80) \end{aligned}$$

and the previous inequality becomes

$$\begin{aligned}
\frac{d}{d\lambda} |r^{(1-\frac{2}{p})} \alpha|_{p,S} &\leq \|\Omega\|_\infty \left(4p|\underline{\omega}|_\infty + \left(\frac{p}{2} - 1\right) |\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}|_\infty \right) |r^{(1-\frac{2}{p})} \alpha|_{p,S} \\
&+ p\|\Omega\|_\infty \frac{1}{r^4|\lambda|^{2+\frac{\epsilon''}{2}}} \|r^5|\lambda|^{2+\frac{\epsilon''}{2}} \nabla \beta\|_\infty + c \frac{1}{r^5|\lambda|^{\frac{7-2\delta}{2}}} \left(\|r^3|\lambda|^{3-\delta}(\rho, \sigma)\|_\infty + \|r^4|\lambda|^{2+\epsilon''} \beta\|_\infty \right) \\
&\leq \frac{c}{r|\lambda|} |r^{(1-\frac{2}{p})} \alpha|_{p,S} + \frac{c}{r^4|\lambda|^{2+\frac{\epsilon''}{2}}} \leq \frac{c}{|\lambda|^2} |r^{(1-\frac{2}{p})} \alpha|_{p,S} + \frac{c}{r^4|\lambda|^{2+\frac{\epsilon''}{2}}} .
\end{aligned}$$

Integrating along $\underline{C}(\nu)$ we obtain

$$|r^{(1-\frac{2}{p})} \alpha|_{p,S}(\lambda, \nu) \leq c \left(|r^{(1-\frac{2}{p})} \alpha|_{p,S}(\lambda_1, \nu) + \frac{1}{r^4|\lambda|^{1+\frac{\epsilon''}{2}}} \right) \quad (2.81)$$

where $\lambda_1 = u_{\Sigma_0 \cap \underline{C}(\nu)}$, and multiplying the inequality by $r^4|\lambda|^{1+\frac{\epsilon''}{2}}$, we obtain

$$|r^{(5-\frac{2}{p})} |\lambda|^{1+\frac{\epsilon''}{2}} \alpha(\hat{\mathcal{L}}_T R)|_{p,S}(\lambda, \nu) \leq c \left(|r^{(5-\frac{2}{p})} |\lambda|^{1+\frac{\epsilon''}{2}} \alpha(\hat{\mathcal{L}}_T R)|_{p,S}(\lambda_1, \nu) + 1 \right) \quad (2.82)$$

which completes the proof of Theorem 2.3.

2.5 The decay estimates for the $\mathcal{L}_T R$ null components

In the final section 2.7 we use the estimates of the null components of $\mathcal{L}_T R$ to get estimates for the asymptotic behaviour of R . Observe that from the boundedness of the $\tilde{\mathcal{Q}}$ norms we have, instead, a control of the null components of the $\hat{\mathcal{L}}_T R$ tensor. Therefore we have to get from these estimates those for $\mathcal{L}_T R$. The two tensors are connected by the following equation,

$$\mathcal{L}_T R = \hat{\mathcal{L}}_T R + \frac{1}{2} {}^{(T)}[R] - \frac{3}{8} (\text{tr} {}^{(T)}\pi) R, \quad (2.83)$$

where

$${}^{(T)}[R]_{\alpha\beta\gamma\delta} = {}^{(T)}\pi_\alpha^\mu R_{\mu\beta\gamma\delta} + {}^{(T)}\pi_\beta^\mu R_{\alpha\mu\gamma\delta} + {}^{(T)}\pi_\gamma^\mu R_{\alpha\beta\mu\delta} + {}^{(T)}\pi_\delta^\mu R_{\alpha\beta\gamma\mu}. \quad (2.84)$$

In the previous section, see Theorem 2.3, we have proved the following decay for the $\hat{\mathcal{L}}_T R$ null components,

$$\begin{aligned}
\sup_{\mathcal{K}} r^5 |u|^{1+\frac{\epsilon''}{2}} |\alpha(\hat{\mathcal{L}}_T R)| &\leq C_1, \quad \sup_{\mathcal{K}} r^4 |u|^{2+\frac{\epsilon''}{2}} |\beta(\hat{\mathcal{L}}_T R)| \leq C_1 \\
\sup_{\mathcal{K}} r^3 |u|^{3+\frac{\epsilon'}{2}} |\rho(\hat{\mathcal{L}}_T R) - \overline{\rho(\hat{\mathcal{L}}_T R)}| &\leq C_1, \quad \sup_{\mathcal{K}} r^3 |u|^{3+\frac{\epsilon'}{2}} |\sigma(\hat{\mathcal{L}}_T R) - \overline{\sigma(\hat{\mathcal{L}}_T R)}| \leq C_1 \\
\sup_{\mathcal{K}} r^2 |u|^{4+\frac{\epsilon}{2}} \underline{\beta}(\hat{\mathcal{L}}_T R) &\leq C_1, \quad \sup_{\mathcal{K}} r |u|^{5+\frac{\epsilon}{2}} |\underline{\alpha}(\hat{\mathcal{L}}_T R)| \leq C_1,
\end{aligned}$$

Therefore from 2.83 and 2.84 it is clear that, due to the fact that for the null components of R the asymptotic estimates at our disposal are those proved in \mathcal{K} , see 2.16, which are weaker than those suggested from the ‘‘Peeling Theorem’’ by

a factor $r^{-\frac{3}{2}}$ for α and a factor $r^{-\frac{1}{2}}$ for β , the decay in r for the $\mathcal{L}_T R$ components are analogous to those for the $\hat{\mathcal{L}}_T R$ ones if the various components of ${}^{(T)}\pi$ decay at least as r^{-2} .⁴² Nevertheless we will see in a moment that a r^{-2} decay for ${}^{(T)}\pi$ is not yet enough to get the decay of the R null components as to get it we have to perform an extra integration bringing from the decay of $\mathcal{L}_T R$ to the decay of R .

Intuitively we expect that the various components of ${}^{(T)}\pi$ have a sufficiently strong decay for the following argument: The T vector field is Killing in the Kerr spacetime, therefore the only reason why here ${}^{(T)}\pi$ is not exactly zero is due to the extra correction of the initial data metric. As this correction is chosen to decay very fast it follows that the decay of the connection coefficients (which are zero in Kerr) should also decay as faster as we need, satisfying our goal. This argument, although correct, has to be proven and is the content of the next subsection.

2.6 The ${}^{(T)}\pi$ estimates

Defining the vector field T as in [Kl-Ni2],

$$T = \Omega(e_3 + e_4) \quad (2.85)$$

we have the following expression for the various components of ${}^{(T)}\pi$:

$$\begin{aligned} {}^{(T)}\mathbf{i}_{ab} &= 2\Omega \left(\hat{\chi}_{ab} + \hat{\chi}_{ab} + \delta_{ab}(\omega + \underline{\omega}) \right) \\ {}^{(T)}\mathbf{j} &= \Omega(\text{tr}\chi + \text{tr}\underline{\chi}) + 4\Omega(\omega + \underline{\omega}) \\ {}^{(T)}\mathbf{m}_a &= -4\Omega\zeta_a, \quad {}^{(T)}\underline{\mathbf{m}}_a = 4\Omega\zeta_a \\ {}^{(T)}\mathbf{n} &= 0, \quad {}^{(T)}\underline{\mathbf{n}} = 0. \end{aligned} \quad (2.86)$$

From Theorem 2.1 we have:

$$\begin{aligned} |r^{2-\frac{2}{p}}|u|^{1-\delta} {}^{(T)}\mathbf{i}|_{p,S} &\leq c\varepsilon_0 \\ |r^{2-\frac{2}{p}}|u|^{1-\delta} {}^{(T)}\mathbf{j}|_{p,S} &\leq c\varepsilon_0 \\ |r^{2-\frac{2}{p}}|u|^{1-\delta} \text{tr}({}^{(T)}\pi)|_{p,S} &\leq c\varepsilon_0 \\ |r^{2-\frac{2}{p}}|u|^{1-\delta} ({}^{(T)}\mathbf{m}, {}^{(T)}\underline{\mathbf{m}})|_{p,S} &\leq c\varepsilon_0 \end{aligned} \quad (2.87)$$

with arbitrary $\delta > 0$. Using estimates 2.87 we can obtain the decay of the $\mathcal{L}_T R$ null components and prove the following result:

Theorem 2.4. *In the spacetimes \mathcal{K} , whose initial conditions are given in Theorem 1.2, the following inequalities hold*

$$\begin{aligned} \sup_{\mathcal{K}} r^5 |u|^{1+\frac{\epsilon'''}{2}} |\alpha(\mathcal{L}_T R)| &\leq C_2, \quad \sup_{\mathcal{K}} r^4 |u|^{2+\frac{\epsilon''}{2}} |\beta(\mathcal{L}_T R)| \leq C_2 \\ \sup_{\mathcal{K}} r^3 |u|^{3+\frac{\epsilon'}{2}} |\rho(\mathcal{L}_T R) - \overline{\rho(\mathcal{L}_T R)}| &\leq C_2, \quad \sup_{\mathcal{K}} r^3 |u|^{3+\frac{\epsilon'}{2}} |\sigma(\mathcal{L}_T R) - \overline{\sigma(\mathcal{L}_T R)}| \leq C_2 \\ \sup_{\mathcal{K}} r^2 |u|^{4+\frac{\epsilon}{2}} \underline{\beta}(\mathcal{L}_T R) &\leq C_2, \quad \sup_{\mathcal{K}} r |u|^{5+\frac{\epsilon}{2}} |\underline{\alpha}(\mathcal{L}_T R)| \leq C_2. \end{aligned}$$

⁴²The connection coefficients decay in r with integer powers.

Proof: From equations 2.83 and 2.84 it follows:

$$\begin{aligned}
\alpha(\mathcal{L}_T R)_{ab} &= \alpha(\hat{\mathcal{L}}_T R)_{ab} + \frac{1}{2} {}^{(T)}[R]_{a4b4} - \frac{3}{8} (\text{tr}^{(T)} \pi) \alpha(R)_{ab} \\
\beta(\mathcal{L}_T R)_a &= \beta(\hat{\mathcal{L}}_T R)_a + \frac{1}{2} {}^{(T)}[R]_{a434} - \frac{3}{8} (\text{tr}^{(T)} \pi) \beta(R)_a \\
\rho(\mathcal{L}_T R) &= \rho(\hat{\mathcal{L}}_T R) + \frac{1}{2} {}^{(T)}[R]_{3434} - \frac{3}{8} (\text{tr}^{(T)} \pi) \rho(R) \\
\sigma(\mathcal{L}_T R) &= \sigma(\hat{\mathcal{L}}_T R) + \frac{1}{2} {}^{(T)}[*R]_{3434} - \frac{3}{8} (\text{tr}^{(T)} \pi) \sigma(R) \\
\underline{\beta}(\mathcal{L}_T R)_a &= \underline{\beta}(\hat{\mathcal{L}}_T R)_a + \frac{1}{2} {}^{(T)}[R]_{a343} - \frac{3}{8} (\text{tr}^{(T)} \pi) \underline{\beta}(R)_a \\
\underline{\alpha}(\mathcal{L}_T R)_{ab} &= \underline{\alpha}(\hat{\mathcal{L}}_T R)_{ab} + \frac{1}{2} {}^{(T)}[R]_{a3b3} - \frac{3}{8} (\text{tr}^{(T)} \pi) \underline{\alpha}(R)_{ab}
\end{aligned} \tag{2.88}$$

and, observing that ${}^{(T)}\pi_{44} = 0$,

$$\begin{aligned}
{}^{(T)}[R]_{a4b4} &= {}^{(T)}\pi_a^\mu R_{\mu 4b4} + {}^{(T)}\pi_4^\mu R_{a\mu b4} + {}^{(T)}\pi_b^\mu R_{a4\mu 4} + {}^{(T)}\pi_4^\mu R_{a4b\mu} \\
&= -\frac{1}{2} {}^{(T)}\pi_{a4} R_{34b4} + {}^{(T)}\pi_{ac} R_{c4b4} - \frac{1}{2} {}^{(T)}\pi_{43} R_{a4b4} + {}^{(T)}\pi_{4c} R_{acb4} \\
&\quad - \frac{1}{2} {}^{(T)}\pi_{b4} R_{a434} + {}^{(T)}\pi_{bc} R_{a4c4} - \frac{1}{2} {}^{(T)}\pi_{43} R_{a4b4} + {}^{(T)}\pi_{4c} R_{a4bc} \\
&= -\frac{1}{2} {}^{(T)}\hat{\pi}_{a4} R_{34b4} + ({}^{(T)}\hat{\pi}_{ac} R_{c4b4} + \frac{1}{4} (\text{tr}^{(T)} \pi) R_{a4b4}) - \frac{1}{2} ({}^{(T)}\hat{\pi}_{43} R_{a4b4} - \frac{1}{2} (\text{tr}^{(T)} \pi) R_{a4b4}) \\
&\quad + {}^{(T)}\hat{\pi}_{4c} R_{acb4} - \frac{1}{2} {}^{(T)}\hat{\pi}_{b4} R_{a434} + ({}^{(T)}\hat{\pi}_{bc} R_{a4c4} + \frac{1}{4} (\text{tr}^{(T)} \pi) R_{a4b4}) \\
&\quad - \frac{1}{2} ({}^{(T)}\hat{\pi}_{43} R_{a4b4} - \frac{1}{2} (\text{tr}^{(T)} \pi) R_{a4b4}) + {}^{(T)}\hat{\pi}_{4c} R_{a4bc} \\
&= -\frac{1}{2} {}^{(T)}\hat{\pi}_{a4} R_{34b4} + ({}^{(T)}\hat{\pi}_{ac} R_{c4b4} + {}^{(T)}\hat{\pi}_{bc} R_{a4c4} + \frac{3}{4} (\text{tr}^{(T)} \pi) R_{a4b4}) \\
&\quad - \frac{1}{2} {}^{(T)}\hat{\pi}_{43} R_{a4b4} + {}^{(T)}\hat{\pi}_{4c} (R_{acb4} + R_{a4bc}) - \frac{1}{2} {}^{(T)}\hat{\pi}_{b4} R_{a434} .
\end{aligned} \tag{2.89}$$

Therefore estimating ${}^{(T)}[R]_{a4b4}$ we obtain

$$\begin{aligned}
\|{}^{(T)}[R]_{a4b4}\|_\infty &\leq c \left((\|{}^{(T)}\mathbf{i}\|_\infty + \|{}^{(T)}\mathbf{j}\|_\infty) \|\alpha(R)\|_\infty + \|{}^{(T)}\mathbf{m}\|_\infty \|\beta(R)\|_\infty \right) \\
&\leq c \left(\frac{1}{r^{2+\frac{7}{2}} |\lambda|^{1-\delta}} (\|r^2 |\lambda|^{1-\delta} {}^{(T)}\mathbf{i}\|_\infty + \|r^2 |\lambda|^{1-\delta} {}^{(T)}\mathbf{j}\|_\infty) \|r^{\frac{7}{2}} \alpha(R)\|_\infty \right. \\
&\quad \left. + \frac{1}{r^{2+\frac{7}{2}} |\lambda|^{1-\delta}} \|r^2 |\lambda|^{1-\delta} {}^{(T)}\mathbf{m}\|_\infty \|r^{\frac{7}{2}} \beta(R)\|_\infty \right) \\
&\leq c \left(\frac{1}{r^{5+\frac{1}{2}} |\lambda|^{1-\delta}} \|r^{\frac{7}{2}} \alpha(R)\|_\infty + \frac{1}{r^{5+\frac{1}{2}} |\lambda|^{1-\delta}} \|r^{\frac{7}{2}} \beta(R)\|_\infty \right) \\
&\leq c \left(\frac{1}{r^{5+\frac{1}{2}} |\lambda|^{\frac{3}{2}-2\delta}} \|r^{\frac{7}{2}} |\lambda|^{\frac{1}{2}-\delta} \alpha(R)\|_\infty + \frac{1}{r^{5+\frac{1}{2}} |\lambda|^{\frac{3}{2}-2\delta}} \|r^{\frac{7}{2}} |\lambda|^{\frac{1}{2}-\delta} \beta(R)\|_\infty \right) \\
&\leq \frac{c}{r^{5+\frac{1}{2}} |\lambda|^{\frac{3}{2}-2\delta}} \leq \frac{c}{r^5 |\lambda|^{2-2\delta}}
\end{aligned} \tag{2.90}$$

so that, finally,

$$\|r^5|\lambda|^{(2-2\delta)(T)}[R]_{a4b4}\|_\infty \leq c. \quad (2.91)$$

Moreover

$$\begin{aligned} \|(\mathrm{tr}^{(T)}\pi)\alpha(R)\|_\infty &\leq \frac{c}{r^{5+\frac{1}{2}}|\lambda|^{\frac{3}{2}-2\delta}}|r^2|\lambda|^{1-\delta}\mathrm{tr}^{(T)}\pi\|_\infty\|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\alpha(R)\|_\infty \\ &\leq \frac{c}{r^5|\lambda|^{2-2\delta}}|r^2|\lambda|^{1-\delta}\mathrm{tr}^{(T)}\pi\|_\infty\|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\alpha(R)\|_\infty \end{aligned} \quad (2.92)$$

and finally

$$\|r^5|\lambda|^{2-2\delta}(\mathrm{tr}^{(T)}\pi)\alpha(R)\|_\infty \leq c. \quad (2.93)$$

using estimates 2.91 and 2.93 we obtain choosing δ such that

$$1 + \frac{\epsilon'''}{2} \leq 2 - 2\delta$$

$$\sup_{\mathcal{K}} |r^5|u|^{1+\frac{\epsilon'''}{2}}\alpha(\mathcal{L}_T R)| \leq C_2. \quad (2.94)$$

Let us repeat the previous estimate for the β term.

$$\begin{aligned} {}^{(T)}[R]_{a434} &= {}^{(T)}\pi_a^\mu R_{\mu 434} + {}^{(T)}\pi_4^\mu R_{a\mu 34} + {}^{(T)}\pi_3^\mu R_{a4\mu 4} + {}^{(T)}\pi_4^\mu R_{a43\mu} \\ &= -\frac{1}{2}{}^{(T)}\pi_{a4}R_{3434} + {}^{(T)}\pi_{ac}R_{c434} - \frac{1}{2}{}^{(T)}\pi_{43}R_{a434} + {}^{(T)}\pi_{4c}R_{ac34} \\ &\quad - \frac{1}{2}{}^{(T)}\pi_{34}R_{a434} + {}^{(T)}\pi_{3c}R_{a4c4} - \frac{1}{2}{}^{(T)}\pi_{43}R_{a434} + {}^{(T)}\pi_{4c}R_{a43c} \quad (2.95) \\ &= -\frac{1}{2}{}^{(T)}\hat{\pi}_{a4}R_{3434} + ({}^{(T)}\hat{\pi}_{ac}R_{c434} + \frac{1}{4}(\mathrm{tr}^{(T)}\pi)R_{a434}) - \frac{1}{2}({}^{(T)}\hat{\pi}_{43}R_{a434} - \frac{1}{2}(\mathrm{tr}^{(T)}\pi)R_{a434}) \\ &\quad + {}^{(T)}\hat{\pi}_{4c}R_{ac34} - \frac{1}{2}{}^{(T)}\hat{\pi}_{34}R_{a434} + ({}^{(T)}\hat{\pi}_{3c}R_{a4c4} + \frac{1}{4}(\mathrm{tr}^{(T)}\pi)R_{a434}) \\ &\quad - \frac{1}{2}({}^{(T)}\hat{\pi}_{43}R_{a434} - \frac{1}{2}(\mathrm{tr}^{(T)}\pi)R_{a434}) \\ &= -\frac{1}{2}{}^{(T)}\hat{\pi}_{a4}R_{3434} + ({}^{(T)}\hat{\pi}_{ac}R_{c434} + {}^{(T)}\hat{\pi}_{3c}R_{a4c4} + \frac{3}{4}(\mathrm{tr}^{(T)}\pi)R_{a434}) \\ &\quad - \frac{1}{2}{}^{(T)}\hat{\pi}_{43}R_{a434} + {}^{(T)}\hat{\pi}_{4c}R_{ac34} - {}^{(T)}\hat{\pi}_{34}R_{a434} + {}^{(T)}\hat{\pi}_{4c}R_{a43c} \\ &= \left(-\frac{1}{2}{}^{(T)}\hat{\pi}_{a4}R_{3434} + {}^{(T)}\hat{\pi}_{3c}R_{a4c4} + {}^{(T)}\hat{\pi}_{4c}R_{ac34} + {}^{(T)}\hat{\pi}_{4c}R_{a43c} \right) \\ &\quad + \left({}^{(T)}\hat{\pi}_{ac}R_{c434} + \frac{3}{4}(\mathrm{tr}^{(T)}\pi)R_{a434} - \frac{1}{2}{}^{(T)}\hat{\pi}_{43}R_{a434} - {}^{(T)}\hat{\pi}_{34}R_{a434} \right) \end{aligned}$$

Therefore estimating ${}^{(T)}[R]_{a434}$ we obtain

$$\begin{aligned}
& \|{}^{(T)}[R]_{a434}\|_\infty \leq c \left((\|{}^{(T)}\mathbf{m}\|_\infty + \|{}^{(T)}\underline{\mathbf{m}}\|_\infty) \|\rho(R)\|_\infty + (\|{}^{(T)}\mathbf{i}\|_\infty + \|{}^{(T)}\mathbf{j}\|_\infty + (\text{tr}({}^{(T)}\pi))) \|\beta\|_\infty \right) \\
& \leq c \left(\frac{1}{r^{4+1}|\lambda|^{1-\delta}} (\|r^2|\lambda|^{1-\delta}{}^{(T)}\mathbf{m}\|_\infty + \|r^2|\lambda|^{1-\delta}{}^{(T)}\underline{\mathbf{m}}\|_\infty) \|r^3\rho(R)\|_\infty \right. \\
& \quad \left. + \frac{1}{r^{2+\frac{7}{2}}|\lambda|^{1-\delta}} (\|r^2|\lambda|^{1-\delta}{}^{(T)}\mathbf{i}\|_\infty + \|r^2|\lambda|^{1-\delta}{}^{(T)}\mathbf{j}\|_\infty + (r^2|\lambda|^{1-\delta}\text{tr}({}^{(T)}\pi))) \|r^{\frac{7}{2}}\beta(R)\|_\infty \right) \\
& \leq c \left(\frac{1}{r^{4+1}|\lambda|^{1-\delta}} \|r^3\rho(R)\|_\infty + \frac{1}{r^{4+\frac{3}{2}}|\lambda|^{\frac{3}{2}-2\delta}} \|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\beta(R)\|_\infty \right) \\
& \leq c \left(\frac{1}{r^4|\lambda|^{2-\delta}} \|r^3\rho(R)\|_\infty + \frac{1}{r^4|\lambda|^{3-2\delta}} \|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\beta(R)\|_\infty \right) \\
& \leq \frac{c}{r^4|\lambda|^{2-\delta}} + \frac{c}{r^4|\lambda|^{3-2\delta}} \leq \frac{c}{r^4|\lambda|^{2-\delta}} \tag{2.96}
\end{aligned}$$

and finally

$$\|r^4|\lambda|^{(2-\delta)(T)}[R]_{a434}\|_\infty \leq c. \tag{2.97}$$

Moreover

$$\begin{aligned}
\|(\text{tr}({}^{(T)}\pi)\beta(R))\|_\infty & \leq \frac{c}{r^{5+\frac{1}{2}}|\lambda|^{\frac{3}{2}-2\delta}} |r^2|\lambda|^{1-\delta}\text{tr}({}^{(T)}\pi)|_\infty \|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\beta(R)\|_\infty \\
& \leq \frac{c}{r^4|\lambda|^{3-2\delta}} |r^2|\lambda|^{1-\delta}\text{tr}({}^{(T)}\pi)|_\infty \|r^{\frac{7}{2}}|\lambda|^{\frac{1}{2}-\delta}\beta(R)\|_\infty \tag{2.98}
\end{aligned}$$

and finally

$$\|r^4|\lambda|^{3-2\delta}(\text{tr}({}^{(T)}\pi)\beta(R))\|_\infty \leq c. \tag{2.99}$$

using estimates 2.97 and 2.99 we obtain choosing δ such that

$$1 + \frac{\epsilon''}{2} \leq \min(2 - \delta, 3 - 2\delta)$$

$$\sup_{\mathcal{K}} |r^4|u|^{1+\frac{\epsilon''}{2}} \beta(\mathcal{L}_T R)| \leq C_2. \tag{2.100}$$

For the other terms there is no need to repeat this computation. In fact the R null components already satisfy the peeling and going from $\hat{\mathcal{L}}_T$ to \mathcal{L}_T the $|u|^\gamma$ weight factor cannot become worst.

If we use only the estimates proved in [Kl-Ni1], see the remark at page 13, we can prove a weaker result, namely

Theorem 2.5. *Using only the estimates proved in [Kl-Ni1] the following inequalities hold*

$$\begin{aligned}
& \sup_{\mathcal{K}} r^5|u|\alpha(\mathcal{L}_T R) \leq C_3, \quad \sup_{\mathcal{K}} r^4|u|^{1+\frac{\epsilon''}{2}} |\beta(\mathcal{L}_T R)| \leq C_3 \\
& \sup_{\mathcal{K}} r^3|u|^{3+\frac{\epsilon'}{2}} |\rho(\mathcal{L}_T R) - \overline{\rho(\mathcal{L}_T R)}| \leq C_2, \quad \sup_{\mathcal{K}} r^3|u|^{3+\frac{\epsilon'}{2}} |\sigma(\mathcal{L}_T R) - \overline{\sigma(\mathcal{L}_T R)}| \leq C_2 \\
& \sup_{\mathcal{K}} r^2|u|^{4+\frac{5}{2}} \underline{\beta}(\mathcal{L}_T R) \leq C_2, \quad \sup_{\mathcal{K}} r|u|^{5+\frac{5}{2}} |\underline{\alpha}(\mathcal{L}_T R)| \leq C_2.
\end{aligned}$$

To prove the peeling for both the components $\alpha(R)$ and $\beta(R)$ we have to use the result of Theorem 2.4 as the decay proved in Theorem 2.5 is not sufficient for getting the “peeling estimate” for $\alpha(R)$. In fact to obtain it, as we explain in detail in the next subsection, we integrate along the integral curves of T for a fixed $r = \bar{r}$ and the decay factor $|u|^{-\gamma}$ makes, if $\gamma > 1$, the integral uniformly bounded. This implies that the estimates of the norms of the null components times their appropriate r weights are bounded if the corresponding initial data are bounded. Using the estimates of Theorem 2.5, the decay factor $|u|^{-1}$ of the $\alpha(\mathcal{L}_T)$ null component is not integrable and therefore it is impossible to obtain the expected decay for $\alpha(R)$. Viceversa the result proved in Theorem 2.4 gives a decay factor $|u|^{-(1+\frac{\epsilon'''}{2})}$ which makes the integration uniformly bounded and the decay of α tied to the that of the associated initial data and, therefore, satisfying the peeling estimate.

2.7 The decay estimates for the R null components

Once we have a control of the decay of the various null components of $\mathcal{L}_T R$ we can easily obtain analogous estimates for the ∂_T derivatives of the (pointwise) norms of the R components. Let us consider the more delicate α term. From the relation

$$\begin{aligned} \alpha(\mathcal{L}_T R)(e_a, e_b) &= (\mathcal{L}_T R)(e_a, e_4, e_b, e_4) \\ &= \partial_T(\alpha(e_a, e_b)) + \sum R([T, e_a], e_4, e_b, e_4) + \sum R(e_a, [T, e_4], e_b, e_4) , \end{aligned} \quad (2.101)$$

observing that in the Kerr spacetime $[T, e_a]$ and $[T, e_4]$ decay sufficiently fast, it follows that these terms cannot have a slower decay in \mathcal{K} due to the extra correction in the metric whose initial data have been chosen to decay faster than the Kerr metric’s ones. In fact

$$\begin{aligned} [T, e_a] &= [\Omega(e_3 + e_4), e_a] = \Omega[e_3, e_a] - e_a(\Omega)e_3 + \Omega[e_4, e_a] - e_a(\Omega)e_4 \\ &= \Omega([e_3, e_a] + [e_4, e_a]) - e_a(\Omega)(e_3 + e_4) \\ &= \Omega(\mathbf{D}_3 e_a + \mathbf{D}_4 e_a) - \Omega(\chi + \underline{\chi})_{ab} e_b - e_a(\Omega)(e_3 + e_4) \end{aligned} \quad (2.102)$$

and

$$\begin{aligned} [T, e_4] &= [\Omega(e_3 + e_4), e_4] = \Omega[e_3, e_4] - e_4(\Omega)e_3 - e_4(\Omega)e_4 \\ &= \Omega(\mathbf{D}_4 \log \Omega)e_3 - \Omega(\mathbf{D}_3 \log \Omega)e_4 + 4\Omega\zeta(e_a)e_a - \Omega(\mathbf{D}_4 \log \Omega)e_3 - \Omega(\mathbf{D}_4 \log \Omega)e_4 \\ &= -\Omega(\mathbf{D}_3 \log \Omega + \mathbf{D}_4 \log \Omega)e_4 + 4\Omega\zeta(e_a)e_a \end{aligned} \quad (2.103)$$

Observing that in Kerr spacetime

$$\mathbf{D}_3 \log \Omega + \mathbf{D}_4 \log \Omega = 0 \quad , \quad (\chi + \underline{\chi}) = 0 \quad , \quad e_a(\Omega) = 0 \quad (2.104)$$

The only terms which survive in Kerr spacetime are $\Omega(\mathfrak{D}_3 e_a + \mathfrak{D}_4 e_a)$ and $4\Omega\zeta(e_a)e_a$ which decay as ⁴³

$$\begin{aligned}\Omega(\mathfrak{D}_3 e_a + \mathfrak{D}_4 e_a) &= \frac{1}{r^3} c_{ab} e_b \\ 4\Omega\zeta(e_a)e_a &= \frac{1}{r^3} c_{ab} e_b .\end{aligned}\quad (2.105)$$

The first line can be proved observing that as shown in the appendix, eq. 4.183, the following estimate holds in Kerr spacetime, see also eq. 4.147 for the definition of the vector field e_λ :

$$[T, e_\lambda] = -2R \sin \theta e_\lambda (\omega_B) e_\phi = O(ar^{-3}) e_\phi \quad , \quad [T, e_\phi] = 0 \quad , \quad (2.106)$$

the second line from an explicit computation of $\zeta(e_\phi)$. Therefore as the corrections to the Kerr metric decay faster also these commutators satisfy in \mathcal{K} a decay at least of the following type

$$|\mathfrak{g}([T, e_a], e_\alpha)| \leq c \frac{1}{r^2 |u|^{1-\delta}} \quad , \quad |\mathfrak{g}([T, e_4], e_\alpha)| \leq c \frac{1}{r^2 |u|^{1-\delta}} \quad (2.107)$$

and observing that $[T, e_4]$ does not produce terms proportional to e_3 ,⁴⁴ it follows that $\partial_T(\alpha(e_a, e_b))$ has the same decay as $\alpha(\mathcal{L}_T R)(e_a, e_b)$.

The same result holds for $\beta(\mathcal{L}_T R)$ and more easily for the remaining components which we already know satisfy the ‘‘Peeling decay’’. Therefore we have the following estimates

$$\begin{aligned}\sup_{\mathcal{K}} r^5 |u|^{1+\frac{\epsilon'''}{2}} |\partial_T(\alpha(R)(e_a, e_b))| &\leq C_3 \\ \sup_{\mathcal{K}} r^4 |u|^{2+\frac{\epsilon'''}{2}} |\partial_T(\beta(R)(e_a))| &\leq C_3 .\end{aligned}\quad (2.108)$$

The final step is to integrate along the integral curves of T at fixed r . It is clear that the $|u|$ weight factors will allow to bound uniformly these integrals. Basically, the details are in the appendix, identifying T with $\frac{\partial}{\partial t}$ and u with $t - r_*$ we have integrating $|\bar{r}^5 \alpha(R)|$ on the curve starting at $r = \bar{r}$,

$$|\bar{r}^5 \alpha(R)|(\bar{t}, \bar{r}) \leq c \left(|\bar{r}^5 \alpha(R)|(0, \bar{r}) + \int_0^{\bar{t} - \bar{r} - r_1} \frac{1}{(\bar{r} - t')^{1+\frac{\epsilon'''}{2}}} dt' \right) , \quad (2.109)$$

where, identifying r with r_* ,

$$\begin{aligned}\bar{r} = r(u, \underline{u}) &= \frac{u - \underline{u}}{2} \quad , \quad \bar{t} = t(u, \underline{u}) = \frac{u + \underline{u}}{2} \\ u = u(\bar{t}, \bar{r}) &= u(0, r_1) = -r_1 \quad , \quad \underline{u} = \underline{u}(\bar{t}, \bar{r}) = \underline{u}(0, r_2) = r_2 \\ \bar{t} = t(u, \underline{u}) &= \frac{r_2 - r_1}{2} = \frac{\bar{t} + \bar{r} - r_1}{2} .\end{aligned}\quad (2.110)$$

⁴³With $\mathfrak{D}_3 e_a, \mathfrak{D}_4 e_a$ we denote the projection of the $\mathbf{D}_4 e_a$ and $\mathbf{D}_3 e_a$ derivatives on TS , see [K1-Nil], Chapter 3.

⁴⁴This will produce a term proportional to $\rho(R)$ whose decay would not be sufficient for the conclusion.

As the exponent in the integrand is greater than one, the integrand is uniformly bounded in \bar{r} and the previous inequality becomes:

$$|\bar{r}^5 \alpha(R)|(\bar{l}, \bar{r}) = |\bar{r}^5 \alpha(R)|(u, \underline{u}) \leq c (|\bar{r}^5 \alpha(R)|(0, \bar{r}) + 1) . \quad (2.111)$$

As the initial data are chosen in such a way that

$$|\bar{r}^5 \alpha(R)|(0, \bar{r}) \leq c ,$$

we conclude that

$$\sup_{\mathcal{K}} |r^5 \alpha(R)| \leq c , \quad (2.112)$$

which is the desired result as it implies the expected asymptotic null decay, that is when r goes to infinity, with $|u|$ constant.

Remark: If we used only the estimates proved in [Kl-Ni1] we would prove the estimate

$$\sup_{\mathcal{K}_1} r^5 |u| |\partial_T(\alpha(R)(e_a, e_b))| \leq C_4$$

and from it we obtain a decay for α which does not satisfy the peeling for the presence of a $\log r$ factor, namely we would obtain

$$\sup_{\mathcal{K}_1} \left| \frac{\bar{r}^5}{\log \bar{r}} \alpha(R) \right| (0, \bar{r}) \leq c . \quad (2.113)$$

3 Conclusions

Let us recall some general aspects of the problem we are considering.

The assumptions done at the beginning on the Kerr mass upper bound allows to treat the Kerr part of the metric as a perturbation and to apply the result of [Kl-Ni1] in the slightly modified version discussed in [Kl-Ni2] . On the other side if the metric correction, $\delta \mathbf{g}$, of the initial Kerr data metric were absent it is clear that the (very external region of the) spacetime would coincide with the one of the corresponding Kerr spacetime. If the initial data metric differ from the Kerr metric by some small corrections then the solutions are still near to Schwarzschild, but also near to Kerr. Therefore, although we are not making any linearization in the metric correction, but treating a true non linear problem, nevertheless these considerations allow us to write the Riemann tensor as

$$R = R_{Kerr} + \delta R . \quad (3.114)$$

This is the reason why to prove the “**expected result**” is more difficult than the “**weaker result**” we are proving here. In that case, in fact, we have to prove a non linear perturbation around a Kerr spacetime and we cannot fully use the machinery in [Kl-Ni1]. The difficulty is, of course, that the rotation generators vector fields cannot be considered in this case as “nearly Killing” and this makes difficult to bound the $\mathcal{Q}(\hat{\mathcal{L}}_O R)$ norms.

4 Appendix

4.1 The initial data \tilde{Q} norms.

Looking at the initial data for one of the \tilde{Q} norms we have that the following quantity has to be bounded

$$\int_{\Sigma_0 \cap \mathcal{K}} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, T)$$

Recalling the expression of $Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, T)$,

$$\begin{aligned} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, T) &= \frac{1}{8} \underline{u}^4 |\alpha|^2 + \frac{1}{8} u^4 |\underline{\alpha}|^2 + \frac{1}{2} (\underline{u}^4 + \frac{1}{2} u^2 \underline{u}^2) |\beta|^2 \\ &+ \frac{1}{2} (u^4 + 4 \underline{u}^2 u^2 + \underline{u}^4) (\rho^2 + \sigma^2) + \frac{1}{2} (u^4 + \frac{1}{2} u^2 \underline{u}^2) |\underline{\beta}|^2 \end{aligned}$$

where $\alpha = \alpha(\mathcal{L}_T R)$, $\beta = \beta(\mathcal{L}_T R)$ Observing that on $\Sigma_0 \cap \mathcal{K}$ u, \underline{u} behave like r , we can write

$$\begin{aligned} \int_{\Sigma_0 \cap \mathcal{K}} |u|^{5+\epsilon} Q(\hat{\mathcal{L}}_O R)(\bar{K}, \bar{K}, T, T) &\cong \\ \int_{R_0}^{\infty} dr r^{2+(5+\epsilon)+4} [|\alpha|^2 + |\underline{\alpha}|^2 + |\beta|^2 + (\rho^2 + \sigma^2) + |\underline{\beta}|^2] \end{aligned}$$

Therefore $\alpha(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T R)$ has to decay, as $r \rightarrow \infty$ as $o(r^{-(6+\tilde{\epsilon})})$ with $\tilde{\epsilon} > \epsilon$ and the same for all the other components. As $\hat{\mathcal{L}}_O$ does not increase the decay this means that all the null components of $\hat{\mathcal{L}}_T R$, with the exception of ρ has to decay in the same way. This implies that, again with the exception of ρ , all the null components of R has to decay toward spatial infinity as $r^{-(5+\tilde{\epsilon})}$. Therefore this requires that the metric correction δg has to decay as $O(r^{-(3+\tilde{\epsilon})})$. To complete this analysis let us look at the more delicate term, namely $\rho(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T R)$. We have

$$\rho(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T R) = \mathcal{L}_O \rho(\hat{\mathcal{L}}_T R) - \frac{1}{8} {}^{(O)}\pi \rho(\hat{\mathcal{L}}_T R) + O\left(\frac{1}{r}\right) (\beta(\hat{\mathcal{L}}_T R) + \underline{\beta}(\hat{\mathcal{L}}_T R)) \quad (4.115)$$

Observe that the term $O\left(\frac{1}{r}\right) (\beta(\hat{\mathcal{L}}_T R) + \underline{\beta}(\hat{\mathcal{L}}_T R))$ has a very good decay as $\beta(\hat{\mathcal{L}}_T R)$ and $\underline{\beta}(\hat{\mathcal{L}}_T R)$ have. The term $\frac{1}{8} {}^{(O)}\pi \rho(\hat{\mathcal{L}}_T R)$ behaves as

$$O(r^{-1}) \rho(\hat{\mathcal{L}}_T R) = O(r^{-1}) r^{-2} |u|^{-\gamma} r^{-3}$$

with $\gamma = \frac{1}{2}$ or $1 - \delta$ depending if we are considering the spacetime \mathcal{K}_1 or \mathcal{K}_2 . In both cases we have $O(r^{-1}) \rho(\hat{\mathcal{L}}_T R) = r^{-3} |u|^{-\gamma}$ which is what we need. We are left to consider the term $\mathcal{L}_O \rho(\hat{\mathcal{L}}_T R) = \partial_O \rho(\hat{\mathcal{L}}_T R)$. Observe that

$$\partial_O \rho(\hat{\mathcal{L}}_T R) = \partial_O \left(\rho(\mathcal{L}_T R) + \frac{3}{8} \text{tr}^{(T)} \pi \rho(R) + \{corrections\} \right) \quad (4.116)$$

Observe that on Σ_0 $\text{tr}^{(T)}\pi = o(r^{-3})$ so that $\frac{3}{8}\text{tr}^{(T)}\pi\rho(R) = o(r^{-6})$ as we need. Finally

$$\partial_O\rho(\mathcal{L}_T R) = \partial_O\left(\partial_T\rho(R) + O(r^{-(3+\tilde{\epsilon})})(\beta(R) + \underline{\beta}(R))\right). \quad (4.117)$$

The last term is a correction decaying very fast and in the first term $\partial_T\partial_O\rho(R)$ the derivatives with respect to O adds a factor r in the decay as the part of ρ which decays as r^{-3} does not depend on the angular variables, the ∂_T derivative does not give zero as ${}^{(T)}\pi$ is not zero due to the corrections to Kerr, but this term goes at least as $r^{-(3+\tilde{\epsilon})}$.⁴⁵ Therefore all the terms decay sufficiently fast and we can conclude that the following estimate holds:

$$|\rho(\hat{\mathcal{L}}_O\hat{\mathcal{L}}_T R)|_{\Sigma_0} = O\left(\frac{1}{r^{6+\tilde{\epsilon}}}\right) \text{ with } \tilde{\epsilon} > \epsilon > 0. \quad (4.118)$$

Therefore the present result is a true extension of the result in [Kl-Ni2] for the situation where in the metric to the $\delta\mathbf{g}$ correction it is added, as a correction, a time independent part which is small and decay as r^{-2} .

4.2 Some proofs

Proof of the last part of Theorem 2.1:

In the spacetime \mathcal{K} , whose existence is proved in Theorem 2.1, the following inequalities hold:

$$\begin{aligned} \sup_{\mathcal{K}} r^2 |u|^{1-\delta} |\text{tr}\chi + \text{tr}\underline{\chi}| &\leq c\epsilon_0 \\ \sup_{\mathcal{K}} r^2 |u|^{1-\delta} |\omega + \underline{\omega}| &\leq c\epsilon_0. \end{aligned} \quad (4.119)$$

Proof: To prove the first line of inequalities 4.119 we have to look at the transport equation for $\text{tr}\chi + \text{tr}\underline{\chi}$. From the structure equations, see [Kl-Ni1], equations (3.1.46),.....,(3.1.48), we have

$$\begin{aligned} \mathbf{D}_3\text{tr}\chi &= -\frac{1}{2}\text{tr}\underline{\chi}\text{tr}\chi - (\mathbf{D}_3\log\Omega)\text{tr}\chi + 2\rho + [-\hat{\chi} \cdot \hat{\chi} + 2|\eta|^2 + 2(\underline{\Delta}\log\Omega + \underline{\mathfrak{d}}\nu\zeta)] \\ \mathbf{D}_3\text{tr}\underline{\chi} &= -\frac{1}{2}(\text{tr}\underline{\chi})^2 + (\mathbf{D}_3\log\Omega)\text{tr}\underline{\chi} + |\hat{\chi}|^2 \end{aligned} \quad (4.120)$$

and from them

$$\begin{aligned} \mathbf{D}_3(\text{tr}\chi + \text{tr}\underline{\chi}) &= -\frac{1}{2}\text{tr}\underline{\chi}(\text{tr}\chi + \text{tr}\underline{\chi}) + (\mathbf{D}_3\log\Omega)(\text{tr}\chi + \text{tr}\underline{\chi}) \\ &\quad + 2(\rho - (\mathbf{D}_3\log\Omega)\text{tr}\chi) + [-\hat{\chi} \cdot \hat{\chi} + 2|\eta|^2 + 2(\underline{\Delta}\log\Omega + \underline{\mathfrak{d}}\nu\zeta) + |\hat{\chi}|^2] \end{aligned}$$

⁴⁵We can see it in a slightly different way: on Σ_0 we can write $\partial_O R = \partial_O R_{Kerr} + \partial_O \delta R$ and $\partial_T = \partial_T R_{Kerr} + \partial_{\delta T}$ so that $\partial_T \partial_O R = \partial_T R_{Kerr} \partial_O \delta R + \partial_{\delta T} \partial_O R_{Kerr}$. Therefore as $\delta T = o(r^{-2})c_\alpha e_\alpha$ it is clear that $\partial_T \partial_O R = o(r^{-6})$.

which we can rewrite

$$\begin{aligned} \frac{\partial}{\partial \lambda}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi}) + \frac{\overline{\Omega \operatorname{tr}\underline{\chi}}}{2}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi}) &= [2^{-1}(\overline{\Omega \operatorname{tr}\underline{\chi}} - \Omega \operatorname{tr}\underline{\chi}) + \Omega(\mathbf{D}_3 \log \Omega)](\operatorname{tr}\chi + \operatorname{tr}\underline{\chi}) \\ &+ 2\Omega(\rho - (\mathbf{D}_3 \log \Omega)\operatorname{tr}\chi) + \Omega[-\hat{\underline{\chi}} \cdot \hat{\chi} + 2|\eta|^2 + 2(\not\Delta \log \Omega + \not{d}iv \zeta) + |\hat{\underline{\chi}}|^2] . \end{aligned} \quad (4.121)$$

Recalling Lemma 4.1.5 of [Kl-Ni1] from the previous equation, observing that $[2^{-1}(\overline{\Omega \operatorname{tr}\underline{\chi}} - \Omega \operatorname{tr}\underline{\chi}) + \Omega(\mathbf{D}_3 \log \Omega)]$ is integrable in λ , we obtain

$$\begin{aligned} |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,S}(\lambda, \nu) &\leq |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,\underline{C}(\nu) \cap \Sigma_0} \\ &+ c \int_{\lambda_1}^{\lambda} d\lambda' |r^{1-\frac{2}{p}}(\rho - (\mathbf{D}_3 \log \Omega)\operatorname{tr}\chi)|_{p,S}(\lambda', \nu) \\ &+ c \int_{\lambda_1}^{\lambda} d\lambda' \left| r^{1-\frac{2}{p}}[-\hat{\underline{\chi}} \cdot \hat{\chi} + 2|\eta|^2 + 2(\not\Delta \log \Omega + \not{d}iv \zeta) + |\hat{\underline{\chi}}|^2] \right|_{p,S} . \end{aligned} \quad (4.122)$$

Observe that the integrand of the last integral can be estimated in the following way, using the results of subsection 4.3.16 in [Kl-Ni1] and recalling that due to the better decay of the initial data we have an extra decay factor $|u|^{\frac{1}{2}-\delta}$, with $\delta > 0$ and arbitrary, we obtain:

$$\begin{aligned} &\left| r^{1-\frac{2}{p}}[-\hat{\underline{\chi}} \cdot \hat{\chi} + 2|\eta|^2 + 2(\not\Delta \log \Omega + \not{d}iv \zeta) + |\hat{\underline{\chi}}|^2] \right|_{p,S} \\ &\leq c\varepsilon_0 r \left(\frac{1}{r^3|u|^{3-2\delta}} + \frac{1}{r^4|u|^{2-2\delta}} + \frac{1}{r^3|u|^{1-\delta}} + \frac{1}{r^2|u|^{4-2\delta}} \right) \leq c \frac{\varepsilon_0}{r|u|^{2-\delta}} . \end{aligned} \quad (4.123)$$

Substituting this estimate in 4.122 we obtain

$$\begin{aligned} |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,S}(\lambda, \nu) &\leq |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,\underline{C}(\nu) \cap \Sigma_0} \\ &+ c \int_{\lambda_1}^{\lambda} d\lambda' |r^{1-\frac{2}{p}}(\rho - (\mathbf{D}_3 \log \Omega)\operatorname{tr}\chi)|_{p,S}(\lambda', \nu) + c \frac{\varepsilon_0}{r|\lambda|^{1-\delta}} \end{aligned} \quad (4.124)$$

and from it, using the following estimate we prove later on

$$\sup_{(\lambda, \nu) \in \mathcal{K}} |r^{3-\frac{2}{p}}|\lambda|^{1-\delta}(\rho - (\mathbf{D}_3 \log \Omega)\operatorname{tr}\chi)|_{p,S}(\lambda, \nu) \leq c\varepsilon_0 , \quad (4.125)$$

$$\begin{aligned} |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,S}(\lambda, \nu) &\leq |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,\underline{C}(\nu) \cap \Sigma_0} \\ &+ c\varepsilon_0 \left(\int_{\lambda_1}^{\lambda} d\lambda' \frac{1}{r^2(\lambda', \nu)|\lambda'|^{1-\delta}} \right) + c \frac{\varepsilon_0}{r|\lambda|^{1-\delta}} \\ &\leq |r^{1-\frac{2}{p}}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,\underline{C}(\nu) \cap \Sigma_0} + c\varepsilon_0 \frac{1}{r(\lambda, \nu)|\lambda|^{1-\delta}} + c \frac{\varepsilon_0}{r(\lambda, \nu)|\lambda|^{1-\delta}} \end{aligned} \quad (4.126)$$

which implies

$$|r^{2-\frac{2}{p}}|\lambda|^{1-\delta}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,S}(\lambda, \nu) \leq |r^{2-\frac{2}{p}}|\lambda|^{1-\delta}(\operatorname{tr}\chi + \operatorname{tr}\underline{\chi})|_{p,\underline{C}(\nu) \cap \Sigma_0} + c\varepsilon_0 \quad (4.127)$$

Therefore the first of inequalities 4.119 is proved if we can prove the estimate 4.125 ⁴⁶

$$\sup_{(\lambda, \nu) \in \mathcal{K}} |r^{3-\frac{2}{p}} |\lambda|^{1-\delta} (\rho - (\mathbf{D}_3 \log \Omega) \text{tr} \chi)|_{p,S}(\lambda, \nu) \leq c \varepsilon_0 .$$

The proof of this last estimate follows the analogous proof in [Kl-Ni2], Proposition 3.1. We write the transport equation for $(\rho - (\mathbf{D}_3 \log \Omega) \text{tr} \chi)$ along an outgoing cone $C(\lambda)$ of the \mathcal{K} spacetime using the structure equations, see [Kl-Ni1] equations (3.1.46), ..., (3.1.48) of [Kl-Ni1] and the null Bianchi equations. We obtain

$$\frac{\partial}{\partial \nu} (\rho - \text{tr} \chi \mathbf{D}_3 \log \Omega) = \Omega \mathbf{D}_4 \rho - [\Omega (\mathbf{D}_3 \log \Omega) \mathbf{D}_4 \text{tr} \chi + \Omega \text{tr} \chi \mathbf{D}_4 \mathbf{D}_3 \log \Omega] . \quad (4.128)$$

where $\frac{\partial}{\partial \nu} = \Omega \mathbf{D}_4$. Recalling the explicit expression of the Bianchi equations, see [Kl-Ni1], equations (3.2.8) we have

$$\Omega \mathbf{D}_4 \rho = -\frac{3}{2} \Omega \text{tr} \chi \rho + [d\text{iv} \beta - (-\frac{1}{2} \hat{\chi} \cdot \alpha - \zeta \cdot \underline{\beta} + 2\eta \cdot \underline{\beta})] \quad (4.129)$$

$$\begin{aligned} &= -\frac{3}{2} \Omega \text{tr} \chi \rho + \varepsilon_0 \left[O\left(\frac{1}{r^{\frac{9}{2}} |\lambda|^{\frac{1}{2}-\delta}}\right) + O\left(\frac{1}{r^{\frac{9}{2}} |\lambda|^{\frac{5}{2}-2\delta}}\right) + O\left(\frac{1}{r^4 |\lambda|^{3-2\delta}}\right) + O\left(\frac{1}{r^4 |\lambda|^{3-2\delta}}\right) \right] \\ &= -\frac{3}{2} \Omega \text{tr} \chi \rho + O\left(\frac{\varepsilon_0}{r^4 |\lambda|^{1-\delta}}\right) \end{aligned} \quad (4.130)$$

The second term in the right hand side of 4.128 can be written, using the transport equation for $\text{tr} \chi$, as:

$$\begin{aligned} &- [\Omega (\mathbf{D}_3 \log \Omega) \mathbf{D}_4 \text{tr} \chi + \Omega \text{tr} \chi \mathbf{D}_4 \mathbf{D}_3 \log \Omega] \\ &= -\Omega \text{tr} \chi (-\rho - (\mathbf{D}_3 \log \Omega) (\mathbf{D}_4 \log \Omega) + [\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega]) \\ &\quad - \Omega \mathbf{D}_3 \log \Omega \left(-\frac{1}{2} \text{tr} \chi^2 + \text{tr} \chi (\mathbf{D}_4 \log \Omega) + |\hat{\chi}|^2 \right) \\ &= -\frac{1}{2} \Omega \text{tr} \chi (-\text{tr} \chi \mathbf{D}_3 \log \Omega) + \Omega \text{tr} \chi \rho - \Omega \text{tr} \chi [\underline{\eta} \cdot \eta - 2\zeta^2 - 2\zeta \cdot \nabla \log \Omega + |\hat{\chi}|^2] \\ &= -\frac{1}{2} \Omega \text{tr} \chi (-\text{tr} \chi \mathbf{D}_3 \log \Omega) + \Omega \text{tr} \chi \rho + O\left(\frac{\varepsilon_0}{r^4 |\lambda|^{2-2\delta}}\right) \end{aligned} \quad (4.131)$$

Collecting 4.129 and 4.131, evolution equation 4.128 can be written as

$$\frac{\partial}{\partial \nu} (\rho - \text{tr} \chi \mathbf{D}_3 \log \Omega) = -\frac{1}{2} \Omega \text{tr} \chi (\rho - \text{tr} \chi \mathbf{D}_3 \log \Omega) + \varepsilon_0 \left[O\left(\frac{1}{r^4 |\lambda|^{1-\delta}}\right) + O\left(\frac{1}{r^4 |\lambda|^{2-2\delta}}\right) \right] .$$

Integrating along an outgoing cone from (λ, ν_*) to (λ, ν) with $\nu_* > \nu$ and applying Gronwall's lemma we obtain ⁴⁷

$$|r^{3-\frac{2}{p}} |\lambda|^{1-\delta} (\rho - \text{tr} \chi \mathbf{D}_3 \log \Omega)|_{p,S}(\lambda, \nu) \leq c |r^{3-\frac{2}{p}} |\lambda|^{1-\delta} (\rho - \text{tr} \chi \mathbf{D}_3 \log \Omega)|_{p,S}(\lambda, \nu_*) + c \varepsilon_0 . \quad (4.132)$$

⁴⁶In fact the slightest weaker one: $\sup_{(\lambda, \nu) \in \mathcal{K}} |r^{2-\frac{2}{p}} |\lambda|^{2-\delta} (\rho - (\mathbf{D}_3 \log \Omega) \text{tr} \chi)|_{p,S}(\lambda, \nu) \leq c \varepsilon_0$ would be sufficient.

⁴⁷Choosing δ sufficiently small.

As the spacetime \mathcal{K} is already at our disposal, we can perform, in the previous inequality the limit $\nu_* \rightarrow \infty$. From [Kl-Nil], Chapter 8, Propositions 8.2.1, 8.3.1, we conclude that

$$\begin{aligned} \lim_{\nu_* \rightarrow \infty} r^3(\rho - \text{tr}\chi \mathbf{D}_3 \log \Omega) &= \lim_{\nu_* \rightarrow \infty} \left[r^3(\rho - \frac{2}{r} \mathbf{D}_3 \log \Omega) + r^3 O\left(\frac{1}{r^2} \underline{\omega}\right) \right] \\ &= \lim_{\nu_* \rightarrow \infty} r^3(\rho - \frac{2}{r} \mathbf{D}_3 \log \Omega) = \lim_{\nu_* \rightarrow \infty} r^3(\rho + \frac{4\underline{\omega}}{r}) = 0 \end{aligned} \quad (4.133)$$

so that 4.132 becomes

$$|r^{3-\frac{2}{p}} |\lambda|^{1-\delta} (\rho - \text{tr}\chi \mathbf{D}_3 \log \Omega)|_{p,S}(\lambda, \nu) \leq c\varepsilon_0, \quad (4.134)$$

proving 4.125. To prove the second inequality of 4.119 we proceed in a similar way. Inequality 4.125 implies immediately the following relation:

$$\underline{\omega} = -(2\text{tr}\chi)^{-1} \rho + O\left(\frac{\varepsilon_0}{r^2 |\lambda|^{1-\delta}}\right). \quad (4.135)$$

Using this relation we can estimate $\omega + \underline{\omega}$ from the estimate of $(\omega - (2\text{tr}\chi)^{-1} \rho)$ as

$$\omega + \underline{\omega} = (\omega - (2\text{tr}\chi)^{-1} \rho) + O\left(\frac{\varepsilon_0}{r^2 |\lambda|^{1-\delta}}\right). \quad (4.136)$$

To estimate of $(\omega - (2\text{tr}\chi)^{-1} \rho)$ we write the evolution equation for this quantity along $\underline{C}(\nu)$. We observe that, from equation (4.3.59) in [Kl-Nil],

$$\begin{aligned} \mathbf{D}_3 \omega &= -\frac{1}{2} \mathbf{D}_3 \mathbf{D}_4 \log \Omega = -\frac{1}{2\Omega} \mathbf{D}_3 \Omega \mathbf{D}_4 \log \Omega + \frac{1}{2} (\mathbf{D}_3 \log \Omega) \mathbf{D}_4 \log \Omega \\ &= -(\mathbf{D}_3 \log \Omega) \omega + \frac{1}{2} \rho - \frac{1}{2\Omega} \hat{F} = -(\mathbf{D}_3 \log \Omega) \omega - \text{tr}\chi \underline{\omega} + \left[O\left(\frac{\varepsilon_0}{r^3 |\lambda|^{1-\delta}} - \frac{1}{2\Omega} \hat{F}\right) \right] \\ &= -(\mathbf{D}_3 \log \Omega) \omega + \text{tr}\chi \underline{\omega} - (\text{tr}\chi + \text{tr}\underline{\chi}) \underline{\omega} + \varepsilon_0 \left[O\left(\frac{1}{r^3 |\lambda|^{1-\delta}}\right) + O\left(\frac{1}{r^4 |\lambda|^{2-2\delta}}\right) \right] \\ &= -(\mathbf{D}_3 \log \Omega) \omega + \text{tr}\chi \underline{\omega} + \left[O\left(\frac{\varepsilon_0}{r^3 |\lambda|^{1-\delta}}\right) \right] \end{aligned} \quad (4.137)$$

where $\hat{F} \equiv 2\Omega \zeta \cdot \nabla \log \Omega + \Omega(\underline{\eta} \cdot \eta - 2\zeta^2)$.

From the structure equations and Bianchi equations, [Kl-Ni1], Chapter 3,

$$\begin{aligned}
\mathbf{D}_3 \left(-(2\text{tr}\chi)^{-1}\rho \right) &= \frac{1}{2} \frac{\rho}{\text{tr}\chi^2} (\mathbf{D}_3 \text{tr}\chi) - \frac{1}{2\text{tr}\chi} \mathbf{D}_3 \rho \\
&= \left(-\underline{\omega} + O\left(\frac{\varepsilon_0}{r^2|\lambda|^{1-\delta}}\right) \right) \frac{1}{\text{tr}\chi} \mathbf{D}_3 \text{tr}\chi - \frac{1}{2\text{tr}\chi} \left(-\frac{3}{2} \text{tr}\chi \rho + \left[-\text{div}\underline{\beta} + O\left(\frac{\varepsilon_0}{r^3|\lambda|^{4-2\delta}}\right) \right] \right) \\
&= -\underline{\omega} \frac{1}{\text{tr}\chi} \mathbf{D}_3 \text{tr}\chi + \frac{1}{2\text{tr}\chi} \frac{3}{2} \text{tr}\chi \rho + \left[O\left(\frac{\varepsilon_0}{r^2|\lambda|^{1-\delta}}\right) \frac{1}{\text{tr}\chi} \mathbf{D}_3 \text{tr}\chi + \frac{1}{2\text{tr}\chi} O\left(\frac{\varepsilon_0}{r^3|\lambda|^{4-2\delta}}\right) \right] \\
&= -\underline{\omega} \frac{1}{\text{tr}\chi} \left(-\frac{1}{2} \text{tr}\chi \text{tr}\chi - (\mathbf{D}_3 \log \Omega) \text{tr}\chi + 2\rho - [\underline{\chi} \cdot \hat{\chi} - 2\text{div}\zeta - 2\Delta \log \Omega - 2|\eta|^2] \right) \\
&\quad + \frac{1}{2\text{tr}\chi} \frac{3}{2} \text{tr}\chi \rho + \varepsilon_0 \left[O\left(\frac{1}{r^2|\lambda|^{1-\delta}}\right) \frac{1}{\text{tr}\chi} \mathbf{D}_3 \text{tr}\chi + \frac{1}{2\text{tr}\chi} O\left(\frac{1}{r^3|\lambda|^{4-2\delta}}\right) \right] \\
&= \frac{1}{2} \text{tr}\chi \underline{\omega} + (\mathbf{D}_3 \log \Omega) \underline{\omega} - \underline{\omega} \frac{1}{\text{tr}\chi} 2\rho + \frac{1}{2\text{tr}\chi} \frac{3}{2} \text{tr}\chi \rho + \varepsilon_0 \left[O\left(\frac{1}{r^2|\lambda|^{2-\delta}}\right) \right] \\
&= \frac{1}{2} \text{tr}\chi \underline{\omega} + (\mathbf{D}_3 \log \Omega) \underline{\omega} - \frac{1}{2} (\mathbf{D}_3 \log \Omega) 4\underline{\omega} - \frac{3}{2} \text{tr}\chi \underline{\omega} + \left[O\left(\frac{\varepsilon_0}{r^2|\lambda|^{2-\delta}}\right) \right] \tag{4.138}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{D}_3 \left(-(2\text{tr}\chi)^{-1}\rho \right) &= \frac{1}{2} \text{tr}\chi \underline{\omega} - (\mathbf{D}_3 \log \Omega) \underline{\omega} - \frac{3}{2} \text{tr}\chi \underline{\omega} + \left[O\left(\frac{\varepsilon_0}{r^2|\lambda|^{2-\delta}}\right) \right] \tag{4.139} \\
&= -\text{tr}\chi \underline{\omega} - (\mathbf{D}_3 \log \Omega) \left(-(2\text{tr}\chi)^{-1}\rho \right) + \left[O\left(\frac{\varepsilon_0}{r^2|\lambda|^{2-\delta}}\right) \right]
\end{aligned}$$

which finally implies, together with 4.137,

$$\mathbf{D}_3 \left(\omega - (2\text{tr}\chi)^{-1}\rho \right) = -(\mathbf{D}_3 \log \Omega) \left(\omega - (2\text{tr}\chi)^{-1}\rho \right) + \left[O\left(\frac{\varepsilon_0}{r^2|\lambda|^{2-\delta}}\right) \right] \tag{4.140}$$

Repeating the same calculation for the first tangential derivatives, applying again Gronwall's lemma and using the initial data assumptions we conclude that

$$\sup_{\mathcal{K}_1} |r^2|\lambda|^{1-\delta} \left(\omega - (2\text{tr}\chi)^{-1}\rho \right) \leq c\varepsilon_0 \tag{4.141}$$

so that, together with 4.136, we obtain

$$\sup_{\mathcal{K}_1} r^2|\lambda|^{1-\delta} |(\omega + \underline{\omega})| \leq c\varepsilon_0 . \tag{4.142}$$

A simplified proof of Theorem 2.1. As said in Section 2 the proof of Theorem 2.1 is a simple adaptation of Theorem 2.1 of [Kl-Ni2]. A direct proof based only on the results in [Kl-Ni1] can be, nevertheless, obtained. The proof of Theorem 2.1 used there requires a different definition of the last slice foliation. This can be avoided proving the result in the following steps:

i) We introduce as in [Kl-Ni2] some \hat{Q} norms relative to the Riemann tensor R , but with an extra weight factor $|u|^{1-2\delta}$ and we prove that using the estimates

4.119 proved before, these $\hat{\mathcal{Q}}$ norms are bounded, provided the initial data decay at least as r^{-2} . This implies that all the null components of the Riemann tensor $\alpha(R), \beta(R), (\rho - \bar{\rho})(R), (\sigma - \bar{\sigma})(R), \underline{\beta}(R), \underline{\alpha}(R)$ have an extra decay factor $|u|^{\frac{1}{2} - \delta}$.

ii) With these improved decay factors for the Riemann components using the transport equations for the connection coefficients one obtains that all the connection coefficients which are identically zero in Kerr spacetime have also the extra decay factor $|u|^{\frac{1}{2} - \delta}$.⁴⁸

Therefore the only thing which has to be proved is that the “Error” for the $\hat{\mathcal{Q}}$ norms is bounded assuming the [Kl-Nil] estimates for the connection coefficients plus inequalities 4.119. Again the problem is similar to the estimate of the error for $\hat{\mathcal{Q}}$ norms relative to $\hat{\mathcal{L}}_T R$, the only terms which have to be controlled are those involving $\rho(R)$ as differently from $(\rho - \bar{\rho})(R)$ the decay of $\bar{\rho}(R)$ cannot be improved.

Proof of Lemma 2.1: As the Kerr spacetime is of type D, following the Petrov classification, in the null frame where the null vector fields are the principal null directions, the only null Riemann component are ρ and σ or, in the Newman-Penrose classification, Ψ_2 and their explicit expression is

$$\rho(R) + i\sigma(R) = \frac{1}{(r - i \cos \theta)^3} = \frac{1}{r^3} + i \frac{3a \cos \theta}{r^4} + O\left(\frac{1}{r^5}\right) \quad (4.143)$$

so that

$$\begin{aligned} \rho(R) &= \frac{1}{r^3} + O\left(\frac{1}{r^5}\right) \\ \sigma(R) &= \frac{3a \cos \theta}{r^4} + O\left(\frac{1}{r^6}\right). \end{aligned} \quad (4.144)$$

The principal null directions l, n defined, for instance, in Chandrasekar book, [Ch], are

$$\begin{aligned} l &= \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{a}{\Delta} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r} \\ n &= \frac{\Delta}{2\Sigma} \left(\frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{a}{\Delta} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right) \\ \tilde{e}_\theta &= \frac{1}{\sqrt{\Sigma}} \frac{\partial}{\partial \theta} \\ \tilde{e}_\phi &= \frac{1}{\sqrt{\Sigma}} \left(a \sin \theta \frac{\partial}{\partial t} + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \end{aligned} \quad (4.145)$$

⁴⁸There is no need of defining a last slice foliation as the spacetime \mathcal{K} is already at our disposal and the transport equations can be used starting from Scri where we know that the relation $\omega - (2\text{tr}\chi)^{-1}\rho = 0$ holds.

where

$$\begin{aligned}
\Delta &= r^2 + a^2 - 2Mr \\
\Sigma &= r^2 + a^2 \cos^2 \theta \\
\Sigma R^2 &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta
\end{aligned} \tag{4.146}$$

Besides the fact that our initial data are not exactly those of the Kerr spacetime due to corrections $\delta\bar{g}$ and $\delta\bar{k}$, one has to recognize that our null orthonormal frame is not the one associated to the principal null directions and, therefore, the Kerr spacetime gives a contribution to all the Riemann components. In fact the frame which, apart from extra small corrections, is adapted to the double null foliation of \mathcal{K} is the one proposed by Israel and Pretorius, see [Is-Pr],

$$\begin{aligned}
e_4 &= 2\Omega L = \sqrt{\frac{R^2}{\Delta}} \left(\frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \phi} \right) \\
e_3 &= 2\Omega \underline{L} = \sqrt{\frac{R^2}{\Delta}} \left(-\frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \phi} \right)
\end{aligned} \tag{4.147}$$

$$\begin{aligned}
e_\lambda &= \frac{R}{\mathcal{L}} \frac{\partial}{\partial \lambda} = \frac{1}{\Sigma R} \left(-\Delta P \frac{\partial}{\partial r} + Q \frac{\partial}{\partial \theta} \right) = \frac{Q}{\Sigma R} \left(-P \frac{\partial}{\partial r_*} + \frac{\partial}{\partial \theta} \right) \\
e_\phi &= \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} .
\end{aligned} \tag{4.148}$$

Its relation with the previous one are

$$\begin{aligned}
l &= \frac{1}{2\sqrt{\Delta}R} \left[(r^2 + a^2 + Q)e_4 + (r^2 + a^2 - Q)e_3 + 2\sqrt{\Delta}a \sin \theta e_\phi \right] \\
n &= \frac{\sqrt{\Delta}}{4\Sigma R} \left[(r^2 + a^2 + Q)e_3 + (r^2 + a^2 - Q)e_4 + 2\sqrt{\Delta}a \sin \theta e_\phi \right] \\
\tilde{e}_\theta &= \frac{\sqrt{\Sigma}R}{Q} e_\lambda + \sqrt{\frac{\Delta}{\Sigma}} \frac{P}{2R} (e_4 - e_3) \\
\tilde{e}_\phi &= \frac{1}{\sqrt{\Sigma}R} (r^2 + a^2) e_\phi + \frac{\sqrt{\Delta}a \sin \theta}{2\sqrt{\Sigma}R} (e_3 + e_4)
\end{aligned} \tag{4.149}$$

and the inverse relation

$$\begin{aligned}
e_4 &= \frac{\sqrt{\Delta}}{2\Sigma R} \left[\left(r^2 + a^2 + \frac{R\Sigma r}{Q} \right) l + \left(r^2 + a^2 - \frac{R\Sigma r}{Q} \right) \frac{2\Sigma}{\Delta} n - 2\sqrt{\Sigma}a \sin \theta \tilde{e}_\phi \right] \\
e_3 &= \frac{\sqrt{\Delta}}{2\Sigma R} \left[\left(r^2 + a^2 - \frac{R\Sigma r}{Q} \right) l + \left(r^2 + a^2 + \frac{R\Sigma r}{Q} \right) \frac{2\Sigma}{\Delta} n - 2\sqrt{\Sigma}a \sin \theta \tilde{e}_\phi \right] \\
e_\lambda &= \frac{Q}{\sqrt{\Sigma}R} \tilde{e}_\theta - \frac{\Delta P}{2\Sigma R} \left(l - \frac{2\Sigma}{\Delta} n \right) \\
e_\phi &= \frac{r^2 + a^2}{\sqrt{\Sigma}R} \tilde{e}_\phi - \frac{\Delta}{2R\Sigma} a \sin \theta \left(l + \frac{2\Sigma}{\Delta} n \right) .
\end{aligned} \tag{4.150}$$

Under the assumption that a and M are small we have approximately

$$\begin{aligned}
l &= e_4 + O\left(\frac{a^2}{r^2}\right) e_3 + O\left(\frac{a \sin \theta}{r}\right) e_\phi \\
n &= e_3 + O\left(\frac{a^2}{r^2}\right) e_4 + O\left(\frac{a \sin \theta}{r}\right) e_\phi \\
\tilde{e}_\theta &= e_\lambda + O\left(\frac{a^2}{r^2}\right) (e_4 - e_3) \\
\tilde{e}_\phi &= e_\phi + O\left(\frac{a \sin \theta}{r}\right) (e_3 + e_4). \tag{4.151}
\end{aligned}$$

These relations show that the ‘‘Kerr part contribution’’ to the null Riemann components different from ρ is negligible and decay always sufficiently fast when we are considering the null components of $\hat{\mathcal{L}}_T R$.

Therefore we can neglect these extra contributions and observe that from the expression of $\tilde{R} = \hat{\mathcal{L}}_T R$ and definition 1.11 it follows that the more delicate contribution to $\rho(\tilde{R})$ is: ${}^{(T)}\pi\rho(R)$ and the explicit expression of ${}^{(T)}\pi$ is given in subsection 2.6. In the spacetime \mathcal{K} , $|{}^{(T)}\pi|$ behaves as $r^{-2}|u|^{1-\delta}$ as proved in subsection 2.6. Therefore from the initial data we have

$$|r^5|u|^{1-\delta}\rho(\hat{\mathcal{L}}_T R)|_{\Sigma_0} \leq c \text{ equivalent to } |r^{6-\delta}\rho(\hat{\mathcal{L}}_T R)|_{\Sigma_0} \leq c. \tag{4.152}$$

4.2.1 Proof of Lemma 2.2:

Proof of 2.49: Using Corollary 4.1.1 of [Kl-Ni1] the following relation holds:

$$\begin{aligned}
&|r^{(3-\frac{2}{3})}|\lambda|^{3-\delta}\nabla\beta|_{p=4,S}(\lambda, \underline{u}) = \left(\int_{S(\lambda, \underline{u})} r^2|\lambda|^2(r^8|\lambda|^{10-4\delta})|\nabla\beta|^4 \right)^{\frac{1}{4}} \\
&\leq \left(\int_{S(u_0, \underline{u})} r^4|r|^{4(3-\delta)}|r^{\frac{3}{2}}\nabla\beta|^4 \right)^{\frac{1}{4}} \\
&+ c \left(\int_{\underline{C}(\underline{u}; [u_0, \lambda])} \left[r^4|\lambda'|^{5-2\delta}|\nabla\beta|^2 + r^6|\lambda'|^{5-2\delta}|\nabla^2\beta|^2 + r^4|\lambda'|^{7-2\delta}|\mathcal{D}_3\nabla\beta|^2 \right] \right)^{\frac{1}{2}} \\
&\leq \left(\int_{S(u_0, \underline{u})} r^4|r|^{4(3-\delta)}|r^{\frac{3}{2}}\nabla\beta|^4 \right)^{\frac{1}{4}} \tag{4.153} \\
&+ c \frac{1}{|\lambda|^{1+\frac{(\epsilon+2\delta)}{2}}} \left(\int_{\underline{C}(\underline{u}; [u_0, \lambda])} |\lambda'|^{(5+\epsilon)} \left[r^4|\lambda'|^2|\nabla\beta|^2 + r^6|\lambda'|^2|\nabla^2\beta|^2 + r^4|\lambda'|^4|\mathcal{D}_3\nabla\beta|^2 \right] \right)^{\frac{1}{2}}.
\end{aligned}$$

We estimate the first of the three integrals in the last inequality of 4.153 as the other ones are treated exactly in the same way; from the inequality, see [Kl-Ni1], Lemma 5.1.1:

$$\int_{S(\lambda', \underline{u})} r^2|\nabla\beta|^2 \leq c \int_{S(\lambda', \underline{u})} |\mathcal{L}_O\beta|^2 \tag{4.154}$$

it follows

$$\int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(5+\epsilon)} r^4 |\lambda'|^2 |\nabla \underline{\beta}|^2 \leq c \int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(5+\epsilon)} r^2 |\lambda'|^2 |\mathcal{L}_O \underline{\beta}|^2. \quad (4.155)$$

From Proposition 5.1.1 of [Kl-Ni1] and the estimates on the deformation tensors and connection coefficients proved there we have

$$\begin{aligned} |\mathcal{L}_O \underline{\beta}(\tilde{R})| &\leq c \left(|\underline{\beta}(\hat{\mathcal{L}}_O \tilde{R})| + \frac{O(\varepsilon_0)}{r} \left(|\underline{\alpha}(\tilde{R})| + |\underline{\beta}(\tilde{R})| + |\rho(\tilde{R}) - \overline{\rho(\tilde{R})}| + |\sigma(\tilde{R}) - \overline{\sigma(\tilde{R})}| \right) \right) \\ &\quad + \frac{O(\varepsilon_0)}{r} (|\overline{\rho(\tilde{R})}| + |\overline{\sigma(\tilde{R})}|) \end{aligned} \quad (4.156)$$

This implies that

$$\begin{aligned} &\int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(5+\epsilon)} r^4 |\lambda'|^2 |\nabla \underline{\beta}(\tilde{R})|^2 \leq c \left(\int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(5+\epsilon)} r^2 |\lambda'|^2 |\underline{\beta}(\hat{\mathcal{L}}_O \tilde{R})|^2 \right. \\ &\quad \left. + \frac{\varepsilon_0^2}{|\lambda|^2} \int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(5+\epsilon)} \left[r^2 |\lambda'|^2 |\underline{\beta}(\tilde{R})|^2 + |\lambda'|^4 |\underline{\alpha}(\tilde{R})|^2 + r^4 [|\rho(\tilde{R}) - \overline{\rho(\tilde{R})}|^2 + |\sigma(\tilde{R}) - \overline{\sigma(\tilde{R})}|^2] \right] \right. \\ &\quad \left. + \varepsilon_0^2 \int_{\underline{C}(\underline{u};[u_0,\lambda])} \frac{|\lambda'|^{(5+\epsilon)}}{|\lambda|^2} r^4 [|\overline{\rho(\tilde{R})}|^2 + |\overline{\sigma(\tilde{R})}|^2] \right) \\ &\leq c \int_{\underline{C}(\underline{u}) \cap V(\lambda, \underline{u})} |\lambda'|^{5+\epsilon} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K}, \bar{K}, T, e_3) + \varepsilon_0^2 \int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(3+\epsilon)} r^4 [|\overline{\rho(\tilde{R})}|^2 + |\overline{\sigma(\tilde{R})}|^2] \\ &\leq c \tilde{Q}(\lambda, \underline{u}) + c \varepsilon_0^2 \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|^2) \int_{\underline{C}(\underline{u};[u_0,\lambda])} |\lambda'|^{(3+\epsilon)} \frac{r^4}{r^6 |\lambda'|^{6-2\delta}} \\ &\leq c \tilde{Q}(\lambda, \underline{u}) + c \frac{\varepsilon_0^2}{|\lambda|^{2-2\delta-\epsilon}} \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}|^2 + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|^2). \end{aligned} \quad (4.157)$$

Therefore going back to 4.153 we have

$$\begin{aligned} &|r^{(3-\frac{\delta}{4})} |\lambda|^{3-\delta} \nabla \underline{\beta}|_{p=4,S}(\lambda, \underline{u}) \leq \left(\int_{S(u_0, \underline{u})} r^4 |r|^{4(3-\delta)} |r^{\frac{3}{2}} \nabla \underline{\beta}|^4 \right)^{\frac{1}{4}} \\ &\quad + c \frac{1}{|\lambda|^{1+\frac{(\epsilon+2\delta)}{2}}} \left(\tilde{Q}^{\frac{1}{2}}(\lambda, \underline{u}) + \frac{\varepsilon_0}{|\lambda|^{1-\delta-\frac{\epsilon}{2}}} \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}| + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|) \right) \end{aligned}$$

which completes the proof of Lemma 2.2.

The first term has also a decay $\frac{c}{|\lambda|^{1+\frac{(\epsilon+2\delta)}{2}}}$ in $|\lambda|$. In fact as it is made by initial data we have $u_0 = u_0(\underline{u}) = -\underline{u} = r(u_0, \underline{u})$ and $|\nabla \underline{\beta}(\tilde{R})| = O(r^{-(7+\frac{\epsilon}{2})})$ with $\tilde{\epsilon} > \epsilon$.⁴⁹ Then

$$|r^{1+(3-\delta)} r^{\frac{3}{2}} \nabla \underline{\beta}|^4 = O(r^{\frac{5}{2}+(3-\delta)-(7+\frac{\epsilon}{2})})^4 = O(r^{-\frac{3}{2}-\frac{(\epsilon+2\delta)}{2}})^4 = O(r^{-6-2(\tilde{\epsilon}+2\delta)}) \quad \text{and}$$

⁴⁹Recall that $\underline{\beta}$ is the component of $\hat{\mathcal{L}}_T R$ and this implies an extra decay factor r^{-1} .

$$\left(\int_{S(u_0, \underline{u})} |r^{1+(3-\delta)} r^{\frac{3}{2}} \nabla \underline{\beta}|^4 \right)^{\frac{1}{4}} = O(r^{-4-2(\tilde{\epsilon}+2\delta)})^{\frac{1}{4}} = O\left(r^{-1-\frac{(\tilde{\epsilon}+2\delta)}{2}}\right) \leq \frac{c\epsilon_0}{|\lambda|^{1+\frac{(\tilde{\epsilon}+2\delta)}{2}}}. \quad (4.158)$$

Therefore

$$\begin{aligned} & |r^{(3-\frac{2}{4})} |\lambda|^{3-\delta} \nabla \underline{\beta}|_{p=4, S(\lambda, \underline{u})} \\ & \leq c \frac{1}{|\lambda|^{1+\frac{(\tilde{\epsilon}+2\delta)}{2}}} \left(\epsilon_0 + \tilde{\mathcal{Q}}^{\frac{1}{2}}(\lambda, \underline{u}) + \frac{\epsilon_0}{|\lambda|^{1-\delta-\frac{\tilde{\epsilon}}{2}}} \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}| + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|) \right). \end{aligned} \quad (4.159)$$

Proof of 2.50: The following inequality holds:

$$\begin{aligned} & |r^{(3-\frac{2}{4})} |\lambda|^{(3-\delta)} \eta \cdot \underline{\beta}|_{p=4, S(\lambda, \underline{u})} = \left(\int_{S(\lambda, \underline{u})} r^2 |\lambda|^{4(3-\delta)} |\underline{\beta}|^4 |r^2 \eta|^4 \right)^{\frac{1}{4}} \\ & \leq |r^2 \eta|_{\infty} \left(\int_{S(\lambda, \underline{u})} r^2 |\lambda|^{4(3-\delta)} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \leq \frac{1}{|\lambda|^{1-\delta}} |r^2 |\lambda|^{1-\delta} \eta|_{\infty} \left(\int_{S(\lambda, \underline{u})} r^2 |\lambda|^{12-4\delta} |\underline{\beta}|^4 \right)^{\frac{1}{4}}. \end{aligned} \quad (4.160)$$

Applying again Corollary 4.1.1 of [Kl-Nil], we have

$$\begin{aligned} & \left(\int_{S(\lambda, \underline{u})} r^2 |\lambda|^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \leq \left(\int_{S(\lambda, \underline{u}_0)} r^2 |\lambda|^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \\ & + \frac{1}{|\lambda|^{2+\frac{\delta+\tilde{\epsilon}}{2}}} \left(\int_{C(\lambda)} |\lambda|^{(5+\tilde{\epsilon})} \left[|\lambda|^2 |\underline{\beta}|^2 + r^2 |\lambda|^2 \nabla \underline{\beta}|^2 + |\lambda|^2 |\mathbf{D}_4 |\lambda|^2 \underline{\beta}|^2 \right] \right)^{\frac{1}{2}}. \end{aligned} \quad (4.161)$$

Proceeding as in the previous case we have

$$\begin{aligned} & \left(\int_{S(\lambda, \underline{u})} r^2 |\lambda|^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \leq \left(\int_{S(\lambda, \underline{u}_0)} r^2 |\lambda|^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \\ & + \frac{1}{|\lambda|^{2+\frac{\delta+\tilde{\epsilon}}{2}}} \left(\sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} \tilde{\mathcal{Q}}(\lambda, \underline{u})^{\frac{1}{2}} + \frac{\epsilon_0}{|\lambda|^{-\delta-\frac{\tilde{\epsilon}}{2}}} \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}| + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|) \right) \\ & \leq \left(\int_{S(\lambda, \underline{u}_0)} r^2 |\lambda|^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \\ & + \frac{1}{|\lambda|^{2-\frac{\delta}{2}}} \left(\sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} \tilde{\mathcal{Q}}(\lambda, \underline{u})^{\frac{1}{2}} + \epsilon_0 \sup_{(\lambda, \underline{u}) \in \mathcal{K}_2} (|r^3 |\lambda|^{3-\delta} \overline{\rho(\tilde{R})}| + |r^3 |\lambda|^{3-\delta} \overline{\sigma(\tilde{R})}|) \right). \end{aligned} \quad (4.162)$$

The estimate of the first term goes in the following way

$$\begin{aligned} & \left(\int_{S(\lambda, \underline{u}_0)} r^2 \tau_-^2 |\lambda|^{\frac{(5-2\delta)}{2}} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \leq c \left(\int_{S_0(\lambda, \underline{u}_0)} r^{2+2+10-4\delta} |\underline{\beta}|^4 \right)^{\frac{1}{4}} \\ & = O\left(r^{\frac{2+4+10-4\delta}{4}}\right) |\underline{\beta}(\hat{\mathcal{L}}_T R)| = O\left(r^{\frac{4+2+10-4\delta}{4}-1-5-\tilde{\epsilon}}\right) \leq O\left(\frac{\epsilon_0}{|\lambda|^{2+(\tilde{\epsilon}+\delta)}}\right). \end{aligned} \quad (4.164)$$

4.2.2 Proof of Lemma 2.3

Observe that from Lemma 5.1.1 in [Kl-Nil] we have

$$\begin{aligned} \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^2 |\lambda|^2 r^2 |\nabla \rho(\tilde{R})|^2 &\leq \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^2 |\lambda|^2 |\mathcal{L}_O \rho(\tilde{R})|^2 \\ &\leq \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^2 |\lambda|^2 |\rho(\hat{\mathcal{L}}_O \tilde{R})|^2 + \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^2 |\lambda|^2 |\text{tr}^{(O)} \pi|^2 |\rho(\tilde{R})|^2 + \{\text{good corrections}\} . \end{aligned} \quad (4.165)$$

and also

$$\begin{aligned} \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^4 r^2 |\nabla \rho(\tilde{R})|^2 &\leq \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^4 |\mathcal{L}_O \rho(\tilde{R})|^2 \\ &\leq \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^4 |\rho(\hat{\mathcal{L}}_O \tilde{R})|^2 + \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^4 |\text{tr}^{(O)} \pi|^2 |\rho(\tilde{R})|^2 + \{\text{good corrections}\} . \end{aligned} \quad (4.166)$$

From the estimate for $\rho(\tilde{R})$ in Theorem 2.2

$$\sup_{\mathcal{K}} |r^3 |\lambda|^{3-\delta} \rho(\tilde{R})| \leq c$$

and from the estimate proved in [Kl-Nil], Chapter 6,

$$\sup_{\mathcal{K}} |r^{(O)} \pi| \leq c ,$$

the second integral in 4.166 can be estimated by

$$\begin{aligned} \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^4 |\text{tr}^{(O)} \pi|^2 |\rho(\tilde{R})|^2 &\leq \int_{S(\lambda, \nu)} |\lambda|^{5+\epsilon} r^2 \frac{1}{r^6 |\lambda|^{6-2\delta}} \\ &\leq c r^2 |\lambda|^{5+\epsilon} r^2 \frac{1}{r^6 |\lambda|^{6-2\delta}} \leq c \frac{1}{r^2 |\lambda|^{1-2\delta-\epsilon}} \end{aligned} \quad (4.167)$$

This implies

$$\begin{aligned} \int_{\underline{\mathcal{C}}(\nu)} |\lambda|^{5+\epsilon} r^4 r^2 |\nabla \rho(\tilde{R})|^2 &\leq \int_{\underline{\mathcal{C}}(\nu)} |\lambda|^{5+\epsilon} r^4 |\rho(\hat{\mathcal{L}}_O \tilde{R})|^2 + c \\ &\leq c \tilde{\mathcal{Q}}_{\Sigma_0} + c \int_{\underline{\mathcal{C}}(\nu)} |\lambda|^{5+\epsilon} r^2 \frac{1}{r^6 |\lambda|^{6-2\delta}} + c \leq c \tilde{\mathcal{Q}}_{\Sigma_0} + c \frac{1}{r |\lambda|^{1-2\delta-\epsilon}} + c \leq c_1 \end{aligned} \quad (4.168)$$

Using Corollary 4.1.1 of [Kl-Nil] we have

$$\left(\int_{S(\lambda, \nu)} r^2 |\lambda|^2 |F|^4 \right)^{\frac{1}{4}} \leq \left(\int_{S(\lambda_1, \nu) = \underline{\mathcal{C}}(\nu) \cap \Sigma_0} r^2 |\lambda|^2 |F|^4 \right)^{\frac{1}{4}} + \left(\int_{\underline{\mathcal{C}}(\nu)} [|F|^2 + \dots] \right)^{\frac{1}{2}} \quad (4.169)$$

Choosing $F = |\lambda|^{\frac{5+\epsilon}{2}} r^3 \nabla \rho(\tilde{R})$ we obtain

$$\begin{aligned}
|r^{4-\frac{2}{p}} |\lambda|^{3+\frac{\epsilon}{2}} \nabla \rho(\tilde{R})|_{p=4,S} &= \left(\int_{S(\lambda,\nu)} ||\lambda|^{3+\frac{\epsilon}{2}} r^{4-\frac{2}{4}} \nabla \rho(\tilde{R})|^4 \right)^{\frac{1}{4}} \tag{4.170} \\
&\leq \left(\int_{S(\lambda_1,\nu)=\underline{C}(\nu)\cap\Sigma_0} ||\lambda|^{3+\frac{\epsilon}{2}} r^{4-\frac{2}{4}} \nabla \rho(\tilde{R})|^4 \right)^{\frac{1}{4}} + \left(\int_{\underline{C}(\nu)} [||\lambda|^{\frac{5+\epsilon}{2}} r^3 \nabla \rho(\tilde{R})|^2 + \dots] \right)^{\frac{1}{2}} \\
&\leq \left(\int_{S(\lambda_1,\nu)=\underline{C}(\nu)\cap\Sigma_0} ||\lambda|^{3+\frac{\epsilon}{2}} r^{4-\frac{2}{4}} \nabla \rho(\tilde{R})|^4 \right)^{\frac{1}{4}} + \left(\int_{\underline{C}(\nu)} |\lambda|^{5+\epsilon} [r^4 |\rho(\hat{\mathcal{L}}_O \tilde{R})|^2 + \dots] \right)^{\frac{1}{2}} \\
&\leq \left(\int_{S(\lambda_1,\nu)=\underline{C}(\nu)\cap\Sigma_0} ||\lambda|^{3+\frac{\epsilon}{2}} r^{4-\frac{2}{4}} \nabla \rho(\tilde{R})|^4 \right)^{\frac{1}{4}} + \tilde{Q}_{\Sigma_0}^{\frac{1}{2}} \leq c,
\end{aligned}$$

where the last inequality uses the boundedness of the \tilde{Q} norms. Observe now that inequality 4.168

$$\int_{\underline{C}(\nu)} |\lambda|^{5+\epsilon} r^4 r^2 |\nabla \rho(\tilde{R})|^2 = \int_{\lambda_1}^{\lambda} ||\lambda|^{\frac{5+\epsilon}{2}} r^3 \nabla \rho(\tilde{R})|_{p=2,S}^2 \leq c_1$$

imply, with $\tilde{\epsilon} > 0$,

$$||\lambda|^{\frac{5+\epsilon}{2}} r^3 \nabla \rho(\tilde{R})|_{p=2,S}^2 \leq c \frac{1}{|\lambda|^{1+\tilde{\epsilon}}} \tag{4.171}$$

and from it

$$||\lambda|^{3+\tilde{\epsilon}+\frac{\epsilon}{2}} r^{4-\frac{2}{p}} \nabla \rho(\tilde{R})|_{p=2,S} \leq c. \tag{4.172}$$

Interpolating between 4.170 and 4.172 we obtain for $p \in [2, 4]$,

$$|r^{4-\frac{2}{p}} |\lambda|^{3+\epsilon} \nabla \rho(\tilde{R})|_{p,S} \leq c \tag{4.173}$$

proving the lemma.

4.3 Some technical details

1. The Kerr metric in the Boyer-Linquist coordinates $\{t, r, \theta, \phi\}$ is:

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 - \frac{4Mar \sin^2 \theta}{\Sigma} d\phi dt + R^2 \sin^2 \theta d\phi^2$$

and its restriction to Σ_0 :

$$\begin{aligned}
ds^2 &= \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + R^2 \sin^2 \theta d\phi^2 \tag{4.174} \\
&= g_S + \left(\frac{a^2 \sin^2 \theta}{r^2} + O\left(\frac{a^2 M}{r^3}\right) + O\left(\frac{a^4}{r^4}\right) \right) dr^2 + \left(\frac{a^2 \cos^2 \theta}{r^2} \right) r^2 d\theta^2 \\
&+ \left(\frac{a^2}{r^2} + O\left(\frac{a^2 M}{r^3}\right) + O\left(\frac{a^4}{r^4}\right) \right) r^2 \sin^2 \theta d\phi^2.
\end{aligned}$$

It follows that the components of the correction to the g_S metric start with terms of order $O(a^2/r^2)$ and if a^2/R_0^2 is $O(\varepsilon_0)$ then it is clear that these terms are corrections compatible with the assumptions in [Kl-Nil].

We expect, therefore, that we can connect to double null foliation of [Is-Pr] to the one introduced in [Kl-Nil] when a is small.

In principle once we have proved that the Riemann components can be written as those of Kerr plus corrections small and decaying fast we could look again at the transport equations for the connection coefficients and show that we can subtract the Kerr part and prove that what remains is small and faster decaying. This is, nevertheless, somewhat delicate as the null frame adapted to the Israel-Pretorius foliation does not coincide exactly with the one used in [Kl-Nil].

Before discussing this point let us recall the main properties of the Israel-Pretorius foliation.

Let us define u, \underline{u} as two functions solutions of the eikonal equation

$$g^{\mu\nu} \partial_\mu w \partial_\nu w = 0 ,$$

$$u = \frac{t - r_*(\theta, r)}{2}$$

$$\underline{u} = \frac{t + r_*(\theta, r)}{2} . \quad (4.175)$$

The Kerr metric in the Boyer-Lindquist coordinates $\{x^\mu\} = \{t, r, \theta, \phi\}$ is

$$\mathbf{g}_{(Kerr)}(\cdot, \cdot) = -\frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dt^2 - a \sin^2 \theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} (dt \otimes d\phi + d\phi \otimes dt)$$

$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2 \quad (4.176)$$

and

$$g^{00} = -\frac{g_{\phi\phi}}{\Delta \sin^2 \theta} , \quad g^{0\phi} = \frac{g_{0\phi}}{\Delta \sin^2 \theta} , \quad g^{\phi\phi} = -\frac{g_{00}}{\Delta \sin^2 \theta}$$

$$g^{rr} = g_{rr}^{-1} , \quad g^{\theta\theta} = g_{\theta\theta}^{-1} .$$

The null (radial) geodesic vector field L, \underline{L}

$$L = L^\rho \frac{\partial}{\partial x^\rho} , \quad \underline{L} = \underline{L}^\rho \frac{\partial}{\partial x^\rho}$$

have the components $\underline{L}^\rho = -g^{\rho\mu} \partial_\mu \frac{(t+r_*)}{2}$, $L^\rho = -g^{\rho\mu} \partial_\mu \frac{(t-r_*)}{2}$, therefore

$$\underline{L}^0 = -\frac{g^{00}}{2} \quad \underline{L}^r = -g^{rr} \frac{(\partial_r r_*)}{2} \quad \underline{L}^\theta = -g^{\theta\theta} \frac{(\partial_\theta r_*)}{2} \quad \underline{L}^\phi = -\frac{g^{\phi 0}}{2}$$

$$L^0 = -\frac{g^{00}}{2} \quad L^r = g^{rr} \frac{(\partial_r r_*)}{2} \quad L^\theta = g^{\theta\theta} \frac{(\partial_\theta r_*)}{2} \quad L^\phi = -\frac{g^{\phi 0}}{2} \quad (4.177)$$

From these definitions with an easy calculation we obtain

$$\begin{aligned}\underline{L} &= \frac{1}{2\Omega^2} \left(\frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \phi} \right) \\ L &= \frac{1}{2\Omega^2} \left(\frac{\partial}{\partial \underline{u}} + \omega_B \frac{\partial}{\partial \phi} \right)\end{aligned}\quad (4.178)$$

where

$$\omega_B = \frac{2Mar}{\Sigma R^2} \quad , \quad \Omega = \sqrt{\frac{\Delta}{R^2}} . \quad (4.179)$$

Let us define

$$\begin{aligned}e_3 &= \frac{1}{\Omega} \left(\frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \phi} \right) \quad , \quad e_4 = \frac{1}{\Omega} \left(\frac{\partial}{\partial \underline{u}} + \omega_B \frac{\partial}{\partial \phi} \right) \\ \underline{N} &= \left(\frac{\partial}{\partial u} + \omega_B \frac{\partial}{\partial \phi} \right) \quad , \quad N = \left(\frac{\partial}{\partial \underline{u}} + \omega_B \frac{\partial}{\partial \phi} \right) .\end{aligned}\quad (4.180)$$

Therefore

$$T = \Omega(e_3 + e_4) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial \underline{u}} \right) + 2\omega_B \frac{\partial}{\partial \phi} = \frac{\partial}{\partial t} + 2\omega_B \frac{\partial}{\partial \phi} \quad (4.181)$$

and , in Kerr,

$$\begin{aligned}[T, e_\lambda] &= \left[\frac{\partial}{\partial t}, e_\lambda \right] + 2\left[\omega_B \frac{\partial}{\partial \phi}, e_\lambda \right] = -e_\lambda(\omega_B) \frac{\partial}{\partial \phi} = -2R \sin \theta e_\lambda(\omega_B) e_\phi \\ [T, e_\phi] &= \left[\frac{\partial}{\partial t}, e_\phi \right] + 2\left[\omega_B \frac{\partial}{\partial \phi}, e_\phi \right] = 0\end{aligned}\quad (4.182)$$

Observe that $\omega_B = O(r^{-3})$ then

$$\begin{aligned}e_\lambda(\omega_B) &= \frac{\partial r}{\partial \lambda} \frac{\partial \omega_B}{\partial r} + \frac{\partial \theta}{\partial \lambda} \frac{\partial \omega_B}{\partial \theta} = \frac{\partial r}{\partial \lambda} O(ar^{-4}) + \frac{\partial \theta}{\partial \lambda} O(a^2 r^{-5}) \\ &= O(1) (O(ar^{-4}) + O(a^2 r^{-5})) = O(ar^{-4})\end{aligned}\quad (4.183)$$

as, see [Is-Pr],

$$\frac{\partial r}{\partial \lambda} = -\frac{\mu P^2 Q \Delta}{\Sigma R^2} = O(1) \quad , \quad \frac{\partial \theta}{\partial \lambda} = \frac{\mu P Q^2}{\Sigma R^2} = O(1) . \quad (4.184)$$

$$\text{where } P^2 = a^2(\lambda - \sin^2 \theta) \quad , \quad Q^2 = (r^2 + a^2)^2 - a^2 \lambda \Delta .$$

Therefore

$$[T, e_\lambda] = -2R \sin \theta e_\lambda(\omega_B) e_\phi = O(ar^{-3}) e_\phi . \quad (4.185)$$

It is immediate to recognize that N and \underline{N} are equivariant vector fields, that is they send $S(u, \underline{u})$ into $S(u + \epsilon, \underline{u})$ and $S(u, \underline{u} + \epsilon)$ respectively.

A simple computation gives the following expressions

$$[N, \underline{N}] = -4\Omega^2 \zeta(e_\phi) e_\phi \quad (4.186)$$

$$\zeta(e_\phi) = -\frac{R \sin \theta}{2\Sigma} Q \frac{\partial \omega_B}{\partial r} \quad , \quad \zeta(e_\lambda) = 0 \quad (4.187)$$

where e_λ is the unit vector which form a null orthonormal frame with e_3, e_4, e_ϕ . In the previous sections we defined $T = \Omega(e_3 + e_4)$ and we expect that $[T, e_4]$ decays fast. Let us compute it in Kerr and check the decay.⁵⁰ Clearly if it is good in this case it will work also in the perturbed Kerr.

$$\begin{aligned} [T, e_4] &= [\Omega(e_3 + e_4), e_4] = [\Omega e_3, e_4] + [\Omega e_4, e_4] = [\underline{N}, \frac{1}{\Omega} N] - e_4(\Omega) e_4 \\ &= \frac{1}{\Omega} [\underline{N}, N] - \frac{1}{\Omega^2} \underline{N}(\Omega) N - e_4(\Omega) e_4 \\ &= \frac{1}{\Omega} [\underline{N}, N] - e_3(\Omega) e_4 - e_4(\Omega) e_4 = -4\Omega \zeta(e_\phi) e_\phi - \Omega(e_3(\log \Omega) + e_4(\log \Omega)) e_4 \\ &\quad - 4\Omega \zeta(e_\phi) e_\phi + 2(\omega + \underline{\omega}) e_4 = -4\Omega \zeta(e_\phi) e_\phi \end{aligned} \quad (4.188)$$

and we know that $\zeta(e_\phi) = O(r^{-3})$ which satisfies our request.

2. Once we have the estimate $\sup_{\mathcal{K}} |r^5 |u|^{1+\epsilon} \partial_T \alpha_{ab}|$ to perform the integral we proceed in the following way: let us consider the integral curves of the vector field $T = \Omega(e_3 + e_4)$, let us denote them $\gamma(s)$ such that

$$\frac{d\gamma^\mu}{ds} = \Omega(e_3 + e_4) = (\underline{N} + N)^\mu$$

where

$$\underline{N} = \Omega e_3 \quad , \quad N = \Omega e_4$$

are the equivariant vector fields associated to the foliation. Mimicking the situation of the exact Kerr case we can define

$$\begin{aligned} N &= \frac{\partial}{\partial u} + X \quad ; \quad \underline{N} = \frac{\partial}{\partial u} + X \\ e_4 &= \frac{1}{\Omega} \left(\frac{\partial}{\partial u} + X \right) \quad ; \quad e_3 = \frac{1}{\Omega} \left(\frac{\partial}{\partial u} + X \right) \end{aligned} \quad (4.189)$$

and we expect that

$$X = \omega_B \frac{\partial}{\partial \phi} + \delta X = X_\phi e_\phi + X_\theta e_\theta \quad (4.190)$$

where

$$X_\phi := \mathbf{g}(X, e_\phi) \quad , \quad X_\theta := \mathbf{g}(X, e_\theta) \quad ,$$

⁵⁰If we had chosen T such that in Kerr becomes $\frac{\partial}{\partial t}$ then the commutator (in Kerr) would be zero.

where X is a vector field tangent to S and δX is the part which is identically zero in Kerr. As we know that

$$[N, \underline{N}] = -4\Omega^2 \zeta(e_a) e_a$$

it follows that

$$\left(\frac{\partial}{\partial \underline{u}} - \frac{\partial}{\partial u}\right)X = -4\Omega^2 \zeta(e_a) e_a .$$

With these definitions

$$\frac{d}{ds}u(\gamma(s)) = \frac{d\gamma^\mu}{ds} \frac{\partial u}{\partial x^\mu} = \frac{d\gamma^u}{ds} = \Omega(e_3 + e_4)^u = 1 \quad (4.191)$$

so that

$$u(\gamma(s; \bar{r})) = u(\gamma(0; \bar{r})) + s . \quad (4.192)$$

In \mathcal{K} it is easy to show that defining e_3, e_4 as in 4.189 and using u, \underline{u} as coordinates we can write the metric in the following way

$$\mathbf{g}(\cdot, \cdot) = -4\Omega^2 du d\underline{u} + \gamma_{ab} (d\omega^a - X^a(du + d\underline{u})) (d\omega^b - X^b(du + d\underline{u})) . \quad (4.193)$$

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