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THE 2^{nd} ORDER RENORMALIZATION GROUP FLOW FOR NON-LINEAR SIGMA MODELS IN 2 DIMENSIONS

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Abstract. We show that for two dimensional manifolds M with negative Euler characteristic there exists subsets of the space of smooth Riemannian metrics which are invariant and either parabolic or backwards-parabolic for the 2^{nd} order RG flow. We also show that solutions exists globally on these sets. Finally, we establish the existence of an eternal solution that has both a UV and IR limit, and passes through regions were the flow is parabolic and backwards-parabolic.

1. Introduction

The world-sheet nonlinear sigma model renormalization group flow arises from quantizing the classical action

$$S(x) = \frac{1}{4\pi\alpha'}\int_\Sigma \gamma^{\alpha\beta} g_{ij}(x)\partial_\alpha x^i \partial_\beta x^j dV(\gamma),$$

where $\alpha' > 0$ is the string coupling constant, (Σ, γ) is a 2-dimensional Riemannian manifold (i.e. world sheet), (M, g) is a n-dimensional Riemannian manifold (i.e. target space), and $x : \Sigma \to M$; $(\theta^1, \theta^2) \mapsto (x^1(\theta)\ , \dots , x^n(\theta))$ is a map. A perturbative quantization of this classical theory requires the introduction of a momentum cutoff $\Lambda > 0$, and gives rise to a one parameter family of quantum field theories indexed by Λ. The requirement that the family of quantum field theories be equivalent on length scales $L \gg 1/\Lambda$ leads to the renormalization group (RG) flow equations

$$\partial_\Lambda g_{ij} = -\beta^g_{ij}\,.$$

In the regime where perturbation theory is valid ($\alpha' \ll 1$), the β -functions β^g_{ij} can be expanded in powers of α' [2, 7]:

$$\beta^g_{ij} = \alpha' R_{ij} + \tfrac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \mathrm{O}(\alpha'^3)\,.$$

Defining a "time" by $t = -\ln(\Lambda)$, the RG flow equations become

$$\partial_t g_{ij} = -\alpha' R_{ij} - \tfrac{\alpha'^2}{2} R_{iklm} R_j{}^{klm} + \mathrm{O}(\alpha'^3). \qquad (1.1)$$

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It is expected that in the perturbative regime, the 1st order truncation

$$\partial_t g_{ij} = -\alpha' R_{ij} \tag{1.2}$$

should provide a "good approximation" to the full flow. However, without an estimate of the error, the notion of a good approximation cannot be quantified. The problem of understanding the error is obstructed by the fact that a mathematically rigorous quantization of the non-linear sigma model is presently unavailable. However, if it ultimately turns out that expansion (1.1) obtained using perturbation theory is valid for the RG flow, even as an asymptotic expansion, then it is not unreasonable to expect that the error between the full flow and (1.2) will be qualitatively described by the 2nd order truncation

$$\partial_t g_{ij} = -\alpha' R_{ij} - \tfrac{\alpha'^2}{2} R_{iklm} R_j{}^{klm}, \tag{1.3}$$

at least for situations where the curvature is not too large. We note that this expectation is borne out in other field theories where it has been established that it is enough to consider the 2nd order truncation of the RG flow to establish the existence of a continuum limit [3]. This reinforces the view that knowing the 1st order flow is not always enough for applications where quantitative control on the error is required.

As has been noted now many times, the 1st order RG flow (1.2) coincides with Ricci flow. It is known that there are many solutions to Ricci flow that become singular at a finite time. As shown by Hamilton [6], a singular time T of Ricci flow is characterized by curvature blow up: $\lim_{t\nearrow T} R_{ijkl}R^{ijkl} = \infty$. This suggests that near a singular time for Ricci flow, the higher order curvature corrections in (1.1) will dominate the behavior of the flow even for $\alpha' \ll 1$. Thus it is natural from this viewpoint to consider the higher order truncations of (1.1) to try and capture the effect of higher order curvature terms.

With the above motivation in mind, the main aim of this article is to continue the study initiated in [5] of the 2nd order RG flow (1.3) using techniques from geometric analysis. To facilitate comparisons with Ricci flow, we rescale the time and metric to bring (1.3) into the form

$$\partial_t g_{ij} = -2R_{ij} - \tfrac{\alpha'}{2} R_{iklm} R_j{}^{klm} \tag{1.4}$$

which makes the leading term consistent with the standard presentation of Ricci flow.

Although Ricci flow may be recovered in the limit $\alpha' \searrow 0$, the 2nd order RG flow differs from Ricci flow in two important respects: it is fully non-linear, and it is not parabolic for all choices of α' and g_{ij}. Therefore, in addition to curvature blow up, loss of parabolicity along the flow presents a possible new mechanism for singularity formulation. To investigate this possibility, we restrict ourselves to the simplest possible setting of a closed two dimensional target space M. In this case, the curvature tensor is given by

$$R_{ijkl} = \tfrac{1}{2}R\big(g_{il}g_{jk} - g_{ik}g_{jl}\big)$$

which implies that the 2nd order RG equations (1.4) reduce to

$$\partial_t g_{ij} = -\mathcal{R} g_{ij}, \tag{1.5}$$

where

$$\mathcal{R} = R + \tfrac{\alpha'}{4}R^2. \tag{1.6}$$

Following the Ricci flow terminology, we will call a solution $g(t)$ to (1.4) (equivalently (1.5)) *ancient*, *immortal*, or *eternal*, if the solution exists on a time interval of the form $-\infty < t < t_0$, $t_0 < t < \infty$, or $-\infty < t < \infty$, respectively. We will also use the physics terminology of a *UV* (*IR*) limit which refers to a fixed point of (1.4) from which an ancient (immortal) solutions originates (terminates). From both a mathematical and physical view point, the existence and classification of the ancient, immortal, and eternal solutions are of fundamental interest. For physical applications, the UV limits are of particular importance as they correspond to cut-off removal for the quantum field theory. In other words, a UV limit identifies quantum field theories with a well defined microscopic limit. The IR limits also have a physical interpretation and correspond to a well defined macroscopic limit.

The main result of this paper is to show that if M has negative Euler characteristic, then there exist large regions in the space of smooth Riemannian metrics $\mathcal{M}$ on M for which the flow is either parabolic or backwards-parabolic, and that these regions remain invariant under the flow. Moreover, on these invariant regions, the flow has good long term existence properties.

We also establish the existence of an eternal solution to (1.4) with both a UV and IR limit that passes from a region of backward-parabolicity into a region of parabolicity. We find this solution particularly interesting as it is a consequence of the curvature correction $\frac{\alpha'}{2}R_{iklm}R_j{}^{klm}$ term to Ricci flow. This solution shows that the lack of uniform parabolicity for all choices of metrics g_{ij} should not necessarily be viewed as a defect. Instead, the notion of uniform parabolicity should be replaced with that of invariant parabolic or backwards-parabolic sets. As this paper demonstrates, the existence of neighboring parabolic and backward-parabolic regions opens up the possiblity for constructing solutions by joining together two solutions at a degenerate parabolic point that joins a backward-parabolic region to a parabolic one.

2. Parabolicity of the 2nd order RG equations

Due to the conformal nature of the 2nd order RG flow (1.5), it is consistent given initial data $g_{ij}\big|_{t=0} = \tilde{g}_{ij}$ to write $g_{ij}(t) = e^{u(t)}\tilde{g}_{ij}$ in which case the initial value problem

$$\partial_t g_{ij} = -\mathcal{R} g_{ij} \quad : \quad g_{ij}(0) = \tilde{g}_{ij} \tag{2.1}$$

is equivalent to

$$\partial_t u = -\mathcal{R}(u) \quad : \quad u(0) = 0, \tag{2.2}$$

where

$$\mathcal{R}(u) = R + \tfrac{\alpha'}{4} R^2 \quad \text{and} \quad R = e^{-u}\big(-\tilde{\Delta} u + \tilde{R}\big). \tag{2.3}$$

Here, $\tilde{R}$ and $\tilde{\Delta}$ denote the Ricci scalar and Laplacian of $\tilde{g}_{ij}$, respectively.

The linearization

$$D\mathcal{R}(u) \cdot v = \big(1 + \tfrac{\alpha'}{2} R\big)\big(-e^{-u}\tilde{\Delta} v - Rv\big),$$

shows that (2.2) is *parabolic* when $1 + \frac{\alpha'}{2}R > 0$ and *backwards parabolic* when $1 + \frac{\alpha'}{2}R < 0$. This and the properties of the evolution equation to be described in the next section motivate us to define the following subsets of the space of smooth Riemannian metrics $\mathcal{M}$:

$$\mathcal{M}_+ = \{\, g \in \mathcal{M} \mid -\tfrac{2}{\alpha'} < R < 0 \,\} \quad \text{and} \quad \mathcal{M}_- = \{\, g \in \mathcal{M} \mid -\tfrac{4}{\alpha'} < R < -\tfrac{2}{\alpha'} \,\} \tag{2.4}$$

To avoid the situation where these sets are empty, we will, as mentioned in the introduction, restrict ourselves to closed manifolds with negative Euler characteristic.

3. Invariance of $\mathcal{M}_\pm$ and global existence

Theorem 3.1. *Suppose $\tilde{g} \in \mathcal{M}_+$. Then there exists a smooth one parameter family of metrics $g(t)$ for $0 \leq t < \infty$ that satisfy the following:*

(i) *$g(t)$ solves the 2^{nd} order RG equation (1.4) with $g(0) = \tilde{g}$,*

(ii) *$g(t) \in \mathcal{M}_+$ for all $t \geq 0$, and*

(iii) *there exists a constant $C_{\tilde{R}}$ depending only on $C^+_{\tilde{R}} = \max_{x\in M} \tilde{R}(x)$ and $C^-_{\tilde{R}} = \min_{x\in M} \tilde{R}(x)$ such that*

$$C^-_{\tilde{R}} \leq R(t,x) \quad \textit{and} \quad |R(t,x)| \leq \frac{C_{\tilde{R}}}{1+t}$$

for all $(t,x) \in [0,\infty) \times M$.

Proof. Since the equation (2.2) is parabolic whenever $g = e^u \tilde{g} \in \mathcal{M}_+$, it follows by standard local existence theorems for parabolic equations (see Proposition 8.1, pg. 338 of [9]) that there exists a smooth solution $u(t)$ to the initial value problem (2.2) defined on some interval $0 \leq t < T$. Given this solution, a short calculation using (1.5) shows that the Ricci scalar satisfies the equation

$$\partial_t R = \Delta \mathcal{R} + \mathcal{R} R, \tag{3.1}$$

or equivalently

$$\partial_t R = (1 + \tfrac{\alpha'}{2}R)\Delta R + \tfrac{\alpha'}{2}|\nabla R|^2 + \mathcal{R}R. \tag{3.2}$$

To control the behavior of $R(t)$, we will use the maximum principle. This requires us to analyze solutions of the ODE $dy/dt = y^2 + \frac{\alpha'}{4}y^3$.

Lemma 3.2. *Suppose $y_0 \in \left(-\frac{4}{\alpha'}, 0\right)$. Then the unique solution $y(t)$ to the initial value problem*

$$\frac{dy}{dt} = y^2 + \tfrac{\alpha'}{4}y^3 \quad : \quad y(0) = y_0 \tag{3.3}$$

exists for all $t \geq 0$ and satisfies

$$y_0 \leq y(t) < 0 \quad \textit{and} \quad |y(t)| \leq \frac{C_0}{1+t} \quad \forall\, t \geq 0, \tag{3.4}$$

where C_0 is a constant that depends only on y_0.

Proof. Since $y = 0$ and $y = -\frac{4}{\alpha'}$ are the only fixed point of (3.3), $dy/dt > 0$ for $-\frac{4}{\alpha'} < y < 0$, and $y(0) = y_0 \in \left(-\frac{4}{\alpha'}, 0\right)$, it follows that the solution $y(t)$ exists for all $t \geq 0$ and satisfies

$$y_0 \leq y(t) < 0 \quad \forall\, t \geq 0, \tag{3.5}$$

and

$$\lim_{t\to\infty} y(t) = 0. \tag{3.6}$$

Next, we observe that (3.3) can be integrated to get

$$\tfrac{\alpha'}{4}\ln\left(\frac{y_0(1+\frac{\alpha'}{4}y(t))}{y(t)(1+\frac{\alpha'}{4}y_0)}\right) + \frac{1}{y_0} - \frac{1}{y(t)} = t. \tag{3.7}$$

Together, (3.5) and (3.7) imply that

$$|y(t)| = \frac{1 - \frac{4}{\alpha'}|y(t)|\ln(|y(t)|)}{t + \frac{1}{|y_0|} - \frac{4}{\alpha'}\ln\left(\frac{|y_0|(1+\frac{4}{\alpha'}y(t))}{1+\frac{4}{\alpha'}y_0}\right)}. \tag{3.8}$$

But $\lim_{t\to\infty}|y(t)|\ln(|y(t)|) = 0$ by (3.6), and so it follows from (3.5) and (3.8) that $|y(t)| \leq C_0(1+t)^{-1}$ $(t \geq 0)$ for some constant C_0 that depends only on y_0. □

Now, set $C^+_{\tilde{R}} = \max_{x\in M}\tilde{R}(x)$ and $C^-_{\tilde{R}} = \min_{x\in M}\tilde{R}(x)$. Then from equation (3.2), Lemma 3.2, and the maximum principle (see Theorem 4.4, pg. 96 in [1]), there exists a constant $C_{\tilde{R}}$ depending only on $C^{\pm}_{\tilde{R}}$ such that

$$C^-_{\tilde{R}} \leq R(t,x) \quad \text{and} \quad |R(t,x)| \leq \frac{C_{\tilde{R}}}{1+t} \tag{3.9}$$

for all $(t,x) \in [0,T) \times M$. Integrating the evolution equation (1.5) in time and applying the inequality (3.9) then yields

$$|u(t,x)| \lesssim \ln(1+t) \quad \forall\, (t,x) \in [0,T) \times M. \tag{3.10}$$

The inequalities (3.9)-(3.10) together with the formulas (2.3) show that

$$(1+t)|\partial_t u(t,x)| + |\tilde{\Delta} u(t,x)| \lesssim 1 \quad \forall\, (t,x) \in [0,T) \times M \tag{3.11}$$

Clearly, the derivative $\partial_t u$ satisfies the equation

$$\partial_t(\partial_t u) = -D\mathcal{R}(u) \cdot \partial_t u.$$

By the estimates (3.9)-(3.11), this equation is uniformly parabolic with bounded continuous coefficients. Consequently, we can apply the Krylov-Safonov estimates [8] to conclude the existence of constants $C_T > 0$ and $0 < \sigma < 1$ such that

$$\|\partial_t u(t)\|_{C^{0,\sigma}(M)} \leq C_T \quad \forall\, t \in [0,T). \tag{3.12}$$

This implies, increasing C_T if necessary, that

$$\|R(t)\|_{C^{0,\sigma}(M)} \leq C_T \quad \forall\, t \in [0,T). \tag{3.13}$$

Viewing (2.2) as an elliptic equation for u with source term $\partial_t u$, the estimates (3.9)-(3.11), (3.12) -(3.13) allow us to apply Schauder estimates (see Lemma 6.16, pg. 103 of [4]) to conclude

$$\|u(t)\|_{C^{2,\sigma}(M)} \leq C_T \quad \forall\, t \in [0,T).$$

Applying the parabolic continuation principle (see Proposition 8.1, pg. 338 of [9]), the solution can be continued for at least a small time past T. Thus we conclude that the solution $u(t)$ exists for all $t \geq 0$. □

Theorem 3.3. *Suppose $\tilde{g} \in \mathcal{M}_-$. Then there exists a smooth one parameter family of metrics $g(t)$ for $-\infty < t \leq 0$ that satisfy the following:*

(i) *$g(t)$ solves the 2^{nd} order RG equation (1.4) with $g(0) = \tilde{g}$,*
(ii) *$g(t) \in \mathcal{M}_-$ for all $t \leq 0$, and*
(iii) *there exists a constant $C_{\tilde{R}}$ depending only on $C^+_{\tilde{R}} = \max_{x\in M} \tilde{R}(x)$ and $C^-_{\tilde{R}} = \min_{x\in M} \tilde{R}(x)$ such that*

$$R(t,x) \leq C^+_{\tilde{R}} \quad \text{and} \quad |R(t,x) + \tfrac{4}{\alpha'}| \leq C_{\tilde{R}} e^t$$

for all $(t,x) \in (-\infty, 0] \times M$.

Proof. Since the equation (2.2) is now backwards parabolic, we instead consider the forward equation obtained be replacing t with $-t$:

$$\partial_t u = \mathcal{R}(u).$$

As in the proof of Theorem 3.1, we can control the curvature by analyzing solutions to the ODE $dy/dt = -y^2 - \frac{4}{\alpha'}y^3$.

Lemma 3.4. *Suppose* $y_0 \in \left(-\frac{4}{\alpha'}, 0\right)$. *Then the unique solution* $y(t)$ *to the initial value problem*

$$\frac{dy}{dt} = -y^2 - \tfrac{\alpha'}{4}y^3 \quad : \quad y(0) = y_0 \tag{3.14}$$

exists for all $t \geq 0$ *and satisfies*

$$-\tfrac{4}{\alpha'} < y(t) \leq y_0 \quad \text{and} \quad |y(t) + \tfrac{4}{\alpha'}| \leq C_0 e^{-t} \quad \forall\, t \geq 0, \tag{3.15}$$

where C_0 *is a constant that depends only on* y_0.

Proof. This proof is essentially the same as the proof of Lemma 3.4 except for the asymptotics. To see the exponential convergence, we replace t with $-t$ in the formula (3.7) and rearrange to get

$$|1 + \tfrac{\alpha'}{4}y(t)| = \frac{|y(t)(1 + \frac{\alpha'}{4}y_0)|}{|y_0|} e^{\frac{4}{\alpha'}(1/|y_0| - 1/|y(t)|)} e^{-t}. \tag{3.16}$$

Since y(t) is bounded by $-\frac{4}{\alpha'} < y(t) \leq y_0 < 0$, the exponential convergence $|y(t) + \frac{4}{\alpha'}| \leq C_0 e^{-t}$ follows directly from (3.16). □

Using this lemma and the evolution equation for the scalar curvature

$$\partial_t R = -\Delta R - \mathcal{R}R,$$

the proof of Theorem 3.3 follows from a simple adaptation of the arguments used in the proof Theorem 3.1. □

4. An eternal solution connecting $\mathcal{M}_-$ to $\mathcal{M}_+$

Directly from the equation (1.5), it is clear that the flat metric $\bar{g}$, and the metric $\hat{g}$ with Ricci scalar equal to $-\frac{4}{\alpha'}$ are fixed points for the 2^{nd} order RG flow (1.5). We now show that there exists an eternal solution that connects the UV fixed point $\hat{g}$ to the IR fixed point $\bar{g}$. Moreover, we show that this solution passes through both the sets $\mathcal{M}_{\pm}$.

To begin, let $\tilde{g}$ be a metric with constant negative curvature

$$\tilde{R} = -1.$$

Next, define

$$g(t, x) := \frac{1}{y(t)}\tilde{g}(x), \tag{4.1}$$

which implies that the Ricci scalar of g is given by

$$R(t) = y(t). \tag{4.2}$$

The ansatz (4.1) is consistent for the 2^{nd} order RG flow and leads to the same equation studied in Lemma 3.2 (i.e. just substitute (4.2) in (3.1)), namely

$$\frac{dy}{dt} = y^2 + \tfrac{\alpha'}{4}y^3. \tag{4.3}$$

Choosing initial data

$$y(0) = -\tfrac{2}{\alpha'}, \tag{4.4}$$

it follows immediately from Lemmas 3.2 and 3.4 that the unique solution $y(t)$ to the initial value problem (4.3)-(4.4) exists for all $t \in (-\infty, \infty)$ and satisfies

$$-\tfrac{4}{\alpha'} < y(t) < -\tfrac{2}{\alpha'}, \qquad |y(t) + \tfrac{4}{\alpha'}| \leq C_- e^t \quad \forall\, t < 0,$$

and

$$-\tfrac{2}{\alpha'} < y(t) < 0, \qquad |y(t)| \leq \tfrac{C_+}{1+t} \quad \forall\, t > 0,$$

for some constants $C_\pm$. In particular, this implies that

$$g(t) \in \mathcal{M}_- \quad \forall\, t < 0 \quad \text{and} \quad g(t) \in \mathcal{M}_+ \quad \forall\, t > 0.$$

Moreover, it is clear that

$$\lim_{t\to-\infty} g(t) = \hat{g} \quad \text{and} \quad \lim_{t\to\infty} R(t) = 0.$$

In this sense, the solution $g(t)$ connects the UV fixed point $\hat{g}$ to the IR fixed point $\bar{g}$.

5. Discussion

We have shown that the 2^{nd} order RG flow admits an eternal solution that connects a constant negative curvature metric to the flat metric. Moreover, we have shown that the existence of the eternal solution is due to the curvature correction term $\frac{\alpha'}{2} R_{iklm} R_j{}^{klm}$ to the 1^{st} order flow (i.e. Ricci flow). More specifically, we demonstrated that the term $\frac{\alpha'}{2} R_{iklm} R_j{}^{klm}$ destroys the uniform parabolicity of Ricci flow, and is it is precisely this lack of uniform parabolicity that allows for the existence of the eternal solution.

The results of this article show that the lack of uniform parabolicity of the 2^{nd} (and higher) order RG flow is not necessarily a defect. Instead, the lack of parabolicity opens up the possibility of constructing solutions by matching together solutions from neighboring regions of backwards and forwards parabolicity. We speculate that this construction could be useful in for both physics and geometry as it suggests a new method for connecting different geometries (i.e. UV and IR fixed points) via a geometric flow.

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