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Abstract

In this paper, we consider the class of generalized Nariai solutions of Einstein's field equations in vacuum with a positive cosmological constant. According to the cosmic no-hair conjecture, generic expanding solutions isotropize and approach the de-Sitter solution asymptotically, at least locally in space. The generalized Nariai solutions, however, show quite unusual asymptotics and hence should be non-generic in some sense. In the first part of the paper, we list the necessary facts and characterize the asymptotic behavior geometrically. In the second part, we study the question of non-genericity, which we are able to confirm within the class of spatially homogeneous solutions. It turns out that perturbations of the three isometry classes of generalized Nariai solutions are related to different mass regimes of Schwarzschild de-Sitter solutions. In subsequent papers, we will continue this research in more general classes of solutions.

1 Introduction

For the understanding of the qualitative properties of cosmological solutions of Einstein's field equations the outstanding issues of strong cosmic censorship and the so-called BKL-conjecture are of particular importance. The reader can find relevant background material in [1]. Recently, there have been several break-throughs, but still many issues are far from being understood. A particular reason is that the structure of singularities can be enormously complicated, as one can already see in the spatially homogeneous case for the Mixmaster solutions where unanswered questions still exist as reviewed in [14].

In this paper, a related fundamental open problem in the qualitative understanding of the dynamics of cosmological solutions, which appears easier at the first glance, but which is also not well understood, will play the main role. Namely loosely speaking, how do generic expanding solutions of the field equations behave asymptotically? It is clear that this question is directly related to the problem of finding the correct interpretation of current cosmological observations which suggest that our universe is in an epoch of its history in which it is expanding more and more rapidly. These observations are surprisingly consistent with spatially homogeneous and isotropic model solutions of

Einstein's field equations with a positive cosmological constant [20, 21]. However, it is unclear how strongly these results depend on the high symmetry of the models. There are various approaches to study more general models. In any case, let us, for the moment, suppose that our universe is really isotropic and homogeneous and expands at an accelerated rate at present. It is then natural to ask whether our universe is special with this property, or if generic models undergoing such strong inflationary expansion phases are qualitatively similar. The conjecture that this is generic for expanding cosmological solutions of Einstein's field equations in vacuum with a positive cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1}$$

is known as the cosmic no-hair conjecture, which was essentially introduced in [13]. In the literature, this conjecture is often formulated for a wider class of models, but let us restrict to vacuum with $\Lambda > 0$ here. It states that, generically, such solutions will approach the de-Sitter solution asymptotically in some way, and hence become more and more homogeneous and isotropic and expand in an accelerated manner eventually. We will continue the discussion of this conjecture later. From this point of view, a particularly interesting solution of Eq. (1) is the so-called Nariai solution, which we introduce in Section 2. There, we also define and describe the notion of Nariai asymptotics and motivate why this solution is believed to be special because it expands in an anisotropic manner not consistent with the cosmic no-hair picture asymptotically. Cosmic no-hair hence suggests that under arbitrary small perturbations of the Nariai solutions, there is either no expansion, or if there is expansion, then it is consistent with the cosmic no-hair picture.

The question whether this is true for Nariai asymptotics is the main motivation for this paper. Several heuristic results in special cases exist as summarized in [5]. For our investigations, we want to study the properties of solutions of Eq. (1) corresponding to fully non-linear perturbations of Nariai initial data; we make this more precise later. In this paper, we will mainly lay the foundation for a series of works to be published in the future by summarizing the known results in a consistent manner, by pointing to some obscurities in the literature, but also by showing several new results. In principle, we are interested in generic perturbations with no symmetries. In practice, however, this is not possible. As a first step, we will hence investigate the spatially homogeneous (but non-isotropic) case in this paper. In our subsequent paper [4], the same question will be investigated in an inhomogeneous class of solutions.

We note that the Nariai solution has often been considered before in the literature; a summary of references is given in [5]. This solution has turned out to be very interesting from the semi-classical point of view because it has certain extremal horizon properties. We will not discuss such issues here. We stress that all our results are purely classical, i.e. all quantum effects are ignored.

Let us list some of our main assumptions. We will always restrict our attention to cosmological solutions by which we mean 4-dimensional globally hyperbolic solutions of Einstein's field equations in vacuum with positive cosmological constant Λ with compact spatial topology. Spatial $\mathbb{S}^1 \times \mathbb{S}^2$-topology will be of particular interest for us, since this is the spatial topology of the Nariai solutions.

This paper is organized as follows. In Section 2 we define and recall several known results concerning the Nariai solution. a geometric characterization of what we call Nariai asymptotics and its relation to the cosmic no-hair picture will be of particular interest to us. In the course of this discussion we prove – to our knowledge for the first time rigorously in the literature – the fact that the Nariai solution does not possess a smooth conformal boundary. In Section 3, we introduce the class of generalized Nariai solutions and discuss their relation to the standard Nariai solution. In particular, we prove that all generalized Nariai solutions can be embedded isometrically into the standard Nariai solution, at least locally. Motivated by our fundamental question, we study fully non-linear spatially homogeneous perturbations of the generalized Nariai solutions in Section 4. In the course of the study, we are able to confirm the conjectured instability within this class and give a precise characterization of it.

2 Standard Nariai spacetime

2.1 Basic properties

Let Λ be an arbitrary positive parameter. Consider the Lorentzian manifold (M, g) given by

$$M = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2), \quad g = \frac{1}{\Lambda}\left(-dt^2 + \cosh^2 t\, d\rho^2 + g_{\mathbb{S}^2}\right), \tag{2}$$

with the standard coordinate ρ on the manifold $\mathbb{S}^1$, the standard round unit metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ and the time coordinate t. This spacetime is called the Nariai spacetime. It was first discussed by Nariai [17, 18] and later on reconsidered in various works; an overview of references is given by [5]. Since we also want to study generalized Nariai spacetimes in Section 3, we will often speak of the standard Nariai spacetime where necessary. We will also often consider the universal cover of the standard Nariai spacetime. This is the simply connected Lorentzian manifold $(\tilde{M}, g)$ with $\tilde{M} = \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2)$ and the metric g above.

The standard Nariai spacetime has the following further properties.

(i) It is an analytic solution of Einstein's field equations in vacuum with a cosmological constant $\Lambda > 0$. Hence we will often also speak of the (standard) Nariai solution.

(ii) It is globally hyperbolic and the $t = \mathit{const}$-hypersurfaces are spacelike Cauchy surfaces with topology $\mathbb{S}^1 \times \mathbb{S}^2$. Since the Cauchy surfaces are hence compact, the standard Nariai solution is a cosmological solution of Eq. (1).

(iii) It is geodesically complete. Note first that (M, g) is the direct product of the 2-dimensional de-Sitter spacetime

$$M_1 = \mathbb{R} \times \mathbb{S}^1, \quad g_1 = \frac{1}{\Lambda}\left(-dt^2 + \cosh^2 t\, d\rho^2\right),$$

and the rescaled round 2-sphere

$$M_2 = \mathbb{S}^2, \quad g_2 = g_{\mathbb{S}^2}/\Lambda;$$

i.e.

$$M = M_1 \times M_2, \quad g = g_1 + g_2.$$

One finds that the geodesic equation on (M, g) splits into two decoupled parts, namely the geodesic equation on (M_1, g_1) and that on (M_2, g_2). The completeness of (M, g) then follows because both (M_1, g_1) and (M_2, g_2) are geodesically complete manifolds.

(iv) When we consider the universal cover, the representation of the standard Nariai spacetime given in Eq. (2) is the maximal analytic extension. Maximality follows from geodesic completeness. Uniqueness is implied by the version of Theorem 6.3 in [15] for Lorentzian metrics, or by Theorem 4.6 from [6] for the special case of geodesic completeness. In the literature one finds other representations of the Nariai metric, see the references above, which are not always geodesically complete and hence not maximal.

(v) It is spatially homogeneous with Kantowski-Sachs symmetry group $G = \mathbb{R} \times \mathrm{SO}(3)$. The action of G can be represented, first, by translations along the $\mathbb{S}^1$-factor of the manifold generated by

$$\xi_3 = \partial_\rho$$

corresponding to the $\mathbb{R}$-factor of G, and, second, by the standard action of SO(3) on $\mathbb{S}^2$ generated by

$$\begin{aligned} W_1 &= \sin\phi \partial_\theta + \cos\phi \cot\theta \partial_\phi, \\ W_2 &= \cos\phi \partial_\theta - \sin\phi \cot\theta \partial_\phi, \\ W_3 &= \partial_\phi. \end{aligned}$$

Here, (θ, ϕ) are standard polar coordinates on $\mathbb{S}^2$. However, the 4-dimensional group G, which yields spatial homogeneity, is just a subgroup of the full 6-dimensional symmetry group of (M, g). The additional symmetry originates in the boost symmetries of the 2-dimensional de-Sitter space (M_1, g_1). Indeed, the algebra of Killing fields generated by (W_1, W_2, W_3, ξ_3) can be extended to the maximal Killing algebra of (M, g) by means of the fields

$$\begin{aligned} \xi_1 &= \sin\rho \, \partial_t + \cos\rho \, \tanh t \, \partial_\rho, \\ \xi_2 &= \cos\rho \, \partial_t - \sin\rho \, \tanh t \, \partial_\rho. \end{aligned}$$

(vi) The Nariai solution (M, g) has Nariai asymptotics; a term which we now explain.

2.2 Nariai asymptotics and cosmic no-hair

A particularly peculiar property of the Nariai solution is its behavior for large $|t|$. In the following we will, without loss of generality, restrict ourselves to the future time direction, i.e. the limit of large positive t. Analogous arguments can be used to analyze the past time direction. While the $\mathbb{S}^1$-factor of the spatial slices expands exponentially for increasing t, the volume of the $\mathbb{S}^2$-factor stays constant. Thus, this solution shows an anisotropic asymptotic expansion. This is extraordinary from the point of view of the standard interpretation of inflation [16]. Namely, we can interpret the expansion to be driven by the cosmological constant – also referred to as dark energy in such discussions – and the usual picture is that the "repulsive gravitational force" given by dark energy is isotropic. Indeed, the standard belief is that generic inflationary solutions of Einstein's field equations in vacuum with $\Lambda > 0$ approach the 4-dimensional de-Sitter solution locally asymptotically in time. This implies that the expansion becomes accelerated and isotropic eventually. This is known as the cosmic no-hair conjecture. Recall that the Nariai solution is geodesically complete and that, when we speak about future asymptotics of any solution, we always assume at least future causal geodesic completeness.

Possibly, the anisotropic asymptotic behavior which we observe for the Nariai solution could be a consequence of a strange choice of foliation. Maybe there exist other slicings which yield an isotropic asymptotic expansion? Indeed, our discussion, which is based so far on looking at the $\mathbb{S}^1$- and $\mathbb{S}^2$-factors of the manifold, is not geometrically invariant. So let us take a more geometrical point of view now. One way of speaking about asymptotics of solutions geometrically is to study the existence and properties of conformal boundaries. We will not define so-called future asymptotically de-Sitter solutions [8, 7, 2] here, but the main point is the existence of a smooth future conformal boundary $\mathcal{J}^+$ in the case of vacuum solutions with $\Lambda > 0$. We refer the reader to the references above for a definition. As is shown for instance in [12], the de-Sitter solution is future asymptotically de-Sitter and hence forms a smooth future conformal boundary $\mathcal{J}^+$. As we prove below, the Nariai solution does not.

A more general way of discussing asymptotics geometrically has been introduced by Ringström [19]. He says that in a spacetime, late time observers are "oblivious to topology" if there is a Cauchy surface Σ such that there are no future directed inextendible causal curves whose causal pasts contain Σ. We just remark that there is a related result in [9] which, however, does not yield further insights for our discussion here. The de-Sitter spacetime is a particular example of a spacetime where late time observers are oblivious to topology. This can be seen most easily in the standard conformal diagram given for instance in [12]. In physical terms the standard interpretation is that due to the rapid expansion in the future of the spacetime, all light rays are confined to a subregion of the spatial slices and hence all observers have a future event horizon.

Indeed in the Nariai solution, late time observers are also oblivious to topology due to the expansion of the 2-dimensional de-Sitter factor (M_1, g_1) introduced above. Consequently, we require a finer characterization of the asymptotics in order to distinguish the asymptotic behavior of the Nariai solution from that of the de-Sitter spacetime geometrically. It turns out that Ringström's notion of spacetimes whose late time observers are "completely oblivious to topology" is useful. He says that late time observers in a spacetime are completely oblivious to topology if there is a Cauchy surface Σ such that for all future inextendible causal curves γ the intersection of Σ with the causal past of γ is contained in a subset of Σ homeomorphic to an Euclidean 3-ball, i.e. in a coordinate neighborhood. In this situation, observers have no information about the manifold properties of Σ whatsoever because they can only see a "topologically trivial" 3-ball at most. We note here that for the de-Sitter spacetime, late time observers are completely oblivious to topology. This is one implication of the following Lemma.

Lemma 1 For any future causal geodesically complete cosmological spacetime (M, g) with a spacelike smooth compact future conformal boundary, late time observers are completely oblivious to topology.

In order to prove this lemma, recall the fact that due to this hypothesis the future conformal boundary $\mathcal{J}^+$ is a Cauchy surface [2] of the conformal extension[1] $(\tilde{M}, \tilde{g})$ of (M, g). In particular $\mathcal{J}^+$ is homeomorphic to all Cauchy surfaces of (M, g). This implies that any given future directed inextendible causal curve of (M, g) can be extended in $(\tilde{M}, \tilde{g})$ to hit $\mathcal{J}^+$. Moreover, there is a neighborhood of $\mathcal{J}^+$ in $(\tilde{M}, \tilde{g})$ with a Gaussian time function τ with respect to $\mathcal{J}^+$ such that the $\tau = 0$-hypersurface equals $\mathcal{J}^+$ and all $\tau = const$-surfaces with $\tau < 0$ are Cauchy surfaces of (M, g). Now if we choose a $\tau_0 < 0$ with $|\tau_0|$ small enough, then the $\tau = \tau_0$-surface Σ_0 has the required properties. Namely, choose any future directed inextendible causal curve γ on (M, g) with future $p \in \mathcal{J}^+$. Choose a neighborhood U of p in $\mathcal{J}^+$ homeomorphic to a 3-ball. Let U_0 be the set in Σ_0 obtained from U via the past flow of the Gaussian time vector field. Then the causal past of p intersected with Σ_0 is contained in U_0 if $|\tau_0|$ is sufficiently small. Now the lemma follows because the causal past of p includes the causal past of the corresponding causal curve γ.

We remark that some of the strong assumptions made here can possibly be relaxed, but we will not elaborate on this.

It is clear that spacetimes whose observers are completely oblivious to topology are also oblivious to topology. But the inverse is not true and the Nariai solution is an example of this. Ringström, in [19], proves an even more detailed statement of that, namely, regardless of the choice of Cauchy surface Σ and future directed inextendible causal curve γ, the intersection of Σ and the causal past of γ is not contained in a set homeomorphic to a 3-ball. In simple words, one could say that although observers cannot have full information about the spatial topology of certain Cauchy surfaces in the Nariai case, due to the rapid expansion, they will always have *some* information because the expansion is so anisotropic. We remark that for Ringström's proof of this statement, compactness of the Cauchy surfaces is crucial. We have not yet studied this, but it is possible that his statement is wrong for the universal cover of the Nariai solution, in particular for Cauchy surfaces on which the Nariai time function t is unbounded.

[1]Note that we use the opposite notation to [4]. In our work here, all quantities related to the conformal metric $\tilde{g}$ are marked with a tilde, while all quantities related to the physical metric g have no mark.

Since, in the case of the de-Sitter solution, late time observers are completely oblivious to topology, the cosmic no-hair conjecture suggests that generic solutions of Eq. (1) have the same property. If this were true, the Nariai solution would be exceptional. This is the precise motivation why we would like to find out if "Nariai asymptotics" is instable under generic perturbations, and if this is true how this instability is realized dynamically. In this paper, we study this question for spatially homogeneous perturbations in Section 4.

An argument similar to the one presented above yields the following corollary which further characterizes the Nariai asymptotics.

Corollary 2 The standard Nariai solution does not have even a patch of a smooth conformal boundary.

Suppose the contrary would be the case, and a smooth future conformal boundary $\mathcal{J}^+$, not necessarily compact, would exist. Since the Nariai solution is a cosmological solution of Eq. (1), $\mathcal{J}^+$ would be spacelike [8]. We cannot argue that $\mathcal{J}^+$ would be a Cauchy surface, as we did in the proof of the previous lemma, because $\mathcal{J}^+$ would not necessarily be compact. But, in any case, there would exist a future directed inextendible causal curve γ of (M, g) which could be extended in the conformally extended spacetime $(\tilde{M}, \tilde{g})$ to hit $\mathcal{J}^+$ in the point p. Let U be a neighborhood of p in $\mathcal{J}^+$ homeomorphic to a 3-ball. There would be a neighborhood of U in $(\tilde{M}, \tilde{g})$ with a Gaussian time function as in the proof of the previous lemma. If U is chosen small enough, the level sets of the time function sufficiently close to $\mathcal{J}^+$ in its past would be spacelike and hence could be extended to Cauchy surfaces of (M, g). The same argument as before would lead to the existence of a Cauchy surface Σ such that intersection of the causal past of γ with Σ would be contained in a 3-ball. However, this is in contradiction to Ringström's statement, discussed above, that no Cauchy surface and causal curve with this property exist in the Nariai solution.

To our knowledge, this is the first rigorous proof of this fundamental fact for the Nariai solution in the literature, although it has been often claimed before, for instance in [5]. This proof relies on Ringström's statement, which might be wrong for the universal cover of the Nariai solution as mentioned before. We have not yet worked this out but we conjecture that nevertheless our Corollary is true for the universal cover. A way of proving this would be to study the behavior of the Weyl tensor along null geodesics.

3 Generalized Nariai spacetimes

3.1 General properties

Consider the Lorentzian manifold (M, g) with

$$g = \frac{1}{\Lambda}\Big(-dt^2 + \Phi(t)^2\, d\rho^2 + g_{\mathbb{S}^2}\Big) \tag{3a}$$

with

$$\Phi(t) = \Phi_0 \cosh t + \Phi_0' \sinh t \tag{3b}$$

for arbitrary $\Lambda > 0$, $\Phi_0 > 0$ and $\Phi_0' \in \mathbb{R}$ and with

$$M = I \times (\mathbb{S}^1 \times \mathbb{S}^2).$$

Here I is the unique maximal connected open interval in $\mathbb{R}$ such that $0 \in I$ and $\Phi|_I > 0$. Note that Φ can change its sign at most once; see below. We call this family of spacetimes generalized Nariai spacetimes. Clearly for $\Phi_0 = 1$ and $\Phi_0' = 0$, we recover the standard Nariai solution. All generalized Nariai spacetimes are analytic solutions of the vacuum Einstein's field equations with a cosmological constant Λ. They are globally hyperbolic, but not necessarily causal geodesically complete; as we discuss further now. They are spatially homogeneous and of Kantowski-Sachs type. Also here we will also often consider the universal cover. This is the simply connected Lorentzian manifold $(\tilde{M}, g)$ with $\tilde{M} = \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2)$ and the metric g above.

3.2 Global behavior

Let us define

$$\sigma_0 := \Phi_0^2 - (\Phi_0')^2.$$

One finds easily that Φ defined in Eq. (3) has a real zero if and only if $\sigma_0 < 0$. There cannot be more than one real zero. In particular, $I = \mathbb{R}$ if and only if $\sigma_0 \geq 0$. Similarly, one shows that Φ has one and only one local minimum if and only if $\sigma_0 > 0$.

Let us consider the universal cover of a generalized Nariai solution in order to avoid problems with identifications on the $\mathbb{S}^1$-factor in the following arguments. Furthermore, let us fix once and for all a value of Λ. The discussion above implies that if $\sigma_0 > 0$, we can shift the time coordinate t so that $t = 0$ is the time of the minimum of Φ and hence $\Phi_0' = 0$ and $\Phi_0 > 0$. Then, by rescaling the ρ-coordinate we can achieve that $\Phi_0 = 1$. Hence the universal cover of any generalized Nariai solutions with $\sigma_0 > 0$ is isometric to the universal cover of the standard Nariai solution with $\sigma_0 = 1$. The isometry involved here is analytic. For $\sigma_0 = 0$, Φ has neither a zero nor a local extremum, and $\Phi_0' = \pm\Phi_0$. By a change of time direction and rescaling of the ρ-coordinate, we can always arrange that $\Phi_0' = \Phi_0 = 1$. Hence, the universal cover of any generalized Nariai solution with $\sigma_0 = 0$ is isometric to the universal cover of the generalized Nariai solution with $\Phi_0' = \Phi_0 = 1$. The isometry is again analytic. Finally, consider the universal cover of a generalized Nariai solutions with $\sigma_0 < 0$. By a rescaling of the ρ-coordinate, we can achieve that $\sigma_0 = -1$. Now, since $\Phi : I \to \mathbb{R}^{>0}$ is surjective, there is a time translation which yields that for instance $\Phi_0 = (e^2 - 1)/2e$; note that σ_0 is invariant under time translation. Since $\sigma_0 = -1$, we can transform $t \mapsto -t$ if necessary in order to get $\Phi_0' = (e^2 + 1)/2e$. Thus the universal cover of any generalized Nariai solutions with $\sigma_0 < 0$ is isometric to the universal cover of the generalized Nariai solution given by $\Phi_0 = (e^2 - 1)/2e$, $\Phi_0' = (e^2 + 1)/2e$, and the isometry is analytic.

Hence, we have identified three analytic isometry classes of generalized Nariai solutions given by the sign of the constant σ_0. Are they distinct, or related for instance to the standard Nariai solution in some way? For the case $\sigma_0 > 0$ we have already shown that the universal cover of the corresponding generalized Nariai solution is globally isometric to the universal cover of the standard Nariai solution. Let us consider the case $\sigma_0 < 0$. Without loss of generality we can assume that $\Phi_0 = (e^2-1)/2e$, $\Phi_0' = (e^2+1)/2e$ and hence that $I =]-1, \infty[$. Let us denote the universal cover of this generalized Nariai solution by $(\hat{M}, \hat{g})$ and its coordinates by $(\hat{t}, \hat{\rho}, \hat{\theta}, \hat{\phi})$. The universal cover of the standard Nariai solution is referred to as (M, g) with coordinates (t, ρ, θ, ϕ). Consider the map

$$\Psi_- : \hat{M} \to M, \quad (\hat{t}, \hat{\rho}, p) \mapsto (t(\hat{t}, \hat{\rho}), \rho(\hat{t}, \hat{\rho}), p)$$

where p denotes an arbitrary point on $\mathbb{S}^2$ and where

$$\begin{aligned} t(\hat{t}, \hat{\rho}) &= \operatorname{arcsinh}\left(\sinh(\hat{t}+1)\sqrt{\sinh^2\hat{\rho}+1}\right), \\ \rho(\hat{t}, \hat{\rho}) &= \arcsin\left(\frac{\sinh(\hat{t}+1)\sinh\hat{\rho}}{\sqrt{\left(\sinh^2\hat{\rho}+1\right)\sinh^2(\hat{t}+1)+1}}\right) + \pi. \end{aligned}$$

One can check that the map Ψ_- is an analytic embedding and, considered as a map onto its image, an isometry. It is obvious that the same statement would be true if we added other constants to $\rho(\hat{t}, \hat{\rho})$. Before we discuss this map further, let us continue with the case $\sigma_0 = 0$. Let us use the same notation as before in this case for the map

$$\Psi_0 : \hat{M} \to M.$$

When we set

$$t(\hat{t}, \hat{\rho}) = \operatorname{arcsinh}\left(\frac{e^{\hat{t}}\left(\hat{\rho}^2+1\right) - e^{-\hat{t}}}{2}\right)$$

$$\rho(\hat{t}, \hat{\rho}) = \begin{cases} -\arccos\left(\frac{e^{-\hat{t}} - e^{\hat{t}}\left(\hat{\rho}^2+1\right) + 2e^{\hat{t}}}{\sqrt{\left(e^{\hat{t}}(\hat{\rho}^2+1) - e^{-\hat{t}}\right)^2 + 4}}\right) & \hat{\rho} < 0, \\ 0 & \hat{\rho} = 0 \\ \arccos\left(\frac{e^{-\hat{t}} - e^{\hat{t}}\left(\hat{\rho}^2+1\right) + 2e^{\hat{t}}}{\sqrt{\left(e^{\hat{t}}(\hat{\rho}^2+1) - e^{-\hat{t}}\right)^2 + 4}}\right) & \hat{\rho} > 0, \end{cases},$$

we find that Ψ_0 is also an analytic isometric embedding.

Hence we have shown for the universal covers, that all generalized Nariai solutions can be embedded isometrically into the standard Nariai solution. We visualize this fact in Fig. 1. In this figure, the horizontal axis represents the ρ-coordinate and the vertical axis the coordinate τ of the universal cover of the standard Nariai solution where

$$\tau := 2\arctan(\tanh(t/2)).$$

This compactified time coordinate has the property that $t \to \pm\infty$ corresponds to $\tau \to \pm\pi/2$. Each point in this diagram determines a 2-sphere in the standard Nariai solution.

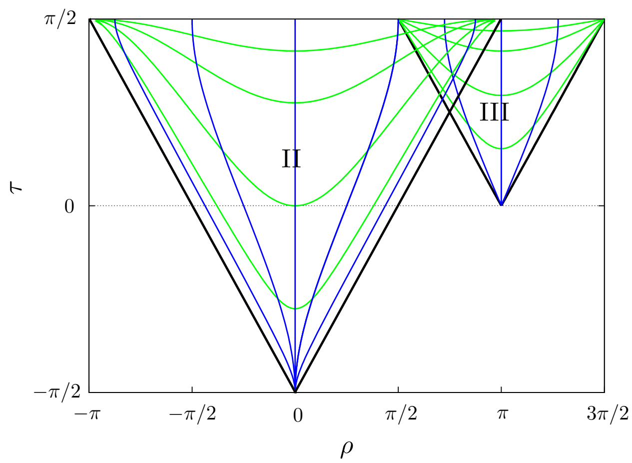

Fig. 1: Embeddings of generalized Nariai solutions into the standard one

In these coordinates, null curves which are constant on the $\mathbb{S}^2$-factor are straight lines and their angles have the Minkowski value. Hence the picture appears like a Penrose diagram. However, note that this would not be the case if we considered general null curves. For the case $\sigma_0 > 0$, the corresponding generalized Nariai solution is isometric globally to the standard Nariai solution. For $\sigma_0 = 0$, the region covered by the embedding Ψ_0 is given by the region II in Fig. 1 for $\Phi_0' > 0$ which we can always assume. The green curves represent $\hat{t} = const$-surfaces and the blue curves $\hat{\rho} = const$-surfaces. The black lines correspond to the limit $\hat{t} \to -\infty$. For the case $\sigma_0 < 0$ and $\Phi_0' > 0$, the region III is covered by the map Ψ_-. Here the black lines correspond to the curves $\hat{t} \to -1$ for our standard choice of parameters.

The above implies that the maximal analytic extensions of the universal covers of all generalized Nariai solutions are isometric to the universal cover of the standard Nariai solution. Our discussion furthermore proves that for $\sigma_0 \leq 0$, all generalized Nariai solutions are causal geodesically incomplete. Finally, after a change of time direction if necessary, all generalized Nariai solutions have future Nariai asymptotics.

All these statements hold for the universal covers and yield globally defined isometric embeddings. Returning to the original manifolds, we see that the maps corresponding to Ψ_0 and Ψ_- are not smooth globally for $\sigma_0 \leq 0$, however are still local isometries.

Note that there is an analogy between the properties of generalized Nariai solutions with their 2-dimensional de-Sitter-like factor similar to (M_1, g_1) and the properties of various forms of the, say, 4-dimensional de-Sitter spacetime as presented for instance in [12].

Since we have shown that all generalized Nariai solutions are "basically the same" at least locally, it is natural to ask why study these spaces at all? One reason is that the instability properties are different for the three isometry classes, as we see below.

4 Instability of Nariai asymptotics within the Kantowski-Sachs family

4.1 Ansatz and general solution

Above, we motivated why we would like to study perturbations of generalized Nariai solutions within the spatially homogeneous class of solutions with spatial $\mathbb{S}^1 \times \mathbb{S}^2$-topology, i.e. within the Kantowski-Sachs family. Our approach is now to first write the general solution of the field equations, second to identify the family of generalized Nariai solutions in this family, and third to study the properties of small perturbations of the Nariai solutions in the spatially homogeneous family of solutions.

More precisely, we consider the class of spacetimes with a manifold given by $M = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$ and a metric g which is invariant under a global smooth effective action of the Kantowski-Sachs group, whose orbits are spacelike Cauchy surfaces. This allows to write the metric in the form

$$g = \frac{1}{\Lambda}\left(-dt^2 + \bar{F}(t)d\rho^2 + \bar{G}(t)g_{\mathbb{S}^2}\right) \tag{4}$$

with similar conventions as before. We factor out $1/\Lambda$ in order that all quantities in the bracket be dimensionless. Recall that, in geometric units, Λ has the dimension Length^{-2} and that the coordinate components of g have dimension Length^2 for dimensionless coordinates. In fact, we will always assume $\Lambda = 1$ in the following. A metric g, which is a solution of Eq. (1) with $\Lambda = 1$, yields a metric $\hat{g} = g/\hat{\Lambda}$ for arbitrary $\hat{\Lambda} > 0$ which solves Eq. (1) with $\Lambda = \hat{\Lambda}$. This means that we can set $\Lambda = 1$, and hence restrict to exclusively dimensionless quantities in the following without loss of information.

The metric given by Eq. (4) is well-defined as long as $\bar{F}$ and $\bar{G}$ are positive. However, in this representation, it is difficult to discuss what happens geometrically when $\bar{F}$ or $\bar{G}$ become zero. This is the reason why we prefer to work with "Eddington-Finkelstein"-like coordinates (s, μ, θ, ϕ) on $\mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$, related to the above Gaussian coordinates by

$$\sqrt{\bar{F}(t)}dt = ds, \quad d\rho = 1/\bar{F}(t)\, ds + d\mu,$$

as long as $\bar{F}$ is positive. In these coordinate the metric takes the form

$$g = 2dsd\mu + F(s)d\mu^2 + G(s)g_{\mathbb{S}^2}$$

where

$$F(s) = \bar{F}(t(s)), \quad G(s) = \bar{G}(t(s)).$$

Note that s is a null coordinate. The symmetry hypersurfaces are the $t = const$-, or equivalently the $s = const$-surfaces. As before for the coordinate ρ, the function μ is the coordinate on the $\mathbb{S}^1$-factor, generated by the translation subgroup of the Kantowski-Sachs symmetry group. The motivation for this choice of coordinates is that the metric is well-behaved when F changes sign, the curvature is bounded as long as F is smooth and $G > 0$, and hence the metric can be extended where the symmetry hypersurfaces change their causal character.

Let us assume in the following that the initial hypersurface for the initial value problem of Eq. (1) under these assumptions is given by $s = 0$ and is spacelike, i.e. $F(0), G(0) > 0$. It is clear that an s-neighborhood of the initial hypersurface is globally hyperbolic and that the $s = const$-hypersurfaces are spacelike compact Cauchy surfaces. One easily computes the mean curvature of the $s = const$-hypersurfaces

$$H(s) = \frac{1}{6}\left(\frac{1}{\sqrt{F(s)}}\frac{dF}{ds} + 2\frac{\sqrt{F(s)}}{G(s)}\frac{dG}{ds}\right), \tag{5}$$

which holds as long as the surface is spacelike.

In the coordinates (s, μ, θ, ϕ), Einstein's field equations in vacuum with $\Lambda = 1$ reduce to

$$0 = \ddot{G}(s) - \frac{\dot{G}^2(s)}{2G(s)} \tag{6a}$$

$$0 = \ddot{F}(s) + \frac{\dot{F}(s)\dot{G}(s)}{G(s)} - 2 \tag{6b}$$

$$0 = 1 - \frac{1}{G(s)} - \frac{\dot{F}(s)\dot{G}(s)}{2G(s)} - \frac{F(s)\dot{G}^2(s)}{4G^2(s)}, \tag{6c}$$

where a dot represents derivatives with respect to s. The initial value problem for these equations is well-posed under our assumptions, and for given data on the $s =$ 0-hypersurface satisfying the constraint Eq. (6c), it has a unique solution maximally extended in the coordinates (s, μ, θ, ϕ). Indeed, all solutions can be written explicitly as

$$F(s) = F_* + \dot{F}_* s \frac{2}{H_*^{(0)} s + 2} + s^2 \frac{H_*^{(0)} s + 6}{3H_*^{(0)} s + 6},$$

$$G(s) = \frac{G_*}{4}(H_*^{(0)} s + 2)^2,$$

with constants $G_* > 0$, $F_* > 0$, $\dot{F}_*$ and $H_*^{(0)}$ corresponding to the data of F and G on the initial hypersurface by

$$F_* = F(0), \quad G_* = G(0), \quad \dot{F}_* = (\partial_s F)(0), \quad H_*^{(0)} = (\partial_s G)(0)/G(0).$$

4.2 Characterization of the solutions

Consider arbitrary data consistent with the constraint with $H_*^{(0)} = 0$. The corresponding solution – denoted (M, g) in the following – is

$$F(s) = F_* + \dot{F}_* s + s^2, \quad G(s) = 1,$$

with $F_* > 0$ and $\dot{F}_*$ arbitrary. We find that (M, g) is an analytic extension of the generalized Nariai solution – denoted by $(\tilde{M}, \tilde{g})$ for now – given by

$$\Phi_0 = \sqrt{F_*}, \quad \Phi_0' = \dot{F}_*/2. \tag{7}$$

The corresponding analytic isometric embedding $\tilde{M} \to M$ is given by

$$s(t) = (\partial_t \Phi)(t) - \Phi_0',$$
$$\mu(\rho, t) = \rho - \begin{cases} \frac{1}{\sqrt{\sigma_0}} \left(\arctan \frac{(\partial_t \Phi)(t)}{\sqrt{\sigma_0}} - \arctan \frac{\Phi_0'}{\sqrt{\sigma_0}} \right) & \sigma_0 > 0, \\ -\frac{1}{(\partial_t \Phi)(t)} + \frac{1}{\Phi_0'} & \sigma_0 = 0, \\ -\frac{1}{\sqrt{|\sigma_0|}} \left(\operatorname{arctanh} \frac{(\partial_t \Phi)(t)}{\sqrt{\sigma_0}} - \operatorname{arctanh} \frac{\Phi_0'}{\sqrt{\sigma_0}} \right) & \sigma_0 < 0. \end{cases}$$

This is why we call data for the Kantowski-Sachs initial value problem with $H_*^{(0)} = 0$ Nariai data. With this we find the following further aspects for generalized Nariai solutions. In these discussions we identify $(\tilde{M}, \tilde{g})$ with the image of the isometric embedding in (M, g).

(i) If $\sigma_0 > 0$ for $(\tilde{M}, \tilde{g})$, then (M, g) and $(\tilde{M}, \tilde{g})$ are equal, and both correspond to the maximal globally hyperbolic development of Nariai data. This follows from geodesic completeness.

(ii) If $\sigma_0 < 0$ for $(\tilde{M}, \tilde{g})$, then $(\tilde{M}, \tilde{g})$ is only defined for $t \in I \neq \mathbb{R}$, and (M, g) is an extension of $(\tilde{M}, \tilde{g})$. From Eq. (5), we see that the mean curvature of the $s = \mathit{const}$-surfaces, which are Cauchy surfaces of $(\tilde{M}, \tilde{g})$, blows up exactly at the boundary of the embedding. This implies that the solution cannot be extended as globally hyperbolic spacetime. We conclude that $(\tilde{M}, \tilde{g})$ is the maximal globally hyperbolic development of Nariai data. However, since the curvature is bounded, it is extendible and the boundary of $(\tilde{M}, \tilde{g})$ in (M, g) is a Cauchy horizon of topology $\mathbb{S}^1 \times \mathbb{S}^2$, generated by closed null curves.

(iii) If $\sigma_0 = 0$ for $(\tilde{M}, \tilde{g})$, then $(\tilde{M}, \tilde{g})$ is defined for all $t \in \mathbb{R}$. After a change of time direction, if necessary, we have that

$$\lim_{t \to -\infty} s(t) = -\Phi_0', \quad \lim_{t \to +\infty} s(t) = +\infty.$$

Hence, (M, g) is an extension of $(\tilde{M}, \tilde{g})$. In this case, the mean curvature of the $s = \mathit{const}$-hypersurfaces stays bounded in the limit $t \to -\infty$. Based on these simple arguments it is not possible to conclude that $(\tilde{M}, \tilde{g})$ is the maximal globally hyperbolic development of Nariai data with $\sigma_0 = 0$, because we cannot exclude the possibility that there exist other extensions which are globally hyperbolic. We will not discuss this issue further here, since, as far as we can see at the moment, the case $\sigma_0 = 0$ is a borderline case of no particular interest for us.

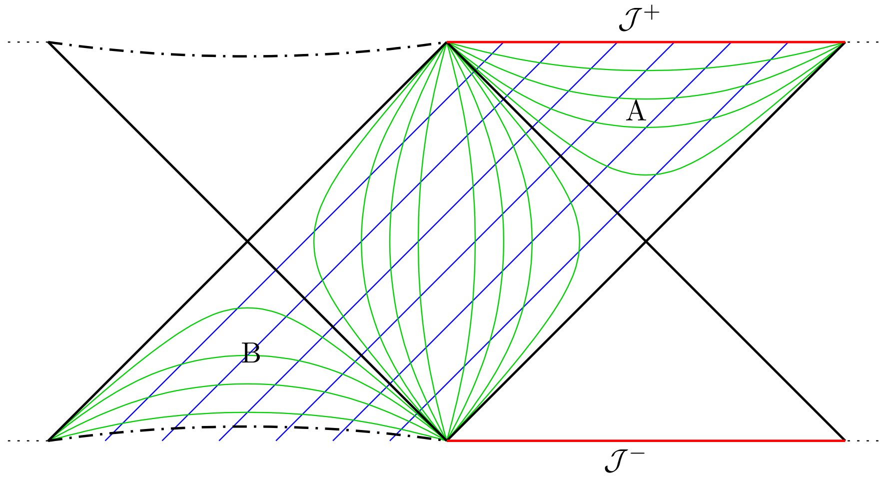

Fig. 2: Penrose diagram of Schwarzschild-de-Sitter for $0 < M < 1/3$ and $\Lambda = 1$

Now let us consider the "generic" case $H_*^{(0)} \neq 0$. Then we can introduce the following new coordinates

$$\hat{s} = \frac{\sqrt{G_*}}{2}(H_*^{(0)} s + 2), \quad \hat{\mu} = \frac{2}{H_*^{(0)}\sqrt{G_*}}\,\mu. \tag{8}$$

In these coordinates, the metric becomes

$$g = 2d\hat{s}d\hat{\mu} + \underbrace{\frac{(H_*^{(0)})^2 G_*}{4} F\left((2\hat{s}/\sqrt{G_*} - 2)/H_*^{(0)}\right)}_{=:\hat{F}(\hat{s})} d\hat{\mu}^2 + \hat{s}^2 g_{\mathbb{S}^2}.$$

After straight forward computations, substituting F_* by means of the constraint, we find

$$\hat{F}(\hat{s}) = -1 + \frac{2S}{\hat{s}} + \frac{1}{3}\hat{s}^2 \tag{9a}$$

with

$$S := G_*^{3/2}\left(\frac{1}{3} - \frac{1}{4}\dot{F}_* H_*^{(0)}\right). \tag{9b}$$

Hence we have shown that the Kantowski-Sachs solutions with $H_*^{(0)} \neq 0$ yield the Schwarzschild-de-Sitter solution with mass S in Eddington-Finkelstein coordinates.

We remark that all the results here are consistent with the Birkhoff theorem for $\Lambda > 0$ proved in [22].

4.3 Perturbations of Nariai within the Kantowski-Sachs family

So far we have shown that all Kantowski-Sachs solutions of our initial value problem with $H_*^{(0)} \neq 0$ are locally isometric to subsets of Schwarzschild-de-Sitter solutions with mass S given by Eq. (9), while for $H_*^{(0)} = 0$, we get the generalized Nariai solutions. In order to simplify the discussion again, we will often go to the universal covers in the following without further comment. The corresponding subsets of Schwarzschild-de-Sitter can be deduced from Fig. 2, which shows the Penrose diagram of the Schwarzschild-de-Sitter solution with mass $0 < S < 1/3$ as an example. Black bold lines denote the event and cosmological horizons, dashed black line refer to the singularity, bold red lines mark the conformal boundary $\mathcal{J}$. Further, green lines represent $s = const$-hypersurfaces and blue lines $\mu = const$-hypersurfaces. Precisely, in regions A and B, the $s = const$-surfaces are spacelike. The cosmological horizon and the event horizon become Cauchy horizons for the Cauchy development of any $s = const$-hypersurface in the regions A and B.

We do not want to give further information, neither locally nor globally, for the general case in this family. This is done elsewhere [12]. Rather, we want to restrict to the case given by $H_*^{(0)} \neq 0$, $|H_*^{(0)}| \ll 1$. Such a case we interpret as a small perturbation of some generalized Nariai solution. More precisely, let us fix a generalized Nariai solution (M, g) either by means of the parameters $\Phi_0 > 0$, $\Phi_0' \in \mathbb{R}$ or by $F_* > 0$, $\dot{F}_* \in \mathbb{R}$, $H_*^{(0)} = 0$ related by Eq. (7). Now, leave all these parameters fixed except for $H_*^{(0)}$ which we give some small but non-vanishing value. We denote the corresponding Kantowski-Sachs solution by $(\tilde{M}, \tilde{g})$ and call it the perturbation of (M, g).

For sufficiently small $|H_*^{(0)}| > 0$, we find that $(\tilde{M}, \tilde{g})$ is a Schwarzschild-de-Sitter solution, in the above sense with mass

$$S = \frac{1}{3} + \frac{\sigma_0}{8}(H_*^{(0)})^2 - \Phi_0' \frac{2(\Phi_0')^2 - 3\sigma_0}{24}(H_*^{(0)})^3 + O((H_*^{(0)})^4).$$

Note that these and some of the following formulas have been derived before, cf. [11], but under more special assumptions and not from this point of view. In particular, the authors there restrict to the standard Nariai solution. In any case, for sufficiently small $|H_*^{(0)}| > 0$ we have the following conclusions:

(i) If $\sigma_0 > 0$ for our choice of generalized Nariai solution (M, g), then $(\tilde{M}, \tilde{g})$ is isometric to a subset of a Schwarzschild-de-Sitter solution with $M > 1/3$.

(ii) If $\sigma_0 < 0$ for (M, g), then $(\tilde{M}, \tilde{g})$ is isometric to a subset of a Schwarzschild-de-Sitter solution with $0 < M < 1/3$.

(iii) If $\sigma_0 = 0$ for (M, g), then $(\tilde{M}, \tilde{g})$ has the following properties

  (a) If $\Phi_0' H_*^{(0)} > 0$: isometric to a subset of a Schwarzschild-de-Sitter solution with $0 < M < 1/3$.

  (b) If $\Phi_0' H_*^{(0)} < 0$: isometric to a subset of a Schwarzschild-de-Sitter solution with $M > 1/3$.

  (c) The case $\Phi_0' H_*^{(0)} = 0$ can be excluded.

Hence we see here explicitly the interesting fact that the three classes of generalized Nariai solutions, which we have shown to be so closely related, yield quite different perturbations.

Furthermore, we find for $(\tilde{M},\tilde{g})$ for given finite s and μ and sufficiently small $|H_*^{(0)}| > 0$,

$$\hat{s} = 1 + \frac{\Phi_0' + s}{2} H_*^{(0)} + O((H_*^{(0)})^2),$$
$$\hat{\mu} = \mu \left(\frac{2}{H_*^{(0)}} - \Phi_0' - \frac{2(\Phi_0') + \sigma_0}{4} H_*^{(0)} + O((H_*^{(0)})^2) \right).$$

Note that this suggests that generalized Nariai solutions can be considered as singular limits of Schwarzschild-de-Sitter solutions with masses as discussed above. Namely, in the limit $H_*^{(0)} \to 0$, we have

$$M \to 1/3, \quad \hat{s} \to 1,$$

and further, for $H_*^{(0)} \searrow 0$, one gets

$$\hat{\mu} \to \begin{cases} -\infty & \mu < 0 \\ 0 & \mu = 0 \\ \infty & \mu > 0. \end{cases}$$

For $H_*^{(0)} \nearrow 0$, the signs for the limits of $\hat{\mu}$ turn around. We will not consider this fact further, but see for instance [11, 10]. However, note that μ and s have to be finite in order to give sense to these expansions, and thus they do not yield new interesting information about the asymptotics of the generalized Nariai solution given by $H_*^{(0)} \to 0$. This fact and the fact that the limit for $\hat{\mu}$ is singular in this "Nariai limit of the Schwarzschild-de-Sitter class" is apparent, but often not pointed out clearly in the literature. For instance, the author of [5] is led to the statement that the Nariai solution is "the largest possible black hole in de Sitter space", i.e. a Schwarzschild-de-Sitter solution with mass $M = 1/3$, by the considerations in [11]. This can be misleading, at least from our point of view.

In order to compare the global properties of $(\tilde{M},\tilde{g})$ with those of (M,g) let us restrict to the case $\sigma_0 < 0$. This is not the standard case in the literature, but will play a particular role in our following works, for instance [4]. Let us consider Fig. 2 again for the Penrose diagram of the Schwarzschild-de-Sitter solution with $0 < M < 1/3$. We can change the time direction such that $\Phi_0' > 0$ (note that $\Phi_0' = 0$ is not allowed in this class) for the chosen (M,g). Now let $H_*^{(0)} > 0$ be sufficiently small. In this case, the initial hypersurface of our Kantowski-Sachs initial value problem given by $s = 0$ corresponds to $\hat{s} = \hat{s}_0 > 1$. From the results derived above and the well-known root structure of $\hat{F}$, it follows that the initial hypersurface must correspond to a $\hat{s} = const$-Cauchy surface of region A in Fig. 2. Hence, we can say the following in this case:

- In the future time direction, while the unperturbed Nariai solution (M,g) has Nariai asymptotics, the perturbation $(\tilde{M},\tilde{g})$ has a smooth future conformal boundary $\mathcal{J}^+$. In other words, while for the original solution the volume of the $\mathbb{S}^2$-factor is constant in time, an arbitrarily small initial positive expansion of the $\mathbb{S}^2$-factor determined by a small $H_*^{(0)} > 0$ leads to accelerated expansion of the $\mathbb{S}^2$-factor.

Together with the accelerated expansion of the $\mathbb{S}^1$-factor present also in the unperturbed solution, this yields the existence of a smooth future conformal boundary $\mathcal{J}^+$.

- In the past time direction, both (M, g) and the perturbation $(\tilde{M}, \tilde{g})$ develop a Cauchy horizon, which in the case of the perturbation corresponds to the cosmological horizon. Thus the maximal globally hyperbolic extension of the data on the $s = 0$-hypersurface is extendible.

The case $H_*^{(0)} < 0$ can be discussed similarly. The relevant region in Fig. 2 is now marked as region B, but note that we have to go backwards with respect to the Schwarzschild-de-Sitter time due to the definition of $\hat{s}$ in Eq. (8). For the perturbation $(\tilde{M}, \tilde{g})$ the initial expansion of the $\mathbb{S}^2$-factor is now negative and the result is that the future Nariai asymptotics of (M, g) turn into a curvature singularity for the perturbation $(\tilde{M}, \tilde{g})$. The singularity is of cigar type according to the classification in [23] because the volume of the $\mathbb{S}^1$-factor becomes infinite at the singularity. In the past, both (M, g) and $(\tilde{M}, \tilde{g})$ develop a Cauchy horizon, which in the case of the perturbation corresponds to the event horizon. By means of similar arguments, the cases $\sigma_0 \geq 0$ can also be studied.

5 Summary and outlook

We have introduced and discussed the family of generalized Nariai solutions in this paper. We have listed known aspects and wrote them in a consistent way for our subsequent work starting with [4]. We have complemented the known results with new ones; for instance the first rigorous proof of the fact that the Nariai solutions do not possess smooth conformal boundaries. We introduced the notion of Nariai asymptotics and discussed its relation to the standard cosmic no-hair picture. This suggests that Nariai asymptotics should be a non-generic phenomenon for solutions of Einstein's field equations in vacuum with a positive cosmological constant. This is why we have discussed the instability of Nariai asymptotics within the spatially homogeneous class of solutions, as the simplest case. This discussion is not completely new in principle. The explicit solution of the field equations in this case is well known, and the classification of these solutions, at least locally, has been given before by proving a Birkhoff theorem in [22]. However, we have successfully reconsidered these results and interpreted them in terms of our fundamental question of interest. In the spatially homogeneous case we were able to make the statement that Nariai asymptotics is instable precise. Namely, under arbitrary small spatially homogeneous perturbations of any Nariai data with, say, future Nariai asymptotics, the corresponding solution either develops a smooth $\mathcal{J}^+$– and hence is consistent with the cosmic no-hair picture – or collapses to a curvature singularity, in the future. In the past time direction, such perturbations never have Nariai asymptotics. A particular new outcome of our investigations is that the precise instability properties are different for the three classes of generalized Nariai solutions and can be characterized by means of the regimes of the Schwarzschild-de-Sitter solutions determined by the mass parameter.

The spatially homogeneous case is clearly special. In principle we would be interested in generic perturbations of Nariai solutions without any symmetries. However, this seems hopeless in practice. A systematic approach would be to reduce the symmetry step by step. In the spatially homogeneous case for spatial topology $\mathbb{S}^1 \times \mathbb{S}^2$ we have four spatial Killing vector fields in general. Hence, it would make sense to proceed with the spherically symmetric case with three spatial Killing vector fields in general; i.e. we would give up the homogeneity along the $\mathbb{S}^1$-factor of the manifold. Much is known about this case locally, for instance by the Birkhoff theorem mentioned above. But to our knowledge, there are no rigorous results of global nature which be relevant to the question under consideration here. Several heuristic results exist, however, for this case as summarized in [5].

We have decided not to proceed with the spherically symmetric case now, but rather with the Gowdy symmetric case, for which one has two spatial Killing vector fields. It is conceivable that it is very difficult to handle this class analytically. One reason is the presence of symmetry axes. Indeed, there are several fundamental open problems in the class of Gowdy symmetric solutions with spatial $\mathbb{S}^1 \times \mathbb{S}^2$-topology which are related to our question of interest, for instance the issue of strong cosmic censorship. This is one motivation to choose this class. A second motivation is that we have already developed numerical techniques which can be applied to the $\mathbb{S}^1 \times \mathbb{S}^2$-Gowdy class of solutions in [3]. A description of our approach and first steps in the investigation of the question of the instability of Nariai asymptotics within the Gowdy class are presented in [4].

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