THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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S. B. Edgar, A. Garcia-Parrado Gómez-Lobo and J. M. Martin-Garcia

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Petrov D vacuum spaces revisited: Identities and Invariant Classification

S. Brian Edgar[1], **Alfonso García-Parrado Gómez-Lobo**[1] **and José M. Martín-García**[1,2,3]

[1] Matematiska institutionen, Linköpings Universitet, SE-581 83 Linköping, Sweden.

[2] Laboratoire Univers et Théories, Observatoire de Paris, CNRS, Univ. Paris Diderot, 5 place Jules Janssen, 92190 Meudon, France.

[3] Institut d'Astrophysique de Paris, Univ. Pierre et Marie Curie, CNRS, 98bis boulevard Arago, 75014 Paris, France.

E-mail: `bredg@mai.liu.se`, `algar@mai.liu.se`, `Jose.Martin-Garcia@obspm.fr`

Abstract. For Petrov D vacuum spaces, two simple identities are rederived and some new identities are obtained, in a manageable form, by a systematic and transparent analysis using the GHP formalism. This gives a complete involutive set of tables for the four GHP derivatives on each of the four GHP spin coefficients and the one Weyl tensor component. It follows directly from these results that the theoretical upper bound on the order of covariant differentiation of the Riemann tensor required for a Karlhede classification of these spaces is reduced to two.

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1. Introduction

In his pioneering work in the NP formalism, which provided definitive results on vacuum Type D spacetimes [18], surprisingly Kinnersley did not exploit, and in fact seems to have been unaware of, two simple NP identities for these spacetimes

$$I_1 \equiv \pi\bar{\pi} - \tau\bar{\tau} = 0 = \rho\bar{\mu} - \bar{\rho}\mu \equiv I_2 \qquad (1)$$

(In the usual NP notation [21], the two tetrad vectors l, n are principal null vectors of the Weyl tensor.) More surprisingly, it seems that it took longer than another decade before these identities were noticed; and, even more surprisingly, they were obtained by chance — for the vacuum case. Debever and McLenaghan [8] had discovered, as the result of a simple calculation, for Type D *electrovac* solutions that

$$\Phi(\pi\bar{\pi} - \tau\bar{\tau}) = 0 = \Phi(\rho\bar{\mu} - \bar{\rho}\mu) \qquad (2)$$

from which (1) immediately follows — *but only for* $\Phi \neq 0$; this led them to investigate if these identities were also true for the vacuum case, and they confirmed the validity of identities (1) for the *vacuum* case — by a direct calculation *using the explicit metrics* given by Kinnersley. Subsequently, Czapor and McLenaghan [6] established (1) for Type D *vacuum* spacetimes — without integrating the equations, but by a much more complicated and much longer calculation than for the electrovac case; the calculation was only achieved with computer support. Later these authors used this calculation as an example of their Maple Package for calculations in the NP formalism [7]. They pointed out that the presentation of some of the intermediate expressions used in their calculations required pages, and that some of the steps 'would be virtually impossible by hand'; furthermore, they stressed that 'even aided by computer, this calculation is far from automatic'. Their calculation was built on the unusual tetrad gauge choice $\pi = \rho$, using a programme involving repeated applications of the commutator equations, judicial substitutions and tricky factorisations.

However, the identities (1) are clearly invariant under spin and boost gauge choices, and so are valid in any gauge; the GHP formalism [14], [15], [9] which is invariant under such gauge choices, should be a more efficient tool, and it is to be expected that the results should come out without the need for such long calculations, and indeed more transparently than in the NP formalism. The GHP formalism has proved very efficient in a number of calculations involving Type D spaces, [16], [22], [10], [11],[2]; indeed, in the latter, Carminati and Vu have re-established these identities (1) as an example of their Maple Package for calculations in the GHP formalism.

These identities do not seem to be well known; for example, an investigation of the invariant classification of vacuum Type D metrics in [4, 5] did not use them in a situation where they would have been particularly useful. So the first purpose of this paper is to highlight these identities (1), as well as to draw attention to some other identities for spin coefficients, some of which we believe to be new. The second purpose is to emphasise the efficiency of the GHP formalism by obtaining all of these identities in a comparatively brief, straightforward, and systematic manner within the GHP formalism — by hand for transparency, but with computer support for accuracy. The third purpose is to reconcile the theoretical and measured upper bounds in the invariant classification for Type D vacuum spaces — without explicitly integrating the equations.

A procedure, for the invariant classification of spacetimes, which originated with Cartan [3], was developed by Brans [1], and refined by Karlhede [17], has established theoretical upper bounds on the order of the Cartan scalars — the Riemann tensor and its covariant derivatives, calculated in a particular frame. At first sight, for some Petrov types, these upper bounds can be as high as *seven*, so it is of great practical significance to find whether these values are sharp, or whether they can be lowered to a more manageable number. In [4, 5] it was shown, for Type D vacuum spacetimes, how the GHP operators can replace the covariant derivatives in the investigation of the different orders of derivatives of the Weyl tensor. By this means, the GHP formalism was exploited to lower the theoretical upper bound to three in the invariant classification of Type D spacetimes — without explicitly integrating the equations; more precisely, it was shown in [4, 5] that the theoretical upper bound was *two* for two subcases, while for a third subcase the bound was shown to be three, but a fourth subcase of Petrov D vacuum spacetimes was overlooked‡. Petrov type D vacuum is the only Petrov subclass where all spacetimes are known explicitly, and the upper bound has been calculated *directly from these metrics* to be *two* [24], [25]. In this paper we shall strengthen the result in [4, 5], and reduce the upper bound to *two* directly for the remaining subcases, by a further exploitation of the GHP formalism, but without explicitly integrating the equations; so the theoretical and measured upper bounds are now in agreement. To achieve this, we exploit a series of identities for vacuum Petrov D spaces, culminating in a complete set of tables for each spin coefficient and the one Weyl tensor component, Ψ_2. These complete tables equate the four GHP derivative operators of each of the spin coefficients and Ψ_2 to expressions built only from the spin coefficients and Ψ_2.

The paper is organised as follows: the complete tables for the spin coefficients and Ψ_2 are computed in section 2. In section 3 we build on the work of [4, 5] and show how these results enable us to derive that the Karlhede bound for the vacuum Petrov type D spacetimes is two, thus improving the result in [4, 5]. Our conclusions and a brief discussion are given in the final section. The appendix contains a classification of vacuum Petrov type D spacetimes adapted to our purposes.

The computer calculations alluded to above have been performed with *xAct* [19]. This is a suite of *Mathematica* packages with many features, including the canonicalisation of tensor expressions by means of powerful algorithms based on permutation group theory [20], and the ability of working simultaneously with several different frames.

2. Identities and complete tables

We follow the conventions and notation of [14] in our presentation. We will first write down the basic equations — the Ricci, Bianchi and commutator equations — of the GHP formalism particularised, in the usual way, for vacuum, type D spacetimes. Because of the simple structure of these spacetimes in the GHP formalism, all equations have $'$, * and $^{*\prime}$ counterparts (some of which may not be independent) but we shall not use * and $^{*\prime}$ explicitly in the presentation, although they are very useful for checking.

<u>**Nonvanishing weighted spin coefficients and curvature components**</u>

$$\mathcal{S} \equiv \{\rho, \quad \tau, \quad \rho', \quad \tau', \quad \Psi_2\}.$$

<u>**Ricci equations**</u>

$$Þ\rho = \rho^2, \tag{3}$$

$$ð\rho = \tau(\rho - \bar{\rho}), \tag{4}$$

‡ The investigation in [4, 5] relied on a result in [12] which claimed that there were only two additional subcases in addition to the generic subcase $\rho\rho'\tau\tau' \neq 0$; this ignored the null orbit possibility $\rho \neq 0, \rho' = 0$ [6], which yields a subset of the Kinnersley Case IIE metrics, which we call Class II in our Appendix.

$$Þ\tau = \rho(\tau - \bar{\tau}'), \tag{5}$$
$$ð\tau = \tau^2, \tag{6}$$
$$Þ'\rho = \rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2 + ð'\tau, \tag{7}$$

and their $'$ counterparts;

Bianchi equations

$$Þ\Psi_2 = 3\rho\Psi_2, \tag{8}$$
$$ð\Psi_2 = 3\tau\Psi_2, \tag{9}$$

and their $'$ counterparts;

Commutator equations

$$[ÞÞ' - Þ'Þ]\eta_{pq} = \Big((\bar{\tau} - \tau')ð + (\tau - \bar{\tau}')ð' - p(\Psi_2 - \tau\tau') - q(\bar{\Psi}_2 - \bar{\tau}\bar{\tau}')\Big)\eta_{pq}, \tag{10}$$
$$[Þð - ðÞ]\eta_{pq} = \Big(\bar{\rho}ð - \bar{\tau}'Þ + q\bar{\rho}\bar{\tau}'\Big)\eta_{pq}, \tag{11}$$
$$[Þð' - ð'Þ]\eta_{pq} = \Big(\rho ð' - \tau'Þ + p\rho\tau'\Big)\eta_{pq}, \tag{12}$$
$$[Þ'ð - ðÞ']\eta_{pq} = \Big(\rho'ð - \tau Þ' - p\rho'\tau\Big)\eta_{pq}, \tag{13}$$
$$[Þ'ð' - ð'Þ']\eta_{pq} = \Big(\bar{\rho}'ð' - \bar{\tau}Þ' - q\bar{\rho}'\bar{\tau}\Big)\eta_{pq}, \tag{14}$$
$$[ðð' - ð'ð]\eta_{pq} = \Big((\bar{\rho}' - \rho')Þ + (\rho - \bar{\rho})Þ' + p(\Psi_2 + \rho\rho') - q(\bar{\Psi}_2 + \bar{\rho}\bar{\rho}')\Big)\eta_{pq}, \tag{15}$$

where we have written out all commutators explicitly acting on a scalar η_{pq} of spin and boost weight $(p-q)/2,\ (p+q)/2$ respectively.

When we substitute the Bianchi equations into the commutators for Ψ_2 and also make use of the Ricci equations, in some cases we obtain the trivial identity; from the other commutators we obtain the

Post-Bianchi equations

$$\Psi_2(-Þ'\rho + Þ\rho' - \tau\bar{\tau} + \tau'\bar{\tau}') = 0,$$
$$\Psi_2(ð'\rho - Þ\tau') = 0,$$

and their $'$ counterparts; and since $\Psi_2 \neq 0$, it follows that

$$Þ'\rho = Þ\rho' - \tau\bar{\tau} + \tau'\bar{\tau}', \tag{16}$$
$$ð'\rho = Þ\tau', \tag{17}$$

and their $'$ counterparts. Both these equations are quoted in [2], but the first (16) is overlooked in [4, 5].

Applying the commutators to the spin coefficients given in (3), (4), (5), (6), yields no new information.

Note that although we now have equations with terms involving all four operators acting on ρ, τ (and hence also, by $'$ symmetry, on ρ', τ') these are not all given in separate equations. Therefore it will be convenient to introduce new symbols; we follow the notation of Carminati and Vu [2] and introduce the zero-weighted Z_1 and the $\{0,2\}$-weighted Z_2,

$$\begin{aligned} Z_1 &= \eth'\tau, \\ &= \text{þ}'\rho - \rho\bar{\rho}' + \tau\bar{\tau} + \Psi_2, \\ &= \text{þ}\rho' - \rho\bar{\rho}' + \tau'\bar{\tau}' + \Psi_2, \\ &= \eth\tau' + \bar{\rho}\rho' - \rho\bar{\rho}', \end{aligned} \tag{18}$$

and

$$\begin{aligned} Z_2 &= \eth'\rho, \\ &= \text{þ}\tau', \end{aligned} \tag{19}$$

where we have used the Ricci equations and post-Bianchi equations to obtain the different alternatives.
Note that, because of the Ricci and post-Bianchi equations,

$$Z_1' = Z_1 + \bar{\rho}'\rho - \rho'\bar{\rho}, \tag{20}$$

whereas

$$\begin{aligned} Z_2' &= \eth\rho', \\ &= \text{þ}'\tau. \end{aligned} \tag{21}$$

We will use $-Z_2'$ in place of the Z_3 used in [2]. Using the Sachs symmetry * we have the properties

$$Z_2{}^{*\prime} = Z_2, \tag{22}$$

$$Z_1{}^* = Z_1 + \rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2. \tag{23}$$

We now have a set of equations involving all four operators acting on ρ, τ and by $'$ symmetry, on ρ', τ'. These are the Ricci equations, the post-Bianchi equations and (18)-(19). Thus we can now apply each of the commutator equations to each of the four spin coefficients. In a number of cases the commutators are trivially satisfied via the Ricci equations, Bianchi equations and post-Bianchi equations. The remaining non-trivial results from the commutators are a set of first-order equations for the new symbols Z_1, Z_2:

$$\text{þ}Z_1 = \rho(2Z_1 - \bar{Z}_1 + \tau'\bar{\tau}' - \bar{\rho}\rho' + \bar{\rho}'\rho) + (\tau - \bar{\tau}')Z_2, \tag{24}$$

$$\eth Z_1 = \tau(2Z_1 + \Psi_2 + \bar{\Psi}_2 + \rho\bar{\rho}' + \bar{\rho}'\bar{\rho}) - (\bar{\rho} - \rho)Z_2' - \bar{\tau}'(\rho\bar{\rho}' - \rho\rho'), \tag{25}$$

and their $'$ counterparts; together with

$$\begin{aligned} \text{þ}Z_2 &= 3\rho Z_2, \\ \eth Z_2 &= \tau Z_2 + 2(\rho - \bar{\rho})Z_1 + \bar{\rho}\Psi_2 - \rho\bar{\Psi}_2 + 2\rho\bar{\rho}'(\rho - \bar{\rho}), \\ \eth' Z_2 &= 3\tau' Z_2, \\ \text{þ}' Z_2 &= \rho' Z_2 + \bar{\Psi}_2\tau' + \Psi_2(\bar{\tau} - 2\tau') - 2(\bar{\tau} - \tau')(Z_1 + \rho\bar{\rho}'), \end{aligned} \tag{26}$$

and their $'$ counterparts.

At this stage we have a complete and involutive set of tables for the eight complex quantities in $\mathcal{S}^{++} = \{\rho, \tau, \rho', \tau', \Psi_2, Z_1, Z_2, Z_2'\}$. This enables us to rederive and complete the result in [4, 5]. All the elements§ of $\mathcal{S}^{++}$ are present at the second order of differentiation of the Weyl tensor, and there are no new quantities introduced at the third order; moreover, there is also no change in the invariance group (including the case overlooked in [4, 5], which corresponds to Class II in the Appendix) at the third order, and therefore the Karlhede algorithm terminates at third order.

However, in the sequel, by continuing the systematic analysis, we shall obtain more identities, and then be able to strengthen this result in the next section.

The next step is to check the integrability conditions for (24)-(25); i.e., apply each of the commutator equations to Z_1 and substitute the appropriate expressions from (24)-(25). From the non-trivial results we obtain the new equation

$$\Psi_2(\tau\bar{\tau} - \tau'\bar{\tau}' + \rho\bar{\rho}' - \bar{\rho}\rho') = 0, \tag{27}$$

from which, by taking real and imaginary parts on the second factor, we get the (GHP version of the) identities (1):

$$I_1 \equiv \tau'\bar{\tau}' - \tau\bar{\tau} = 0 = \bar{\rho}\rho' - \rho\bar{\rho}' \equiv I_2. \tag{28}$$

Note that these identities are invariant under the $'$ symmetry (and also the * symmetry). These identities cause simplification in some of the post-Bianchi equations (16), and in some of the definitions (18) and (20) (where $Z_1 = Z_1'$); we will make frequent use of these simplifying identities in the sequel.

In the same way we check the integrability conditions for (26); from the non-trivial results we obtain the new equation

$$\Psi_2(Z_2 + \rho\bar{\tau} - 2\rho\tau' + \tau'\bar{\rho}) = 0, \tag{29}$$

from which we obtain Z_2,

$$Z_2 = -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho} \tag{30}$$

and hence from (19) the four new identities:

$$\begin{aligned} \eth'\rho &= -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho}, \\ þ\tau' &= -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho}, \end{aligned} \tag{31}$$

together with their $'$ counterparts.

The results so far obtained duplicate those of Carminati and Vu [2]; but we shall now continue the analysis even further.

For consistency it is now necessary to substitute the value for Z_2 into (26); making use of the Ricci equations, the post-Bianchi equations and the identities (28), yields the

§ As noted above, in [4, 5], the first of the post-Bianchi equations (16) was overlooked; this is why in that paper the corresponding expressions for $D^2\Psi$ and $D^3\Psi$ contain *nine* variables: there is an additional variable ($þ\rho'$), which corresponds to Z_1' in our notation, but which we know is related to Z_1 because of (16).

following linear equations for Z_1, $\bar{Z}_1$

$$\bar{\tau}' Z_1 - \tau \bar{Z}_1 = \tau\tau'\bar{\tau}' - \tau'\bar{\tau}'^2, \tag{32}$$

$$\bar{\rho}' Z_1 - \rho' \bar{Z}_1 = 2\rho'\bar{\rho}'(\rho - \bar{\rho}) + \tau'\bar{\tau}'(\bar{\rho}' - \rho') + \bar{\rho}'\Psi_2 - \rho'\bar{\Psi}_2, \tag{33}$$

together with the $'$ counterpart of (32) (which is simply its complex conjugate), and the $'$ counterpart of (33) given by

$$\bar{\rho} Z_1 - \rho \bar{Z}_1 = 2\rho\bar{\rho}(\rho' - \bar{\rho}') + \tau\bar{\tau}(\bar{\rho} - \rho) + \bar{\rho}\Psi_2 - \rho\bar{\Psi}_2, \tag{34}$$

Note that (33) and (34) are not independent when $\rho' \neq 0 \neq \rho$, but in the cases $\rho' = 0$ or $\rho = 0$ they may provide different information.
In addition, it is also necessary to apply each of the four operators to the identities (28) (in practice, only one of the operators needs to be checked because of symmetry); the result is a duplication of the existing information.

At this stage we have a complete and involutive set of tables for the six complex quantities in $\mathcal{S}^+ = \{\rho, \tau, \rho', \tau', \Psi_2, Z_1\}$. However, this does not permit us yet to lower the Karlhede bound; in order to do that we need to be able to substitute Z_1 in terms exclusively of the elements in $\mathcal{S}$.

Until now, we have been looking at the whole class of Petrov D vacuum spaces together: for Ψ_2 we have a *complete table of its derivatives*, i.e., expressions for all four of its GHP derivatives; but for each spin coefficient we have incomplete tables (from the point of view of elements of $\mathcal{S}$), since we have obtained explicit expressions built from $\mathcal{S}$, for only *three* of its derivatives. These expressions, contained in (3)-(6), (31), involve only algebraic products of spin coefficients, and for the sake of clarity are gathered in the next expression

$$\begin{aligned} &\text{Þ}\rho = \rho^2, \quad \eth\rho = \tau(\rho - \bar{\rho}), \quad \eth'\rho = -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho}, \\ &\text{Þ}\tau = \rho(\tau - \bar{\tau}'), \quad \text{Þ}'\tau = -\rho'\bar{\tau}' + 2\rho'\tau - \tau\bar{\rho}', \quad \eth\tau = \tau^2; \end{aligned} \tag{35}$$

From here we also get the derivatives of ρ', τ' by taking the prime of each equation. There remains one derivative for each spin coefficient which contains terms involving Z_1; in the *generic case* (see **Class IIIB**, below), we can obtain Z_1 from (32)-(34), and it is found to consist of only algebraic products of spin coefficients and Ψ_2, but there are cases in which (32)-(34) are not independent, and hence we must proceed in a different way. To continue the analysis requires that we now separate into the different classes which are presented in the Appendix.

Class I: $\rho\rho' \neq 0$; $\tau = 0 = \tau'$.
Equation (18) implies $Z_1 = 0$ and hence adding (18) to (35) it follows immediately that *all the GHP derivatives of all non-zero elements of $\mathcal{S}$ are given by expressions constructed algebraically from elements of $\mathcal{S}$: a complete involutive set of tables for the action of the GHP operators on all of the nonvanishing elements of $\mathcal{S}$.*
Note that in this case we also have an extra relation involving the spin coefficients which is obtained by setting $Z_1 = 0$ in (33),

$$\bar{\Psi}_2\rho' + 2(\bar{\rho}' - \rho')\bar{\rho}\rho' - \Psi_2\bar{\rho}' = 0, \tag{36}$$

together with its $'$ counterpart which is obtained from (34). However, the latter is not independent of (36), since we are assuming $\rho' \neq 0 \neq \rho$ in this case.
The integrability conditions arising from (35) give a set of conditions which depends functionally on (36). Similarly if we differentiate (36) and replace the derivatives of the spin coefficients and Ψ_2 by their expressions we get a condition which is functionally dependent on (36) and thus gives nothing new.

Class II: $\tau\tau' \neq 0$; $\rho' = 0 \neq \rho$.
The third line of (18) gives $Z_1 = \Psi_2 + \tau'\bar{\tau}'$ and hence adding (18) to (35) it follows immediately that *all the GHP derivatives of all non-zero elements of* $\mathcal{S}$ *are given by expressions constructed algebraically from elements of* $\mathcal{S}$*: a complete involutive set of tables for the action of the GHP operators on all of the nonvanishing elements of* $\mathcal{S}$.
Also (32) becomes

$$\bar{\Psi}_2\tau + 2(\tau - \bar{\tau}')\bar{\tau}\tau - \Psi_2\bar{\tau}' = 0, \tag{37}$$

but (33) and (34) are identically satisfied. The substitution of the values of Z_1 and Z_2 in (24)-(25) yields the extra restriction (since $\rho \neq 0$)

$$\rho(\Psi_2 + \bar{\Psi}_2 - 2\tau\tau' + 2\tau\bar{\tau} - 2\bar{\tau}'\bar{\tau}) = 0. \tag{38}$$

The last pair of equations can be regarded as a linear system in the variables Ψ_2, $\bar{\Psi}_2$ from which it can be deduced that

$$(\tau + \bar{\tau}')\Psi_2 = 2\tau^2\tau' \tag{39}$$

and so

$$\Psi_2 = \frac{2\tau^2\tau'}{\tau + \bar{\tau}'}, \tag{40}$$

where, from (39), $\tau' \neq 0$, $\tau \neq 0$ imply that $\tau + \bar{\tau}' \neq 0$.
Next we insert (40) into (8)-(9) and their $'$ counterparts, and use the expressions already found for the derivatives of the spin coefficients. The result is the restriction

$$I_3 \equiv \rho\bar{\tau} + \bar{\rho}\tau' = 0. \tag{41}$$

Neither the integrability conditions of (35) nor the differentiation of (41) give further conditions. Finally we note that the combination of (40) and (41) yields the relation

$$\Psi_2\bar{\rho}^3 + \bar{\Psi}_2\rho^3 = 0. \tag{42}$$

Class II$'$: $\tau\tau' \neq 0$; $\rho = 0 \neq \rho'$.
This class does not need to be considered separately, since, by the interchange of the null vectors $l \leftrightarrow n$, this class is transformed into the previous one.

Class IIIA: $\tau\tau'\rho'\rho \neq 0$; $\bar{\rho}'\tau - \rho'\bar{\tau}' = 0 = \bar{\rho}\tau' - \rho\bar{\tau}$.
When we differentiate the conditions $\bar{\rho}'\tau - \rho'\bar{\tau}' = 0 = \bar{\rho}\tau' - \rho\bar{\tau}$ we obtain

$$\rho = \bar{\rho}, \quad \rho' = \bar{\rho}', \quad \tau = \bar{\tau}', \quad \Psi_2 = \bar{\Psi}_2, \quad Z_1 = \bar{Z}_1, \tag{43}$$

We take as independent variables ρ, ρ', τ, and Ψ_2. Combining (43) with (29) and (35) gives respectively

$$Z_2 = 0 = Z_2', \tag{44}$$

and

$$þ\tau = 0 = þ'\tau, \qquad ð\rho = 0 = ð\rho', \quad ð'\rho' = 0 = ð'\rho. \tag{45}$$

The only remaining equations where we can get information about Z_1 are (24)-(25), which simplify to

$$þZ_1 = \rho(Z_1 + \tau\bar{\tau}), \tag{46}$$
$$ðZ_1 = 2\tau(Z_1 + \Psi_2 + \rho\bar{\rho}'). \tag{47}$$

Putting

$$3A = (Z_1 + \tau\bar{\tau})/\Psi_2^{1/3}, \qquad 3B = \Psi_2^{1/3} + (Z_1 + \rho\bar{\rho}')/\Psi_2^{2/3} \tag{48}$$

we find directly from the above equations that

$$þA = 0 = þ'A, \qquad þB = 0 = þ'B, \tag{49}$$
$$ðA = 2\tau B\Psi_2^{2/3}/3,\ ð'A = 2\tau' B\Psi_2^{2/3}/3,\ \ ðB = 3\tau\Psi_2^{1/3},\ ð'B = 3\tau'\Psi_2^{1/3} \tag{50}$$

From the last set of equations it follows that

$$ð(B^2 - 9A) = 0 = ð'(B^2 - 9A) \tag{51}$$

For zero-weighted quantities, such as A, B, we have the relationship

$$\nabla_a = n_a þ + l_a þ' - \bar{m}_a ð - m_a ð'$$

and so this implies,

$$\nabla_a(B^2 - 9A) = 0 \quad \Rightarrow \quad (B^2 - 9A) = k, \quad \text{real constant.} \tag{52}$$

Therefore

$$(Z_1 + \Psi_2 + \rho\bar{\rho}')^2 - 3(Z_1 + \tau\bar{\tau})\Psi_2 - k\Psi_2^{4/3} = 0, \tag{53}$$

and so Z_1 can be obtained explicitly in terms of the spin coefficients and Ψ_2 (and the constant k).
Therefore, by substitution for Z_1 from (53) into (18), and then adding (18) to (35), it follows immediately that *all the GHP derivatives of all non-zero elements of $\mathcal{S}$ are given by expressions constructed from elements of $\mathcal{S}$: a complete involutive set of tables for the action of the GHP operators on all of the nonvanishing elements of $\mathcal{S}$.*
The integrability conditions of this set of equations are identically fulfilled and thus they give no further restrictions.

Classes IIIB,C: $\tau\tau'\rho'\rho \neq 0$; $\bar{\rho}'\tau - \rho'\bar{\tau}' \neq 0 \neq \bar{\rho}\tau' - \rho\bar{\tau}$.
In this case from (32)-(33) it follows that

$$(\rho'\bar{\tau}' - \bar{\rho}'\tau)Z_1 = \tau\Big(\bar{\Psi}_2\rho' + 2(\bar{\rho}' - \rho')\rho'\bar{\rho} - \Psi_2\bar{\rho}' + (2\rho'\tau - \bar{\rho}'\tau - \rho'\bar{\tau}')\bar{\tau}\Big) \tag{54}$$

Given that $\bar\rho'\tau-\rho'\bar\tau'\neq 0$ this equation enables us to obtain Z_1, and by inserting such a result in (18), we obtain the expressions for the one remaining derivative of each spin coefficient. These expressions are

$$\eth'\tau=\frac{\tau\Big(2\bar\rho\rho'^2-\bar\Psi_2\rho'+((\bar\tau'-2\tau)\bar\tau-2\bar\rho'\bar\rho)\rho'+\Psi_2\bar\rho'+\bar\rho'\tau\bar\tau\Big)}{\bar\rho'\tau-\rho'\bar\tau'},$$
$$\text{Þ}'\rho=\frac{\rho'\Big(\bar\Psi_2\tau+(\bar\rho'\bar\rho+2(\tau-\bar\tau')\bar\tau)\tau-\Psi_2\bar\tau'+\rho'\bar\rho(\bar\tau'-2\tau)\Big)}{\rho'\bar\tau'-\bar\rho'\tau}. \tag{55}$$

together with their $'$ counterparts.

Hence, adding (55) to (35), it follows immediately that *all the GHP derivatives of all non-zero elements of $\mathcal{S}$ are given by expressions constructed algebraically from elements of $\mathcal{S}$: a complete involutive set of tables for the action of the GHP operators on all of the nonvanishing elements of $\mathcal{S}$.*

The integrability conditions of this set of equations are identically fulfilled and thus they give no further restrictions.

Class IV: $\tau\tau'\neq 0$; $\rho'=0=\rho$.

The results from Class II are still valid and we only need to set $\rho=0$ in them; therefore *all the GHP derivatives of all non-zero elements of $\mathcal{S}$ are given by expressions constructed algebraically from elements of $\mathcal{S}$: a complete involutive set of tables for the action of the GHP operators on the non-vanishing spin coefficients.*

In this case, (38) does not arise, and hence we only need to consider (37). Again, neither the differentiation of this condition nor the integrability conditions of (35) result in new relations.

3. Invariant Classification

The Karlhede algorithm [17] for an invariant classification of geometries involves calculating — in a particular frame — the (zeroth derivative of) Riemann tensor and then computing successively higher orders of its frame covariant derivatives ('Cartan scalars'), together with possible changes of frame. The algorithm terminates at that particular order where (i) no *new* functionally independent Cartan scalar is supplied by the frame derivatives of the Riemann tensor; and (ii) no *further fixing* of the frame is possible using these frame derivatives, i.e., the invariance group of the frame is unchanged. This particular order is called the *Karlhede bound.*

Collins et al [4, 5] have used the GHP formalism to establish a relationship between the Cartan scalars and the GHP derivatives of the Weyl tensor, and hence provided an efficient approach to the Karlhede classification of vacuum Petrov type D spacetimes. We shall now exploit their results directly, and strengthen their final conclusion. It is shown in [4, 5], and in their notation, that:

- the zeroth-order covariant derivative of the Weyl tensor is given by $D^0\Psi(\Psi_2)$;
- the first covariant derivatives of Ψ_2 are shown to be — via the Bianchi equations — algebraic expressions involving only all elements of $\mathcal{S}$, and so are given by

$D^1\Psi(\Psi_2, \rho, \rho', \tau, \tau')$; for each of the different classes, the dimension of the invariance subgroup changes.

This means that we have to continue to the next order of differentiation. This is where we strengthen the earlier result because our analysis in the last section shows that derivatives of the spin coefficients do not in fact introduce any new quantities (since we have found explicit expressions for Z_1 exclusively in terms of the elements of $\mathcal{S}$, in all cases.) Hence,

• the second covariant derivatives of Ψ_2 — via Bianchi equations, Ricci equations, (35) and (18) substituted with the various explicit expressions for Z_1 — are expressions involving exclusively elements of $\mathcal{S}$ and are given by $D^2\Psi(\Psi_2, \rho, \rho', \tau, \tau')$; also the dimension of the invariance subgroup does not change for any of the different classes.

(For completeness, we note that in [4] it was pointed out explicitly — for the cases corresponding to our Classes I and IV — that since all components of $D^2\Psi$ can be expressed algebraically in terms of first order quantities, the frame cannot be fixed any further at second order; in addition — for the case corresponding to our Class III — the frame was fixed completely at first order, and so no further fixing is possible at second order. For our Class II — which was overlooked in [4, 5] — the frame can be fixed completely at first order by choosing, e.g., $|\rho| = 1$ and $\tau = \bar{\tau}$; and so no further fixing is possible.)

In summary, all elements of $\mathcal{S}$ occur at first derivative order, and no new quantities appear at second derivative order; moreover, the invariance group remains unchanged from first to second derivative order: therefore the Karlhede algorithm terminates at second order,

4. Conclusion and Discussion

Thanks to the efficiency of the GHP formalism, we have rederived the identities (28), and derived some new identities (32), (33), (34), (55) for the spin coefficients, in a transparent manner; using these results it has been easy to lower to *two* the upper bound for the invariant classification of vacuum Petrov D spaces.

In addition to showing in Section 2 that all GHP derivatives of all non-zero elements in $\mathcal{S}$ are given by expressions built exclusively from elements of $\mathcal{S}$, we have obtained a complete and involutive set of explicit tables for the action of all four GHP derivatives on the elements of $\mathcal{S}$ — in a manageable form, by a systematic and transparent analysis using the GHP formalism. These identities (28), as well as *partial tables* (35) (for *three* GHP derivative operators acting on each spin coefficient) have already been obtained by Carminati and Vu [2] in the GHP formalism, but we believe that this is the first time that the spin coefficient identities (55) are presented and that all possible compatibility and integrability conditions have been explicitly checked for these spaces. These tables will be the basis for a systematic and efficient integration of the different type D vacuum classes in a future paper.

In their NP Maple investigation [7], Czapor and McLenaghan state that they have obtained the complete tables for all of the NP spin coefficients (including the 'badly-behaved' ones), but that these tables, together with a lot of the intermediate calculations, were too long to print out explicitly in the NP formalism. Therefore, in their work [7], there was perhaps the potential to deduce the result that the theoretical upper bound was *two* in the invariant classification (at least, in the generic case); however, this possibility was obscured by the presence of the badly behaved spin coefficients which led to the very long and unmanageable expressions, and checking for possible tetrad changes would probably have been virtually impossible. In comparison, the GHP calculations and results are manageable and transparent, and it is very instructive to compare how the calculations develop in the GHP procedure of this paper compared with the NP version in [7].
The classification given in the Appendix improves on, and ties up a few loose ends regarding earlier classifications.

The system *xAct* was used to cross-check all the computations derived by hand and to derive others which due to their complication were difficult to obtain without the aid of the computer. A *Mathematica* notebook in which all the equations of the GHP formalism are obtained from scratch is available in [13].

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Appendix: Classification of vacuum Petrov type D spacetimes

We shall give a classification for vacuum Petrov type D spacetimes which is well suited to our investigations; since this classification eventually turns out to be closely related to the Kinnersley classification we shall label it similarly to that classification.
As well as the GHP zero-weighted quantitiies I_1 and I_2 we shall use the GHP versions

$$I_3 \equiv \bar{\rho}\tau' + \rho\bar{\tau}, \qquad I_4 \equiv \bar{\rho}'\tau + \rho'\bar{\tau}'$$

$$I_5 \equiv \bar{\rho}\tau' - \rho\bar{\tau}, \qquad I_6 \equiv \bar{\rho}'\tau - \rho'\bar{\tau}'$$

of the NP quantities I_3, I_4, I_5 which have also been defined in [6], and in addition, for completeness, we have added I_6.

These are not invariants under tetrad transformations, but since they are of GHP weight, they are rather rescaled; so, if for example $I_3 = 0$, then under arbitrary spin and boost tetrad transformations, $I_3' = 0$. By making direct use of the results $I_1 = 0 = I_2$, the simple lemma follows

Lemma 1. *If* $\rho \neq 0$, *then:*
(i) $I_3 = 0 \;\Rightarrow\; I_4 = 0$;
(ii) $I_5 = 0 \;\Rightarrow\; I_6 = 0$.
If $\rho' \neq 0$, *then:*
(i) $I_4 = 0 \;\Rightarrow\; I_3 = 0$;
(ii) $I_6 = 0 \;\Rightarrow\; I_5 = 0$.

From the earlier work, we obtain two other simple lemmas:

Lemma 2.

$$(i)\ \rho = 0 \ \Rightarrow \tau \neq 0 \neq \tau', \tag{56}$$

$$(ii)\ \rho' = 0 \Rightarrow \tau \neq 0 \neq \tau'. \tag{57}$$

Proof: Assume that $\rho = 0$; then (7) becomes

$$0 = -\tau\bar{\tau} - \Psi_2 + \eth'\tau,$$

which implies $\tau \neq 0$ in a type D spacetime. Combining this with the condition $I_1 = 0$ shown in (28) we conclude that $\tau' \neq 0$. If $\rho' = 0$ the argument is similar. □

Lemma 3.

$$(i)\ \rho \neq 0 \neq \rho', \tau = 0 \qquad \Rightarrow \qquad \tau' = 0$$

$$(ii)\ \rho \neq 0 \neq \rho', \tau' = 0 \qquad \Rightarrow \qquad \tau = 0$$

Proof: Apply (5) and its prime counterpart. □

From these lemmas it follows that any vacuum type D spacetime falls within one of the following mutually exclusive subclasses:

Class I. $\rho\rho' \neq 0$; $\tau = 0 = \tau'$.

Class II.‖ $\tau\tau' \neq 0$; $\rho \neq 0 = \rho'$.

[**Class II$'$.** $\tau\tau' \neq 0$; $\rho' \neq 0 = \rho$.]

Class III. $\rho\rho'\tau\tau' \neq 0$.

Class IV. $\tau\tau' \neq 0$; $\rho = 0 = \rho'$.

Class III can be further subdivided:

‖ This was the case overlooked in [12] and [4], [5].

Class IIIA. $I_6 = 0 = I_5$; $I_4 \neq 0 \neq I_3$.

Class IIIB. $I_6 \neq 0 \neq I_5$; $I_4 \neq 0 \neq I_3$.

Class IIIC. $I_6 \neq 0 \neq I_5$; $I_4 = 0 = I_3$.

We have used these labels because of the substantial overlap with the Kinnersley cases: by examining the different explicit metrics obtained by Kinnersley, we deduce the following:

- Class I coincides with Kinnersley Case I.
- Class II coincides with a subcase ($\rho' = 0$) of Kinnersley Case IIE.
- Class IIIA coincides with Kinnersley Case IIIA.
- Class IIIB coincides with Kinnersley Case IIIB.
- Class IIIC coincides with Kinnersley IIA-D, F, together with the other subcase ($\rho' \neq 0$) of Kinnersley Case IIE.
- Class IV coincides with Kinnersley Case IV.

A partial classification of the vacuum Petrov D metrics with non-zero cosmological constant, using $I_1, I_2 \ldots I_5$, was given in [6], and the above results are more complete. Note that the table on p2166 of [6] permits $\rho' = 0 \neq I_3$, which is ruled out above; and so the only Kinnersley case with null orbits (in the vacuum case with zero cosmological constant) is IIE. our attention has been drawn to a classification given in [23] which is very similar to the one in this paper, and it also notes these additional properties.

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