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ON ISOPERIMETRIC SURFACES IN GENERAL RELATIVITY, II

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ABSTRACT. We determine the optimal way to enclose volume in a class of domains inside certain Friedmann-Robertson-Walker metrics. The method employed is an adaptation of the Bray-Morgan isoperimetric comparison procedure to the Lorentzian setting. We also make some remarks on isoperimetric comparison in the Riemannian setting, for rotationally-symmetric space-like slices in non-vacuum space-times.

1. INTRODUCTION

A classical problem in geometry is to determine how to enclose a given volume V with a hypersurface of optimal area. The condition for a (smooth) hypersurface to be critical for area with a volume constraint is that its mean curvature be constant. In a given explicit geometry, one may be able to determine a class of constant mean curvature surfaces (for example, in case the surfaces are orbits of a subgroup of the isometry group) which stand as candidates for the optimization problem. It generally takes considerably more work to argue that the local condition of having constant mean curvature implies the hypersurface satisfies a global optimization problem.

Isoperimetric surface techniques were employed by Bray in this thesis [3] to obtain partial results on the Riemannian Penrose conjecture from general relativity. These techniques were codified by Bray and Morgan [6], and further explored in [7]. The main idea of the argument, as we will recall in more detail below, is that in certain special circumstances, one can glean the isoperimetric profile of a space by comparison to a space where the isoperimetric profile is already established. In the relativistic context, appropriate comparison to the Euclidean space establishes that the spherically symmetric spheres in the standard space-like slice of the (positive mass) Schwarzschild space-time minimize area for given volume enclosed with the horizon [6]. The analogous result holds for the Reissner-Nördstrom space-time, while comparison to the hyperbolic space proves the analogous result for Schwarzschild-anti-DeSitter [7]. Bray showed a connection between the isoperimetric profiles and the Penrose inequality, using a Hawking mass quantity associated to the profile. More recently, the relation between isoperimetric problems and the mass-energy of space-times has been developed by Huisken [11], who uses the fact that the scalar curvature measures the top-order deviation of the volumes of small geodesic balls from that of their Euclidean counterparts (cf. (4.3)). In the case of the time-symmetric Einstein constraint equations (with vanishing cosmological

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constant), the scalar curvature of the space-like slice measures the energy density of the physical fields, and thus the scalar curvature arises in the study of the energy content of isolated systems (the Positive Mass Theorem and the Penrose Inequality). However, Bray notes in [4] that the "potential energy contributions between matter (to the extent this is well-defined in certain examples) tends to make a negative contribution to the total mass" of the isolated system. We will see in examples in Section 4 that negative scalar curvature (energy density) provides a stabilizing effect for constant mean curvature hypersurfaces in certain geometries, which is in fact consistent with the effect of the scalar curvature function on the volumes of geodesic balls.

In the Lorentzian context, much is known about the class of space-like hypersurfaces with constant mean curvature in some special space-times, such as Minkowski space-time [1]. Moreover, the work of Bahn-Ehrlich settles the isoperimetric problem for a class of domains in Minkowski space [2]. We will extend the Bray-Morgan approach for an analogous Lorentzian isoperimetric comparison to Minkowski geometry. In contrast to the Riemannian case, the unit sphere in a Lorentzian vector space is non-compact, and compact space-like hypersurfaces of Minkowski space necessarily have nonempty boundary. This impacts the domains we use for the isoperimetric problem, which in this case are tied to the causal structure. We remark that a similar amount of care in defining domains on which to measure volume is needed to achieve certain Lorentzian analogues of the classical volume comparison theorems, cf. [9], [8].

2. Preliminaries

We begin by recalling the isoperimetric inequality of Bahn and Ehrlich. Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Minkowski space with metric signature $(-,+,\cdots,+)$. Recall that a set A in a time-oriented Lorentzian space-time is *achronal* if no future-pointing time-like curve intersects A in more than one point. Let S be a compact, simply connected, achronal hypersurface contained in the causal future $I^+(O)$ of the origin of Minkowski space-time. We assume without further remark that S is (piecewise) smooth with (piecewise) smooth boundary. $I^+(O)$ is foliated by umbilic space-like hypersurfaces $\mathbb{H}(r)=\{x\in I^+(O): d(O,x)=r\}$, $r>0$, each of which is isometric to a hyperbolic space (of curvature $K=-\frac{1}{r^2}$). Here d is the Lorentzian distance function. We let $\|x\|^2=-\eta(x,x)$, where η is the standard Minkowski metric. We can rescale a hypersurface S to $\mu(S)=\{\frac{x}{\|x\|}: x\in S\}\subset\mathbb{H}(1)$. We let the closed *cone* of S be given by $C(S)=\{\lambda x: x\in S, \lambda\in[0,1]\}$.

Theorem 2.1 (Bahn-Ehrlich [2]). *Let $S\subset I^+(O)\subset\mathbb{L}^{n+1}$ be a compact, simply connected, achronal, space-like hypersurface. Let $t^*=d(O,S)$. Let A_0 be Lorentzian area, and let V_0 be the Lorentzian volume, and let $\omega_S=V_0(C(\mu(S)))$. Then*

$$(A_0(S))^{n+1}\leq(n+1)^{n+1}\omega_S(V_0(C(S)))^n.$$

Equality holds if and only if $S\subset\mathbb{H}(t^*)$.

As a corollary, we see that for hypersurfaces with a given $\mu(S)$, only subsets of the hyperbolic leaves *maximize* area for given volume enclosed by the cone $C(S)$.

We now turn to spaces for which we want to generalize the above result. Let $I=(0,b)\subseteq(0,+\infty)$ be an open interval. First, we note that the metric η on $I^+(O)$ can be written as a warped product metric on $(0,+\infty)\times\mathbb{H}^n$ as $\eta=-dt^2+t^2g_{\mathbb{H}^n}$.

On $I \times \mathbb{H}^n$ we consider other warped product metrics $g = -dt^2 + (a(t))^2 g_{\mathbb{H}^n}$. Such metrics form a class of Friedmann-Robertson-Walker (FRW) metrics, which more generally take the form $g = -dt^2 + (a(t))^2 g_\kappa$ on $M = I \times N$, where g_κ is a Riemannian metric on N of constant sectional curvature κ. Such metrics are used to model homogeneous and isotropic cosmologies, and the warping factor $a(t)$ encodes the dynamics of the space-time.

We will consider smooth maps $F : I \times \mathbb{H}^n \to I^+(O) \approx (0, +\infty) \times \mathbb{H}^n$ of the form $F(t, \omega) = (\psi(t), \omega)$, for some suitably chosen ψ. We take ψ to strictly increasing, so that F is a diffeomorphism onto its image.

With the Bahn-Ehrlich Theorem in mind, we want to define the analogous subsets in FRW space-times and examine how they behave under F. Consider a compact, space-like, achronal hypersurface S. We define a cone-like set $\Gamma(S)$ on S as follows: $\Gamma(S) = \{(t, \omega) : (\tau, \omega) \in S \text{ and } 0 < t < \tau\}$. We also want to define the analogous *shadow set* $\hat{\mu}_t(S) \subset \{t\} \times \mathbb{H}^n$ by $\hat{\mu}_t(S) = \{(t, \omega) : \text{for some } \tau > 0,\ (\tau, \omega) \in S\}$. Note that since ψ is increasing, $F(\hat{\mu}_t(S)) = \mu_{\psi(t)}(F(S))$. These sets are related to the causal structure. We note that curves $\alpha(s) = (s, \omega)$ are time-like geodesics. Indeed since $\alpha'(s) = \frac{\partial}{\partial t}\Big|_{\alpha(s)}$, the acceleration vector is given by $\alpha''(s) := D_{\alpha'(s)}\alpha'(s) = \Gamma^k_{00}\frac{\partial}{\partial x^k}$, where $x^0 = t$ and $(x^1, \ldots, x^n)$ are coordinates on the hyperbolic factor, and $\Gamma^k_{00} = \frac{1}{2}g^{km}(2g_{m0,0} - g_{00,m}) = 0$. Thus the sets $\Gamma(S)$ and $\hat{\mu}_t(S)$ are constructed by using time-like geodesics passing through S. In the Minkowski case, the geodesics emanate from an origin O; in the general FRW case, if $\lim\limits_{t \to 0^+} a(t) = 0$, we see that the diameter of the shadow $\hat{\mu}_t(S)$ of any such surface S (compact) shrinks to zero as $t \to 0^+$. The geodesics might thus be interpreted as emanating from a single point; in case $\lim\limits_{t \to 0^+} a'(t) = +\infty$, one may interpret this as a *big-bang* singularity.

3. Lorentzian isoperimetric comparison

We will now discuss an adaptation of the Bray-Morgan isoperimetric comparison technique to our setting. We let $(M, g) = (I \times \mathbb{H}^n, -dt^2 + (a(t))^2 g_{\mathbb{H}^n})$ be an FRW space-time as above, and let $(I^+(O), \eta)$ be the future of O in the Minkowski space-time, which we identify with $((0, +\infty) \times \mathbb{H}^n, -dt^2 + t^2 g_{\mathbb{H}^n})$. We consider a diffeomorphism $F : M \to I^+(O)$ which has the form $F(t, \omega) = (\psi(t), \omega)$ with respect to the indicated identifications, with ψ increasing in t. We define the area stretch factor α_Σ for a space-like hypersurface $\Sigma \subset M$ by $F^*(dA_{F(\Sigma)}) = \alpha_\Sigma dA_\Sigma$, where dA_Σ and $dA_{F(\Sigma)}$ are the induced area forms on the surfaces, and where we assume $F(\Sigma)$ is also space-like; a similar definition holds for time-like hypersurfaces. The volume stretch factor β is given by $F^*(dV_{M_0}) = \beta dV_M$, where dV_M and dV_{M_0} denote the respective volume forms.

To estimate the stretch factors, we first compute them for two types of surfaces. We let $D \subset \mathbb{H}^n$ be any regular domain, and let $D_t = \{t\} \times D \subset M$. Let $\alpha_1(t) = \alpha_{D_t}$, so that

$$\alpha_1(t) = \frac{\int\limits_D \psi^n(t)\, dA_{\mathbb{H}^n}}{\int\limits_D a^n(t)\, dA_{\mathbb{H}^n}} = \frac{\psi^n(t)}{a^n(t)}.$$

We define the stretch factor $\alpha_0(t)$ for the annular time-like surface determined by flowing an $(n-1)$-dimensional submanifold $\Gamma \subset \mathbb{H}^n$ along the time-like direction field $\frac{\partial}{\partial t}$, and we let $d\sigma_\Gamma$ be the $g_{\mathbb{H}^n}$-induced area element along Γ:

$$\alpha_0(t) = \frac{\frac{d}{dt}\int\limits_{\psi(t_0)}^{\psi(t)}\int\limits_{\Gamma} \tau^{n-1}\, d\sigma_\Gamma\, d\tau}{\frac{d}{dt}\int\limits_{t_0}^{t}\int\limits_{\Gamma} a^{n-1}(\tau)\, d\sigma_\Gamma\, d\tau} = \frac{\psi^{n-1}(t)\psi'(t)}{a^{n-1}(t)}.$$

Finally the volume stretch is given by

$$\beta(t) = \frac{\frac{d}{dt}\int\limits_{\psi(t_0)}^{\psi(t)}\int\limits_{D} \tau^{n}\, dA_{\mathbb{H}^n}\, d\tau}{\frac{d}{dt}\int\limits_{t_0}^{t}\int\limits_{D} a^{n}(\tau)\, dA_{\mathbb{H}^n}\, d\tau} = \frac{\psi^{n}(t)\psi'(t)}{a^{n}(t)} = \sqrt[n]{\alpha_1(t)}\;\alpha_0(t).$$

Let $\Sigma \subset M$ be a space-like hypersurface; at a (smooth) point $p \in \{t\} \times \mathbb{H}^n$, $T_p\Sigma \cap T_p(\{t\}\times\mathbb{H}^n)$ is at least $(n-1)$ dimensional. Let $E_2, \ldots, E_n$ be an orthonormal set in this intersection. Let E_1 complete these to form an orthonormal basis for $T_p(\{t\} \times \mathbb{H}^n)$, and let $E_0 = \frac{\partial}{\partial t}$. Let $V \in T_p\Sigma$ be a unit vector orthogonal to E_j, $j = 2, \ldots, n$. Then there exist $\xi_0, \xi_1 \in \mathbb{R}$ with $\xi_1^2 - \xi_0^2 = 1$ so that $V = \xi_0 E_0 + \xi_1 E_1$. Therefore we see

$$\begin{aligned}
\alpha_\Sigma &= dA_{F(\Sigma)}(F_*(E_2), \ldots, F_*(E_n), F_*(\xi_0 E_0 + \xi_1 E_1)) \\
&= |F_*(E_2)| \cdots |F_*(E_n)||\xi_0 F_*(E_0) + \xi_1 F_*(E_1)| \\
&= \frac{\psi^{n-1}(t)}{a^{n-1}(t)}\sqrt{\xi_1^2 \frac{\psi^2(t)}{a^2(t)} - \xi_0^2(\psi'(t))^2}.
\end{aligned}$$

Therefore, $\alpha_\Sigma^2 = \xi_1^2\alpha_1^2 - \xi_0^2\alpha_0^2 = \alpha_1^2 + \xi_0^2(\alpha_1^2 - \alpha_0^2)$.

Theorem 3.1. *Let S be a compact, simply-connected, space-like, achronal hypersurface in an FRW spacetime $(M, g) \approx (I \times \mathbb{H}^n, g = -dt^2 + (a(t))^2 g_{\mathbb{H}^n})$, with $V(\Gamma(S)) < +\infty$. Let $T(S) = \{t : S \cap (\{t\} \times \mathbb{H}^n) \neq \emptyset\}$, and let $t^* > 0$ be such that the volume $V(\Gamma(\hat{\mu}_{t^*}(S)))$ from the shadow $\hat{\mu}_{t^*}(S)$ is precisely $V(\Gamma(S))$. Furthermore, suppose a map F as above can be constructed so that $\beta(t) \leq \beta(t')$ for $t, t' \in T(S)$ with $t \geq t^* \geq t'$ (e.g. in case $\beta(t)$ is non-increasing for $t \in T(S)$), so that the average value of α_S over S is at least $\alpha_1(t^*) = \frac{A_0(F(\hat{\mu}_{t^*}(S)))}{A(\hat{\mu}_{t^*}(S))}$, and so that $F(S)$ is space-like and achronal. Then the area $A(S)$ of S is at most the area of the shadow $\hat{\mu}_{t^*}(S)$, with equality if and only if $S = \hat{\mu}_{t^*}(S)$.*

Proof. Suppose S and $t^* > 0$ are as in the theorem, and moreover that $A(S) \geq A(\hat{\mu}_{t^*}(S))$. Then we have by change of variables and the fact that $\frac{1}{A(S)}\int\limits_S \alpha_S\, dA_S \geq \alpha_1(t^*)$,

$$A_0(F(S)) = \int_{F(S)} dA_{F(S)} = \int\limits_S \alpha_S\, dA_S \geq A_0(F(\hat{\mu}_{t^*}(S)) = A_0(\mu_{\psi(t^*)}(F(S))).$$

Since $\beta(t)$ is decreasing in t, the volume $V_0(C(F(S)) \cap \{(\tau, \omega) : \tau > \psi(t^*)\})$ is less than the volume $V_0(C(F(S)) \cap \{(\tau, \omega) : \tau < \psi(t^*)\})$. We may thus conclude that $V_0(C(F(S))) \leq V_0(C(\mu_{\psi(t^*)}(F(S))))$. By assumption on the map F, $F(S)$ is achronal and space-like, so that we may apply the Bahn-Ehrlich theorem to prove $(A_0(F(S)))^{n+1} = (n+1)^{n+1}\omega_{F(S)}(V_0(C(F(S))))^n$, and thus conclude $F(S) \subset \{\psi(t^*)\} \times \mathbb{H}^n$. Pulling back to M, we can conclude the theorem. $\square$

Remark 3.2. We remark that the assumption $V(\Gamma(S)) < +\infty$ will be satisfied in our applications below. In any case, we could instead choose $t_0 > 0$ and replace $V(\Gamma(S))$ with $V(\Gamma(S), t_0) = V(\Gamma(S) \cap \{(t, \omega) \in M : t > t_0\})$, which is always finite for compact S, and state an analogous theorem.

3.1. **Applications.** It remains to be shown to what extent the theorem can be applied, in other words, under what restrictions on the FRW metric we can find ψ so that F satisfies the conditions of the theorem. We begin with the following proposition.

Proposition 3.3. *Consider an FRW metric $g = -dt^2 + (a(t))^2 g_{\mathbb{H}^n}$ on $I \times \mathbb{H}^n$ with $0 < a'(t) \leq 1$, and $a''(t) \leq 0$. For any compact, simply connected, space-like, achronal hypersurface S, we let $t^* > 0$ be chosen so that $V(\Gamma(\hat{\mu}_{t^*}(S))) = V(\Gamma(S))$. Then the area $A(S)$ of S is at most the area $A(\hat{\mu}_{t^*}(S))$ of the shadow, with equality if and only if $S = \hat{\mu}_{t^*}(S)$.*

Proof. We verify the conditions of Theorem 3.1. We define F by taking $\psi(t) = a(t)$. Then the area stretch $\alpha_1(t) = 1$ is constant, while $\alpha_0(t) = a'(t)$. Thus $\alpha_1(t) \geq \alpha_0(t)$ if and only if $a'(t) \leq 1$. The volume stretch in this case is $\beta(t) = a'(t)$, so that β is non-increasing precisely for $a''(t) \leq 0$.

We now consider how F affects the causal nature of vectors. Let $v = \frac{\partial}{\partial t} + \lambda u$, where u is a $g_{\mathbb{H}^n}$-unit vector. Then $g(v, v) = -1 + \lambda^2 a^2(t)$, and $F_*(v) = \psi'(t)\frac{\partial}{\partial t} + \lambda u$, so that

$$\eta(F_*(v), F_*(v)) = -(\psi'(t))^2 + \lambda^2 \psi^2(t) = -(a'(t))^2 + \lambda^2 a^2(t).$$

Under the condition $0 < a'(t) \leq 1$, we obtain

$$\eta(F_*(v), F_*(v)) \geq g(v, v) = -1 + \lambda^2 a^2(t).$$

Thus we see F_* maps space-like vectors to space-like vectors, so that $F(S)$ is space-like. This inequality also shows that $F(S)$ is achronal: if there were a future-pointing time-like curve $\gamma = F \circ \alpha$ from $F(p)$ to $F(q)$ on $F(S)$, then α is a future-pointing time-like curve from p to q on S, which contradicts achronality of S. $\square$

A standard calculation [13] gives the Ricci tensor and scalar curvature $R(g)$ (the metric trace of the Ricci tensor) of an FRW metric. Let $N = \frac{\partial}{\partial t}$, and let V and W be tangent to $\{t\} \times \mathbb{H}^n$:

$$\begin{aligned} Ric(g)(N, N) &= -\frac{na''(t)}{a(t)} \\ Ric(g)(N, V) &= 0 \\ Ric(g)(V, W) &= \big(a(t)a''(t) + (n-1)\big((a'(t))^2 - 1\big)\big) \\ R(g) &= \frac{2na(t)a''(t) + n(n-1)\big((a'(t))^2 - 1\big)}{(a(t))^2}. \end{aligned}$$

The metrics covered by the proposition, then, have nonnegative Ricci curvature in the time direction (orthogonal to the hyperbolic space-like slices), and nonpositive Ricci curvature in the spatial directions.

Let us interpret the metrics covered by this proposition as solutions of Einstein's equation in the case $n = 3$. Using the above curvature formulas, one easily obtains the equation

$$Ric(g) - \frac{1}{2}R(g)g + \Lambda g = 8\pi T,$$

where Λ is a constant (cosmological constant), and the tensor T has the form $T_{ab} = \mu N_a N_b + P(g_{ab} + N_a N_b)$, for functions $\mu(t)$ and $P(t)$ which will be specified below. T has the form of the stress-energy tensor of a perfect fluid with velocity N, density μ and pressure P. The density and pressure are related to the metric function $a(t)$ as follows, where $h(t) = \frac{a'(t)}{a(t)}$ is called the *Hubble parameter* [13]:

$$(h(t))^2 := \frac{(a'(t))^2}{(a(t))^2} = \frac{1}{3}\Big(8\pi\mu + \frac{3}{a^2(t)} + \Lambda\Big) \tag{3.1}$$

$$\frac{a''(t)}{a(t)} = -\frac{4\pi}{3}(\mu + 3P) + \frac{1}{3}\Lambda. \tag{3.2}$$

We note the condition that $\mu \geq 0$ is equivalent to $h^2 \geq a^{-2} + \Lambda/3$, i.e. $(a'(t))^2 \geq 1 + \frac{1}{3}\Lambda(a(t))^2$. Observe that in the classical case of vanishing cosmological constant, the metrics covered by the preceding proposition have $\mu \leq 0$; for the proposition to accommodate models with positive matter density, one must use an Einstein equation with $\Lambda < 0$.

We remark that in the proof of the proposition, we constructed a specific comparison map F, and that it might be possible to find a different comparison map that may capture a different regime of FRW metrics, and in particular, may allow models with $\mu \geq 0$ in case $\Lambda = 0$. For example, consider the condition $\alpha_0(t) = \alpha_1(t)$; under this condition we see for any space-like S, $F(S)$ is also space-like (see (3.3) below), and $\alpha_S \geq \alpha_1 = \alpha_0$. By the above calculations, this condition is equivalent to

$$\frac{\psi'(t)}{\psi(t)} = \frac{1}{a(t)}.$$

The solutions to this equation are determined up to a constant factor.

We note that with this choice of ψ, F preserves the causal nature of vectors, and hence $F(S)$ is space-like and achronal if S is. In fact from the above we see for $v = \frac{\partial}{\partial t} + \lambda u$,

$$\eta(F_*(v), F_*(v)) = -(\psi'(t))^2 + \lambda^2\psi^2(t) = \frac{\psi^2(t)}{a^2(t)} g(v, v). \tag{3.3}$$

Furthermore, this choice of ψ yields a monotone volume stretch $\beta(t) = \frac{\psi^{n+1}(t)}{a^{n+1}(t)}$. Indeed we find

$$\beta'(t) = (n+1)\beta^{\frac{n}{n+1}}(t)\frac{\psi'(t)a(t) - \psi(t)a'(t)}{a^2(t)} = (n+1)\beta^{\frac{n}{n+1}}(t)\frac{\psi(t)(1 - a'(t))}{a^2(t)}.$$

Thus, β is non-increasing precisely if $a'(t) \geq 1$, which corresponds to nonnegative density $\mu \geq 0$ in case $\Lambda = 0$.

However there is a catch: the last item to check is the area stretch factor for S. We note that since $\alpha_1(t) = \beta^{\frac{n}{n+1}}(t)$, the sign of $\alpha_1'(t)$ is the same as the sign of $\beta'(t)$. This is problematic, since one way to satisfy the area stretch condition in Theorem 3.1 is to arrange $\alpha_S \geq \alpha_1(t^*)$, which would be satisfied if α_1 has a minimum at t^*. This simple statement will not hold in this case, but at least the monotone behavior of $\alpha_1(t)$ does give us a way to interpret the condition on the area of S used in Theorem 3.1: in trying to minimize volume and maximize area (for a given shadow set), the only possible way to improve upon a hyperboloidal hypersurface is to try to arrange for a majority of the surface to lie in $\{t > t^*\}$, while at the same time adhering to the volume constraint. We conjecture this cannot improve on the hyperboloidal hypersurface.

4. Remarks on energy density and stability of CMC surfaces

The Lorentzian isoperimetric comparison in Proposition 3.3 can be interpreted to work in certain FRW space-times with $\Lambda = 0$ and $\mu \leq 0$. In this section we consider the Riemannian setting, and in the context of the Einstein constraint equations, we remark that negative energy density tends to promote stability of CMC spheres in rotationally symmetric perturbations of the standard space-like slices in Schwarzschild space-times.

In particular, we will consider metrics on domains $E_R = \{x \in \mathbb{R}^3 : |x| > R\}$ of the form $g = u^4 g_E$, where g_E is the Euclidean metric, and $u(x) = u(r) > 0$ ($r = |x|$). Please see the Appendix for useful formulas (Christoffel symbols, curvatures) for these metrics. An important family of such metrics is given by the Schwarzschild metrics: $g^S = \left(1 + \frac{m}{2r}\right)^4 g_E$. The parameter m measures the deviation of the metrics from the model Euclidean metric. This parameter is called the *mass*, and indeed it has an interpretation in terms of the energy of isolated gravitational systems [5]. For $m > 0$, there is a unique minimal (in fact totally geodesic) sphere $r = m/2$, which in the context of general relativity is called the (apparent) horizon. The metric can be smoothly reflected across the horizon (using the isometric inversion $r \mapsto \frac{m^2}{4r}$ in the horizon sphere with respect to the metric distance along radial geodesics) to produce complete metrics with two ends. We note that we can do a radial change of coordinates so that half of the Schwarzschild metric can be written $\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$, where $d\Omega^2$ is the standard metric on the sphere $\mathbb{S}^2$, and $r \geq 2m$; in these coordinates $r = 2m$ corresponds to the horizon. When the mass m is negative, the metric $(1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega^2$ is defined on $r > 0$, and is an inextendible metric with no minimal sphere. The coordinates are only singular at the origin; in fact the metric is incomplete, as radial geodesics have finite length as $r \to 0^+$, but the Ricci tensor blows up on approach to the origin.

Of present interest for us is the fact that the stability of the area functional at the constant mean curvature (CMC) spheres S_r of constant r depends on the sign of the mass m. Let N be a unit normal field to S_r, and let $A(\tau)$ be the area function induced from a variation of S_r with normal variation field $V = \eta N$; we assume that the volume is preserved (at least through the second order), so that $\int_{S_r} \eta \, dA = 0$, and the second variation formula (see Appendix A) reduces to

$$A''(0) = \int_{S_r} \left[|\nabla^\Sigma \eta|^2 - \eta^2 \|\mathrm{II}\|^2 - \eta^2 Ric(N, N) \right] dA. \tag{4.1}$$

We let $\lambda_1 > 0$ be the lowest non-zero eigenvalue for the Laplacian, whose variational characterization gives us the Poincaré inequality: for all η with $\int_\Sigma \eta \, dA = 0$, $\lambda_1 \int_\Sigma \eta^2 \, dA \leq \int_\Sigma |\nabla^\Sigma \eta|^2 \, dA$, where equality holds precisely for functions in the λ_1-eigenspace. For a round two-sphere of area $4\pi\rho^2$, $\lambda_1 = \frac{2}{\rho^2}$, with the λ_1-eigenspace spanned by the restrictions of the coordinate functions x, y, z to the sphere (isometrically embedded in $\mathbb{R}^3$ centered at the origin).

If we use coordinates for which the Schwarzschild metric takes the form $g^S = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$, then by applying the preceding discussion with $\Sigma = S_r$, we obtain

$$A''(0) \geq \int_{S_r} \frac{6m}{r^3}\eta^2 \, dA, \tag{4.2}$$

with equality if and only if η is in the λ_1-eigenspace. We see from this that in the *positive* mass Schwarzschild case, the second variation must be positive for (nontrivial) volume-preserving deformations; note that by applying Theorem 4.1 below, one can show the spheres are in fact *isoperimetric* for $m > 0$. In the negative mass case, however, we see that for η a first eigenfunction, the right-hand side of Equation (4.2) is negative, so the spheres are *unstable* for $m < 0$. Of course in the Euclidean ($m = 0$) borderline case, the spheres are stable, but not strictly stable, as the translations are isometries.

The Schwarzschild metrics have zero scalar curvature. A natural follow-up to the above observations is to determine geometries that are in some sense perturbations of Schwarzschild geometries, for the which the scalar curvature is non-zero and the rotationally symmetric spheres enjoy some stability or isoperimetry. We will in fact deform the area profiles of Schwarzschild geometries to obtain rotationally symmetric geometries in which the spheres S_r (for some range of r values) have a desired stability property, while keeping the sign of the scalar curvature fixed. In the context of general relativity, this may be interpreted as specifying the sign of the energy density of the matter fields, as we now recall. Consider a spacetime $(\mathcal{S}, \bar{g})$ satisfying the Einstein equation $Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\bar{g} = 8\pi T$. Then the Gauss and Codazzi equations (together with the Einstein equation) imply constraint equations on the geometry (intrinsic and extrinsic) of space-like slices. If g is the induced metric and $I\!I$ the second fundamental form (with trace H) of a spacelike slice, then using the Einstein equation along with the Gauss equation, we obtain the Hamiltonian constraint, which yields $R(g) - \|I\!I\|^2 + H^2 = 2T_{00} = 16\pi\mu$, where μ is the local energy density of the matter fields. In the totally geodesic ($I\!I = 0$) case (also known as the *time-symmetric* case), these constraints reduce to the condition $R(g) = 16\pi\mu$.

We now construct examples of time-symmetric initial data for the Einstein constraint equations of the form $g = u^4 g_E$ with scalar curvature (energy density) of fixed sign. The examples will point out that non-positive energy density has a stabilizing effect on the rotationally symmetric spheres. Though this might seem strange from a physical point of view, that a natural variational problem seems to prefer exotic matter, it is quite natural from a geometric point of view. Indeed, recall the well-known expansion for the volume of small geodesic balls $B_r(p)$ about p in a Riemannian three-manifold compared to that of Euclidean space [10]:

$$V(B_p(r)) = \frac{4\pi r^3}{3}\left(1 - \frac{R(p)}{6}r^2 + O(r^3)\right). \tag{4.3}$$

We see that negative scalar curvature implies that small geodesic balls contain more volume than their Euclidean counterparts. Of course, volume is just one part of the isoperimetric problem, and this volume behavior needs to be compared with the area profile of the rotationally symmetric spheres S_r.

Before we state our results, we recall a simple form of the Bray-Morgan comparison theorem we will use for our setting.

Theorem 4.1 (Bray-Morgan [6]). *Consider a Riemannian three-manifold $M = I \times \mathbb{S}^2$ with a rotationally symmetric metric (so the metric can be written in the* *form* $dr^2 + f^2(r)d\Omega^2$*, or equivalently* $h^2(r)dr^2 + r^2 d\Omega^2$*), and let* $S_r = \{r\} \times \mathbb{S}^2$ *be a radially symmetric sphere. Suppose* M *has nonpositive radial Ricci curvature, and that* M *has nonnegative tangential sectional curvature (with respect to the spheres* S_r*). Suppose furthermore that* S_r *has nonnegative mean curvature in the* $-\frac{\partial}{\partial r}$ *direction, for all* $r \in I$*. Then the radially symmetric spheres* S_r *minimize surface area among smooth surfaces enclosing the same volume (against* S_{r_0}*, say).*

We can now list the quantities of interest for the metric $g = u^4 g_E$: the area profile $A(r) = A(S_r)$ (and in particular $A'(r)$), the scalar curvature $R(g)$, the radial Ricci curvature $Ric^M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})$, the mean curvature H of S_r with respect to $N = -u^{-2}\partial_r$, the tangential section curvature $K^M = K^M(T_pS_r)$, as well as a quantity Ξ which satisfies $\Xi(r) \geq 0$ if and only if S_r is stable (under volume-preserving deformations). These quantities are given by (see Appendix B):

$$\begin{aligned}
A'(r) &= 4\pi \frac{d}{dr}(r^2 u^4) \\
R(g) &= -8u^{-5}\Delta_{g_E} u = -8u^{-5}\Big(u'' + \frac{2}{r}u'\Big) \\
Ric^M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) &= \frac{-4u''}{u} + \frac{4u'^2}{u^2} - \frac{4u'}{ru} \\
K^M &= \frac{-4}{u^6}\Big((u')^2 + \frac{uu'}{r}\Big) \\
H &= \frac{2}{u^2}\Big(\frac{2u'}{u} + \frac{1}{r}\Big) \\
\Xi(r) &= uu'' - \frac{uu'}{r} - 3(u')^2. \qquad (4.4)
\end{aligned}$$

We will produce some examples by picking appropriate functions u. We first note $u_0(r) = 1 + \frac{m}{2r}$ gives the general solutions (up to a constant factor) of $u'' + \frac{2}{r}u' = 0$; in particular u_0 are the rotationally invariant harmonic functions (up to scale). With an eye on (4.4), we will look for solutions of the equation

$$u'' + \frac{2}{r}u' = -\frac{C}{r^p}, \qquad (4.5)$$

where C is a constant whose sign gives the sign of μ, and where p gives the decay rate of μ as $r \to +\infty$. Solutions of this equation are given by

$$u(r) = u_0(r) + \frac{C}{(2-p)(p-3)}\frac{1}{r^{p-2}} = 1 + \frac{m}{2r} + \frac{C}{(2-p)(p-3)}\frac{1}{r^{p-2}},$$

where again u_0 is determined up to a constant factor. We take $p > 3$ so that u_0 gives the top-order part of u. We now proceed by specifying the parameters m, C and p, and then checking whether and where the corresponding metric $g = u^4(r)g_E$, defined on some $E_R = \{u > 0\}$, satisfies the conditions (4.4)-(4.4) for stability and/or isoperimetry.

We let $p = 4$, and for reference we collect here the following identities:

$$A'(r) = \frac{4\pi(r(2r+m)-C)^3(r(2r-m)+3C)}{8r^7}$$

$$Ric^M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = -\frac{4(2mr^2 - C(8r+m))}{r(r(2r+m)-C)^2} \tag{4.6}$$

$$K^M = \frac{64r^6(mr-2C)(2r^2+C)}{(r(2r+m)-C)^6} \tag{4.7}$$

$$H = \frac{8r^3(r(2r-m)+3C)}{(r(2r+m)-C)^3} \tag{4.8}$$

$$\Xi(r) = \frac{6mr^3 - Cr(16r-m) - 4C^2}{4r^6}. \tag{4.9}$$

4.1. $m = 0$ **case.** We take $m = 0$ and $p = 4$ in the definition of u, so $u = 1 - \frac{C}{2r^2}$, and so for large r, then, the metric $g = u^4 g_E$ approaches the flat metric, with top-order deviation $O(r^{-2})$. We have the following identities for the values of various quantities of interest:

$$A'(r) = \frac{4\pi(2r^2-C)^3(2r^2+3C)}{8r^7}$$

$$Ric^M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = \frac{32C}{(2r^2-C)^2}$$

$$K^M = -\frac{128Cr^6(2r^2+C)}{(2r^2-C)^6}$$

$$H = \frac{8r^3(2r^2+3C)}{(2r^2-C)^3}$$

$$\Xi(r) = -\frac{C(4r^2+C)}{r^6}.$$

If we let $C < 0$, then the metric is defined on $\{r > 0\}$, and there is a unique minimal sphere at $r_0 = \sqrt{\frac{-3C}{2}}$: note that the spheres for $r > r_0$ are convex. The scalar curvature $R(g)$ is negative in this case, and $R(g) \in L^1(\{r > r_0\}, d\mu_g)$, where $d\mu_g$ is the metric volume measure, which is u^6 times the Euclidean volume measure. We note that $N = u^{-2}\frac{\partial}{\partial r}$ is the outward unit normal to S_r, and $\lim_{r \to 0^+} A(r) = +\infty$, and that $\lim_{r\to 0^+} Ric^M(N,N) = 0 = \lim_{r\to 0^+} K^M(T_pS_r)$. Indeed, the metric is asymptotically flat with *two* ends; however as $r \to 0^+$, the difference of the metric from the flat metric is only $O(s^{-2/3})$, where s is the intrinsic distance along radial geodesics approaching $r = 0$; in fact, $R(g) \notin L^1(\{r_0 > r > 0\}, d\mu_g)$. In any case, the above identities show that the Bray-Morgan comparison proves that the spheres S_r for $r > r_0$ are isoperimetric. Moreover, these isoperimetric spheres are *strictly* stable, unlike in the Euclidean case. In fact, $\Xi(r) > 0$ if and only if $r > r_c = \frac{\sqrt{-C}}{2}$, and $r_0 > r_c$.

The Penrose Inequality ([5],[4],[12]) states that in an asymptotically flat metric with *nonnegative* scalar curvature, the total mass measured at infinity in an asymptoticaly flat end is always at least $\sqrt{\frac{A}{16\pi}}$, where A is the area of an *outermost* minimal surface. If we let $(M, g) = (\{r \geq r_0\}, u^4 g_E)$ in our example above, then we see the Penrose Inequality does not hold; of course, the scalar curvature is negative in this example.

If we now let $C > 0$, so that the scalar curvature (energy density) is positive, we note that the metric is defined on $\{r > \sqrt{\frac{C}{2}}\}$. There are no minimal spheres: this is no surprise, in light of the Penrose inequality. The hypotheses of the Bray-Morgan theorem above do not hold, so we cannot use that to conclude the spheres are isoperimetric. In fact, clearly $\Xi(r) < 0$, so the spheres S_r are *unstable*.

4.2. $m < 0$ **case.** We let $m = -2$ and $p = 4$, so that $g = u^4 g_E$ with $u(r) = 1 - \frac{1}{r} - \frac{C}{2r^2}$.

Suppose we try to stabilize spheres S_r by increasing C from zero to positive (thus by increasing the scalar curvature from zero to positive); the metric in this case is defined for $r > \frac{1+\sqrt{1+2C}}{2} = r_c$. Then by (4.9), $\Xi(r) = -\frac{6r^3 + Cr(8r+1) + 2C^2}{2r^6} < 0$, and so S_r is unstable.

Suppose instead we seek to stabilize some range of spheres S_r by making the scalar curvature negative. Note from (4.9), that no matter what C is, $\Xi(r) < 0$ for large r; this makes sense, since the spheres S_r are unstable in the negative mass case, and the mass term dominates the scalar curvature term for large r. From the form of r_c above, we see that for $C < -\frac{1}{2}$, the metric is defined on $\{r > 0\}$. By (4.8), $H = \frac{8r^3(2r^2+2r+3C)}{(2r^2-2r-C)^3}$, so there is a unique minimal sphere at $r_0 = \frac{1}{2}(-1 + \sqrt{1-6C})$. For C just below $-\frac{1}{2}$, the radius r_0 of the minimal sphere is just above $\frac{1}{2}$, and there is a small range of r values near r_0 about which the spheres are stable; the range becomes smaller as $C \to -\frac{1}{2}^-$. For example, if let $C = -1$, we produce an example of a metric g on $\{r > 0\}$ with negative scalar curvature, and with a unique minimal sphere S_{r_0} with $r_0 = \frac{\sqrt{7}-1}{2} \approx 0.823$. We note that $\Xi(r)$ is positive on an interval $I \supset (0.557, 1.254)$ around r_0, so that the spheres S_r are stable in this range. Moreover, on the interval $J = (r_0, 1)$, we have negative radial Ricci curvature, positive (inward) mean curvature, and positive tangential sectional curvature. Hence by Bray-Morgan, the spheres S_r are isoperimetric in $(J \times \mathbb{S}^2, g)$.

4.3. $m > 0$ **case.** We finally consider the case where $m > 0$, say $m = 2$. We let $p = 4$ again, so that $g = u^4 g_E$ with $u(r) = 1 + \frac{1}{r} - \frac{C}{2r^2}$.

When $C < 0$ (negative scalar curvature), the metric is defined on $\{r > 0\}$, and by (4.8), $H = \frac{8r^3(2r^2-2r+3C)}{(2r^2+2r-C)^3}$, we see there is a unique minimal sphere S_{r_0} with $r_0 = \frac{1}{2}(1 + \sqrt{1-6C})$. Since $2r_0^2 + C > 0$, a glance at (4.6)-(4.8) shows that the Bray-Morgan comparison can be applied to show that each S_r for $r > r_0$ is isoperimetric.

When $C > \frac{1}{6}$, however, the geometry does not resemble the Schwarzschild geometry of mass $m = 2$ for small r. Indeed, the metric is defined for $r > r_c = \frac{1}{2}(-1 + \sqrt{1+2C})$, in which range there are no minimal spheres. As $r \to r_c^+$, the metric becomes singular (consider the intrinsic curvature quantities in (4.6) and (4.7)), and the spheres S_r can be unstable: for example if $C = 4$, then $r_c = 1$ and $\Xi(r_c) < 0$.

4.4. **Conclusion.** The above examples illustrate the stabilizing effect of negative scalar curvature. The borderline ($m = 0$) case may illustrate this the best: the effect of the scalar curvature is shown no matter how small C is, and the metric $u^4 g_E$ converges uniformly on compacts on $\{r > r_c\}$ (where r_c may be 0) to the

Euclidean metric as C tends to 0. We also emphasize that in the negative mass cases, positive scalar curvature promoted further instability, while in the positive mass case, negative scalar curvature promoted further stability. When one tries to change the stability properties, one must add enough scalar curvature of the appropriate sign to make a large enough deviation in the geometry to change the stability properties of the spheres in some interval; though the geometry may change a lot, we maintain control on the sign of the scalar curvature.

Appendix A. The Second Variation of Area

In this section we sketch the derivation of the second variation of area formula, which we use to determine whether hypersurfaces under consideration, domains inside the hyperbolic slices or the radial spheres in a rotationally symmetric metric, are stable critical points for the constrained variational problem. Indeed, we let Σ be either a smooth domain contained in some $\{t_0\} \times \mathbb{H}^n$ inside an FRW space (M, g) as above, or $\Sigma = S_r = \{r\} \times \mathbb{S}^2$ in a rotationally symmetric metric g on $M = I \times \mathbb{S}^2$. In either case, Σ has constant mean curvature in M, and in fact is totally umbilic. This is easy to see in case $\Sigma = S_r$, since the ambient metric is rotationally invariant. In the other case, one can see that the slice $\{t_0\} \times \mathbb{H}^n$ inside an FRW space-time is umbilic by noting the ambient metric is preserved by hyperbolic isometries, or by direct calculation of the second fundamental form with respect to the unit normal $\frac{\partial}{\partial t}$ to the slice (and using coordinates (x^i) on $\mathbb{H}^n$):

$$\mathrm{II}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g\left(\frac{\partial}{\partial t}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) = -\Gamma^0_{ij} = -\frac{a'(t_0)}{a(t_0)} g_{ij} = -h(t_0) g_{ij}.$$

We now compute the relevant formulas for the first and second variation of area of Σ (in either case posed above). Consider a variation $\Phi : (-\epsilon, \epsilon) \times \Sigma \to M$ with variation field $V = (\Phi_\tau)|_{\tau=0}$, where $\Phi_\tau := \Phi_*(\frac{\partial}{\partial \tau})$. Let γ be the induced metric on $\Phi(\tau, \Sigma)$, and let the area of the embedding $\Phi(\tau, \Sigma)$ be $A(\tau)$. Let $\{E_i\}$ be a local trivialization of $T\Phi(\tau, \Sigma)$ which commutes with the variation field Φ_τ along Φ, and which is a local orthonormal frame field on Σ, and let ν be the outward-pointing co-normal on $\partial\Sigma$ (in the FRW case). The mean curvature vector field of Σ is then $\vec{H} = \sum_{i=1}^{n} (\nabla_{E_i} E_i)^N$, so that for any normal vector field W on Σ, $div_\Sigma W = -g(W, \vec{H})$. Let the scalar-valued mean curvature H be defined by $\vec{H} = HN$. For example, in the case of $\Sigma \subset \{t_0\} \times \mathbb{H}^n$ in FRW as above, $H = nh(t_0)$.

Recall the basic calculation for the first variation (where a dot denotes differentiation with respect to τ):

$$\frac{d}{d\tau}\sqrt{\det \gamma} = \frac{1}{2}\sqrt{\det \gamma}\left(\gamma^{ij}\dot{\gamma}_{ij}\right) = (div_{\Sigma_\tau} \Phi_\tau)\sqrt{\det \gamma}. \tag{A.1}$$

If we evaluate this at $\tau = 0$ and split $V = V^T + V^N$ into tangential and normal parts, we can apply the divergence theorem to $div_\Sigma V^T$ to derive the first variation of area formula

$$A'(0) = -\int_\Sigma g(V, \vec{H})\, dA_\Sigma + \int_{\partial\Sigma} g(V, \nu) d\sigma_{\partial\Sigma}. \tag{A.2}$$

For the second variation we consider

$$\frac{d^2}{d\tau^2}\sqrt{\det\gamma} = \frac{1}{4}\sqrt{\det\gamma}\,(\gamma^{ij}\dot\gamma_{ij})^2 - \frac{1}{2}\sqrt{\det\gamma}\,\gamma^{ik}\dot\gamma_{kl}\gamma^{lj}\dot\gamma_{ij} + \frac{1}{2}\sqrt{\det\gamma}\,\gamma^{ij}\ddot\gamma_{ij}. \tag{A.3}$$

We evaluate each of these at $\tau = 0$. By (A.1)

$$\frac{1}{4}\sqrt{\det\gamma}\,(\gamma^{ij}\dot\gamma_{ij})^2 = (div_\Sigma(V))^2\sqrt{\det\gamma}. \tag{A.4}$$

Using the fact that V commutes with E_i, the second term in (A.3) is

$$-\frac{1}{2}\sqrt{\det\gamma}\sum_{i,j=1}^{n}\Big(g(\nabla_{E_i}V, E_j) + g(\nabla_{E_j}V, E_i)\Big)^2. \tag{A.5}$$

The third term in (A.3) involves second derivatives of the induced metrics, and can be expressed in terms of curvature quantities:

$$\begin{aligned}
\frac{1}{2}\gamma^{ij}\ddot\gamma_{ij} &= \frac{1}{2}\gamma^{ij}\Big(g(\nabla_V\nabla_{\Phi_\tau}E_i, E_j) + 2g(\nabla_V E_i, \nabla_V E_j) + g(E_i, \nabla_V\nabla_{\Phi_\tau}E_j)\Big)\\
&= \sum_{i=1}^{n}\Big[g(\nabla_V E_i, \nabla_V E_i) + g(\nabla_V\nabla_{E_i}\Phi_\tau, E_i)\Big]\\
&= \sum_{i=1}^{n}\Big[g(\nabla_V E_i, \nabla_V E_i) + g(\nabla_{E_i}\nabla_V\Phi_\tau, E_i) + g(R(E_i, V, V), E_i)\Big].
\end{aligned}$$

We used the fact that E_i commutes with Φ_τ, and we have used the curvature convention $R(X, Y, Z) = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z$.

Thus the third term in (A.3) is just

$$\sqrt{\det\gamma}\Big(\sum_{i=1}^{n}\Big[g(\nabla_{E_i}V, \nabla_{E_i}V) - g(R(V, E_i)V, E_i) + div_\Sigma(\nabla_V\Phi_\tau)\Big]\Big). \tag{A.6}$$

In the Riemannian case, we consider $\Phi_\tau = f(\tau, x)N(\tau, x)$ to be normal to the embedding. In the Lorentzian case, we let the variation $\Phi_\tau = f(\tau, x)\frac{\partial}{\partial t}\Big|_{\Phi(\tau,x)}$ be in the t-direction. Moreover, we also assume the volume $V(\tau)$ enclosed by $\Phi(\tau, \Sigma)$ (either the cone volume, or the homological volume, depending on the case) is constant. In the FRW case, we let $D(\Sigma) \subset \mathbb{H}^n$ be the projection of Σ onto the second factor of $I \times \mathbb{H}^n$. There is a function $t(\tau, p)$ so that $V(\tau) = \int\limits_{p\in D(\Sigma)}\int\limits_{0}^{t(\tau,p)} a(s)^n\, ds\, dA_{\mathbb{H}^n}$. In either case, $V'(0) = \int_\Sigma f\, dA_\Sigma$, and by applying (A.1) and (A.2) we obtain

$$V''(0) = \int\limits_\Sigma\left(\frac{\partial f}{\partial\tau} - \epsilon H f^2\right)\, dA_\Sigma = 0,$$

where $\epsilon = g(N,N) = \pm 1$. We use the fact that $g(\nabla_V \Phi_\tau, \vec{H}) = \epsilon H \frac{\partial f}{\partial \tau}$, and $div_\Sigma V = -g(V, \vec{H})$ since V is normal to Σ in either case, to write

$$\begin{aligned} -\epsilon H V''(0) &= \int_\Sigma \left[(g(V,H))^2 - g(\nabla_V \Phi_\tau, \vec{H}) \right] \, dA_\Sigma \\ &= \int_\Sigma \left[(div_\Sigma V)^2 - g(\nabla_V \Phi_\tau, \vec{H}) \right] \, dA_\Sigma. \end{aligned} \tag{A.7}$$

We now have to add the terms in (A.4)-(A.6) together and integrate over Σ. We first use the volume constraint to handle the term in (A.4) and the divergence term in (A.6). This latter term can be written

$$div_\Sigma(\nabla_V \Phi_\tau) = div_\Sigma(\nabla_V \Phi_\tau)^T + div_\Sigma(\nabla_V \Phi_\tau)^N = div_\Sigma(\nabla_V \Phi_\tau)^T - g(\nabla_V \Phi_\tau, \vec{H}).$$

We apply the divergence theorem to get

$$\int_\Sigma div_\Sigma(\nabla_V \Phi_\tau)^T \, dA_\Sigma = \int_{\partial\Sigma} g(\nu, \nabla_V \Phi \tau) \, d\sigma_{\partial\Sigma} = 0,$$

since it is easy to show $\nabla_V \Phi_\tau$ is normal to Σ. Using this and (A.7) along with the volume constraint, we see the term in (A.4) and the divergence term in (A.6) do not contribute to the second variation.

We have some more simplification for normal variations:

$$\sum_{i=1}^n g(\nabla_{E_i} V, \nabla_{E_i} V) = \sum_{i=1}^n g(\nabla^N_{E_i} V, \nabla^N_{E_i} V) + \sum_{i,j=1}^n (g(\nabla_{E_i} V, E_j))^2.$$

Since V is normal, $g(\nabla_{E_i} V, E_j) = -g(V, \nabla_{E_i} E_j)$. This allows us to combine terms and arrive at the well-known second variation of area for normal variations preserving volume:

$$A''(0) = \int_\Sigma \Big(\sum_{i=1}^n g(\nabla^N_{E_i} V, \nabla^N_{E_i} V) - \sum_{i,j=1}^n (g(\nabla_{E_i} E_j, V))^2 - g(R(V, E_i)V, E_i) \Big) \, dA_\Sigma.$$

Since $\nabla^N_{E_i} V = E_i[f] \, N$ (note that $\frac{\partial}{\partial t} = N$ on $\{t_0\} \times \mathbb{H}^n$), we have

$$A''(0) = \int_\Sigma \Big(\epsilon |\nabla^\Sigma f|^2 - \sum_{i,j=1}^n (g(\nabla_{E_i} E_j, V))^2 - f^2 Ric(N,N) \Big) \, dA_\Sigma.$$

We observe that the FRW metrics with $a''(t) \leq 0$ satisfy $A''(0) \leq 0$. More generally, if g satisfies the *time-like convergence condition* (nonnegative Ricci curvature in time-like directions), then we see that $A''(0) \leq 0$.

Appendix B. Metric Calculations

Let $g = u^4 g_E = u^4(r)(dr^2 + r^2(d\varphi^2 + \sin^2\varphi \, d\theta^2))$. We summarize below calculations of the Christoffel symbols and curvatures used in Section 4. For this purpose, we index (r, φ, θ) as $(1,2,3)$ respectively

CHRISTOFFEL SYMBOLS: The non-vanishing Christoffel symbols are as follows.

$$\Gamma^1_{11} = \frac{2u'}{u}, \quad \Gamma^1_{22} = -(\frac{2u'}{u}r^2 + r),$$
$$\Gamma^1_{33} = -\sin^2\varphi(\frac{2u'}{u}r^2 + r), \quad \Gamma^2_{33} = -\sin\varphi\ \cos\varphi,$$
$$\Gamma^3_{23} = \cot\varphi, \quad \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{2u'}{u} + \frac{1}{r}.$$

RADIAL RICCI CURVATURE:

$$\begin{aligned} R^2_{121} = R^3_{131} &= \frac{-2u''}{u} + \frac{2u'^2}{u^2} - \frac{2u'}{ru} \\ \mathrm{Ric}^M(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = R^1_{111} + R^2_{121} + R^3_{131} &= \frac{-4u''}{u} + \frac{4u'^2}{u^2} - \frac{4u'}{ru} \end{aligned}$$

TANGENTIAL SECTIONAL CURVATURE:

$$\begin{aligned} R(\partial_\varphi, \partial_\theta, \partial_\varphi) &= \nabla_{\partial_\theta}\nabla_{\partial_\varphi}\partial_\varphi - \nabla_{\partial_\varphi}\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\theta}(\Gamma^1_{22}\partial_r) - \nabla_{\partial_\varphi}(\Gamma^3_{23}\partial_\theta) \\ &= \Big(1 - r^2(\frac{1}{r} + \frac{2u'}{u})^2\Big)\partial_\theta \\ K^M &= \frac{\langle R(\partial_\varphi, \partial_\theta, \partial_\varphi), \partial_\theta\rangle}{||\partial_\theta||^2||\partial_\varphi||^2} = \frac{-4}{u^6}\Big((u')^2 + \frac{uu'}{r}\Big). \end{aligned}$$

SECOND FUNDAMENTAL FORM: Let $N = -u^{-2}\partial_r$ be normal to S_r. We now compute the second fundamental form of S_r with respect to N.

$$\begin{aligned} \mathrm{II}_{\theta\theta} &= \mathrm{II}(\partial_\theta, \partial_\theta) = \langle -\nabla_{\partial_\theta}N, \partial_\theta\rangle = \big\langle u^{-2}\Gamma^k_{31}\partial_k, \partial_\theta\big\rangle \\ &= u^{-2}(\frac{2u'}{u} + \frac{1}{r})g_{\theta\theta} = u^2r^2\sin^2\varphi(\frac{2u'}{u} + \frac{1}{r}) \\ \mathrm{II}_{\varphi\varphi} &= \mathrm{II}(\partial_\varphi, \partial_\varphi) = \big\langle -\nabla_{\partial_\varphi}N, \partial_\varphi\big\rangle = \big\langle u^{-2}\Gamma^k_{21}\partial_k, \partial_\varphi\big\rangle \\ &= u^{-2}(\frac{2u'}{u} + \frac{1}{r})g_{\varphi\varphi} = u^2r^2(\frac{2u'}{u} + \frac{1}{r}) \\ ||\mathrm{II}||^2 &= g^{ik}_{S_r}g^{jl}_{S_r}\mathrm{II}_{ij}\mathrm{II}_{kl} = g^{\theta\theta}g^{\theta\theta}\mathrm{II}^2_{\theta\theta} + g^{\varphi\varphi}g^{\varphi\varphi}\mathrm{II}^2_{\varphi\varphi} = \frac{2}{u^4}(\frac{2u'}{u} + \frac{1}{r})^2. \end{aligned}$$

MEAN CURVATURE: The mean curvature of S_r measured with respect to $N = -u^{-2}\partial_r$ is given by

$$H = g^{\phi\phi}\mathrm{II}_{\phi\phi} + g^{\theta\theta}\mathrm{II}_{\theta\theta} = \frac{2}{u^2}\Big(\frac{2u'}{u} + \frac{1}{r}\Big).$$

STABILITY: The sphere S_r has $\lambda_1 = \frac{2}{u^4r^2}$, so that the second variation (4.1) (with $V = \eta N$) and the Poincaré inequality, along with the identities above yield

$$\begin{aligned} A''(0) &\geq \int_{S_r} \eta^2\left(\frac{2}{u^4r^2} - \frac{2}{u^4}\Big(\frac{2u'}{u} + \frac{1}{r}\Big)^2 - u^{-4}\left(\frac{-4u''}{u} + \frac{4u'^2}{u^2} - \frac{4u'}{ur}\right)\right) dA \\ &= 4u^{-6}\left(uu'' - \frac{uu'}{r} - 3(u')^2\right)\int_{S_r}\eta^2\, dA. \end{aligned}$$

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