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REPORT No. 40, 2008/2009, fall

ISSN 1103-467X
ISRN IML-R- -40-08/09- -SE+fall

The light-cone theorem

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March 31, 2009

Abstract

We prove that the area of cross-sections of light-cones, in space-times satisfying suitable energy conditions, is smaller than or equal to that of the corresponding cross-sections in Minkowski, or de Sitter, or anti-de Sitter space-time, with equality if and only if the metric coincides with the corresponding model to the future of the light-cone.

Contents

*PTC and YCB are grateful to the Mittag-Leffler Institute, Djursholm, Sweden, for hospitality and financial support during the major part of work on this paper.

1 Introduction

It is a well known fact in general relativity that gravitation tends to focus null geodesics; this fact lies at the heart of, e.g., the singularity theorems of Hawking and Penrose [10]. In this work we wish to point a simple and striking illustration of this fact, which seems to have been overlooked in the literature, concerning the area of sections of light-cones: We prove that areas of sections of light-cones, in a space-time satisfying the Einstein equations with vanishing cosmological constant Λ, and with the energy-momentum satisfying the dominant energy condition, are smaller than the corresponding areas of sections of light-cones in Minkowski space-time. Moreover, under supplementary restrictions on the energy-momentum tensor, equality of areas for a section S implies that the space-time is Minkowski in the domain of dependence of the part of the light-cone which lies to the past of the section S. A similar result holds when $\Lambda \neq 0$, in the statement just given one needs to replace the Minkowski space-time by the de Sitter or anti-de Sitter space-time. The precise statements can be found in Section 2.

The idea of the argument is to show, using the dominant energy condition, that the divergence of the light-cone is smaller than that of the model space; this implies the area inequality. The rigidity part of our statement is based on an analysis, closely following that in [13], of the associated characteristic Cauchy problem; see also [2, 5, 6, 8, 12, 14] and references therein.

2 The theorem

Consider an $(n+1)$–dimensional space-time $(\mathscr{M}, g)$, $n \geq 2$, satisfying the *dominant energy-condition*,

$$T_{\mu\nu}X^\mu Y^\nu \geq 0 \text{ for all future oriented timelike vectors } X \text{ and } Y. \tag{2.1}$$

This will be the only condition needed for our comparison result. However, to obtain rigidity, more conditions will be needed. We shall say that the *rigid dominant energy condition holds at* q if (2.1) holds, together with the implication:

$$T_{\mu\nu}X^\mu X^\nu = 0 \text{ for some causal vector } X \text{ at } q \implies T_{\mu\nu}X^\nu = 0 \text{ at } q. \quad (2.2)$$

(It is well known that the implication is always true for timelike vectors by (2.1) (compare Appendix B), so this is only a restriction for null X's.)

General relativistic fluids with timelike flow vector u^μ, with $0 \le |p| \le \rho$, and with an equation of state which excludes the possibility $p = -\rho$ except when $\rho = 0$, provide energy-momentum tensors satisfying (2.2) everywhere. Another example is provided by the energy-momentum $T_{\mu\nu} = \rho \ell_\mu \ell_\nu$, where $\rho \ge 0$ and ℓ_μ is null.

Examples of energy-momentum tensor satisfying the dominant energy condition and which do *not* satisfy (2.2) are given by $T_{\mu\nu} = -\rho g_{\mu\nu}$, $\rho \ge 0$,[1] or by massless scalar fields, or by the Maxwell energy-momentum tensor, as discussed in Appendix A.

There is, however, a version of (2.2) which applies to both massless scalar fields and Maxwell fields; see Propositions A.2 and A.3 below; we emphasize that the argument there is *non-local* (as it requires integration), and *non-algebraic* (as it makes use of the field equations): To define this, let ℓ be a field of null tangents to a null hypersurface $\mathscr{N}$. We shall say that the *rigid dominant energy condition holds on* $\mathscr{N}$ if (2.1) holds together with the implication:

$$T_{\mu\nu}\ell^\mu \ell^\nu = 0 \text{ on } \mathscr{N} \implies T_{\mu\nu}\ell^\nu = 0 \text{ on } \mathscr{N}. \quad (2.3)$$

Let $p \in \mathscr{M}$ and let $\mathscr{C}_p^+$ be the future light-cone emanating from p. Let T be any unit timelike vector at p, and normalize all null vectors ℓ at p by requiring that $g(\ell, T) = -1$. This defines an affine parameter, denoted by s, on the future null geodesics $s \mapsto \gamma_\ell(s)$ with $\gamma_\ell(0) = p$ and with initial tangent ℓ. Let $\mathscr{A}(s)$ denote the $(n-1)$–dimensional surface reached by these geodesic after affine time s:

$$\mathscr{A}(s) = \{\gamma_\ell(s)\} \subset \mathscr{C}_p^+ , \quad (2.4)$$

where the vectors ℓ run over all null future vectors at p normalized as above; see Figure 2. We denote by $\mathscr{C}(t)$ the subset of the light-cone covered by all

[1] Our signature is $(-, + \dots, +)$.

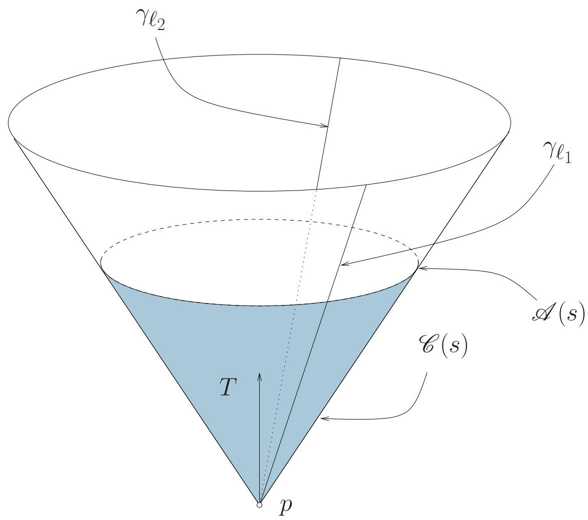

Figure 2.1: The section $\mathscr{A}(s)$ of the light-cone $\mathscr{C}_p^+$; $\mathscr{C}(s)$ is the shaded blue region. Two generators γ_{ℓ_1} and γ_{ℓ_2} are also shown.

the geodesics up to affine time t:

$$\mathscr{C}(t) = \cup_{0\le s\le t}\mathscr{A}(s)\ . \tag{2.5}$$

Note that $\gamma_\ell(s)$ might not be defined for all s. Further, $\mathscr{A}(s)$ might not be a smooth surface. However, for every point p there exists a maximal $s_0 > 0$ such that $\mathscr{A}(s)$ is defined and smooth for all $s < s_0$. We restrict ourselves to $s < s_0$, though it is rather clear that this can be relaxed using the methods of [4]; we have, however, not attempted to verify all details of that.

Let $|\mathscr{A}(s)|_g$ denote the area of $\mathscr{A}(s)$. So if g is the Minkowski metric, which we denote by η, then we have

$$|\mathscr{A}(s)|_\eta = \omega_{n-1}s^{n-1}\ ,$$

where ω_{n-1} is the area of the unit round sphere in $\mathbb{R}^n$.

We consider metrics satisfying the Einstein equations with cosmological constant $\Lambda\in\mathbb{R}$ and sources. We assume smoothness of the metric for simplicity, though our result can be proved under weaker differentiability conditions:

THEOREM 2.1 *Let $(\mathscr{M}, g)$ be a smooth globally hyperbolic space-time satisfying the Einstein equations with source a stress energy tensor satisfying the dominant energy condition. We restrict attention to s such that $\mathscr{C}(s)$ lies within the domain of injectivity of the exponential map at p.*

1. *The area $|\mathscr{A}(s)|_g$ satisfies the inequality*

$$|\mathscr{A}(s)|_g \le |\mathscr{A}(s)|_\eta \, . \tag{2.6}$$

2. *If equality is attained at some $s = s_2$, and if moreover* either
   (a) *the rigid dominant energy holds at $\mathscr{C}(s_2)$,* or
   (b) *the energy-momentum tensor is traceless,*

   then the domain of dependence of $\mathscr{C}(s_2)$ is isometric to the corresponding domain of dependence in Minkowski space-time, or de Sitter space-time, or anti-de Sitter space-time.

PROOF: Let θ denote the rate of change of area along the null geodesic generators of $\mathscr{C}_p^+$, and let σ denote the shear of $\mathscr{C}_p^+$ (see, e.g., [9]). Let γ be such a generator, and recall the Raychaudhuri equation in space-time dimension $n+1$ [9]:

$$\frac{d\theta}{ds} = -\sigma^{AB}\sigma_{AB} - \frac{1}{n-1}\theta^2 - R_{\sigma\mu}\dot\gamma^\sigma\dot\gamma^\mu \, . \tag{2.7}$$

Here s is an affine parameter along the generators: $\nabla_{\dot\gamma}\dot\gamma = 0$.[2]

Before giving a detailed proof, it might be useful to present an outline: Let θ_0 denote the divergence of a light-cone in Minkowski space-time:

$$\theta_0 := \frac{n-1}{s} \, ;$$

then θ_0 satisfies (2.7) with vanishing Ricci tensor and σ. Since θ approaches $(n-1)/s$ as the tip of the light-cone is approached, a comparison argument using (2.7) shows that θ is smaller than its Minkowskian value. This, subsequently, implies the area inequality. Equality holds on $\mathscr{A}(s_2)$ if and only if σ

[2]For the proof of rigidity we will be using a coordinate system (u, r, x^A), where $s = r$, with a wave-map condition imposed on the extension of the coordinates away from the light-cone. However, no such condition is needed for the comparison argument.

and $R_{\sigma\mu}\dot\gamma^\sigma\dot\gamma^\mu$ vanish along all geodesic generators of $\mathscr{C}_p^+$ until these generators reach $\mathscr{A}(s_2)$, i.e., on $\mathscr{C}(s_2)$. When the rigid dominant energy condition holds (in either its local or nonlocal form), the usual energy calculation implies that the metric is vacuum in the domain of dependence of $\mathscr{C}(s_2)$. Under the traceless condition a more detailed analysis is necessary. This, together with the vanishing of σ on $\mathscr{C}(s)$, is used to show that the metric tensor takes the model-metric values on $\mathscr{C}(s)$, and the result follows by uniqueness of solutions of the characteristic initial value problem.

Let us pass now to the details of the above. Since θ_0 satisfies the equation

$$\frac{d\theta_0}{ds} = -\frac{\theta_0^2}{n-1} ,$$

from (2.7) we have

$$\begin{aligned}\frac{d(\theta-\theta_0)}{ds} &= \frac{\theta_0^2-\theta^2}{n-1} - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \\ &= -\frac{(\theta-\theta_0)^2}{n-1} - \frac{2}{s}(\theta-\theta_0) - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \\ &\leq -\frac{2}{s}(\theta-\theta_0) - \sigma_{AB}\sigma^{AB} - R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \ . \end{aligned} \tag{2.8}$$

Hence, for $s > s_1 > 0$,

$$s^2(\theta-\theta_0)(s) \leq s_1^2(\theta-\theta_0)(s_1) - \int_{s_1}^{s} (\sigma_{AB}\sigma^{AB} + R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu)s^2 ds \ . \tag{2.9}$$

Now, for a smooth metric we have

$$\theta = \frac{(n-1)+o(1)}{s} \tag{2.10}$$

for small s, so we can pass to the limit $s_1 \to 0$ to obtain

$$(\theta-\theta_0)(s) \leq -\frac{1}{s^2}\int_0^s (\sigma_{AB}\sigma^{AB} + R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu)s^2 ds \ . \tag{2.11}$$

Since the dominant energy condition has been assumed to hold, the right-hand-side of (2.11) is non-positive and we conclude that

$$\theta(s) \leq \frac{n-1}{s} \tag{2.12}$$

as long as the geodesic exists. Furthermore, equality holds for some $s_2 > 0$ if and only if

$$\forall s \text{ satisfying } 0 < s < s_2 \;, \quad \sigma_{AB} = 0 = R_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu \;. \qquad (2.13)$$

The area inequality follows from (2.12) in a standard way, we give the details for completeness. In a coordinate system adapted to the light-cone we can write the metric on the cone in the form

$$g = -\alpha du^2 + 2\nu_A dx^A du - 2e^{2\beta} du dr + g_{AB} dx^A dx^B \;, \qquad (2.14)$$

where $\mathscr{C}_p^+ = \{u = 0\}$, where r is an affine parameter along the generators of $\mathscr{C}_p^+$, vanishing at the vertex, denoted by s in the previous equations. Then

$$\theta = \frac{1}{\sqrt{\det g_{AB}}}\partial_r(\sqrt{\det g_{AB}}) \;. \qquad (2.15)$$

Let us denote by $\mathring{g}_{AB} dx^A dx^B$ the $(n-1)$–dimensional corresponding metric arising on a light-cone in the $(n+1)$–dimensional Minkowski space-time. Then our analysis so far shows that

$$\theta \equiv \partial_r \log \sqrt{\det g_{AB}} \le \theta_0 \equiv \partial_r \log \sqrt{\det \mathring{g}_{AB}} \;.$$

Thus $\log(\det g_{AB}/\det \mathring{g}_{AB})$ is decreasing. By elementary considerations the quotient $\det g_{AB}/\det \mathring{g}_{AB}$ tends to one as r tends to zero, and we conclude

$$\log \sqrt{\det g_{AB}} \le \log \sqrt{\det \mathring{g}_{AB}}, \quad \text{hence} \quad \sqrt{\det g_{AB}} \le \sqrt{\det \mathring{g}_{AB}} \;.$$

The areas $|\mathscr{A}(r)|_g$ and $|\mathscr{A}(r)|_\eta$ are

$$|\mathscr{A}(r)|_g \equiv \int_{S^{n-1}} \sqrt{\det g_{AB}}\, dx^2 \ldots dx^n, \qquad |\mathscr{A}(r)|_\eta \equiv \int_{S^{n-1}} \sqrt{\det \mathring{g}_{AB}}\, dx^2 \ldots dx^n \;,$$

therefore

$$|\mathscr{A}(r)|_g \le |\mathscr{A}(r)|_\eta \;,$$

which establishes part 1. of the theorem.

Assume, now, that equality in this last equation holds at $s = s_2$. Equation (2.13) implies the vanishing of $T_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu$ on $\mathscr{C}(s_2)$.

If we assume that the energy-momentum tensor T satisfies the rigid dominant energy condition, as in (2.2), or the rigid dominant energy condition on $\mathscr{C}(s_2)$, as in (2.3), we can conclude that $T_{\mu\nu}\dot\gamma^\nu$ vanishes on $\mathscr{C}(s_2)$. The proof that the metric is vacuum in the domain of dependence of $\mathscr{C}(s_2)$ is then standard, and proceeds as follows:

Consider the manifold

$$\widehat{\mathscr{M}} := \mathscr{M} \setminus J^+(\mathscr{A}(s_2)) ,$$

with the metric obtained from g by restriction, still denoted by g. Then $(\widehat{\mathscr{M}}, g)$ is globally hyperbolic, with

$$\mathscr{D}^+(\mathscr{C}(s_2), \widehat{\mathscr{M}}) = \mathscr{D}^+(\mathscr{C}(s_2), \mathscr{M}) , \tag{2.16}$$

where $\mathscr{D}^+(\Omega, \mathscr{M})$ denotes the domain of dependence of an achronal set Ω within a space-time $(\mathscr{M}, g)$. The equality in (2.16) means that the manifolds, equipped with the obvious metrics, are isometric.

Let t be any Cauchy time function on $\widehat{\mathscr{M}}$, i.e., a time function ranging over $\mathbb{R}$, the level sets of which are Cauchy surfaces. Replacing t by $t - t(p)$, we can without loss of generality assume that $t(p) = 0$. Let

$$E(s) = -\int_{\mathscr{D}^+(\mathscr{C}(s_2))\cap\{t=s\}} T^\mu{}_\nu n^\nu dS_\mu , \tag{2.17}$$

where n^μ is the field of future directed unit normals to the level sets of t; E is positive in our signature $(-, +, \ldots, +)$. The divergence identity on the set bounded by $\mathscr{C}(s_2) \cap \{t \le s\}$ and $\mathscr{D}^+(\mathscr{C}(s_2)) \cap \{t = s\}$ (compare (D.35), Appendix D, and Lemma B.1, Appendix B) shows that, for any time interval $[0, T]$, there exists a constant $C = C(T)$ such that

$$E(s) \le C \int_0^s E(t)dt - \underbrace{\int_{\mathscr{C}(s)} T^\mu{}_\nu n^\nu dS_\mu}_{=0} , \tag{2.18}$$

where the boundary integrand vanishes by the rigid dominant energy condition, being proportional to $T^\mu{}_\nu n^\nu \dot\gamma^\mu$. Since $E(s)$ approaches zero as s tends to zero, from Gronwall's lemma we obtain

$$E(s) = 0 \text{ for } 0 < s < s_2.$$

Positivity of the integrand implies

$$T_{\mu\nu} n^\mu n^\nu = 0 \quad \text{on } \mathscr{D}^+(\mathscr{C}(s_2)) . \tag{2.19}$$

From (2.19) and Lemma B.1, Appendix B, we conclude that an energy-momentum tensor satisfying the rigid dominant energy condition must vanish on every level set of t within the domain of dependence of $\mathscr{C}(s_2)$. As $\mathscr{D}^+(\mathscr{C}(s_2))$ is covered by these level sets, the vanishing of $T_{\mu\nu}$ on $\mathscr{D}^+(\mathscr{C}(s_2))$ follows.

The proof of (2.19) for tensors that *do not* satisfy the rigid dominant energy condition requires more care. In view of (2.13), at this stage of the analysis we can only conclude that

$$R_{rr} = 8\pi T_{rr} = 0 = 8\pi T_{rA} = R_{Ar} \tag{2.20}$$

on $\mathscr{C}(s_2)$. Indeed, to see the vanishing of T_{Ar}, set $\ell = \partial_r$. Then, by the dominant energy condition, the vector field $T^\mu{}_\nu \ell^\nu \partial_\mu$ is causal, and has vanishing scalar product with ℓ, hence is proportional to ℓ. So $T^\mu{}_r\partial_\mu$ is proportional to ∂_r; subsequently

$$T_{Ar} = g_{A\mu} \underbrace{T^\mu{}_r}_{0 \text{ unless } \mu = r} = \underbrace{g_{Ar}}_{0} T^r{}_r = 0 \ ,$$

as desired.

(Actually, we can further show that *a traceless* $T_{\mu\nu}$ must vanish at the vertex of the light-cone: for this, by continuity and (2.13) we find that $T_{\mu\nu}\ell^\mu\ell^\nu = 0$ at p for every null vector $\ell \in T_p\mathscr{M}$. By [3, Lemma 2.8], $T_{\mu\nu}$ is proportional to the metric at p, and tracelessness implies the claim. But this fact does not seem to be useful in the analysis that follows.)

Let x^μ denote normal coordinates centered at p, let $R > 0$ denote the largest number so that the exponential map at p is a diffeomorphism from a truncated solid cone $\Omega(R) \subset T_p\mathscr{M}$, defined as

$$\Omega(R) := \{0 \le x^0 \le R \ , \ r := \sqrt{\sum (x^i)^2} \le x^0\} \ ,$$

to its image in $\mathscr{M}$. Note that this image is included in $\mathscr{D}^+(\mathscr{C}^+_p(R))$ when the level sets of x^0 are timelike within $\Omega(R)$.

If $\Lambda = 0$, we let the functions y^μ be solutions of the following characteristic Cauchy problem:

$$\Box_g y^\mu = 0 \ , \tag{2.21}$$

$$y^\mu|_{\mathscr{C}^+_p(R)} = x^\mu \quad . \tag{2.22}$$

For non-zero Λ, we impose again the boundary conditions (2.22), but we require instead that the map $x^\alpha \mapsto y^\mu(x^\alpha)$ satisfies the wave-map equation, with the (anti)-de Sitter metric in the target,

$$\mathring{g} = -\left(1 - \frac{2\Lambda}{n(n-1)} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2\Lambda}{n(n-1)} r^2} + r^2 \mathring{h}_{AB} dx^A dx^B \ , \qquad (2.23)$$

where $\mathring{h}_{AB} dx^A dx^B$ is the round unit metric on S^{n-1}. Thus, in both cases, the functions y^μ satisfy the set of equations

$$g^{\alpha\beta}\left(\partial_\alpha \partial_\beta y^\mu - \Gamma^\sigma_{\alpha\beta} \frac{\partial y^\mu}{\partial x^\sigma} + \mathring{\Gamma}^\mu_{\nu\rho} \frac{\partial y^\nu}{\partial x^\alpha} \frac{\partial y^\rho}{\partial x^\beta}\right) = 0 \ , \qquad (2.24)$$

where the $\mathring{\Gamma}^\mu_{\nu\rho}$'s are the Christoffel symbols of $\mathring{g}$, except that (2.24) is linear when $\Lambda = 0$, and thus the solutions exist globally on the domain of dependence of the smooth part of the light-cone, while for $\Lambda \neq 0$ the solutions might exist only for some neighborhood of the tip of the light-cone.

By [7, Theorem 5.4.2] (compare [1]) the functions y^μ are smooth up-to-boundary on $\mathscr{D}^+(\mathscr{C}_p^+(R))$. Decreasing R if necessary, the functions y^μ form a smooth coordinate system on $\mathscr{D}^+(\mathscr{C}_p^+(R))$. Let $g_{\mu\nu}$ denote the components of the metric in the coordinates y^μ, then the $g_{\mu\nu}$'s are smooth up-to-boundary on $\mathscr{D}^+(\mathscr{C}_p^+(R))$. If we pass to a coordinate system so that

$$u := y^0 - |\vec{y}| \ ,$$

and $r = |\vec{y}|$, and where the x^A's are local coordinates on S^{n-1}, then the cone is given by the equation $u = 0$, and the metric on $\mathscr{C}_p^+(R)$ takes the form (2.14):

$$g = -\alpha du^2 + 2\nu_A dx^A du - 2e^{2\beta} du\, dr + h_{AB} dx^A dx^B \ . \qquad (2.25)$$

We emphasize that we *do not* assume that the metric takes the form (2.25) *away* from $\{u = 0\}$, so care must be taken when ∂_u–derivatives are taken.

By definition [9], σ_{AB} is the trace-free part of

$$g(\nabla_A \partial_r, \partial_B) = g_{BC} \Gamma^C_{Ar} = \frac{1}{2} \partial_r g_{AB} \ ,$$

so from the vanishing of σ_{AB}, and from the the explicit formula for $\theta = \theta_0$ we obtain

$$\partial_r h_{AB} = \frac{2}{r} h_{AB} \quad \Longleftrightarrow \quad \partial_r (r^{-2} h_{AB}) = 0 \ . \qquad (2.26)$$

Since $r^{-2}h_{AB}$ tends to the unit round metric $\mathring{h}_{AB}$ on S^{n-1} as r tends to zero, we conclude that

$$h_{AB} = r^2 \mathring{h}_{AB} \ .$$

We continue by showing that $\beta = 0$. For this note that, by definition of normal coordinates, r is an affine parameter along the geodesics generators of $\mathscr{C}_p^+$. So $\nabla_{\partial_r}\partial_r = 0$, which is equivalent to $0 = \Gamma^\mu_{rr}$. But

$$\Gamma^\mu_{rr} = \delta^\mu_r(2\partial_r\beta + \frac{1}{2}e^{-2\beta}\partial_u g_{rr}) \ , \quad \text{and we conclude that} \quad e^{-2\beta}\partial_u g_{rr} = -4\partial_r\beta \ . \tag{2.27}$$

We set

$$\lambda^\mu := -g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} \ , \tag{2.28}$$

$$\mathring{\lambda}^\mu := -g^{\alpha\beta}\mathring{\Gamma}^\mu_{\alpha\beta} \ , \tag{2.29}$$

where $\mathring{\Gamma}^\mu_{\alpha\beta}$ denotes the Christoffel symbols of $\mathring{g}_{\mu\nu}$. The wave map condition $\lambda_r = g_{r\mu}\mathring{\lambda}^\mu$ can be shown to read

$$\underbrace{\frac{1}{2}h^{AB}\partial_r h_{AB}}_{\frac{n-1}{r}} + \underbrace{e^{-2\beta}\partial_u g_{rr}}_{-4\partial_r\beta} \equiv \lambda_r = g_{r\mu}\mathring{\lambda}^\mu \equiv \frac{n-1}{r}e^{2\beta} \ .$$

Writing $y = e^{2\beta}$, this is the same as

$$\partial_r y = \frac{n-1}{2r}y(1-y) \ .$$

Integrating, we obtain either $y \equiv 1$, or

$$y = \frac{C(x^A)r^{(n-1)/2}}{1 + C(x^A)r^{(n-1)/2}} \ ,$$

for some function $C(x^A)$. But, in normal coordinates, β approaches zero as r goes to zero, and we conclude that $y \equiv 1$; equivalently, $\beta \equiv 0$.

In Appendix C we show that the vanishing of R_{rA} is equivalent to

$$\begin{aligned} 0 &= \frac{(n-2)(n-3)}{2r^2}\nu_A + \frac{3n-5}{2r}\partial_r\nu_A + \partial_r\partial_r\nu_A \\ &= \frac{1}{r^{n-1}}\partial_r\left[r^{n-1}\left(\partial_r\nu_A + \frac{n-3}{2r}\nu_A\right)\right] \ . \end{aligned} \tag{2.30}$$

Integrating (2.30) in r once we obtain, for some smooth functions $\hat\nu_A = \hat\nu_A(x^B)$,

$$\hat\nu_A r^{1-n} = \partial_r \nu_A + \frac{n-3}{2r}\nu_A = r^{\frac{-n+3}{2}}\partial_r(r^{\frac{n-3}{2}}\nu_A) \; .$$

Integrating again, we conclude that there exist smooth functions $\mathring{\nu}_A(x^B)$ such that, for $n>1$,

$$\nu_A(r,x^B) = r^{\frac{3-n}{2}}\mathring{\nu}_A(x^B) - \frac{2}{(n-1)} r^{2-n}\hat\nu_A(x^B) \; . \tag{2.31}$$

But from the definition of our coordinate system it is elementary to show that ν_A approaches zero as $r\to 0$, which implies that $\nu_A \equiv 0$.

We are ready now to establish (2.19) for traceless energy-momentum tensors. For this let

$$\Omega(s_*) := J^+(p)\cap\{t<s_*\} \; , \tag{2.32}$$

where $t = x^0$ is a normal coordinate, and where we define $s_* \le R$ to be the largest number smaller than or equal to s_2 such that $\Omega(s_*)$ lies within the domain of definition of normal coordinates, assuming moreover that ∂_t and ∇t are timelike on $\Omega(s_*)$, and that the functions y^μ, defined as solutions of (2.24), form a coordinate system on $\Omega(s_*)$. The proof of the vanishing of $T_{\mu\nu}$, to be found in Appendix D, is again an energy calculation, using instead the energy functional defined as

$$E(s) = -\int_{\mathscr{D}^+(\mathscr{C}(s_2))\cap\{t=s\}} T^\mu{}_\nu X^\nu dS_\mu \; , \tag{2.33}$$

where $X = X^\mu\partial_\mu$ is the vector field with normal–coordinates components

$$X^\mu = x^\mu \; . \tag{2.34}$$

This choice of X^μ ensures the vanishing of the boundary term that arises on $\mathscr{C}(s_*)$ in the divergence identity (D.35) of Appendix D. However, this leads to a difficulty because X^μ is null at $\mathscr{C}(s)$, which implies that the integrand of (2.33) does not control uniformly the energy as the boundary $\mathscr{C}(s_*)$ of $\Omega(s_*)$ is approached. Thus, the standard energy argument requires a careful reinspection. The price to pay is the need to impose tracelessness of $T_{\mu\nu}$. Moreover the argument does not guarantee that the metric is vacuum throughout $\mathscr{D}^+(\mathscr{C}(s_2))$, but only on $\mathscr{D}^+(\mathscr{C}(s_*))$,[3] and we will return to this issue at the end of the proof.

We let s_* be the number defined in the paragraph after (2.32) when $T_{\mu\nu}$ is traceless, and we set $s_* = s_2$ if the rigid dominant energy condition holds on $\mathscr{C}(s_2)$. Since the metric is now vacuum on $\mathscr{D}^+(\mathscr{C}(s_*))$, we have

$$g^{AB}R_{AB} = 2\Lambda \tag{2.35}$$

there. We shall use (2.35) to prove that $\alpha = 1 - 2\Lambda r^2/n(n-1)$ on $\mathscr{C}(s_*)$.

Recall that, at this stage, on $\mathscr{C}(s_*)$ the metric takes the form

$$g = -\alpha du^2 - 2dudr + r^2 \mathring{h}_{AB} dx^A dx^B \ . \tag{2.36}$$

In Appendix E we show that

$$g^{AB}R_{AB} = 4\Lambda\frac{n+1}{n-1} + 2\partial_r\partial_r\alpha + \frac{3(n-1)}{r}\partial_r\alpha + \frac{(n-1)(n-2)}{r^2}(\alpha - 1). \tag{2.37}$$

This, together with (2.35), provides a Fuchsian ODE for $\alpha - 1$, with characteristic exponents λ which solve the equation

$$2\lambda(\lambda-1) + 3(n-1)\lambda + (n-1)(n-2) = 0 \ ,$$

and thus the solutions are

$$\alpha = 1 - \frac{2\Lambda}{n(n-1)}\ r^2 + \alpha_+(x^A)r^{\lambda_+} + \alpha_-(x^A)r^{\lambda_-} \ ,$$

where $\alpha_\pm$ are smooth functions on S^{n-1}, and

$$\lambda_\pm \in \left\{ \frac{1-n}{2}, 2-n \right\} \ .$$

Since both characteristic exponents are negative, the only regular solution is $\alpha \equiv 1 - \frac{2\Lambda}{n(n-1)} r^2$.

We have therefore shown that $g_{\mu\nu}$ takes the Minkowski, or (anti)-de Sitter form on $\mathscr{C}(s_*)$. Note that the energy argument above can be used to prove

[3]Strictly speaking, the argument presented in the Appendix D only proves that the metric is vacuum in $\Omega(s_*)$. But $\{t = s_*\} \cap J^+(\mathscr{C}(s_*))$ is a Cauchy surface for $\mathscr{D}^+(\mathscr{C}(s_*))$, so a standard argument proves then that the metric is vacuum in $\mathscr{D}^+(\mathscr{C}(s_*))$.

uniqueness of solutions of the reduced Einstein equations, with the components of the metric in the wave-map gauge prescribed on the light-cone, in the usual way (compare [6,12,14] and references therein). It follows that $g_{\mu\nu}$ equals the corresponding reference metric on the domain of dependence of $\mathscr{C}(s_*)$.

So, we have that $x^\mu = y^\mu$ on $\Omega(s_*)$, with $g_{\mu\nu} = \mathring{g}_{\mu\nu}$ there. If $s_* < s_2$, then one can repeat the argument of Appendix D to obtain the above conclusions on $\Omega(\hat{s}_*)$, for some $\hat{s}_*$ satisfying $s_* < \hat{s}_* \le s_2$. Using this observation, an easy open-closed argument shows that $s_* = s_2$, which had to be established. □

For further reference we note the following result, which follows immediately from (2.8) and (2.11):

Proposition 2.2 *The divergence $\theta(s)$ will become negative along a generator γ of $\mathscr{C}_p^+$ at some value of s strictly smaller than s_2 whenever*

$$\int_0^{s_2} (\sigma_{AB}\sigma^{AB} + R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu) s^2 ds \ge (n-1)s_2 \, . \tag{2.38}$$

□

Once $\theta(s)$ has become negative, standard arguments imply that θ will diverge in finite time, so that either γ will be incomplete, or will leave $\dot{J}^+(p)$ in finite time.

A The rigid dominant energy condition on the null cone: Maxwell and scalar fields

We start by verifying:

Proposition A.1 *Both the Maxwell energy-momentum tensor and the massless scalar field energy-momentum tensors satisfy the dominant energy condition.*

Proof: It suffices to show that if n^μ is unit and timelike, then $P_\mu := T_{\mu\nu}n^\mu$ is causal. Now, in an orthonormal frame e_μ with $n^\mu\partial_\mu = e_0$ we have, for the massless scalar field,

$$T_{00} = \frac{1}{2}(e_0(\phi))^2 + \frac{1}{2}\sum_i (e_i(\phi))^2 \, , \quad T_{0i} = e_0(\phi)e_i(\phi) \, ,$$

and the causal character of $P_\mu = T_{0\mu}$ follows from $a|\vec{b}| \le \frac{1}{2}(a^2 + |\vec{b}|^2)$.

For the Maxwell field, further rotating the frame so that $F_{0i} \sim \delta^1_i$, it holds that

$$\begin{aligned} T_{00} &= \frac{1}{2}\sum_j F_{0j}^2 + \frac{1}{4}\sum_{i,j} F_{ij}^2 = \frac{1}{2}F_{01}^2 + \frac{1}{2}\sum_j F_{1j}^2 + \frac{1}{4}\sum_{i,j\neq 1} F_{ij}^2 \ , \\ T_{0i} &= F_{01}F_{i1} \ , \end{aligned}$$

and the result follows as for the scalar field. □

Now we show that the scalar and Maxwell fields do not obey the rigid dominant energy condition in its local form (2.2) at a point q. For a scalar field ϕ, define $k_\mu \equiv \partial_\mu \phi|_q$. Then the energy-momentum tensor at q can be expressed as

$$T_{\mu\nu} = k_\mu k_\nu - \frac{1}{2}|k|_g^2 g_{\mu\nu}$$

For spacelike k_μ the associated tensor $T^\mu{}_\nu$ has null eigenvectors (which are orthogonal to k) with nonzero eigenvalue $-\frac{1}{2}|k|_g^2$, which implies that $T_{\mu\nu}$ does not obey the rigid dominant energy condition (2.2).

The Maxwell stress energy tensor of an electromagnetic field is, whatever the space-dimension $n \ge 2$,

$$T_{\alpha\beta} = F^\lambda{}_\alpha F_{\lambda\beta} - \frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu} \ .$$

At any point at which $F_{\mu\nu}$ is of the form $Y_{[\mu}Z_{\nu]}$, for some *spacelike* vectors Y and Z, there exist null vectors l^μ for which $F_{\mu\nu}l^\nu = 0$. Such vectors are eigenvectors of $T^\alpha{}_\beta$ with nonzero eigenvalue. This implies that the Maxwell field does not obey the rigid dominant energy condition (2.2).

Next, let $\mathscr{C}(s_2)$ be the subset of the future null cone $\mathscr{C}_p^+$ defined by (2.4)-(2.5). We have:

Proposition A.2 *In space-times satisfying the Einstein-Maxwell field equations, the rigid dominant energy condition holds on $\mathscr{C}(s_2)$.*

Proof: We use a coordinate system in which the metric takes the form (2.25). The condition $T_{\mu\nu}\ell^\mu\ell^\nu = 0$ in (2.3) reads in those coordinates

$$T_{rr} = F^\lambda{}_r F_{\lambda r} = 0 \ ,$$

with, by antisymmetry, $F_{rr} = F^u{}_r = 0$. Hence

$$T_{rr} = g^{AB} F_{Ar} F_{Br} = 0 \ ,$$

which implies

$$F_{Ar} = 0, \quad \text{hence also} \ \ F^{Au} = 0 \ .$$

Keeping in mind $g_{rr} = g_{rA} = 0$, we obtain a direct, alternative justification of (2.20):

$$T_{rA} = F^\lambda{}_r F_{\lambda A} = 0 \ .$$

The Maxwell identity $dF = 0$ shows that

$$\partial_r F_{AB} = 0 \ .$$

Because of the polar character of the coordinates x^A, regularity of F at the vertex gives the vanishing of $F_{AB} = 0$ there, and hence everywhere.

The Maxwell equation

$$\partial_\mu (\sqrt{|\det g_{\alpha\beta}|} F^{\mu u}) = 0$$

reduces in our coordinates to

$$\partial_r (e^{-2\beta} \sqrt{\det h_{AB}} F_{ru}) = 0 \ .$$

Since $e^{-2\beta}\sqrt{\det h_{AB}} F_{ru}$ tends to zero as $r \to 0$, we conclude that $F_{ur} \equiv 0$. Now (recall that $g_{ur} = -e^{2\beta}$ and $F_{Ar} = F^{Au} = 0$),

$$T_{ur} = F^u{}_u F_{ur} + \frac{e^{2\beta}}{2} F^{ur} F_{ur} + \frac{e^{2\beta}}{4} F^{AB} F_{AB} \ ,$$

and so

$$T_{ur} = 0 \ .$$

Hence $T_{\mu\nu}\ell^\mu$ vanishes on $\mathscr{C}(s_2)$, as desired. □

Similarly, we have

PROPOSITION A.3 *In space-times satisfying the Einstein — massless scalar field equations, the rigid dominant energy condition holds on* $\mathscr{C}(s_2)$.

Proof: In this case

$$T_{\alpha\beta} = \partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}g_{\alpha\beta}\partial_\lambda\phi\nabla^\lambda\phi\ .$$

Hence

$$T_{rr} = (\partial_r\phi)^2,$$

and $T_{rr} = 0$ implies $\partial_r\phi = 0$. So ϕ is constant on $\mathscr{C}(s_2)$; uniqueness of solutions of the wave equation implies that ϕ is constant in the domain of dependence of $\mathscr{C}(s_2)$, and so $T_{\mu\nu}$ vanishes there. □

B The dominant energy condition and its consequences

We will write $|Z|_g$ for $\sqrt{|g(Z,Z)|}$. Given a timelike vector n we shall write $|Z|_{g,n}$ for the square root of

$$g(Z,Z) + 2\frac{g(Z,n)^2}{|g(n,n)|} \geq 0\ . \tag{B.1}$$

Note that $|n|_g = |n|_{g,n}$, and also $|Z|_g = |Z|_{g,n}$ when Z is orthogonal to n.

We recall a well known result, which we prove for completeness:

Lemma B.1 *Suppose that a symmetric two-covariant tensor T satisfies the dominant energy condition (2.1), and let n be a timelike vector.*[4] *Then for any vectors W, Z we have*

$$|T(W,Z)| \leq \frac{|W|^2_{g,n} + |Z|^2_{g,n}}{|n|^2_g}T(n,n)\ . \tag{B.2}$$

Furthermore, for any causal vector X we also have

$$T(X,X) \leq \frac{2|X|_{g,n}}{|n|_g}T(X,n)\ . \tag{B.3}$$

[4] We hope that the clash of notation with the space-dimension n, as used elsewhere in this paper, will not lead to confusions.

REMARK B.2 Denoting by $|T|_{g,n}$ the norm of T with respect to the Riemannian metric associated with the quadratic form (B.1), (B.2) implies

$$|T|_{g,n} \leq \frac{2}{|n|_g^2} T(n,n) \ . \tag{B.4}$$

PROOF: Let, first W be orthogonal to n. As $|W|_{g,n} = |W|_g$, the vectors $W_\pm := |W|_{g,n} n \pm |n|_g W$ are null consistently time-oriented, thus

$$0 \leq T(W_+, W_-) = |W|_{g,n}^2 T(n,n) - |n|_g^2 T(W,W) \ ,$$

giving, for $W \perp n$,

$$T(W,W) \leq \frac{|W|_{g,n}^2}{|n|_g^2} T(n,n) \ . \tag{B.5}$$

Adding the two equations obtained by writing explicitly $T(W_+, W_+) \geq 0$ and $T(W_-, W_-) \geq 0$ gives

$$T(W,W) \geq -\frac{|W|_{g,n}^2}{|n|_g^2} T(n,n) \quad \Longrightarrow \quad |T(W,W)| \leq \frac{|W|_{g,n}^2}{|n|_g^2} T(n,n) \ . \tag{B.6}$$

We also have,

$$0 \leq T(n, W_\pm) = T(n, |W|_g n) \pm T(n, |n|_g W) \ ,$$

giving, again for $W \perp n$,

$$|T(W,n)| \leq \frac{|W|_{g,n} |n|_{g,n}}{|n|_g^2} T(n,n) \ . \tag{B.7}$$

Next, if both W and Z are orthogonal to n, using (B.6) we find

$$\begin{aligned} |T(W,Z)| &= \frac{1}{4} |T(W+Z, W+Z) - T(W-Z, W-Z)| \\ &\leq \frac{|W+Z|_{g,n}^2 + |W-Z|_{g,n}^2}{4|n|_g^2} T(n,n) \qquad \text{(B.8)} \\ &= \frac{|W|_{g,n}^2 + |Z|_{g,n}^2}{2|n|_g^2} T(n,n) \ . \qquad \text{(B.9)} \end{aligned}$$

Finally, for general vectors W and Z we can write

$$W = w \frac{n}{|n|_g} + W^\perp \ , \quad Z = z \frac{n}{|n|_g} + Z^\perp \ ,$$

with both $W^\perp$ and $Z^\perp$ orthogonal to n. Then

$$|W|^2_{g,n} = w^2 + |W^\perp|^2_{g,n} \;, \quad |Z|^2_{g,n} = z^2 + |Z^\perp|^2_{g,n} \;,$$

and, from what has been said so far,

$$\begin{aligned}
|T(W,Z)| &= \left|\frac{wz}{|n|^2_g}T(n,n) + \frac{w}{|n|_g}T(n,Z^\perp) + \frac{z}{|n|_g}T(n,W^\perp) + T(W^\perp,Z^\perp)\right| \\
&\le \frac{|wz| + |wZ^\perp|_{g,n} + |zW^\perp|_{g,n} + \frac{1}{2}(|W^\perp|^2_{g,n} + |Z^\perp|^2_{g,n})}{|n|^2_g}T(n,n) \\
&\le \frac{w^2 + z^2 + |W^\perp|^2_{g,n} + |Z^\perp|^2_{g,n}}{|n|^2_g}T(n,n) \\
&= \frac{|W|^2_{g,n} + |Z|^2_{g,n}}{|n|^2_g}T(n,n) \;.
\end{aligned}$$

This proves (B.2).

For (B.3), set $Z^\mu = -T^\mu{}_\nu X^\nu$; the dominant energy condition implies that Z^μ is causal future directed. Let e_a, $a \in \{0,\ldots,n\}$, be any orthonormal frame such that $n = |n|_g e_0$, and let X^a denote the components of X in this frame, thus $X = X^a e_a$, similarly for Z^a. Then (B.3) is equivalent to

$$-g(Z,X) \le \frac{2|X|_{g,n}}{|n|_g}(-g(Z,n)) \;. \tag{B.10}$$

Now, since both Z and X are causal and future directed we have $|\sum_i X^i Z^i| \le Z^0 X^0$, so

$$\begin{aligned}
-g(Z,X) &= Z^0X^0 - \sum_i X^i Z^i \le 2Z^0X^0 = 2\frac{X^0}{n^0}(-g(Z,n)) \\
&= 2\frac{X^0}{|n|_g}(-g(Z,n)) \;,
\end{aligned}$$

and (B.3) follows. (The constant is optimal, with the inequality becoming an equality when Z is null, $T_{\mu\nu} = Z_{(\mu}n_{\mu)}$, and $X^0 = Z^0$, $X^i = -Z^i$.) □

C R_{rA}

In this appendix we calculate the components R_{rA} of the Ricci tensor of a metric which on a null hypersurface $\mathscr{N} = \{u = 0\}$ takes the form

$$g = -\alpha du^2 + 2\nu_A dx^A du + 2\varepsilon du dr + \underbrace{r^2 \mathring{h}_{AB}}_{h_{AB}} dx^A dx^B \; . \tag{C.1}$$

Here we allow $\varepsilon = \pm 1$, according to whether a future ($\varepsilon = -1$) or a past ($\varepsilon = 1$) light-cone is considered. We emphasize that the above form of the metric is only assumed at $\{u = 0\}$, so all the $g_{\mu\nu}$'s are allowed a priori to be non-zero away from $\mathscr{N}$; similarly for their derivatives.

The equations in this appendix, and in appendix E, have been checked with the *xAct* system for tensor computer algebra [11].

Writing $g^\sharp$ for the inverse metric, we have

$$g^\sharp = \psi \partial_r^2 + 2\mu^A \partial_r \partial_A + 2\varepsilon \partial_u \partial_r + \frac{1}{r^2} \mathring{h}^{AB} \partial_A \partial_B \; , \tag{C.2}$$

$$g^{rA} \equiv \mu^A = -\varepsilon \frac{1}{r^2} \mathring{h}^{AB} \nu_B \; , \qquad g^{rr} \equiv \psi = \alpha + \frac{1}{r^2} \mathring{h}^{AB} \nu_A \nu_B \; . \tag{C.3}$$

We reserve the symbols ν_A, μ^A, α, ψ and h_{AB} for objects defined on $\{u = 0\}$, so that e.g. $\partial_u \nu_A$ does not make sense (but $\partial_u g_{uA}$ does, and might a priori be non-zero).

The Levi-Civita connection of the metric h_{AB} will be denoted as D_A and will have Christoffel symbols γ^C_{AB} with respect to the derivative ∂_A.

All the equations that follow are on $\mathscr{N}$.

We have the following Christoffel symbols (the remaining ones can be obtained by symmetry):

$$\Gamma^u_{uu} = \frac{\varepsilon}{2}(\partial_r\alpha + 2\partial_u g_{ur}) \,, \tag{C.4}$$

$$\Gamma^u_{ur} = \frac{\varepsilon}{2}\partial_u g_{rr} \,, \tag{C.5}$$

$$\Gamma^u_{rr} = 0 \,, \tag{C.6}$$

$$\Gamma^r_{uu} = \frac{1}{2}\mu^A\partial_A\alpha + \frac{1}{2}\psi(\partial_r\alpha + 2\partial_u g_{ur}) + \mu^A\partial_u g_{uA} + \frac{\varepsilon}{2}\partial_u g_{uu} \,, \tag{C.7}$$

$$\Gamma^r_{ur} = -\frac{\varepsilon}{2}\partial_r\alpha + \frac{1}{2}\mu^A\partial_r\nu_A + \frac{1}{2}\mu^A\partial_u g_{rA} + \frac{1}{2}\psi\partial_u g_{rr} \,, \tag{C.8}$$

$$\Gamma^r_{rr} = -\frac{\varepsilon}{2}\partial_u g_{rr} \,, \tag{C.9}$$

$$\Gamma^u_{Au} = \frac{\varepsilon}{2}(\partial_u g_{rA} - \partial_r\nu_A) \,, \tag{C.10}$$

$$\Gamma^u_{Ar} = 0 \,, \tag{C.11}$$

$$\begin{aligned}\Gamma^r_{Au} = {} & -\frac{\varepsilon}{2}\partial_A\alpha + \frac{1}{2}\mu^B(D_A\nu_B - D_B\nu_A + \partial_u g_{AB}) \\ & + \frac{1}{2}\psi(\partial_u g_{rA} - \partial_r\nu_A) \,, \end{aligned} \tag{C.12}$$

$$\Gamma^r_{Ar} = -\frac{\varepsilon}{2}(\partial_u g_{rA} - \partial_r\nu_A) + \frac{1}{2}\mu^B\partial_r h_{AB} \,, \tag{C.13}$$

$$\Gamma^u_{AB} = -\frac{\varepsilon}{2}\partial_r h_{AB} \,, \tag{C.14}$$

$$\Gamma^r_{AB} = \frac{\varepsilon}{2}(D_A\nu_B + D_B\nu_A - \partial_u g_{AB}) - \frac{1}{2}\psi\partial_r h_{AB} \,, \tag{C.15}$$

$$\Gamma^C_{uu} = \frac{1}{2}h^{CA}\partial_A\alpha + \frac{1}{2}\mu^C\partial_r\alpha + h^{CA}\partial_u g_{uA} + \mu^C\partial_u g_{ur} \,, \tag{C.16}$$

$$\Gamma^C_{ur} = \frac{1}{2}h^{CA}(\partial_u g_{rA} + \partial_r\nu_A) + \frac{1}{2}\mu^C\partial_u g_{rr} \,, \tag{C.17}$$

$$\Gamma^C_{rr} = 0 \,, \tag{C.18}$$

$$\Gamma^C_{Au} = \frac{1}{2}\mu^C(\partial_u g_{rA} - \partial_r\nu_A) + \frac{1}{2}h^{BC}(D_A\nu_B - D_B\nu_A + \partial_u g_{AB}) \,, \tag{C.19}$$

$$\Gamma^C_{Ar} = \frac{1}{2}h^{BC}\partial_r h_{AB} \,, \tag{C.20}$$

$$\Gamma^C_{AB} = \gamma^C_{AB} - \frac{1}{2}\mu^C\partial_r h_{AB} \,. \tag{C.21}$$

The traces of the Christoffel symbols read:

$$\begin{aligned}
\Gamma^{\mu}_{u\mu} &= \varepsilon\partial_u g_{ur} + \frac{1}{2}\psi\partial_u g_{rr} + \mu^A\partial_u g_{rA} + \frac{1}{2}h^{AB}\partial_u g_{AB} \;, && \text{(C.22)}\\
\Gamma^{\mu}_{r\mu} &= \frac{1}{2}h^{AB}\partial_r h_{AB} \;, && \text{(C.23)}\\
\Gamma^{\mu}_{A\mu} &= \frac{1}{2}h^{BC}\partial_A h_{BC} \;. && \text{(C.24)}
\end{aligned}$$

Let λ^μ be defined by (2.28), we have

$$\begin{aligned}
\lambda^u &= -\partial_u g_{rr} + \frac{\varepsilon}{2}h^{AB}\partial_r h_{AB} \;, && \text{(C.25)}\\
\lambda^r &= -\varepsilon h^{AB}D_B\nu_A + \partial_r\alpha - \mu^A\mu^B\partial_r h_{AB} + \frac{1}{2}h^{AB}\psi\partial_r h_{AB} \\
&\quad -2\varepsilon\mu^A\partial_r\nu_A + \frac{\varepsilon}{2}h^{AB}\partial_u g_{AB} - \frac{\varepsilon}{2}\psi\partial_u g_{rr} && \text{(C.26)}\\
&= -\varepsilon h^{AB}D_B\nu_A + \frac{\partial_r(\psi\sqrt{\det h_{EF}})}{\sqrt{\det h_{EF}}} + \frac{\varepsilon}{2}h^{AB}\partial_u g_{AB} - \frac{\varepsilon}{2}\psi\partial_u g_{rr} \;, && \text{(C.27)}\\
\lambda^A &= -h^{CD}\gamma^A_{CD} + \frac{1}{2}h^{BC}\mu^A\partial_r h_{BC} - h^{AC}\mu^B\partial_r h_{BC} \\
&\quad -\varepsilon(h^{AB}\partial_r\nu_B + h^{AB}\partial_u g_{rB} + \mu^A\partial_u g_{rr}) \;, && \text{(C.28)}\\
\lambda_u &= -h^{AB}\partial_B\nu_A + \varepsilon\partial_r\alpha - \mu^A\partial_r\nu_A \\
&\quad +\frac{1}{2}h^{AB}\partial_u g_{AB} + \mu^A\partial_u g_{rA} + \frac{1}{2}\psi\partial_u g_{rr} \;, && \text{(C.29)}\\
\lambda_r &= \frac{1}{2}h^{AB}\partial_r h_{AB} - \varepsilon\partial_u g_{rr} \;, && \text{(C.30)}\\
\lambda_A &= -h^{BC}h_{AD}\gamma^D_{BC} - \mu^B\partial_r h_{AB} - \varepsilon(\partial_r\nu_A + \partial_u g_{rA}) \;. && \text{(C.31)}
\end{aligned}$$

We choose the metric (2.23) as model metric, expressed in the following coordinate system:

$$\begin{aligned}
\mathring{g} &= -\underbrace{\left(1 - \frac{2\Lambda}{n(n-1)}\,r^2\right)}_{=:\mathring{\alpha}}\underbrace{dt^2}_{(du-\frac{\varepsilon}{\mathring{\alpha}}dr)^2} + \frac{dr^2}{1-\frac{2\Lambda}{n(n-1)}r^2} + r^2\mathring{h}_{AB}dx^Adx^B \\
&= -\mathring{\alpha}du^2 + 2\varepsilon du dr + r^2\mathring{h}_{AB}dx^Adx^B \;. && \text{(C.32)}
\end{aligned}$$

Its non-vanishing Christoffel symbols are, up to symmetry,

$$\mathring{\Gamma}^u_{uu} = -\frac{2\varepsilon\Lambda r}{n(n-1)} \;, \quad \mathring{\Gamma}^u_{BC} = -\varepsilon r \mathring{h}_{BC} \;, \quad \mathring{\Gamma}^r_{uu} = -\frac{2\Lambda r}{n(n-1)}\mathring{\alpha} \;,$$
$$\mathring{\Gamma}^r_{ur} = \frac{2\varepsilon\Lambda r}{n(n-1)} \;, \quad \mathring{\Gamma}^r_{BC} = -r\mathring{h}_{BC}\mathring{\alpha} \;, \quad \mathring{\Gamma}^A_{Br} = \frac{1}{r}\delta^A_B \;, \quad \mathring{\Gamma}^A_{BC} = \mathring{\gamma}^A_{BC} \;. \tag{C.33}$$

We shall shortly assume that the metric g satisfies the wave-map conditions

$$\lambda^\mu = \mathring{\lambda}^\mu \;,$$

with $\mathring{\lambda}^\mu$ defined in (2.29). We find

$$\mathring{\lambda}^u = -g^{\mu\nu}\mathring{\Gamma}^u_{\mu\nu} = r\varepsilon g^{AB}\mathring{h}_{AB} = \varepsilon\frac{n-1}{r} \;, \tag{C.34}$$

$$\mathring{\lambda}^r = -g^{\mu\nu}\mathring{\Gamma}^r_{\mu\nu} = \frac{n-1}{r} - \frac{2(n+1)\Lambda r}{n(n-1)} \;, \tag{C.35}$$

$$\begin{aligned}\mathring{\lambda}^A &= -g^{\mu\nu}\mathring{\Gamma}^A_{\mu\nu} = -2g^{rB}\mathring{\Gamma}^A_{rB} - g^{BC}\mathring{\Gamma}^A_{BC} = -\frac{2}{r}g^{rA} - \frac{1}{r^2}\mathring{h}^{BC}\mathring{\gamma}^A_{BC} \\ &= -\frac{2}{r}\mu^A + \frac{1}{r^2\sqrt{\det \mathring{h}_{EF}}}\partial_B(\sqrt{\det \mathring{h}_{EF}}\mathring{h}^{AB}) \;.\end{aligned} \tag{C.36}$$

Using $\lambda^u = \mathring{\lambda}^u$, from (C.25) and (C.34) we obtain

$$\partial_u g_{rr} = 0 \;, \text{ hence also } \; \partial_u g^{uu} = 0 \;.$$

From $\lambda^A = \mathring{\lambda}^A$ we deduce that

$$\partial_u g_{Cr} = -\frac{(n-1)}{r}\nu_C - \partial_r \nu_C \;,$$

and finally $\lambda^r = \mathring{\lambda}^r$ gives

$$\frac{1}{2}\mathring{h}^{AB}\partial_u g_{AB} = \mathring{h}^{AB}D_A\nu_B - \varepsilon\frac{\partial_r(r^{n-1}\psi)}{r^{n-3}} + (n-1)\varepsilon r - \frac{2(n+1)\Lambda\varepsilon r^3}{n(n-1)} \;.$$

Now,

$$R_{Ar} = \partial_\gamma \Gamma^\gamma_{Ar} - \partial_r \Gamma^\gamma_{A\gamma} + \Gamma^\gamma_{\sigma\gamma}\Gamma^\sigma_{Ar} - \Gamma^\gamma_{\sigma r}\Gamma^\sigma_{A\gamma} \;,$$

and from what has been said so far, in particular using the harmonicity conditions, we obtain

$$\begin{aligned} R_{Ar} &= \partial_u\Gamma^u_{Ar} + \partial_r\Gamma^r_{Ar} + \partial_B\Gamma^B_{Ar} - \partial_r\Gamma^\gamma_{A\gamma} + \Gamma^\gamma_{u\gamma}\underbrace{\Gamma^u_{Ar}}_{0} + \underbrace{\Gamma^\gamma_{r\gamma}}_{\Gamma^B_{rB}}\Gamma^r_{Ar} + \Gamma^\gamma_{B\gamma}\Gamma^B_{Ar} \\ &\quad - \underbrace{\Gamma^u_{ur}}_{0}\Gamma^u_{Au} - \underbrace{\Gamma^u_{rr}}_{0}\Gamma^r_{Au} - \underbrace{\Gamma^u_{Br}}_{0}\Gamma^B_{Au} - \underbrace{\Gamma^i_{ur}\Gamma^u_{Ai}}_{\Gamma^B_{ur}\Gamma^u_{AB}} - \underbrace{\Gamma^i_{rr}}_{0}\Gamma^r_{Ai} - \Gamma^i_{Br}\Gamma^B_{Ai} \\ &= \partial_u\Gamma^u_{Ar} + \partial_B\Gamma^B_{Ar} - \partial_r\Gamma^u_{Au} - \partial_r\Gamma^B_{AB} + \Gamma^B_{rB}\Gamma^r_{Ar} + \Gamma^\gamma_{B\gamma}\Gamma^B_{Ar} \\ &\quad - \Gamma^B_{ur}\Gamma^u_{AB} - \Gamma^r_{Br}\Gamma^B_{Ar} - \Gamma^C_{Br}\Gamma^B_{AC}\ . \end{aligned}$$

We have:

$$\partial_u\Gamma^u_{Ar} = \frac{1}{2}\partial_u\Big(g^{u\mu}(\partial_A g_{\mu r} + \partial_r g_{\mu A} - \partial_\mu g_{Ar})\Big) = \frac{1}{2}\partial_u g^{uB}\partial_r g_{BA}\ .$$

On the null surface it holds that

$$\partial_u g^{uB} = -\varepsilon h^{AB}\partial_u g_{rA} - \varepsilon\mu^B\partial_u g_{rr}\ , \tag{C.37}$$

so, using the harmonicity conditions, we are led to

$$\partial_u g^{uB} = -\frac{n-1}{r}\mu^B + \varepsilon h^{AB}\partial_r\nu_A\ . \tag{C.38}$$

Hence

$$\partial_u\Gamma^u_{Ar} = \varepsilon\frac{1}{r^2}\left((n-1)\nu_A + r\partial_r\nu_A\right). \tag{C.39}$$

With some work, using the formulae derived so far, one similarly obtains

$$\partial_B\Gamma^B_{rA} - \partial_r\Gamma^B_{AB} = -\frac{\varepsilon}{r}\partial_r\nu_A + \frac{\varepsilon}{r^2}\nu_A\ , \tag{C.40}$$

$$-\partial_r\Gamma^u_{Au} = -\frac{\varepsilon}{2}\partial_r\{\partial_u g_{rA} - \partial_r\nu_A\}\ , \tag{C.41}$$

$$\Gamma^r_{rA}\Gamma^B_{rB} = \varepsilon\frac{n-1}{r}\left\{\frac{1}{2}(\partial_r\nu_A - \partial_u g_{rA}) - \frac{1}{r}\nu_A\right\}\ , \tag{C.42}$$

$$\Gamma^B_{rA}(\Gamma^u_{Bu} + \Gamma^r_{Br} + \Gamma^C_{BC}) = \frac{1}{r}\mathring{\gamma}^C_{AC}\ , \tag{C.43}$$

$$-\Gamma^B_{ru}\Gamma^u_{AB} = \frac{\varepsilon}{2r}(\partial_r\nu_A + \partial_u g_{rA})\ , \tag{C.44}$$

$$-\Gamma^r_{rB}\Gamma^B_{Ar} - \Gamma^B_{rC}\Gamma^C_{AB} = -\frac{\varepsilon}{2r}(\partial_r\nu_A - \partial_u g_{rA}) - \frac{1}{r}\mathring{\gamma}^B_{AB} \ . \tag{C.45}$$

Adding, we are led to

$$\begin{aligned}\varepsilon R_{Ar} &= \frac{(n-2)(n-3)}{2r^2}\,\nu_A + \frac{3n-5}{2r}\,\partial_r\nu_A + \partial_r\partial_r\nu_A \\ &= \frac{1}{r^{n-1}}\partial_r\left[r^{n-1}\left(\partial_r\nu_A + \frac{n-3}{2r}\nu_A\right)\right] .\end{aligned} \tag{C.46}$$

D An energy inequality for traceless $T_{\mu\nu}$

We use normal coordinates centred on the vertex p_0 of the future light-cone, denoted by x^μ, thus restricting our considerations to the region where those are well defined.Passing to a subset of the domain of normal coordinates if necessary, we can (and will) assume that ∂_t and ∇t are timelike. Throughout this appendix the symbol X denotes the vector field $X^\mu\partial_\mu$ with normal–coordinates components as in (2.34),

$$X^\mu = x^\mu \ .$$

Now, because X is null at the boundary, the integrand of (2.33) does *not* control all components of $T_{\mu\nu}$ *uniformly* as one approaches the light-cone, and we need to quantify that. So we start by showing that there exist $\epsilon > 0$ and a constant $C > 0$ such that, for any $T \in [0,\infty)$, and for all points for which $|1+g_{00}| \le \epsilon$ and $0 < r < t \le T$ we have

$$T_{\mu\nu}n^\mu X^\nu \ge C^{-1}(t-r)T_{0\nu}n^\nu \ . \tag{D.1}$$

Note that for $0 < r \le t/2 \le T/2$, the inequality follows immediately from the fact that all three vectors X^μ, ∂_t, and n^μ are timelike there, and from (B.3). So it remains to consider points for which $|1+g_{00}| \le \epsilon$ and $0 < t/2 < r < t$.

We let Z^μ be a future directed null vector of the form $Z^\mu = \lambda\delta^\mu_0 + X^\mu$. Then

$$\begin{aligned}T_{\mu\nu}X^\mu n^\nu &= (tT_{0\nu} + T_{i\nu}x^i)n^\nu \\ &= (t-r)T_{0\nu}n^\nu + \Big((\lambda+t)T_{0\nu} + T_{i\nu}x^i\Big)n^\nu - (\lambda+t-r)T_{0\nu}n^\nu \\ &= (t-r)T_{0\nu}n^\nu + \underbrace{T_{\mu\nu}Z^\mu n^\nu}_{\ge 0} - (\lambda+t-r)T_{0\nu}n^\nu \ .\end{aligned}$$

The middle term is positive by the dominant energy condition. Equation (D.1) will follow if we can show that $|\lambda + t - r| \leq \frac{1}{2}(t-r)$.

Now, the condition that Z^μ is null reads:

$$0 = g_{\mu\nu}Z^\mu Z^\nu = g_{00}\lambda^2 + 2g_{0\mu}X^\mu\lambda + g_{\mu\nu}X^\mu X^\nu \ . \tag{D.2}$$

But, as is well known [15], in normal coordinates we have

$$g_{\mu\nu}(x)x^\nu = \eta_{\mu\nu}x^\nu \ . \tag{D.3}$$

So

$$g_{0\mu}X^\mu = \eta_{0\mu}X^\mu = -t \ , \quad g_{\mu\nu}X^\mu X^\nu = -t^2 + r^2 \ , \tag{D.4}$$

and (D.2) can be rewritten as

$$0 = g_{00}\lambda^2 - 2t\lambda + r^2 - t^2 \ . \tag{D.5}$$

Solving (D.5) for λ, we choose the solution

$$\begin{aligned}
\lambda &= \frac{1}{g_{00}}\left(t - r + (r - \sqrt{(1+g_{00})t^2 - g_{00}r^2})\right) \\
&= \frac{1}{g_{00}}\left(t - r + \frac{(1+g_{00})(r^2-t^2)}{r + \sqrt{(1+g_{00})t^2 - g_{00}r^2}}\right) \\
&= \left(-\frac{1}{g_{00}}\right)\left(1 - \frac{(1+g_{00})(t+r)}{r + \sqrt{(1+g_{00})t^2 - g_{00}r^2}}\right)(r-t) \ .
\end{aligned} \tag{D.6}$$

Hence

$$\lambda + t - r = \frac{1+g_{00}}{-g_{00}}\left[1 - \frac{(t+r)}{r + \sqrt{(1+g_{00})t^2 - g_{00}r^2}}\right](r-t) \ ,$$

from which the result immediately follows, keeping in mind that $0 < t/2 < r < t$.

The above considerations applied to general metrics. In the remainder of the proof we restrict attention to metrics which behave as in the proof of Theorem 2.1: Thus, we assume that, on $\mathscr{C}(s_*)$, the metric takes the form (2.36),

$$g = -\alpha du^2 - 2du\,dr + r^2\mathring{h}_{AB}dz^A dz^B \ , \tag{D.7}$$

We denote by $(z^\mu) \equiv (u, r, z^A)$ these coordinates, and we assume that $\{u = 0\}$ is $\mathscr{C}(s_*)$. We suppose that $u = y^0 - |\vec{y}|$, $r = |\vec{y}|$, and that the z^A's are angular coordinates parameterising the unit vector $\vec{y}/r$. Furthmore the coordinates y^μ are required to coincide with the normal coordinates x^μ on the light-cone, and we assume that the map $x^\mu \mapsto y^\mu$ is a smooth diffeomorphism in a neighbourhood of the future light-cone of p_0. Note that we write here z^A for what is denoted by x^A elsewhere in the paper since, to avoid confusions, in the considerations below we reserve the symbol x^μ for normal coordinates.

We will also need the hypothesis that the u-derivatives of the metric at the light-cone satisfy

$$\partial_u g_{rr} = \partial_u g_{rA} = 0 \ .$$

As already pointed out, all those conditions will be the satisfied by the wave-map coordinates from the main body of the paper at the current stage of the argument. But we emphasize we do not need to assume that the coordinates y^μ satisfy more conditions than the ones just listed.

The argument is considerably simpler if we assume that the metric takes the form (D.7) with $y^\mu = x^\mu$ in a neighbourhood of the light cone, rather than on the light-cone only. For then we have $\mathscr{L}_X \eta = 2\eta$, where η is the Minkowski metric. Further

$$\mathscr{L}_X du = d(\mathscr{L}_X u) = d(X(u)) = d\big((t\partial_t + r\partial_r)(t-r)\big) = d\big(t-r\big) = du \ . \quad \text{(D.8)}$$

Along the light-cone we have $g = \eta + (1-\alpha)du^2$ which implies, again along the light-cone,

$$\mathscr{L}_X g \;\; = \;\; 2g - r\partial_r \alpha \, du^2 \ . \qquad \text{(D.9)}$$

This gives directly Proposition D.1 below with $Y \equiv 0$. The calculations at the end of the proof get simplified accordingly.

It is conceivable that (D.7) together with our remaining hypotheses implies the same form of the metric in normal coordinates, in which case we would be done. However, we have not been able to find a simple argument for this. Instead we proceed as follows:

PROPOSITION D.1 *On any bounded interval of t, say $0 \le t \le T$, and assuming as before that we are working within the domain of definition of normal coordinates, there exists a causal-or-zero vector field Y and a constant C such* *that, for* $0 < r \le t \le T$,

$$\Big|\mathscr{L}_X g_{\mu\nu} - \Big(2g_{\mu\nu} - \frac{\partial_r \alpha}{r} X_\mu X_\nu + (X_\mu Y_\nu + Y_\mu X_\nu)\Big)\Big|_b \le C(t-r) \;, \tag{D.10}$$

where the norm $|\cdot|_b$ *is taken with respect to some (arbitrarily chosen) Riemannian metric* b.

PROOF: If X is a vector field, we write $X|_p$ to indicate the fact the X is considered at a point p. We write p_0 for the vertex of the light-cone, thus $x^\mu(p_0) = 0$. In normal coordinates, by definition we have $X|_p = x^\mu(p)\partial_{x^\mu}|_p$. Changing to any other coordinates $x^\alpha \mapsto z^\mu(x^\alpha)$, the transformation rule for vectors gives

$$X|_p = x^\mu(p) \frac{\partial z^\alpha}{\partial x^\mu}(x^\gamma(p)) \frac{\partial}{\partial z^\alpha}\Big|_p \;. \tag{D.11}$$

Hence, on the light cone, when $(z^\alpha) = (u, r, z^A)$,

$$X = r \frac{\partial z^\alpha}{\partial r} \frac{\partial}{\partial z^\alpha} = r\partial_r \;.$$

Further, from (D.11) in the coordinate system z^μ,

$$\begin{aligned} \frac{\partial X^\alpha}{\partial u} &= \frac{\partial}{\partial u}\left(x^\mu \frac{\partial z^\alpha}{\partial x^\mu}\right) = \frac{\partial x^\mu}{\partial u}\frac{\partial z^\alpha}{\partial x^\mu} + x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 z^\alpha}{\partial x^\lambda \partial x^\mu} \\ &= \delta^\alpha_u + x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 z^\alpha}{\partial x^\lambda \partial x^\mu} \;. \end{aligned} \tag{D.12}$$

By definition of Lie derivative,

$$\mathscr{L}_X dz^\mu|_{u=0} = \delta^\mu_r dr + \partial_u X^\mu du \;. \tag{D.13}$$

So, for a metric of the form (D.7) we obtain,

$$\begin{aligned} \mathscr{L}_X g &= -(X(\alpha) + 2\alpha\partial_u X^u)du^2 - 2(\partial_u X^u + \partial_u X^r)du\,dr + 2r^2 \mathring{h}_{AB}\partial_u X^A dz^B \\ &\quad \underbrace{-2du\,dr + 2r^2\mathring{h}_{AB}dz^A dz^B}_{2g+2du\,dr+2\alpha du^2} \\ &= 2g + (2\alpha - X(\alpha) - 2\alpha\partial_u X^u)du^2 + 2(1-\partial_u X^u - \partial_u X^r)du\,dr \\ &\quad +2r^2\mathring{h}_{AB}\partial_u X^A dz^B du \;. \end{aligned} \tag{D.14}$$

Clearly, we need information about $\partial_u X^\mu$.

Now, from the definition of normal coordinates, $X^\mu\partial_\mu$ is the value, at $s=1$, of the vector tangent to the geodesic starting from p_0 and with initial tangent $x^\mu\partial_\mu$ at p_0. Thus, in the coordinate system (u,r,z^A), we have $X^\mu = dz^\mu/ds|_{s=1}$, where

$$\gamma(s) \equiv (z^\mu(s)) := (u(s), r(s), z^A(s))$$

is the solution of the geodesic equation,

$$\frac{d^2 z^\nu}{ds^2} + \Gamma^\nu_{\alpha\beta}\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds} = 0 \ , \tag{D.15}$$

with initial values

$$u(0) = 0 = r(0) \ , \quad z^A(0) = z^A \ , \quad \dot u(0) = 0 = \dot z^A(0) \ , \quad \dot r(0) = r \ .$$

We want to calculate $\partial_u X^\nu$, for this we need to understand the relevant initial conditions for the geodesic deviation equation. It is a standard property of normal coordinates that the curves

$$s \mapsto \gamma^\mu(s) = s x^\mu$$

are geodesics, with initial tangent vector $x^\mu(p)\partial_\mu|_{p_0}$, connecting the origin at $s=0$ with the point p with normal coordinates $x^\mu = x^\mu(p)$ at $s=1$. (We emphasise that this is a tangent vector at p_0 with components $x^\mu(p)$ in the basis $\partial_\mu|_{p_0} \in T_{p_0}M$.) Suppose that we use the coordinates z^β to parameterise the initial tangent vector $\dot\gamma(0)$; thus, in the coordinate system z^μ we have

$$\dot\gamma^\nu(0) = x^\alpha(z^\beta)\frac{\partial z^\nu}{\partial x^\alpha}(0) \ . \tag{D.16}$$

Let us denote by $J^\nu(s)$ the solution of the equation obtained by formally differentiating (D.15)‘with respect to u,

$$\frac{d^2 J^\nu}{ds^2} + \partial_u\Gamma^\nu_{\alpha\beta}\frac{dz^\alpha}{ds}\frac{dz^\beta}{ds} + 2\Gamma^\nu_{\alpha\beta}\frac{dz^\alpha}{ds}\frac{dJ^\beta}{ds} = 0 \ , \tag{D.17}$$

along the geodesic above.[5] By usual considerations we have $\partial_u X^\nu = J^\nu(1)$.

[5]Equation (D.17) is of course equivalent to the geodesic deviation equation.

In the (u, r, z^A) coordinate system we have $u(s) = 0$, $r(s) = rs$, $z^A(s) = z^A(0)$. Along such geodesics, (D.17) on $\mathscr{C}(s_*)$ with $\nu = u = \mu$ reads

$$\frac{d^2 J^u}{ds^2} + r^2 \underbrace{\partial_u \Gamma^u_{rr}}_{0} + 2r \underbrace{\Gamma^u_{r\beta}}_{0} \frac{dJ^\beta}{ds} = 0 \ , \qquad \text{(D.18)}$$

where the last two terms vanish by (E.3)-(E.4).

From (D.12)

$$\gamma^\alpha(s) = s x^\mu \frac{\partial z^\alpha}{\partial x^\mu}(s x^\beta) \ , \qquad \text{(D.19)}$$

and by differentiation

$$\begin{aligned} J^\alpha(s) \ &:= \ \frac{\partial \gamma^\alpha}{\partial u}(s) = \frac{\partial}{\partial u}\left(s x^\mu \frac{\partial z^\alpha}{\partial x^\mu}(s x^\beta)\right) \\ &= \ s \frac{\partial x^\mu}{\partial u}\frac{\partial z^\alpha}{\partial x^\mu}(s x^\beta) + s x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial z^\alpha}{\partial x^\mu}(s x^\beta)\right) \\ &= \ s \delta^\alpha_u + s^2 x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 z^\alpha}{\partial x^\lambda \partial x^\mu}(s x^\beta) \ . \end{aligned} \qquad \text{(D.20)}$$

Equation (D.18) with $\alpha = u$ shows that there exist functions $a = a(x^\mu)$, $b = b(x^\mu)$ such that

$$s^2 x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(s x^\beta) = a + bs \ .$$

Keeping in mind that $u = y^0 - |\vec{y}|$, where y^μ are smooth functions of x^μ, the left-hand-side if $O(s)$. (A simple way to check this is to note that this is true when the coordinates x^μ are replaced by the y^μ's, and use the fact that the map $x^\mu \mapsto y^\mu$ is a smooth diffeomorphism.) So $a \equiv 0$ and we conclude that

$$s x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(s x^\beta) = b(x^\mu) \ .$$

Equivalently, for $s \in (0, 1]$,

$$s x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(s x^\beta) = x^\mu \frac{\partial x^\lambda}{\partial u}\frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(x^\beta) \ . \qquad \text{(D.21)}$$

Now, in normal coordinates x^μ we have

$$\nabla_\mu X^\mu = \partial_\mu x^\mu + \frac{1}{\sqrt{\det |g_{\alpha\beta}|}} x^\mu \partial_\mu \sqrt{\det |g_{\alpha\beta}|} = n + 1 + O(|x|^2) \ .$$

On the other hand, in the coordinates z^μ we have on $\mathscr{C}(s_*)$

$$\begin{aligned}\nabla_\mu X^\mu &= \partial_u X^u + \underbrace{\partial_r X^r}_{1} + \underbrace{\partial_A X^A}_{0} + \frac{1}{\sqrt{\det |g_{\alpha\beta}|}} r\partial_r \underbrace{\sqrt{\det |g_{\alpha\beta}|}}_{r^{n-1}} \\ &= \partial_u X^u + n\ .\end{aligned}$$

We conclude that $\partial_u X^u = 1 + O(|x|^2)$, and (D.12) gives

$$x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(x^\beta) = O(r^2)\ .$$

Therefore also

$$sx^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(sx^\beta) = O(s^2 r^2)$$

This is compatible with (D.21) if and only if

$$x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 u}{\partial x^\lambda \partial x^\mu}(x^\beta) \equiv 0\ ,$$

and we obtain

$$\partial_u X^u|_{u=0} = 1\ .$$

Next, (D.17) with $\nu = A$ and $\mu = u$ becomes

$$\frac{d^2 J^A}{ds^2} + r^2 \partial_u \Gamma^A_{rr} + 2r\Gamma^A_{r\beta} \frac{dJ^\beta}{ds} = 0\ . \tag{D.22}$$

By (E.3)-(E.4) we have $\partial_u \Gamma^A_{rr} = 0 = \Gamma^A_{ru} = \Gamma^A_{rr}$ and $\Gamma^A_{rB} = \delta^A_B/sr$ on the light-cone (keeping in mind that in (D.17) the Christoffels are calculated at points with coordinates $(z^\mu(s)) = (0, rs, z^A(0)))$, which implies that

$$\frac{d^2 J^A}{ds^2} + \frac{2}{s}\frac{dJ^A}{ds} = 0\ . \tag{D.23}$$

Integrating, there exist functions $\alpha^A(x^\mu)$ and $\beta^A(x^\mu)$ such that

$$J^A(s) = \alpha^A + \frac{\beta^A}{s}\ .$$

From (D.20) we obtain

$$s^2 x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(sx^\beta) = \alpha^A + \frac{\beta^A}{s}\ .$$

Keeping in mind that the z^A's are spherical coordinates, the left-hand-side is bounded when s approaches zero, which gives

$$\beta^A \equiv 0 \ .$$

We conclude that

$$s^2 x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(s x^\beta) = x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(x^\beta) \ . \tag{D.24}$$

Now, in normal coordinates we have

$$\begin{aligned} \nabla_\mu X_\nu + \nabla_\nu X_\mu &= \mathscr{L}_X g_{\mu\nu} = x^\sigma \partial_\sigma g_{\mu\nu} + 2 g_{\mu\nu} \\ &= 2 g_{\mu\nu} + O(|x|^2) \ . \end{aligned} \tag{D.25}$$

This implies that the same is true in the y^μ coordinates. Taking into account the angular character of the coordinates z^A, we then obtain in the coordinates (u, r, z^A)

$$\nabla_A X_u + \nabla_u X_A = O(|x|^3) \ .$$

Equivalently, along the light-cone,

$$g_{ru} \underbrace{\nabla_A X^r}_{=\Gamma^r_{Ar} X^r = 0} + g_{AB} \underbrace{\nabla_u X^B}_{=\partial_u X^B + \Gamma^B_{ur} X^r = \partial_u X^B} = O(|x|^3) \ . \tag{D.26}$$

Thus, $\partial_u X^B = O(|x|)$, and from (D.12)

$$x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(x^\beta) = O(r) \ .$$

This implies

$$s^2 x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(s x^\beta) = O(s^2 r) \ ,$$

which is compatible with (D.21) if and only if

$$x^\mu \frac{\partial x^\lambda}{\partial u} \frac{\partial^2 z^A}{\partial x^\lambda \partial x^\mu}(x^\beta) \equiv 0 \ ,$$

and (D.12) allows us to conclude that

$$\partial_u X^A|_{u=0} = 0 \ . \tag{D.27}$$

Equation (D.14) at $\{u=0\}$ thus reads

$$\mathscr{L}_X g \;=\; 2g - r\partial_r\alpha\, du^2 - 2\partial_u X^r du\, dr\ . \tag{D.28}$$

The estimate

$$|r\partial_r\alpha| \le Ct^2 \ \text{ for }\ 0 < r \le t \le T\ , \tag{D.29}$$

is straightforward. Moreover, from (D.25) and (C.4),

$$O(|x|^2) = \nabla_u X_u = \partial_u(g_{r\mu}X^\mu) - \underbrace{\Gamma^u_{uu}}_{O(r)} X_u = -\partial_u X^r + O(|x|^2)\ .$$

Hence

$$\partial_u X^r = O(|x|^2)\ . \tag{D.30}$$

On the light cone we have

$$du = -\frac{1}{r}X_\mu dx^\mu\ , \quad dr = -g_{ru}dr = -g_{\mu\nu}(\partial_u)^\nu dx^\mu$$

with the vector field ∂_u being *timelike* there. It is then easily seen that there exists a *causal* vector field, say $Y = Y^\mu\partial_\mu$, which coincides with ∂_u at $\{u=0\}$ away from the vertex, with normal-coordinates components satisfying

$$\frac{\partial Y^\nu}{\partial x^\mu} = O(|x|^{-1})\ .$$

So we can rewrite (D.28) as

$$\mathscr{L}_X g_{\mu\nu}|_{u=0} \;=\; 2g_{\mu\nu} - \frac{1}{r}\partial_r\alpha\ X_\mu X_\nu - \frac{1}{r}\partial_u X^r(X_\mu Y_\nu + Y_\mu X_\nu)\ . \tag{D.31}$$

Let us return now to normal coordinates. In those coordinates we have the standard estimate

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |x|_b|\partial_\sigma g_{\mu\nu}| \le C|x|_b^2\ . \tag{D.32}$$

which gives

$$\begin{aligned}
|\partial_\lambda(\mathscr{L}_X g_{\mu\nu})| &= |\partial_\lambda(X^\rho\partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\rho\mu})| \\
&= |\partial_\lambda(x^\rho\partial_\rho g_{\mu\nu} + 2g_{\mu\nu})| \le C|x|_b\ .
\end{aligned}$$

Integrating along inward directed rays, at $t = \text{const}$, from $t = r$ towards the center, one finds:

$$\begin{aligned} \mathscr{L}_X g_{\mu\nu}(t,r,z^A) - \mathscr{L}_X g_{\mu\nu}(t,t,z^A) &= -\int_r^t \frac{x^i}{r}\partial_i(\mathscr{L}_X g_{\mu\nu})(t,s,z^A)dx \\ &= O\big(t(t-r)\big) \ . \end{aligned}$$

A similar Taylor expansion of the remaining terms gives

$$\left|\mathscr{L}_X g_{\mu\nu} - \left(2g_{\mu\nu} + \frac{1}{r}\partial_r\alpha \ X_\mu X_\nu - \frac{1}{r}\partial_u X^r(X_\mu Y_\nu + Y_\mu X_\nu)\right)\right|_b \leq C(t-r) \ . \tag{D.33}$$

A redefinition of Y gives (D.10). $\square$

Let $E(s)$ be defined as in (2.33), except that t there is taken now to be a normal coordinate x^0 within its domain of definition. Recall that

$$\Omega(s) := J^+(p) \cap \{t \leq s\} \ . \tag{D.34}$$

We consider the divergence identity on $\Omega(s)$:

$$\begin{aligned} E(s) + \int_{\mathscr{C}(s)} T^\mu{}_\nu X^\nu dS_\mu &= -\int_{\partial\Omega(s)} T^\mu{}_\nu X^\nu dS_\mu = -\int_{\Omega(s)} \nabla_\mu(T^\mu{}_\nu X^\nu) \\ &= -\int_{\Omega(s)} T^{\mu\nu}\mathscr{L}_X g_{\mu\nu} \ . \end{aligned} \tag{D.35}$$

Since $T_{\mu\nu}$ is traceless by hypothesis, from (D.10) and from (B.2) we obtain

$$|T^{\mu\nu}(\mathscr{L}_X g_{\mu\nu} - \frac{\partial_r\alpha}{r}X_\mu X_\nu - 2X_\mu Y_\nu)|_b \leq C(t-r)T_{\mu\nu}n^\mu n^\nu \ . \tag{D.36}$$

Since $T(X,\cdot)$ is causal while both n and Y are timelike we have

$$|T_{\mu\nu}X^\mu Y^\nu| \leq C|T_{\mu\nu}X^\mu n^\nu| \ . \tag{D.37}$$

As $\partial_r\alpha/r$ is bounded, (D.36)-(D.37) together with (B.3) imply

$$\begin{aligned} |T^{\mu\nu}\mathscr{L}_X g_{\mu\nu}| &\leq C\Big(T_{\mu\nu}X^\mu X^\nu + T_{\mu\nu}X^\mu n^\nu + (t-r)T_{\mu\nu}n^\mu n^\nu\Big) \\ &\leq C'\Big(T_{\mu\nu}n^\mu X^\nu + (t-r)T_{0\nu}n^\nu\Big) \ . \end{aligned} \tag{D.38}$$

By (D.1) the right-hand-side is bounded by a multiple of $T_{\mu\nu}n^\mu X^\nu$, and we can conclude that

$$E(s) \le C \int_0^s E(t)dt - \underbrace{\int_{\mathscr{C}(s)} T^\mu{}_\nu X^\nu dS_\mu}_{=0} ,$$

where the vanishing of the last integral follows from the fact that X^ν is tangent to the generators of $\mathscr{C}$, hence null there, and from (2.13). Since $E(s)$ approaches zero as s tends to zero, from Gronwall's lemma we obtain

$$E(s) = 0 \text{ for } 0 < s < s_*.$$

Positivity of the integrand implies

$$T_{\mu\nu}X^\mu n^\nu = 0 \quad \text{on } \Omega(s_*) \ . \tag{D.39}$$

Since X is timelike on the interior of $\Omega(s_*)$, from (B.3) we conclude that the space-time is vacuum in $\Omega(s_*)$.

E $g^{AB}R_{AB}$

In this appendix we continue our analysis for a metric which, in addition to the hypotheses of Appendix C, satisfies further $\nu_A = 0$ at $\{u = 0\}$; thus, there we have

$$g = -\alpha du^2 + 2\varepsilon du dr + h_{AB}dx^A dx^B \ , \tag{E.1}$$

with

$$g^\sharp = \alpha\partial_r^2 + 2\varepsilon\partial_u\partial_r + h^{AB}\partial_A\partial_B \ . \tag{E.2}$$

As in Appendix C, all calculations are done on the null hypersurface $\{u = 0\}$.

In addition to the previous list of vanishing Christoffel symbols,

$$\Gamma^u_{rr} = \Gamma^A_{rr} = \Gamma^u_{rA} = 0 \ , \tag{E.3}$$

we now also have, due to the wave-map conditions and the vanishing of ν_A,

$$\partial_u g_{rr} = \partial_u g_{rA} = \Gamma^u_{ur} = \Gamma^u_{uA} = \Gamma^r_{rr} = \Gamma^r_{rA} = \Gamma^A_{ru} = 0 \ . \tag{E.4}$$

The remaining Christoffel symbols can be obtained from those listed in appendix C by setting $\nu_A = 0$ there.

We will need the following traces:

$$\Gamma^\alpha_{u\alpha} = \frac{1}{2r^2}\mathring{h}^{AB}\partial_u g_{AB} + \varepsilon\partial_u g_{ru} \,, \tag{E.5}$$

$$\Gamma^\alpha_{r\alpha} = \frac{n-1}{r} \,, \tag{E.6}$$

$$\Gamma^\alpha_{A\alpha} = \frac{1}{2}\mathring{h}^{BC}\partial_A\mathring{h}_{BC} \,. \tag{E.7}$$

In view of (C.35),

$$\lambda^r = \frac{\varepsilon}{2r^2}\mathring{h}^{AB}\partial_u g_{AB} + \frac{n-1}{r}\alpha + \partial_r\alpha \,, \qquad \mathring{\lambda}^r = \frac{n-1}{r} - 2\Lambda r\frac{n+1}{n(n-1)} \,, \tag{E.8}$$

and of the wave-map condition $\lambda^r = \mathring{\lambda}^r$, we conclude that

$$\frac{\varepsilon}{2r^2}\mathring{h}^{AB}\partial_u g_{AB} = \frac{n-1}{r}(1-\alpha) - 2\Lambda r\frac{n+1}{n(n-1)} - \partial_r\alpha \,. \tag{E.9}$$

We want, next, to calculate

$$\begin{aligned} g^{AB}R_{AB} &= g^{AB}(\partial_\alpha\Gamma^\alpha_{AB} - \partial_B\Gamma^\alpha_{A\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{AB} - \Gamma^\alpha_{\sigma B}\Gamma^\sigma_{A\alpha}) \\ &= g^{AB}\Big(\partial_u\Gamma^u_{AB} + \partial_r\Gamma^r_{AB} + \partial_C\Gamma^C_{AB} - \partial_B\Gamma^\alpha_{A\alpha} + \Gamma^\alpha_{u\alpha}\Gamma^u_{AB} \\ &\quad + \Gamma^\alpha_{r\alpha}\Gamma^r_{AB} + \Gamma^\alpha_{C\alpha}\Gamma^C_{AB} - \Gamma^D_{CB}\Gamma^C_{AD} - 2\Gamma^r_{CB}\Gamma^C_{Ar} - 2\Gamma^u_{CB}\Gamma^C_{Au}\Big) \,. \end{aligned} \tag{E.10}$$

We will calculate separately various terms above, in random order, starting with the two last ones:

$$-2h^{AB}\Gamma^C_{uA}\Gamma^u_{BC} = \frac{\varepsilon}{r^3}\mathring{h}^{AB}\partial_u g_{AB} \,, \tag{E.11}$$

$$-2h^{AB}\Gamma^C_{rA}\Gamma^r_{BC} = \frac{\varepsilon}{r^3}\mathring{h}^{AB}\partial_u g_{AB} + \frac{2(n-1)}{r^2}\alpha \,, \tag{E.12}$$

$$\begin{aligned} h^{AB}\Big(\partial_u\Gamma^u_{AB} + \partial_r\Gamma^r_{AB}\Big) &= -\frac{\varepsilon}{r^2}\mathring{h}^{AB}\partial_r\partial_u g_{AB} \\ &\quad -\frac{n-1}{r^2}\partial_r(r\alpha) + \frac{n-1}{r}\partial_u g_{ur} \,, \end{aligned} \tag{E.13}$$

$$\begin{aligned} & h^{AB}\Big(\Gamma^\alpha_{u\alpha}\Gamma^u_{AB} + \Gamma^\alpha_{r\alpha}\Gamma^r_{AB}\Big) \\ &= -\varepsilon\frac{n-1}{r^3}\mathring{h}^{AB}\partial_u g_{AB} - \frac{(n-1)^2}{r^2}\alpha - \frac{n-1}{r}\partial_u g_{ur} \,. \end{aligned}$$

The remaining terms are

$$g^{AB}\Big(\partial_C\Gamma^C_{AB}-\partial_B\Gamma^\alpha_{A\alpha}+\Gamma^\alpha_{C\alpha}\Gamma^C_{AB}-\Gamma^D_{CB}\Gamma^C_{AD}\Big)=h^{AB}\mathcal{R}_{AB}\ ,$$

where $\mathcal{R}$ is the Ricci tensor of the metric h_{AB}. Adding, one is led to the simple identity

$$\begin{aligned} g^{AB}R_{AB} \ &= \ -\frac{\varepsilon}{r^2}\mathring{h}^{AB}\partial_r\partial_u g_{AB}-\frac{\varepsilon}{r^3}(n-3)\mathring{h}^{AB}\partial_u g_{AB} \\ &\quad -\frac{(n-1)(n-3)}{r^2}\alpha-\frac{n-1}{r^2}\partial_r(r\alpha)+h^{AB}\mathcal{R}_{AB}\ . \end{aligned} \qquad \text{(E.14)}$$

But, in view of the wave-map condition (E.9),

$$\begin{aligned} &-\frac{\varepsilon}{r^2}\mathring{h}^{AB}\partial_r\partial_u g_{AB}-\frac{\varepsilon}{r^3}(n-3)\mathring{h}^{AB}\partial_u g_{AB} \\ &\quad = 4\Lambda\frac{n+1}{n-1}+2\partial_r\partial_r\alpha+\frac{4(n-1)}{r}\partial_r\alpha\ +\frac{2(n-1)(n-2)}{r^2}(\alpha-1)\ . \end{aligned} \qquad \text{(E.15)}$$

For the model metrics (2.23) we have $h^{AB}\mathcal{R}_{AB}=(n-1)(n-2)/r^2$, so

$$g^{AB}R_{AB}=4\Lambda\frac{n+1}{n-1}+2\partial_r\partial_r\alpha+\frac{3(n-1)}{r}\partial_r\alpha+\frac{(n-1)(n-2)}{r^2}(\alpha-1). \qquad \text{(E.16)}$$

Acknowledgements: YCB and PTC acknowledge useful discussions with Vince Moncrief. Similarly, YCB wishes to thank Thibault Damour for making available his detailed manuscript calculations leading to equations (22) of [5]. JMM thanks the University of Oxford for hospitality. He was supported by the A.N.R. Grant BLAN07-1_201699 entitled "LISA Science", and also in part by the Spanish MICINN project FIS2008-03221.

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