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Constructions of N-body time-symmetric initial data sets in general relativity

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Abstract

Given a collection of N solutions of the Einstein constraint equations which are asymptotically Euclidean and time symmetric, we show how to construct a new solution of the constraints which is itself asymptotically Euclidean and time symmetric, and which contains specified sub-regions of each of the N given solutions.

1 Introduction.

The work of Corvino and Schoen [8, 14] shows that one can glue an interior region of any asymptotically Euclidean (AE) solution of the vacuum constraints to an exterior Kerr or Schwarzschild solution in the following sense: For any given AE solution of the vacuum Einstein constraint equations, and for any chosen interior region (bounded away from infinity) in this solution, there is a new solution of the vacuum constraints which has an interior region that is isometric to the original interior region, and has an exterior region which is isometric to an exterior region in a space-like slice of a Kerr spacetime. If the original solution of the constraints is time symmetric, then the exterior is isometric to a region in a slice of a Schwarzschild spacetime.

This work shows, remarkably, that one can geometrically shield the details of the gravitational field in the interior region, so that sufficiently far away, all that one sees is the effective total energy-momentum and total angular momentum of the interior.

A natural question to pose is whether one can produce solutions of the constraints which contain exact copies of the interior regions of two or more solutions. Using the gluing techniques of [4, 5, 12, 13] one can indeed do this [6]; however the solutions resulting from these gluing methods contain two or more exterior regions. There are gluing results [4] which do incorporate multiple

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interiors, but all such results to date require that the multiple regions be copies of each other, and be placed in certain symmetric configurations.

We show here that in fact one can glue multiple interior regions of solutions of the vacuum constraint equations into a solution with a single asymptotic (Schwarzschild) end, without imposing any matching conditions on the solutions being glued, and without imposing any symmetry restrictions on the placement of the various interior regions. Our main restriction here is that we consider only time symmetric solutions of the constraint equations (and therefore Schwarzschild rather than Kerr exteriors), and in the solutions constructed, the given interior regions must be sufficiently far from each other. In future work, we remove the time symmetric restriction. It is not likely that we can weaken the condition that the regions be sufficiently distant from each other, or get a more effective bound on the scale of the final configuration.

The motivation for the present work is to set up initial data for the general relativistic (Einsteinian) version of the gravitational N body problem. We recall that one of the signature results of eighteenth century physics is the complete and explicit solution of the Newtonian two body problem for point particles. There is no such complete solution for more than two bodies, or for bodies with finite extent and interior dynamics, like stars. However, for any number of bodies, and for most standard equations of state, choosing N body initial data for Newtonian gravity involves solving a single (linear) Poisson equation. While the non linearity and intricacy of the Einstein initial value constraint equations [1] has long suggested that choosing N body initial data for Einsteinian gravity would be much harder, this work (and its followup) indicate that in fact one can set up fairly general N body initial data in general relativity.

As noted above, in this paper our focus is on time symmetric solutions of the vacuum constraints, in which case the gravitational field is described completely by a Riemannian metric g, which satisfies the single equation $R(g) = 0$, where $R(g)$ is the scalar curvature of g. The aim is to start with a collection of N asymptotically Euclidean time symmetric solutions, and choose a fixed interior region in each, thereby specifying each of the "bodies." Then after choosing a collection of points which roughly locate the N bodies on a fiducial flat background, one shows that there is a solution of the constraints which: i) contains N regions which are isometric to chosen bodies; ii) is a solution of the time symmetric constraint $R = 0$ everywhere; iii) is identical to a slice of Schwarzschild sufficiently far from the bodies; iv) has the centers of the bodies in a configuration which is a scaled version of the chosen configuration. Here the scale factor can be chosen arbitrarily above a certain threshold. For example, if $N = 2$, the distance between the bodies can be arbitrarily chosen above a certain threshold value.

To carry out such a construction, we start by recalling that a Riemannian metric g on the exterior of a ball in $\mathbb{R}^n$ is an *asymptotically flat end* (to order ℓ) provided there are coordinates in which, for multi-indices $|\alpha| \le \ell$,

$$|\partial^\alpha(g_{ij} - \delta_{ij})(x)| \;=\; O(|x|^{-|\alpha|-(n-2)})\,. \tag{1.1}$$

We now state our main theorem in the time-symmetric case.

THEOREM 1.1 *Suppose $n \geq 3$. For each $k = 1, \ldots, N$, let (E_k, g^k) be an n-dimensional asymptotically flat end with zero scalar curvature and ADM mass m_k, and let $U_k \subset E_k$ be pre-compact. Then for each $\epsilon > 0$, there is an asymptotically flat manifold (M, g_ϵ) containing a region U isometric to the disjoint union $\bigcup_{k=1}^{N} (U_k, g^k)$, so that (M, g_ϵ) has zero scalar curvature and one asymptotically flat end which is identical to a Schwarzschild metric with mass $m(g_\epsilon)$ satisfying $\left| m(g_\epsilon) - \sum_{i=1}^{k} m_k \right| < \epsilon$.*

The construction allows us to glue together any number of asymptotically flat ends which solve the time-symmetric vacuum constraint equation (*i.e.* vanishing scalar curvature), and the construction is local near infinity in each end, *i.e.* any given compact subset of the end can be realized isometrically in the final metric g_ϵ. Since the construction is local in any end, we can allow the N original solutions to have multiple ends, and we can allow nonzero scalar curvature compactly supported inside U, as we indicate in the following corollary.

COROLLARY 1.2 *Let (M_k, g^k), $k = 1, \ldots, N$, be asymptotically flat n-manifolds ($n \geq 3$), let $E_k \subset M_k$ be chosen asymptotically flat ends of respective ADM mass m_k, and let $U_k \supset M_k \setminus E_k$ be chosen subdomains with $E_k \cap U_k$ precompact. Suppose that on each $M_k \setminus U_k$, $R(g^k) = 0$, and moreover suppose that one can choose asymptotically flat coordinates so that g^k satisfies (1.1). Then for each $\epsilon > 0$, there is an asymptotically flat manifold (M, g_ϵ) containing a region U isometric to $\bigcup_{k=1}^{N} (U_k, g^k)$, so that $(M \setminus U, g_\epsilon)$ has zero scalar curvature and one asymptotically flat end which is identical to a Schwarzschild metric with mass $m(g_\epsilon)$ satisfying $\left| m(g_\epsilon) - \sum_{i=1}^{k} m_k \right| < \epsilon$.*

REMARK 1.3 We will see in the construction that g_ϵ converges as $\epsilon \to 0^+$ to the Euclidean metric on compact subsets of $M \setminus U$.

2 Preliminaries

We begin with a proposition which follows from [4] or [10]. The metrics introduced here give a family of metrics into which the ends will be glued to prove the main theorem.

PROPOSITION 2.1 *There is a family of metrics γ_t, $t \in (0, \delta)$ on $\mathbb{R}^n$ of zero scalar curvature which converge as $t \to 0^+$ to the Euclidean metric in C^k on compact subsets (for any k), and are exactly Schwarzschild on $\{x : |x| \geq 1\}$ of positive mass, with $\lim_{t \to 0^+} m(\gamma_t) = 0$.*

We also recall two results from [8], cf. [4, 14]. Let K be the span of the set of linear and constant functions $K := \operatorname{span}\{1, x^1, \ldots, x^n\}$.

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let $\zeta \in C_c^\infty(\Omega)$ be a cutoff function. There is a neighbourhood $\mathcal{U}$ of the flat metric in $C^{4,\alpha}(\overline{\Omega})$ so that for any $\Omega_0 \subset\subset \Omega$, there is an $\epsilon_0 0$ so that for smooth metrics $g \in \mathcal{U}$ and functions $S \in C_c^\infty(\Omega_0)$ with $\|S\|_{C^{0,\alpha}} < \epsilon_0$, there is a smooth metric $g+h$ so that h is supported in $\overline{\Omega}$ and satisfies a bound $\|h\|_{C^{2,\alpha}} \leq C\|S\|_{C^{0,\alpha}}$, and also satisfies*

$$R(g+h) - (R(g)+S) \in \zeta K.$$

The map $(g,S) \mapsto h$ is continuous.

We will see in the proof of Theorem 1.1 in the next section how this proposition plays a key role in the proof of the following theorem.

Theorem 2.3 *Let (M,g) be an asymptotically flat n-manifold with zero scalar curvature. Choose any asymptotically flat end $\mathcal{E}$, and let $\mathcal{E}_r \subset \mathcal{E}$ be the region corresponding to $\{x : |x| > r\}$ in asymptotically flat coordinates. Let k be a nonnegative integer. Then for any $\epsilon > 0$, there is an $R > 0$ and a (smooth) metric $\bar{g}$ with zero scalar curvature and $\|g - \bar{g}\|_{C^k(\mathcal{E})} < \epsilon$ (norm measured with respect to the Euclidean metric in the asymptotically flat coordinate chart), with $|m(g) - m(\bar{g})| < \epsilon$, and so that $\bar{g}$ is equal to g on $M \setminus \mathcal{E}_R$, and $\bar{g}$ is identical to an asymptotically flat end of a standard Schwarzschild slice on $\mathcal{E}_{2R}$.*

Remark 2.4 If we suppose that g has the following form on N: $g_{ij}(x) = \left(1 + \frac{m}{(n-1)|x|^{n-2}}\right)^{4/(n-2)} \delta_{ij} + O_3(|x|^{-(n-1)})$, or more generally that g has enough parity symmetry so that the center of mass is well-defined [5, 14], then the center of the Schwarzschild end can be chosen close to that of g.

3 Proof of the main theorem

We now use the results assembled in the preceding section to prove the main theorem. We will prove the result in dimension $n = 3$; the argument in other dimensions is identical.

Proof of Theorem 1.1: Choose $c_1, \dots, c_N \in \mathbb{R}^3$, with $\sum_{k=1}^{N} m_k c_k = 0$ and with $|c_k| > 5$. Moreover we assume that the closed balls $\{x : |x - c_k| \leq 4\}$ are pairwise disjoint. Let $B_0 \subset \mathbb{R}^3$ be a Euclidean ball of radius $R_0 = 5 + \max\{|c_1|, \dots, |c_N|\}$, centered at the origin. Let $m_T = \sum_{k=1}^{N} m_k$.

For $0 < \epsilon << 1$, $1/2 < \mu < 3/2$, and $|c| < 1$, we consider the metric $\tilde{g}_{(\epsilon,\mu,c)}$ on $B_0 \setminus \left(\bigcup_{k=1}^{N} \{x : |x - c_k| \leq 1\}\right)$, constructed as follows: let $\bar{\gamma}_\epsilon = \gamma_{t_\epsilon}$ be one of the metrics from the Proposition 2.1, with mass $m(\bar{\gamma}_\epsilon) = \epsilon m_T$. Let ψ be a smooth nondecreasing function so that $\psi(t) = 0$ for $t < 9/4$ and $\psi(t) = 1$ for $t > 11/4$.

- on each $A'_k = \{x : 1 < |x - c_k| < 2\}$, $\tilde{g}_{(\epsilon,\mu,c)}(x) = (1 + \frac{\epsilon m_k}{2|x - c_k|})^4 g_{Eucl}(x)$

- on each $A_k = \{x : 2 \le |x - c_k| \le 3\}$,
  $\tilde{g}_{(\epsilon,\mu,c)}(x) = (1-\psi(|x-c_k|))(1+\frac{\epsilon m_k}{2|x-c_k|})^4 g_{Eucl} + \psi(|x-c_k|)\bar{\gamma}_\epsilon(\mu^{-1}(x-c))$
- on $B_0 \setminus \Big(\bigcup_{k=1}^{N} \{x : |x - c_k| < 3\} \Big)$, $\tilde{g}_{(\epsilon,\mu,c)}(x) = \bar{\gamma}_\epsilon(\mu^{-1}(x-c))$.

Let $\Omega = B_0 \setminus \Big(\bigcup_{k=1}^{N} \{x : |x - c_k| \le 2\} \Big)$. Let ζ be a smooth cutoff function, which we can define as follows:

- $\zeta(x) = 1$ on $\{x : |x| \le R_0 - 1\} \setminus \Big(\bigcup_{k=1}^{N} \{x : |x - c_k| \le 3\} \Big)$
- $\zeta(x) = \psi(|x - c_k|)$ on $A_k = \{x : 2 \le |x - c_k| \le 3\}$
- $\zeta(x) = \psi(R_0 + 2 - |x|)$ on $\{x : R_0 - 1 \le |x| \le R_0\}$.

There is an $\epsilon_1 > 0$ so that for all $(\epsilon, \mu, c) \in \Theta = \{(\epsilon, \mu, c) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 : 0 < \epsilon < \epsilon_1,\ 1/2 < \mu < 3/2,\ |c| < 1\}$, $g_{(\epsilon,\mu,c)}$ is in the neighborhood $\mathcal{U}$ from Proposition 2.2, $\|g_{(\epsilon,\mu,c)} - g_{Eucl}\|_{C^5} \le C\epsilon$, and moreover $\|R(\tilde{g}_{(\epsilon,\mu,c)})\|_{C^{0,\alpha}} < \epsilon_0$. Thus by Proposition 2.2 (with $S = -R(g)$), there is a constant $C > 0$ so that for any $\theta := (\epsilon, \mu, c) \in \Theta$, there is a small deformation h^θ which is smooth on B_0 and supported on $\overline{\Omega}$ so that $R(\tilde{g}_{(\epsilon,\mu,c)} + h^\theta) = \sum_{j=0}^{3} a_j \zeta x^j$, where $x^0 := 1$ and $\|h^\theta\|_{C^3(\Omega)} + \sum_{j=0}^{3} |a_j| \le C\epsilon$. Furthermore, given ϵ, each a_j is continuous in (μ, c). We argue that for ϵ small enough, by choosing μ and c appropriately, we can arrange all $a_j = 0$. This will be accomplished by showing that $\int_\Omega x^i R(\tilde{g}_{\epsilon,\mu,c)} + h^\theta)\, dx$ vanishes for $i = 0, 1, 2, 3$. Let $\tilde{g}_{(\epsilon,\mu,c)} = \tilde{g}^\theta$ for notational convenience, and let $\tilde{h}^\theta = \tilde{g}^\theta - g_{Eucl}$. Note that we can choose $C > 0$ so that $\|\tilde{h}^\theta\|_{C^3(\Omega)} \le C\epsilon$ for all $\theta \in \Theta$.

Now let $L(h) = -\Delta(tr(h)) + div(div(h)) = \sum_{i,j=1}^{3} (h_{ij,ij} - h_{ii,jj})$ be the linearization at the Euclidean metric of the scalar curvature operator $g \mapsto R(g)$. Note that $L^*(x^\ell) = 0$ for $\ell = 0, 1, 2, 3$. We also note the coordinate formula for the scalar curvature

$$R(g) = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + (\Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ik})\right) = g^{ij}g^{km}(g_{ik,jm} - g_{ij,km}) + Q(\partial g),$$

where $Q(\partial g)$ is a quadratic function in ∂g contracted with the metric g. We can plug in $g = \tilde{g}^\theta + h^\theta$ into the preceding to obtain the following: there is a $C_1 > 0$ so that for small ϵ, we have

$$\begin{aligned} R(\tilde{g}^\theta + h^\theta) &= \sum_{i,j=1}^{3} (\tilde{g}^\theta_{ij,ij} - \tilde{g}^\theta_{ii,jj}) + \sum_{i,j=1}^{3} (h^\theta_{ij,ij} - h^\theta_{ii,jj}) + B(\tilde{h}^\theta, h^\theta) + Q(\tilde{h}^\theta) \\ &= (\tilde{g}^\theta_{ij,ij} - \tilde{g}^\theta_{ii,jj}) + L(h^\theta) + B(\tilde{h}^\theta, h^\theta) + Q(\tilde{h}^\theta), \qquad (3.1) \end{aligned}$$

where

$$\begin{aligned} |B(\tilde{h}^\theta, h^\theta)| &\leq C_1 \|\tilde{h}^\theta\|_{C^2(\Omega)} \cdot \|h^\theta\|_{C^2(\Omega)} \leq C_1 C^2 \epsilon^2 \\ |Q(\tilde{h}^\theta)| &\leq C_1 \|\tilde{h}^\theta\|^2_{C^2(\Omega)} \leq C_1 C^2 \epsilon^2. \end{aligned}$$

We also note that for a Schwarzschild metric g^S of mass m and center c, i.e. $g^S(x) = \left(1 + \frac{m}{2|x-c|}\right)^4 g_{Eucl}$, the leading order term in the expansion $\sum_{i,j} \left(g^S_{ij,i} - g^S_{ii,j}\right) \nu^j$ is just

$$-4(1 + \frac{m}{2|x|})^3 (\frac{m}{2|x|^2}) \sum_{i,j} \left(\frac{x^i}{|x|} \delta_{ij} - \frac{x^j}{|x|} \right) \frac{x^j}{|x|}.$$

Let $d\mu_e$ be Euclidean surface measure. Then there are functions $F_\ell(m, c, r)$, with $|F_\ell(m, c, r)| \leq C_2 m^2/r$, where C_2 is independent of $|c| \leq r/2$, so that

$$\int\limits_{|x|=r} \sum_{i,j} \left(g^S_{ij,i} - g^S_{ii,j}\right) \nu^j_e \, d\mu_e = 16\pi m + F_0(m, c, r), \tag{3.2}$$

and for $\ell = 1, 2, 3$,

$$\int\limits_{|x|=r} \left[\sum_i x^\ell \left(g^S_{ij,i} - g^S_{ii,j}\right) \nu^j_e d\mu_e - \sum_i (g^S_{i\ell} \nu^i_e - g^S_{ii} \nu^\ell_e) d\mu_e \right] = 16\pi m c^\ell + F_\ell(m, c, r). \tag{3.3}$$

Using integration by parts along with (3.1) and (3.2), and the fact that ∂h^θ vanishes at $\partial\Omega$, we obtain

$$\int_\Omega R(\tilde{g}^\theta + h^\theta) \, dx = 16\pi\epsilon(\mu - 1) m_T + O(\epsilon^2).$$

Let $\Sigma_k = \{x : |x - c_k| = 2\}$ and $\Sigma_0 = \partial B_0$, and let $c_0 = 0$. We now use (3.1), integration by parts along with $L^*(x^\ell) = 0$, as well as (3.2) and (3.3) to obtain for $\ell = 1, 2, 3$,

$$\begin{aligned} \int_\Omega x^\ell R(\tilde{g}^\theta + h^\theta) \, dx &= \int_\Omega x^\ell \sum_{i,j=1}^3 (\tilde{g}^\theta_{ij,ij} - \tilde{g}^\theta_{ii,jj}) \, dx + O(\epsilon^2) \\ &= \sum_{k=0}^N \int\limits_{\Sigma_k} \left(x^\ell \sum_{i,j=1}^3 (\tilde{g}^\theta_{ij,i} - \tilde{g}^\theta_{ii,j}) \nu^j_e - \sum_{i=1}^3 (\tilde{g}^\theta_{i\ell} \nu^i_e - \tilde{g}^\theta_{ii} \nu^\ell_e) \right) d\mu_e + O(\epsilon^2) \\ &= \sum_{k=0}^N \int\limits_{\Sigma_k} \left((x^\ell - c^\ell_k) \sum_{i,j=1}^3 (\tilde{g}^\theta_{ij,i} - \tilde{g}^\theta_{ii,j}) \nu^j_e - \sum_{i=1}^3 (\tilde{g}^\theta_{i\ell} \nu^i_e - \tilde{g}^\theta_{ii} \nu^\ell_e) \right) d\mu_e \\ &\quad + 16\pi \sum_{k=1}^N \epsilon m_k c^\ell_k + O(\epsilon^2) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Sigma_0} \Big(x^\ell \sum_{i,j=1}^{3} (\tilde g^\theta_{ij,i} - \tilde g^\theta_{ii,j})\nu_e^j - \sum_{i=1}^{3} (\tilde g^\theta_{i\ell}\nu_e^i - \tilde g^\theta_{ii}\nu_e^\ell)\Big) d\mu_e + O(\epsilon^2) \\
&= 16\pi\epsilon\mu m_T c^\ell + O(\epsilon^2),
\end{aligned}
$$

where we have used the condition $\sum_{k=1}^{N} m_k c_k = 0$.

Thus we see by continuity that for ϵ small enough, for some μ and c, with $(\mu - 1) = O(\epsilon)$ and $c = O(\epsilon)$, that we can arrange the above integrals to vanish. This means for $\theta = (\epsilon, \mu, c)$, the resulting metric $\bar g_\epsilon := \tilde g^\theta + h^\theta$ is scalar-flat and is exactly Schwarzschild (with the respective masses) near $\partial\Omega$.

To finish the proof of the theorem, we note that by applying Theorem 2.3, we can assume without loss of generality that on E_k we can choose coordinates x and $R_k > 1$ so that on $\{|x| \geq R_k\}$, g^k has the Schwarzschild form $g^k(x) = (1 + \frac{m_k}{2|x|})^4 \delta_{ij}$. Indeed, in applying the construction in [8] to the given metrics in Theorem 1.1, the original masses can be made to change by an arbitrarily small amount (in our case by less than $\frac{\epsilon}{2N}$, say), by taking $R_k = R$ sufficiently large; we can assume as well that $|x| < R$ on each U_k. We assume this has been done and the masses have been re-labelled to $m_1, \ldots, m_N$.

We can now scale $\bar g_\epsilon$ by pulling back a scaling of coordinates by scale factor $1/\epsilon$, for $\epsilon < \frac{1}{R}$, so that in the resulting metric the masses of the Schwarzschild neighborhoods of the boundaries $|x - c_k/\epsilon| = \frac{1}{\epsilon} > R$ are $m_1, m_2, \ldots, m_N$, and near the outer boundary is μm_T. Finally we can glue in the metrics (U_k, g^k) isometrically. □

4 Apparent Horizons

A natural question to ask is whether the configurations produced by Theorem 1.1 contain apparent horizons (minimal spheres), apart from any minimal spheres present in U. The point is to assure that we do not introduce any new horizons in the construction, which would shield one or more of the N bodies from the others. In this section we show that for small enough ϵ, there are no minimal surfaces outside U. This can be proved by various means (*cf.* [7]), and one such way is to use the following from [9], which relies on the Riemannian Penrose Inequality [2, 11]; for simplicity we consider $n = 3$ (compare [3]). For completeness we record the proof here.

Proposition 4.1 *Let (M, g) be an asymptotically flat three-manifold with non-negative scalar curvature. Suppose that the sectional curvatures are bounded above by $C > 0$. Then if the ADM mass m satisfies $m\sqrt{C} < \frac{1}{2}$, there are no closed minimal surfaces in M.*

Proof: If there were a minimal surface in M, then by the analysis in [11] (which uses the convex barrier spheres near infinity in an end), there would exist an outermost minimal sphere Σ. Let $A(\Sigma)$ denote its area.

We note that the Gauss equation implies an upper bound on the curvature of the horizon Σ as follows. Let p be a point in Σ, and let $\{e_1, e_2\}$ be a basis of $T_p(M)$ in which the second fundamental form II is diagonal. Let $\kappa_1 = \mathrm{II}(e_1, e_1), \kappa_2 = \mathrm{II}(e_2, e_2)$ be the principal curvatures of Σ at p. Since Σ is minimal, we have $\kappa_1 + \kappa_2 = 0$, and so $\kappa_1\kappa_2 \leq 0$. The Gauss equation then yields

$$\kappa \equiv R(e_1, e_2, e_1, e_2) = \overline{R}(e_1, e_2, e_1, e_2) + \kappa_1\kappa_2 \leq C$$

where R denotes the curvature tensor of Σ, and $\overline{R}$ the curvature tensor of M.

We now invoke the Gauss-Bonnet theorem, which together with the preceding inequality yields

$$4\pi = \int_\Sigma \kappa \, d\mu_\Sigma \leq C \, A(\Sigma).$$

Combining this with the Penrose Inequality $m \geq \sqrt{\frac{A(\Sigma)}{16\pi}}$, we obtain

$$m\sqrt{C} \geq \frac{1}{2} \, .$$

□

PROPOSITION 4.2 *For sufficiently small ϵ, any closed minimal surface in (M, g_ϵ) must be contained in U.*

PROOF: Using the notation from the proof of Theorem 1.1, we identify the set $B_0 \setminus \left(\bigcup_{k=1}^{N} \{x : |x - c_k| \leq 1\} \right)$ as a subset of M, and we identify $M \setminus U$ with the closure of $(\mathbb{R}^3 \setminus B_0) \cup \Omega$. Let $U' = U \setminus \left(\bigcup_{k=1}^{N} \{x \in B_0 : 1 \leq |x - c_k| \leq 2\} \right)$. If there were a minimal surface whose intersection with $M \setminus U$ were nonempty, then applying arguments in [11], there is an outermost minimal sphere Σ with respect to the end into which we have glued U, with $\Sigma \cap (M \setminus U)$ nonempty. Now it follows by applying the argument in Proposition 4.1 that for small ϵ there cannot be an outermost horizon contained in $M \setminus U'$. Thus Σ will clearly have to intersect both U' and $M \setminus U$. Scale the metric down by a factor of ϵ, so that the metric on Ω is $\bar{g}_\epsilon$ (which is uniformly close to the Euclidean metric), and let Σ_ϵ be Σ in this metric. By the Penrose inequality, and the estimate of μ, there is a constant C_2 so that $A(\Sigma_\epsilon) \leq C_2\epsilon^2$.

On the other hand, let $p \in \Sigma_\epsilon \cap (M \setminus U)$. There is an $r \in (0, 1)$ so that if $B_r^M(p)$ is the geodesic ball of radius r about p, then $S = \Sigma_\epsilon \cap B_r^M(p)$ is a smooth surface with nonempty boundary (since $\Sigma_\epsilon \cap U'$ is nonempty). We can now apply Schoen's curvature estimates for the stable hypersurface S [15]: for given $r_0 \in (0, r)$, there is a C so that for all small ϵ, the second fundamental form on $\Sigma_\epsilon \cap B_{r_0}^M(p)$ has norm bounded by C. Hence near p, $\Sigma_\epsilon \cap B_{r_0/2}^M(p)$ is graphical with bounded gradient. Thus the area of Σ_ϵ is uniformly bounded from below. This contradicts the area estimate from the Penrose Inequality.

Thus $\Sigma_\epsilon \subset \bar{U}$. But then $\Sigma \subset U$, since $\partial\Omega \cap \bar{U}$ is a union of N convex spheres.
□

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