

THE ROYAL
SWEDISH
ACADEMY OF
SCIENCES



**INSTITUT
MITTAG-LEFFLER**

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

**Hamiltonian description of radiation
phenomena: Trautman-Bondi energy and
corner conditions**

W. Chmielowiec and J. Kijowski

REPORT No. 42, 2008/2009, fall

ISSN 1103-467X

ISRN IML-R- -42-08/09- -SE+fall

Hamiltonian description of radiation phenomena: Trautman-Bondi energy and corner conditions

Witold Chmielowiec, Jerzy Kijowski
Center for Theoretical Physics, Polish Academy of Sciences,
Al. Lotników 32/46, 02-668 Warsaw, Poland
e-mail: wchmiel@cft.edu.pl, kijowski@cft.edu.pl

Abstract

Cauchy initial value problem on a hyperboloid is proved to define a Hamiltonian system, provided the radiation data at null infinity are also taken into account, as a part of Cauchy data. The “Trautman-Bondi mass”, supplemented by the “already radiated energy” assigned to radiation data, plays role of the Hamiltonian function. This approach leads to correct description of the corner conditions.

Keywords: Trautman-Bondi energy, wave equation, initial-characteristic value problem, Hamiltonian field theory.

1 Introduction

The notion of energy in the radiation regime (in our paper referred to as the Trautman-Bondi energy) has been introduced in Einstein’s theory of gravity by Trautman [14] and independently by Bondi [1]. It measures that part of the gravitational energy of an isolated system, which “has not yet been radiated”. In conformal spacetime compactification, the Trautman-Bondi energy may be assigned to any space-like hypersurface having a regular intersection with the conformal boundary \mathcal{I} (null infinity, or the *scri*) [4]. Due to radiation, the Trautman-Bondi energy – unlike the total (ADM) energy – is not conserved, but is *decreasing* because it may be partially radiated in a form of gravitational waves.

As shown in [3, 10], the validity of the Trautman-Bondi energy goes far beyond the gravitational context and may be used in any hyperbolic field theory, also special-relativistic one. In particular, it has a beautiful Hamiltonian interpretation. The goal of the present paper is to apply this idea to the scalar field theory, where the Cauchy data of the system are assigned to a hyperboloid. Field evolution on hyperboloids is proved to be a Hamiltonian system, if we complete Cauchy data by the *radiation data* at the *scri*, and supplement the Trautman-Bondi energy with the corresponding *radiation energy*. The sum of the two defines the total Hamiltonian function of the “matter + radiation” system. But to “tailor” the two disjoint objects: 1) the field Cauchy data on a hyperboloid and 2) the radiation data on the *scri*, into a single object, appropriate compatibly conditions (often called “corner conditions” [5, 6]) must be imposed. We propose a universal approach which solves all these issues.

2 Hamiltonian description of Cauchy initial value problem on a hyperboloid

Hamiltonian description of any field dynamics is based on a “3 + 1” foliation of spacetime. Leaves of the foliation are labeled by a parameter called the “time variable”. Phase space of this dynamics is composed of all the possible field Cauchy data on a given leaf. The “3 + 1” decomposition provides also an identification between different leaves of the foliation, which makes the field dynamics and its Hamiltonian function uniquely defined. Here, we consider the Hamiltonian description within a simple model: the massless scalar field, satisfying the wave equation in two- or four-dimensional Minkowski spacetime. This means that the space is one-dimensional: $x \in \mathbb{R}$ or three-dimensional: $x \in \mathbb{R}^3$. Contrary to the standard (“ADM”) formulation, the field initial data will not be assigned to spatially flat Cauchy surfaces, but to hypersurfaces which extend to null infinity, namely spacelike hyperboloids. Naively, such a Cauchy problem cannot be described as a Hamiltonian system: future evolution of the system is well defined, but the past evolution is absolutely non-unique and may be arbitrarily modified by radiation. Nevertheless, we are going to show, that the Hamiltonian formulation of the field evolution is possible. For this purpose we have to complete Cauchy data on a hyperboloid by appropriate *radiation data* at light infinity.

For pedagogical reasons, we begin our analysis with a (much simpler) finite case, where we restrict field dynamics to a finite light-cone and describe initial data on its (characteristic!) boundary and on the finite part of the hyperboloid, contained within the cone. At the end, we may consider the limiting case (in our notation, this corresponds to $\epsilon \rightarrow 0$), where the finite cone is shifted to infinity. This way we obtain the Hamiltonian description of initial data on the entire hyperboloid and on the *scri* \mathcal{S}^+ (conformal boundary of the spacetime). In the subsequent Section we describe the two-dimensional “toy model”. The complete, four-dimensional theory is analyzed in Section 2.2.

2.1 Two-dimensional Minkowski spacetime

Let us consider a one-parameter family of past oriented light cones in the two-dimensional Minkowski space time:

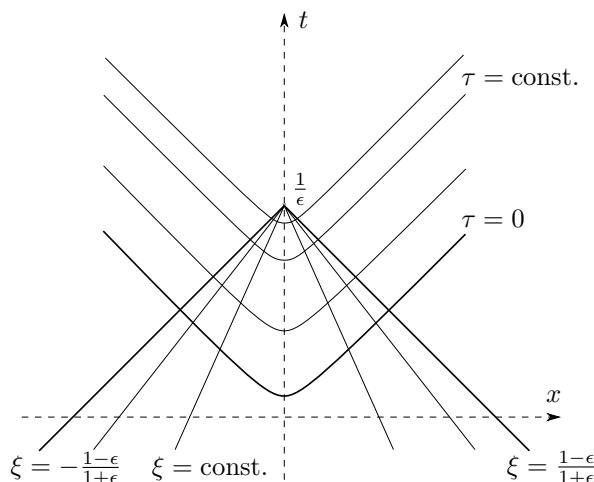
$$\mathcal{C}_\epsilon^- := \left\{ (t, x) : x \in \mathbb{R}, \frac{1}{\epsilon} - t > |x| \right\}, \quad (1)$$

where $\frac{1}{\epsilon} > 1$ is the time coordinate of the vertex of \mathcal{C}_ϵ^- . We introduce new coordinates (τ, ξ) connected with Minkowski coordinates (t, x) in the following way:

$$t = \frac{1}{\epsilon} + \left(\frac{1 + \xi^2}{1 - \xi^2} - \frac{1}{\epsilon} \right) e^{-\epsilon\tau}, \quad (2)$$

$$x = \frac{2\xi}{1 - \xi^2} e^{-\epsilon\tau}, \quad (3)$$

where $\tau \in \mathbb{R}^1$. For $|\xi| \leq \frac{1-\epsilon}{1+\epsilon}$ new coordinates parameterize the whole cone \mathcal{C}_ϵ^- . Moreover, surfaces $\{\tau = \text{const.}\}$ correspond to hyperboloids. In order



to describe field dynamics in a Hamiltonian way, we begin with the standard, relativistic Lagrangian for wave equation:

$$L = -\frac{1}{2}\sqrt{|\det g|}g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) = \frac{1}{2}\left\{(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right\}. \quad (4)$$

Expressing the Lagrangian density “ $L \cdot d^2x$ ” in new coordinates we obtain:

$$L \cdot d^2x = \mathcal{L} \cdot d^2\xi, \quad (5)$$

where

$$\mathcal{L} = \frac{\left(\frac{\partial\varphi}{\partial\tau} + \xi\frac{\partial\varphi}{\partial\xi}\right)^2}{1 - \epsilon + (1 + \epsilon)\xi^2} - \frac{1}{4}\left[1 - \epsilon + (1 + \epsilon)\xi^2\right]\left(\frac{\partial\varphi}{\partial\xi}\right)^2. \quad (6)$$

Denoting

$$\kappa := \frac{1}{2}\left[1 - \epsilon + (1 + \epsilon)\xi^2\right]$$

we obtain the autonomous (i.e. τ -independent) Lagrangian function:

$$\mathcal{L} = \frac{1}{2\kappa}(\partial_\tau\varphi + \xi\partial_\xi\varphi)^2 - \frac{1}{2}\kappa(\partial_\xi\varphi)^2. \quad (7)$$

Now, the standard procedure leading from the Lagrangian to the Hamiltonian description may be applied. We first define conjugate momenta:

$$\pi^\mu := \frac{\partial\mathcal{L}}{\partial\varphi_\mu}, \quad (8)$$

where $\varphi_\mu := \partial_\mu\varphi$, and calculate the variation of the Lagrangian:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \pi^\mu\delta\varphi_\mu = \partial_\mu(\pi^\mu\delta\varphi) + \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\pi^\mu\right)\delta\varphi. \quad (9)$$

Field equation (in our case it is always the wave equation $\square\varphi = 0$) is equivalent to vanishing of the Euler-Lagrange term in (9), hence, equivalent to the following equation

$$\delta\mathcal{L} = \partial_\mu(\pi^\mu\delta\varphi). \quad (10)$$

Integrating (10) over the volume $V_\epsilon := \{\xi : |\xi| < \frac{1-\epsilon}{1+\epsilon}\}$ on the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$, we obtain an identity valid for fields satisfying wave equation

$$\delta \int_{V_\epsilon} \mathcal{L} d\xi = \int_{V_\epsilon} (\pi \delta \phi) \cdot d\xi + \int_{\partial V_\epsilon} (\pi^1 \delta \phi) d\sigma_1 ,$$

where "dot" denotes derivative with respect to the new time variable τ , while ϕ is the restriction of the field φ to the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$:

$$\begin{aligned} \phi(\tau, \xi) &= \varphi(t(\tau, \xi), x(\tau, \xi)) \Big|_\Sigma \\ &= \varphi\left(\frac{1}{\epsilon} + \left(\frac{1+\xi^2}{1-\xi^2} - \frac{1}{\epsilon}\right)e^{-\epsilon\tau}, \frac{2\xi}{1-\xi^2}e^{-\epsilon\tau}\right), \end{aligned} \quad (11)$$

for $|\xi| \leq \frac{1-\epsilon}{1+\epsilon}$. The time component of the momentum: $\pi := \pi^0$, provides, together with ϕ , the complete description of Cauchy data on this surface. Legendre transformation between $\dot{\phi}$ and π gives us:

$$-\delta H_{V_\epsilon} = \int_{V_\epsilon} (\dot{\pi} \delta \phi - \dot{\phi} \delta \pi) d\xi + [\pi^1 \delta \phi]_{\partial V_\epsilon}, \quad (12)$$

where the Hamiltonian H_{V_ϵ} is defined by

$$H_{V_\epsilon}(\phi, \pi) := \int_{V_\epsilon} (\pi \dot{\phi} - \mathcal{L}) d\xi . \quad (13)$$

Canonical structure in the space of Cauchy data is given by the standard symplectic form:

$$\omega_{V_\epsilon} := \int_{V_\epsilon} (\delta \pi \wedge \delta \phi) d\xi . \quad (14)$$

Formulae (7) and (8) imply the following relations between "velocity" $\dot{\phi}$ and momentum π :

$$\pi := \pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \kappa^{-1} \left(\dot{\phi} + \xi \partial_\xi \phi \right) , \quad (15)$$

Taking into account (7) and (15), the Hamiltonian (13) may be written explicitly in terms of the Cauchy data on V_ϵ

$$H_{[-\frac{1-\epsilon}{1+\epsilon}, \frac{1-\epsilon}{1+\epsilon}]}(\phi, \pi) = \frac{1}{2} \int_{-\frac{1-\epsilon}{1+\epsilon}}^{\frac{1-\epsilon}{1+\epsilon}} \left\{ \kappa \left(\pi - \kappa^{-1} \xi \frac{\partial \phi}{\partial \xi} \right)^2 + (\kappa - \kappa^{-1} \xi^2) \left(\frac{\partial \phi}{\partial \xi} \right)^2 \right\} d\xi . \quad (16)$$

The factor $\kappa - \kappa^{-1} \xi^2$ is positive for $\kappa - |\xi| > 0$. Moreover, we have:

$$\kappa - |\xi| = \frac{1}{2}(1 + \epsilon)(1 - |\xi|) \left(\frac{1 - \epsilon}{1 + \epsilon} - |\xi| \right) . \quad (17)$$

This implies positivity of the Hamiltonian (16) for $|\xi| < \frac{1-\epsilon}{1+\epsilon}$, i.e inside the cone \mathcal{C}_ϵ^- . The following Hamiltonian equations (equivalent to Euler-Lagrange equations) may be derived from the Hamiltonian (16):

$$\dot{\phi} = \kappa \pi - \xi \partial_\xi \phi , \quad (18)$$

$$\dot{\pi} = \partial_\xi (\kappa \partial_\xi \phi) - \partial_\xi (\xi \pi) , \quad (19)$$

provided no boundary terms remain after the integration by part of its variation, which, *a priori*, is not true! This apparent paradox is implied by the fact, that evolution of Cauchy data on a hyperboloid is well defined only forward in time and, whence, does not correspond *a priori* to any Hamiltonian system. To overcome this difficulty, we take into account missing data on the light cone below hyperboloid and treat it as a part of Cauchy data (cf. [3]). For this purpose we extend parametrization (2), (3) beyond the volume $V_\epsilon = \{\xi : |\xi| < \frac{1-\epsilon}{1+\epsilon}\}$, taking into account also the corresponding points on the boundary of the cone:

$$t = \frac{1}{\epsilon} - x := \frac{1}{\epsilon} - \frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau + \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}} \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \quad (20)$$

$$t = \frac{1}{\epsilon} + x := \frac{1}{\epsilon} - \frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau - \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}} \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}, \quad (21)$$

and consider the data (ϕ, π) on the entire surface $\Sigma = \{\tau = \text{const.}, \xi \in \mathbb{R}\}$. Thus

$$\begin{aligned} \phi(\tau, \xi) &= \varphi(t(\tau, \xi), x(\tau, \xi)) \Big|_{\partial\mathcal{H}_\epsilon^-} \\ &= \varphi\left(\frac{1}{\epsilon} - \frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau + \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}}, \frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau + \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}}\right) \end{aligned} \quad (22)$$

for $\xi \geq \frac{1-\epsilon}{1+\epsilon}$,

$$\begin{aligned} \phi(\tau, \xi) &= \varphi(t(\tau, \xi), x(\tau, \xi)) \Big|_{\partial\mathcal{H}_\epsilon^-} \\ &= \varphi\left(\frac{1}{\epsilon} - \frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau - \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}}, -\frac{1}{2}\left(\frac{1}{\epsilon} - \epsilon\right)e^{-\epsilon\tau - \epsilon\xi - \frac{\epsilon(1-\epsilon)}{1+\epsilon}}\right) \end{aligned} \quad (23)$$

for $\xi \leq -\frac{1-\epsilon}{1+\epsilon}$. Equation (20) implies that $X := \partial_\tau = -\partial_\xi$, for $\xi \geq \frac{1-\epsilon}{1+\epsilon}$, whereas (21) implies $X := \partial_\tau = \partial_\xi$ for $\xi \leq -\frac{1-\epsilon}{1+\epsilon}$. Within these regions of the Cauchy surface, the dynamics consists in transporting the field data (ϕ, π) over Σ along the field X , according to following equations:

$$\begin{aligned} \mathcal{L}_X \phi &= \partial_\tau \phi = -\partial_\xi \phi \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \\ \mathcal{L}_X \phi &= \partial_\tau \phi = \partial_\xi \phi \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}, \\ \mathcal{L}_X \pi &= \partial_\tau \pi = -\partial_\xi \pi \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \\ \mathcal{L}_X \pi &= \partial_\tau \pi = \partial_\xi \pi \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}, \end{aligned}$$

where \mathcal{L}_X denotes the Lie derivative along the vector field X . The above equations can be derived from the following Hamiltonians (generators of space translations):

$$H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)}(\phi, \pi) := \int_{\frac{1-\epsilon}{1+\epsilon}}^{\infty} (\pi \dot{\phi} - \mathcal{L}) d\xi = \int_{\frac{1-\epsilon}{1+\epsilon}}^{\infty} (-\pi \partial_\xi \phi) d\xi, \quad (24)$$

$$H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]}(\phi, \pi) := \int_{-\infty}^{-\frac{1-\epsilon}{1+\epsilon}} (\pi \dot{\phi} - \mathcal{L}) d\xi = \int_{-\infty}^{-\frac{1-\epsilon}{1+\epsilon}} (\pi \partial_\xi \phi) d\xi, \quad (25)$$

where \mathcal{L} vanishes identically as a pull-back of the scalar density L via the degenerate coordinate transformation (20)-(21), and the momentum π on the Cauchy surface Σ is equal to the pull-back of the (odd) differential form $\pi^\mu \partial_\mu \rfloor d\xi^0 \wedge d\xi^1$ to $\partial\mathcal{C}_\epsilon^-$. Moreover, momentum π^1 coincides with $\pi = \pi^0$ for $\xi \geq \frac{1-\epsilon}{1+\epsilon}$, and with $-\pi^0$ for $\xi \leq -\frac{1-\epsilon}{1+\epsilon}$, as the pull-back of the same form to the hypersurface $\{\xi^1 = \text{const.}\} = \{\xi^0 = \text{const.}\} = \Sigma$. Hence, we obtain the following constraints:

$$\pi = \dot{\phi} = -\partial_\xi \phi \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \quad (26)$$

$$\pi = \dot{\phi} = \partial_\xi \phi \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}. \quad (27)$$

The phase space of Cauchy data on the entire Σ is described by the pairs (ϕ, π) defined on the whole \mathbb{R} and fulfilling constraint (26) or (27) in the corresponding regions.

The total Hamiltonian function H_ϵ on the entire phase space $\mathcal{P} = \{(\phi, \pi)\}$ is equal to the sum of these partial Hamiltonians:

$$H_\epsilon := H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon})} + H_{[-\frac{1-\epsilon}{1+\epsilon}, \frac{1-\epsilon}{1+\epsilon}]} + H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)}. \quad (28)$$

Variation of H_ϵ gives

$$\begin{aligned} -\delta H_\epsilon(\phi, \pi) &= \int_\Sigma (\mathcal{L}_X \pi \delta \phi - \mathcal{L}_X \phi \delta \pi) d\xi \\ &+ [\pi^1 \delta \phi]_{-\infty}^{-\frac{1-\epsilon}{1+\epsilon}} + [\pi^1 \delta \phi]_{\frac{1-\epsilon}{1+\epsilon}}^{\frac{1-\epsilon}{1+\epsilon}} + [\pi^1 \delta \phi]_{\frac{1-\epsilon}{1+\epsilon}}^\infty \end{aligned} \quad (29)$$

with appropriate values for $(\mathcal{L}_X \phi, \mathcal{L}_X \pi)$ in the respective regions of Σ . The functional H_ϵ defines the Hamiltonian dynamics of the total system, if the boundary terms in formula (29) cancel. This requires corner conditions at $\xi = -\frac{1-\epsilon}{1+\epsilon}$ and $\xi = \frac{1-\epsilon}{1+\epsilon}$ and sufficient strong fall-off condition at infinity. To analyse these conditions it is useful to reformulate our Hamiltonian description. Taking into account constraints (26) and (27) and using (24) and (25) we have

$$H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)} = \int_{\frac{1-\epsilon}{1+\epsilon}}^\infty (\partial_\xi \phi)^2 d\xi, \quad (30)$$

$$H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]} = \int_{-\infty}^{-\frac{1-\epsilon}{1+\epsilon}} (\partial_\xi \phi)^2 d\xi, \quad (31)$$

and the corresponding symplectic structures

$$\omega_{[\frac{1-\epsilon}{1+\epsilon}, \infty)} = \int_{\frac{1-\epsilon}{1+\epsilon}}^\infty (-\partial_\xi \delta \phi \wedge \delta \phi) d\xi \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \quad (32)$$

$$\omega_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]} = \int_{-\infty}^{-\frac{1-\epsilon}{1+\epsilon}} (\partial_\xi \delta \phi \wedge \delta \phi) d\xi \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}. \quad (33)$$

Changing variables in the following way:

$$\begin{aligned} \lambda &:= \tau + \frac{1-\epsilon}{1+\epsilon} - \xi \quad \text{for } \xi \geq \frac{1-\epsilon}{1+\epsilon}, \\ \chi &:= \tau + \frac{1-\epsilon}{1+\epsilon} + \xi \quad \text{for } \xi \leq -\frac{1-\epsilon}{1+\epsilon}, \end{aligned}$$

and denoting:

$$\begin{aligned} x_\epsilon(\chi) &:= \phi\left(\tau, \chi - \tau - \frac{1-\epsilon}{1+\epsilon}\right), \\ y_\epsilon(\lambda) &:= \phi\left(\tau, \tau + \frac{1-\epsilon}{1+\epsilon} - \lambda\right), \end{aligned}$$

we see that functions x_ϵ and y_ϵ do not depend upon τ (see (22) and (23)), hence they are single variable functions. Now we can write formulas (30)-(31) and (32)-(33) jointly

$$\begin{aligned} H_{\text{ext},\epsilon} &:= H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]} + H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)} \\ &= \int_{-\infty}^0 \{(\partial_\lambda f_\epsilon^-)^2 + (\partial_\lambda g_\epsilon^-)^2\} d\lambda \end{aligned} \quad (34)$$

$$\begin{aligned} \omega_{\text{ext},\epsilon} &:= \omega_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]} + \omega_{[\frac{1-\epsilon}{1+\epsilon}, \infty)} , \\ &= \int_{-\infty}^0 \{\partial_\lambda \delta f_\epsilon^- \wedge \delta f_\epsilon^- + \partial_\lambda \delta g_\epsilon^- \wedge \delta g_\epsilon^-\} d\lambda , \end{aligned} \quad (35)$$

where

$$f_\epsilon^-(\tau, \lambda) := x_\epsilon(\lambda + \tau) , \quad (36)$$

$$g_\epsilon^-(\tau, \lambda) := y_\epsilon(\lambda + \tau) . \quad (37)$$

Formula (35) shows, that the canonical structure of “external data” (i.e. data outside of the hyperboloid) can be described by the “ $\int \delta f' \wedge \delta f$ ” symplectic form. To find the appropriate functional-analytic framework for the problem and, in particular, to obtain correct formulation of the corner condition, it is useful to reformulate also the “internal data” (on the hyperboloid) in a similar way. Indeed, we shall prove in the sequel that the transition between the hyperboloidal data and that part of the light-cone data, which lies *above the hyperboloid* is a canonical transformation, which converts the canonical form “ $\int \delta \pi \wedge \delta \phi$ ” into the Faddeev form “ $\int \delta f' \wedge \delta f$ ”, (cf. [9]).

For this purpose, assume that we know the light-cone data (f, g) , where f is a function which lives on the left piece of the light-cone, whereas function g lives on the right one. We use convenient coordinates

$$\begin{aligned} u &= t - x , \\ v &= t + x , \end{aligned}$$

and in particular the left piece of $\Gamma_\epsilon := \partial \mathcal{C}_\epsilon^-$ is given by $\{u = \frac{1}{\epsilon}\}$, while the right one is given by $\{v = \frac{1}{\epsilon}\}$. Therefore

$$\varphi\left(\frac{1}{2}\left(v + \frac{1}{\epsilon}\right), \frac{1}{2}\left(v - \frac{1}{\epsilon}\right)\right) = \Phi\left(\frac{1}{\epsilon}\right) + \Psi(v) = f(v) , \quad (38)$$

$$\varphi\left(\frac{1}{2}\left(\frac{1}{\epsilon} + u\right), \frac{1}{2}\left(\frac{1}{\epsilon} - u\right)\right) = \Phi(u) + \Psi\left(\frac{1}{\epsilon}\right) = g(u) , \quad (39)$$

where φ is the general solution of the wave equation, i.e.

$$\varphi(t, x) = \Phi(t - x) + \Psi(t + x) , \quad (40)$$

where Φ and Ψ are functions of one variable. Due to (38) and (39), we can express the general solution (40) in terms of the light-cone data (f, g) :

$$\varphi(t, x) = f(t + x) + g(t - x) - \varphi\left(\frac{1}{\epsilon}, 0\right) , \quad (41)$$

where $\varphi(\frac{1}{\epsilon}, 0) = \Phi(\frac{1}{\epsilon}) + \Psi(\frac{1}{\epsilon}) = f(\frac{1}{\epsilon}) = g(\frac{1}{\epsilon})$. Formula (41) implies the following transformation between the hyperboloidal data (ϕ, π) and the light-cone data (f, g) :

$$\begin{aligned} \phi(\tau, \xi) &= \varphi|_{\tau=\text{const.}} \\ &= f\left(\frac{1}{\epsilon} + \left(\frac{1+\xi}{1-\xi} - \frac{1}{\epsilon}\right)e^{-\epsilon\tau}\right) + g\left(\frac{1}{\epsilon} + \left(\frac{1-\xi}{1+\xi} - \frac{1}{\epsilon}\right)e^{-\epsilon\tau}\right) - \varphi\left(\frac{1}{\epsilon}, 0\right), \end{aligned} \quad (42)$$

$$\begin{aligned} \pi(\tau, \xi) &= \kappa^{-1}(\partial_\tau\varphi + \xi\partial_\xi\varphi)|_{\tau=\text{const.}} \\ &= \frac{2e^{-\epsilon\tau}}{(1-\xi)^2}f'\left(\frac{1}{\epsilon} + \left(\frac{1+\xi}{1-\xi} - \frac{1}{\epsilon}\right)e^{-\epsilon\tau}\right) + \frac{2e^{-\epsilon\tau}}{(1+\xi)^2}g'\left(\frac{1}{\epsilon} + \left(\frac{1-\xi}{1+\xi} - \frac{1}{\epsilon}\right)e^{-\epsilon\tau}\right), \end{aligned} \quad (43)$$

where we used definition (15) of the momentum π . Integrating (43) over intervals $[-\frac{1-\epsilon}{1+\epsilon}, \xi]$ and $[\xi, \frac{1-\epsilon}{1+\epsilon}]$, we obtain, due to (42), the following transformation:

$$f(v) = \frac{1}{2}\left\{\phi\left(-\frac{1-v e^{\epsilon\tau} - \frac{1}{\epsilon}(1-e^{\epsilon\tau})}{1+v e^{\epsilon\tau} + \frac{1}{\epsilon}(1-e^{\epsilon\tau})}\right) + \phi\left(-\frac{1-\epsilon}{1+\epsilon}\right)\right\} + \frac{1}{2}\int_{-\frac{1-\epsilon}{1+\epsilon}}^{-\frac{1-v e^{\epsilon\tau} - \frac{1}{\epsilon}(1-e^{\epsilon\tau})}{1+v e^{\epsilon\tau} + \frac{1}{\epsilon}(1-e^{\epsilon\tau})}} \pi(\eta)d\eta, \quad (44)$$

and

$$g(u) = \frac{1}{2}\left\{\phi\left(\frac{1-\epsilon}{1+\epsilon}\right) + \phi\left(\frac{1-u e^{\epsilon\tau} - \frac{1}{\epsilon}(1-e^{\epsilon\tau})}{1+u e^{\epsilon\tau} + \frac{1}{\epsilon}(1-e^{\epsilon\tau})}\right)\right\} + \frac{1}{2}\int_{\frac{1-u e^{\epsilon\tau} - \frac{1}{\epsilon}(1-e^{\epsilon\tau})}{1+u e^{\epsilon\tau} + \frac{1}{\epsilon}(1-e^{\epsilon\tau})}}^{\frac{1-\epsilon}{1+\epsilon}} \pi(\eta)d\eta. \quad (45)$$

Substitution of (42), (43) into the symplectic form (14) defined on the space of Cauchy data (ϕ, π) over the hyperboloid V_ϵ gives us:

$$\begin{aligned} \omega_{\text{int}, \epsilon} &= \int_{-\frac{1-\epsilon}{1+\epsilon}}^{\frac{1-\epsilon}{1+\epsilon}} \delta\pi(\tau, \xi) \wedge \delta\phi(\tau, \xi)d\xi \\ &= \int_{\frac{1}{\epsilon} + (\epsilon - \frac{1}{\epsilon})e^{-\epsilon\tau}}^{\frac{1}{\epsilon}} \{\partial_z \delta f(z) \wedge \delta f(z) + \partial_z \delta g(z) \wedge \delta g(z)\} dz. \end{aligned} \quad (46)$$

Changing variables in (46) in the following way

$$z = \frac{1}{\epsilon} + \left(\epsilon - \frac{1}{\epsilon}\right)e^{-\epsilon\lambda - \epsilon\tau}$$

and denoting functions

$$f_\epsilon^+(\tau, \lambda) := f\left(\frac{1}{\epsilon} + \left(\epsilon - \frac{1}{\epsilon}\right)e^{-\epsilon\lambda - \epsilon\tau}\right), \quad (47)$$

$$g_\epsilon^+(\tau, \lambda) := g\left(\frac{1}{\epsilon} + \left(\epsilon - \frac{1}{\epsilon}\right)e^{-\epsilon\lambda - \epsilon\tau}\right), \quad (48)$$

(we use superscript ”+”, because these functions are defined over the positive half-line $\lambda \in [0, \infty)$) we can write the formula (46) in the following form

$$\omega_{\text{int}, \epsilon} = \int_0^\infty \{\partial_\lambda \delta f_\epsilon^+ \wedge \delta f_\epsilon^+ + \partial_\lambda \delta g_\epsilon^+ \wedge \delta g_\epsilon^+\} d\lambda. \quad (49)$$

Equation (49) shows that, indeed, the "internal" part of the canonical structure can also be described by the " $\int \delta f' \wedge \delta f$ " symplectic form defined on the light-cone data. This ends our proof.

Now, the corner conditions at points $\xi = \pm \frac{1-\epsilon}{1+\epsilon}$ (necessary for cancellation of boundary terms in the Hamiltonian formula (29)) are expressed into the compatibility condition between the external (given by (35)) and the internal (given by (49)) structures, which must be satisfied at the point $\lambda = 0$. An obvious condition is that the total symplectic form:

$$\omega_\epsilon := \omega_{\text{ext},\epsilon} + \omega_{\text{int},\epsilon} = \int_{-\infty}^{\infty} \{ \delta f'_\epsilon \wedge \delta f_\epsilon + \delta g'_\epsilon \wedge \delta g_\epsilon \} d\lambda, \quad (50)$$

where f_ϵ, g_ϵ are equal to $f_\epsilon^-, g_\epsilon^-$ for $\lambda < 0$ and to $f_\epsilon^+, g_\epsilon^+$ for $\lambda > 0$, respectively, must be well defined. In particular, a step discontinuity is excluded, because its derivative would produce the Dirac delta, which cannot be integrated with a non-continuous function. We see that, due to constrains (26)–(27), the Faddeev-Takhtajan-Dubrovin symplectic form, typical for the Hamiltonian theory of solitons, arises in a natural way in Hamiltonian description of characteristic value problem for standard, hyperbolic equations. Transition between the standard Cauchy data and the radiation (characteristic) data is a symplectomorphism.

We conclude that the product of f_ϵ and g_ϵ by their derivatives must be integrable on real line \mathbb{R} , so must belong to $L^1(\mathbb{R})$. Moreover, functions f_ϵ, g_ϵ and f'_ϵ, g'_ϵ have to belong to mutually dual spaces because they represent "positions" and "momenta" correspondingly. This implies: that $f_\epsilon, g_\epsilon \in H^{\frac{1}{2}}(\mathbb{R})$ and, then, $f'_\epsilon, g'_\epsilon \in H^{-\frac{1}{2}}(\mathbb{R})$. Equations (42)–(43) imply that we have also: $\phi \in H^{\frac{1}{2}}(\mathbb{R})$ and $\pi \in H^{-\frac{1}{2}}(\mathbb{R})$.

Due to equations (36)–(37) and (47)–(48), we have the following relation between the previous and the final representation of Cauchy data:

$$\begin{aligned} f_\epsilon^-(\tau, \lambda) &= \phi\left(\tau, \lambda - \frac{1-\epsilon}{1+\epsilon}\right), \\ g_\epsilon^-(\tau, \lambda) &= \phi\left(\tau, \frac{1-\epsilon}{1+\epsilon} - \lambda\right), \end{aligned}$$

and

$$\begin{aligned} f_\epsilon^+(\tau, \lambda) &= \frac{1}{2} \left\{ \phi\left(\tau, -\frac{(1-\epsilon)(1+\epsilon-e^{\epsilon\lambda})}{(1+\epsilon)(-1+\epsilon+e^{\epsilon\lambda})}\right) + \phi\left(\tau, -\frac{1-\epsilon}{1+\epsilon}\right) \right\} + \frac{1}{2} \int_{-\frac{1-\epsilon}{1+\epsilon}}^{-\frac{(1-\epsilon)(1+\epsilon-e^{\epsilon\lambda})}{(1+\epsilon)(-1+\epsilon+e^{\epsilon\lambda})}} \pi(\tau, \xi) d\xi, \\ g_\epsilon^+(\tau, \lambda) &= \frac{1}{2} \left\{ \phi\left(\tau, \frac{1-\epsilon}{1+\epsilon}\right) + \phi\left(\tau, \frac{(1-\epsilon)(1+\epsilon-e^{\epsilon\lambda})}{(1+\epsilon)(-1+\epsilon+e^{\epsilon\lambda})}\right) \right\} + \frac{1}{2} \int_{\frac{1-\epsilon}{1+\epsilon}}^{\frac{(1-\epsilon)(1+\epsilon-e^{\epsilon\lambda})}{(1+\epsilon)(-1+\epsilon+e^{\epsilon\lambda})}} \pi(\tau, \xi) d\xi. \end{aligned}$$

The Hamiltonian system we have obtained is conservative, because the total Hamiltonian (28) does not depend explicitly on time. Hence, the total energy is conserved:

$$\frac{d}{d\tau} H_\epsilon = 0.$$

But, due to (30) and (31), the external energy $H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]}$ and $H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)}$ are monotonically increasing function on time. Indeed, they are obtained by integration of a non-negative function $(\partial_\xi \phi)^2$ over a part of the boundary $\partial \mathcal{C}_\epsilon^-$

of the cone which increases when the time increases. We conclude that the Trautman-Bondi internal $H_{[-\frac{1-\epsilon}{1+\epsilon}, \frac{1-\epsilon}{1+\epsilon}]}$ on the hyperboloid must be monotonically decreasing function on time.

We have also showed that $H_{[-\frac{1-\epsilon}{1+\epsilon}, \frac{1-\epsilon}{1+\epsilon}]}$ is positive inside the cone. It represents the amount of energy still remaining in the system, whereas $H_{(-\infty, -\frac{1-\epsilon}{1+\epsilon}]}$ and $H_{[\frac{1-\epsilon}{1+\epsilon}, \infty)}$ describe already radiated energy.

2.2 Four-dimensional Minkowski spacetime

The above construction, with appropriate modifications, is valid also in four-dimensional Minkowski spacetime. Consider a one-parameter family of past-oriented light cones:

$$\mathcal{C}_\epsilon^- := \left\{ (t, x) : x \in \mathbb{R}^3, \frac{1}{\epsilon} - t > \|x\| \right\},$$

where $\frac{1}{\epsilon} > 1$ is the time coordinate of the vertex of \mathcal{C}_ϵ^- . We introduce new coordinates $(\xi^\mu) = (\tau, \xi^k)$ ($\mu = 0, \dots, 3$), related to Minkowskian coordinates $(x^\mu) = (t, x^k)$ in a way analogous to (2)-(3):

$$t = \frac{1}{\epsilon} + \left(\frac{1 + \|\xi\|^2}{1 - \|\xi\|^2} - \frac{1}{\epsilon} \right) e^{-\epsilon\tau}, \quad (51)$$

$$x^k = \frac{2\xi^k}{1 - \|\xi\|^2} e^{-\epsilon\tau}, \quad (52)$$

where $\tau \in \mathbb{R}^1$. For $\|\xi\| \leq \frac{1-\epsilon}{1+\epsilon}$, the new coordinates parameterize the entire cone \mathcal{C}_ϵ^- and the surfaces $\{\tau = \text{const.}\}$ correspond to hyperboloids. To derive the Hamiltonian description of the wave equation in these coordinates, we begin with the Lagrangian:

$$\hat{L} = -\frac{1}{2} \sqrt{|\det g|} g^{\mu\nu} (\partial_\mu \hat{\varphi}) (\partial_\nu \hat{\varphi}) = \frac{1}{2} \left\{ (\partial_t \hat{\varphi})^2 - (\nabla \hat{\varphi})^2 \right\}. \quad (53)$$

Expressing it in terms of new coordinates we obtain:

$$\hat{L} \cdot d^4x = \hat{\mathcal{L}} \cdot d^4\xi, \quad (54)$$

where

$$\hat{\mathcal{L}} = \left\{ \frac{\left(\frac{\partial \hat{\varphi}}{\partial \tau} + \xi^k \frac{\partial \hat{\varphi}}{\partial \xi^k} \right)^2}{1 - \epsilon + (1 + \epsilon) \|\xi\|^2} - \frac{1}{4} [1 - \epsilon + (1 + \epsilon) \|\xi\|^2] \delta^{kl} \frac{\partial \hat{\varphi}}{\partial \xi^k} \frac{\partial \hat{\varphi}}{\partial \xi^l} \right\} \left(\frac{2e^{-\epsilon\tau}}{1 - \|\xi\|^2} \right)^2. \quad (55)$$

This, apparently non-autonomous (i.e. τ -dependent), Lagrangian becomes autonomous after an appropriate re-scaling of the field variable:

$$\varphi := \frac{2e^{-\epsilon\tau}}{1 - \|\xi\|^2} \hat{\varphi}. \quad (56)$$

Indeed, we obtain the formula analogous to (6):

$$\begin{aligned} \hat{\mathcal{L}} = & \frac{1}{2\kappa} \left[\partial_\tau \varphi + \xi^k \partial_{\xi^k} \varphi + \left(\epsilon - \frac{2\|\xi\|^2}{1 - \|\xi\|^2} \right) \varphi \right]^2 \\ & - \frac{1}{2} \kappa \delta^{kl} \left(\partial_{\xi^k} \varphi - \frac{2\xi_k}{1 - \|\xi\|^2} \varphi \right) \left(\partial_{\xi^l} \varphi - \frac{2\xi_l}{1 - \|\xi\|^2} \varphi \right), \end{aligned} \quad (57)$$

where we denote:

$$\kappa := \frac{1}{2} [1 - \epsilon + (1 + \epsilon)\|\xi\|^2] .$$

Observe that the function

$$\begin{aligned} \mathcal{L} = & \frac{1}{2\kappa} (\partial_\tau \varphi + \xi^k \partial_{\xi^k} \varphi)^2 - \frac{1}{2} \kappa \delta^{kl} (\partial_{\xi^k} \varphi) (\partial_{\xi^l} \varphi) \\ & - \frac{(1 - \epsilon)(3 + \epsilon^2) - 2(1 - \epsilon)(1 + \epsilon)^2 \|\xi\|^2 - (1 + \epsilon)^3 \|\xi\|^4}{2(1 - \epsilon + (1 + \epsilon)\|\xi\|^2)} \varphi^2 , \end{aligned} \quad (58)$$

differs from the original Lagrangian $\hat{\mathcal{L}}$ by a complete divergence:

$$\hat{\mathcal{L}} = \mathcal{L} + \partial_\mu \mathcal{Z}^\mu ,$$

where

$$\mathcal{Z}^0 = \frac{1}{2\kappa} \left(\epsilon - \frac{2\|\xi\|^2}{1 - \|\xi\|^2} \right) \varphi^2 ,$$

and

$$\mathcal{Z}^k = \frac{1}{4\kappa} \xi^k [1 + \epsilon^2 - (1 + \epsilon)^2 \|\xi\|^2] \varphi^2 .$$

This implies that both Lagrangians $\hat{\mathcal{L}}$ and \mathcal{L} lead to the same equation of motion for the field φ , so we use the latter in the sequel.

To derive the Hamiltonian description, we integrate equation (10) over the volume $V_{\text{int},\epsilon} := \{\xi : \|\xi\| \leq \frac{1-\epsilon}{1+\epsilon}\}$ in the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$. We obtain an identity valid for fields satisfying wave equation:

$$\delta \int_{V_{\text{int},\epsilon}} \mathcal{L} d^3\xi = \int_{V_{\text{int},\epsilon}} (\pi \delta \phi) \cdot d^3\xi + \int_{\partial V_{\text{int},\epsilon}} \pi^k \delta \phi d^2\sigma_k ,$$

where "dot" denotes derivative with respect to the new time variable τ , while ϕ is the restriction of the field φ to the Cauchy surface $\Sigma = \{\tau = \text{const.}\}$. Moreover, we have introduced momentum $\pi := \pi^0$, which is a part of Cauchy data on this surface. Performing Legendre transformation between $\dot{\phi}$ and π we get:

$$-\delta H_{\text{int},\epsilon} = \int_{V_{\text{int},\epsilon}} (\dot{\pi} \delta \phi - \dot{\phi} \delta \pi) d^3\xi + \int_{\partial V_{\text{int},\epsilon}} \pi^\perp \delta \phi , \quad (59)$$

where the Hamiltonian $H_{\text{int},\epsilon}$ is defined by

$$H_{\text{int},\epsilon}(\phi, \pi) := \int_{V_{\text{int},\epsilon}} (\pi \dot{\phi} - \mathcal{L}) d^3\xi , \quad (60)$$

and the symplectic form in phase space of Cauchy data is given by:

$$\omega_{\text{int},\epsilon} := \int_{V_{\text{int},\epsilon}} (\delta \pi \wedge \delta \phi) d^3\xi . \quad (61)$$

The Lagrangian (58) implies the following relation between "velocity" $\dot{\phi}$ and momentum π :

$$\pi = \pi^0 = \frac{\partial \mathcal{L}}{\partial \varphi_0} = \kappa^{-1} \left(\partial_\tau \varphi + \xi^k \frac{\partial \varphi}{\partial \xi^k} \right) = \kappa^{-1} \left(\dot{\phi} + \xi^k \frac{\partial \phi}{\partial \xi^k} \right) . \quad (62)$$

Thus, the Hamiltonian (60) may be written explicitly in terms of the Cauchy data on $V_{\text{int},\epsilon}$

$$H_{\text{int},\epsilon}(\phi, \pi) = \frac{1}{2} \int_{V_{\text{int},\epsilon}} \left\{ \kappa \left(\pi - \kappa^{-1} \xi^l \frac{\partial \phi}{\partial \xi^l} \right)^2 + (\kappa \delta^{kl} - \kappa^{-1} \xi^k \xi^l) \frac{\partial \phi}{\partial \xi^k} \frac{\partial \phi}{\partial \xi^l} + \mu \phi^2 \right\} d^3 \xi, \quad (63)$$

where

$$\mu := \frac{(1-\epsilon)(3+\epsilon^2) - 2(1-\epsilon)(1+\epsilon)^2 \|\xi\|^2 - (1+\epsilon)^3 \|\xi\|^4}{(1-\epsilon + (1+\epsilon) \|\xi\|^2)^2}.$$

One can check that the quadratic form

$$\kappa \delta^{kl} - \kappa^{-1} \xi^k \xi^l$$

is positive definite for $\kappa - \|\xi\| > 0$. But, we have:

$$\kappa - \|\xi\| = \frac{1}{2} (1+\epsilon)(1-\|\xi\|) \left(\frac{1-\epsilon}{1+\epsilon} - \|\xi\| \right).$$

This implies that Hamiltonian (63) is positive for $\|\xi\| < \frac{1-\epsilon}{1+\epsilon}$, i.e. inside the cone \mathcal{C}_ϵ^- . The Euler-Lagrange equation coincides with the following Hamiltonian equations, derived directly from (63):

$$\dot{\phi} = \kappa \pi - \xi^k \frac{\partial \phi}{\partial \xi^k}, \quad (64)$$

$$\dot{\pi} = -\partial_{\xi^k} (\xi^k \pi) + \partial_{\xi^k} (\kappa \delta^{kl} \partial_{\xi^l} \phi) - \mu \phi, \quad (65)$$

provided no boundary terms remain after the integration by part of its variation. To assure their cancellation we proceed in a way analogous to the previous section: we take into account missing radiation data on the light cone. Therefore, we extend parametrization (51)–(52) beyond the volume $V_{\text{int},\epsilon}$, taking into account also the corresponding points on the boundary of the cone:

$$t := \frac{1}{\epsilon} - \frac{1}{2} \left(\frac{1}{\epsilon} - \epsilon \right) e^{-\epsilon \tau + \epsilon \|\xi\| - \frac{\epsilon(1-\epsilon)}{1+\epsilon}} \quad \text{for } \|\xi\| \geq \frac{1-\epsilon}{1+\epsilon}, \quad (66)$$

$$x^k := \frac{1}{2} \left(\frac{1}{\epsilon} - \epsilon \right) \frac{\xi^k}{\|\xi\|} e^{-\epsilon \tau + \epsilon \|\xi\| - \frac{\epsilon(1-\epsilon)}{1+\epsilon}} \quad \text{for } \|\xi\| \geq \frac{1-\epsilon}{1+\epsilon}, \quad (67)$$

and consider the data (ϕ, π) on the entire surface $\Sigma = \{\tau = \text{const.}, \xi \in \mathbb{R}^3\}$. Equations (66)–(67) imply that $X := \partial_\tau = -\frac{\xi^k}{\|\xi\|} \partial_{\xi^k}$, for $\|\xi\| \geq \frac{1-\epsilon}{1+\epsilon}$. The dynamics consists in transporting the field data (ϕ, π) over the surface Σ according to the following field equations:

$$\mathcal{L}_X \phi = \partial_\tau \phi = -\frac{\xi^k}{\|\xi\|} \partial_{\xi^k} \phi, \quad (68)$$

$$\mathcal{L}_X \pi = \partial_\tau \pi = -\partial_{\xi^k} \left(\frac{\xi^k}{\|\xi\|} \pi \right), \quad (69)$$

where (69) follows from the fact that the momentum π is not a scalar field (like ϕ) but a scalar density. The above equations can be also derived from the standard Hamiltonian formula:

$$H_{\text{ext},\epsilon}(\phi, \pi) := \int_{V_{\text{ext},\epsilon}} (\pi \dot{\phi} - \mathcal{L}) d^3\xi = \int_{V_{\text{ext},\epsilon}} \left(-\pi \frac{\xi^k}{\|\xi\|} \partial_{\xi^k} \phi \right) d^3\xi, \quad (70)$$

where $V_{\text{ext},\epsilon} := \{\xi : \|\xi\| \geq \frac{1-\epsilon}{1+\epsilon}\}$ and \mathcal{L} vanishes identically as a pull-back of the scalar density L via the degenerate coordinate transformation (66)–(67). Variation of the above Hamiltonian gives:

$$-\delta H_{\text{ext},\epsilon}(\phi, \pi) = \int_{V_{\text{ext},\epsilon}} (\dot{\pi} \delta \phi - \dot{\phi} \delta \pi) d^3\xi + \int_{\partial V_{\text{ext},\epsilon}} \pi^\perp \delta \phi, \quad (71)$$

where $(\dot{\phi}, \dot{\pi})$ are given by (68) and (69), whereas the boundary term comes from integration by parts. The momentum π on the Cauchy surface Σ is equal to the pull-back of the differential (odd) form $\pi^\mu \partial_\mu \rfloor d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ to $\partial \mathcal{C}_\epsilon^-$. Moreover, momentum π^\perp coincides with π , as the pull-back of the same form to the hypersurface $\{\|\xi\| = \text{const.}\} = \{\tau = \text{const.}\} = \Sigma$, so we obtain the following constraints

$$\pi = -\frac{\left(\frac{1-\epsilon}{1+\epsilon}\right)^2}{\|\xi\|^2} \frac{\xi^k}{\|\xi\|} \partial_{\xi^k} \phi \quad \text{for } \|\xi\| \geq \frac{1-\epsilon}{1+\epsilon}. \quad (72)$$

The phase space of Cauchy data on the entire Σ is described by the pairs (ϕ, π) defined on the whole \mathbb{R}^3 fulfilling constraints (72) outside of the hyperboloid. Moreover, functions ϕ and π should satisfy compatibility conditions ("corner conditions") at points $\|\xi\| = \frac{1-\epsilon}{1+\epsilon}$, because otherwise the total dynamics is not well defined. To formulate these conditions we proceed as in the previous section. Summing up formula (60) for $H_{\text{int},\epsilon}$ and formula (70) for $H_{\text{ext},\epsilon}$ we define the total energy H_ϵ , defined on the total phase space $\mathcal{P} = \{(\phi, \pi)\}$

$$H_\epsilon := H_{\text{int},\epsilon} + H_{\text{ext},\epsilon}.$$

Variation of H_ϵ gives us:

$$-\delta H_\epsilon(\phi, \pi) = \int_\Sigma (\mathcal{L}_X \pi \delta \phi - \mathcal{L}_X \phi \delta \pi) d^3\xi + \int_{\partial V_{\text{int},\epsilon}} \pi^\perp \delta \phi + \int_{\partial V_{\text{ext},\epsilon}} \pi^\perp \delta \phi, \quad (73)$$

where the dynamics $(\mathcal{L}_X \phi, \mathcal{L}_X \pi)$ is given by (64)–(65) (inside) and by (68)–(69) outside of the sphere $\|\xi\| = \frac{1-\epsilon}{1+\epsilon}$. The global dynamics generated by H_ϵ is well defined if the boundary terms in formula (73) vanish. To analyze the resulting corner conditions we reformulate our Hamiltonian description as follows. Taking into account constraint (72) we obtain from (70):

$$\begin{aligned} H_{\text{ext},\epsilon} &= \int_{V_{\text{ext},\epsilon}} \frac{\left(\frac{1-\epsilon}{1+\epsilon}\right)^2}{\|\xi\|^2} \left(\frac{\xi^k}{\|\xi\|} \partial_{\xi^k} \phi \right)^2 d^3\xi \\ &= \int_{S^2} \int_{\rho \geq \frac{1-\epsilon}{1+\epsilon}} (\partial_\rho \phi)^2 \left(\frac{1-\epsilon}{1+\epsilon} \right)^2 d\rho d^2\sigma, \end{aligned} \quad (74)$$

and the corresponding symplectic structure

$$\begin{aligned}\omega_{\text{ext},\epsilon} &= - \int_{V_{\text{ext},\epsilon}} \frac{\left(\frac{1-\epsilon}{1+\epsilon}\right)^2}{\|\xi\|^2} \left(\frac{\xi^k}{\|\xi\|} \partial_{\xi^k} \delta\phi \right) \wedge \delta\phi \, d^3\xi \\ &= - \int_{S^2} \int_{\rho \geq \frac{1-\epsilon}{1+\epsilon}} (\partial_\rho \delta\phi \wedge \delta\phi) \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 d\rho d^2\sigma ,\end{aligned}\quad (75)$$

where $\rho = \|\xi\|$ and $d^2\sigma$ denotes the volume element on the two-dimensional unit sphere $S^2 := \{\xi \in \mathbb{R}^3 : \|\xi\| = 1\}$. Changing variables in the integral (75)

$$\rho = \tau + \frac{1-\epsilon}{1+\epsilon} - \lambda ,$$

the remaining variables being unchanged, and denoting

$$y_\epsilon(\lambda, \dots) := \phi\left(\tau + \frac{1-\epsilon}{1+\epsilon} - \lambda, \dots\right)$$

we obtain that y_ϵ does not depend on variable τ (see formulas (66) and (67)). Hence, we have:

$$\omega_{\text{ext},\epsilon} = \int_{S^2} \int_{\lambda \leq 0} (\partial_\lambda \delta f_\epsilon^- \wedge \delta f_\epsilon^-) \, d\lambda d^2\sigma ,\quad (76)$$

where

$$f_\epsilon^-(\lambda, \dots) := \frac{1-\epsilon}{1+\epsilon} y_\epsilon(\lambda + \tau, \dots) .$$

Expression (76) for “external” symplectic form suggests to consider, instead of the phase space $\mathcal{P} = \{(\phi, \pi)\}$ with constraint (72), the phase space of functions defined on the half of the tube $\mathbb{R} \times S^2$, corresponding to negative values of $\lambda \in \mathbb{R}$. We will show in the sequel that “internal” data (ϕ, π) on the hyperboloid can also be represented by canonically equivalent data on the remaining half-tube. The proof is based on the Euler-Lagrange equations, equivalent to the following identity:

$$\delta\mathcal{L} = \partial_\mu(p^\mu \delta\varphi) ,\quad (77)$$

where by p^μ we denote generalized momenta. Integrating equation (77) over any region \mathcal{V} in spacetime we obtain identity:

$$\delta \int_{\mathcal{V}} \mathcal{L} = \int_{\partial\mathcal{V}} p^\perp \delta\varphi ,\quad (78)$$

which holds for any configuration φ satisfying the field equations. In particular, let \mathcal{V} be the set of points lying between the boundary of the cone $\Gamma_\epsilon := \partial\mathcal{C}_\epsilon^-$ and the hyperboloid $V_{\text{int},\epsilon}$, then

$$\delta \int_{\mathcal{V}} \mathcal{L} = \int_{\Gamma_\epsilon} p^\perp \delta\varphi - \int_{V_{\text{int},\epsilon}} p^\perp \delta\varphi ,\quad (79)$$

where the signs come from the orientation of both surfaces Γ_ϵ and $V_{\text{int},\epsilon}$. Now, we treat these expressions as exterior one-forms on the space of Cauchy data

on Γ_ϵ and $V_{\text{int},\epsilon}$ respectively, and calculate exterior derivative of both sides. Because the left-hand-side is already an exterior derivative, its further exterior differentiation gives zero. This way we prove the identity:

$$\int_{V_{\text{int},\epsilon}} \delta p^\perp \wedge \delta \varphi = \int_{\Gamma_\epsilon} \delta p^\perp \wedge \delta \varphi. \quad (80)$$

Equation (80) means that the transition from the space of Cauchy data on the hyperboloid $V_{\text{int},\epsilon}$ to the space of boundary data on the cone Γ_ϵ , defined by the field dynamics, is a canonical transformation (a *symplectomorphism*).

Using (58) on the Cauchy surface $\Sigma = \{\tau = \text{const.}\} \supset V_{\text{int},\epsilon}$ we obtain:

$$\begin{aligned} \varphi \Big|_{V_{\text{int},\epsilon}} &= \phi, \\ p^\perp &= \pi^\mu \partial_\mu \rfloor d^4 \xi \Big|_{V_{\text{int},\epsilon}} = \pi d^3 \xi. \end{aligned}$$

On the other hand, on $\Gamma_\epsilon = \{\|\xi\| = \frac{1-\epsilon}{1+\epsilon}\}$ we have:

$$\begin{aligned} \varphi \Big|_{\Gamma_\epsilon} &=: f, \\ p^\perp &= \pi^\mu \partial_\mu \rfloor d^4 \xi \Big|_{\Gamma_\epsilon} = \pi^k \frac{\xi^k}{\|\xi\|} \|\xi\|^2 d^2 \xi d\tau = (\partial_\tau \varphi) \|\xi\|^2 d^2 \xi d\tau = (\partial_\tau f) \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 d^2 \xi d\tau, \end{aligned}$$

where f is a function which lives on Γ_ϵ . Thus equation (80) takes the form

$$\begin{aligned} \omega_{\text{int},\epsilon} &:= \int_{V_{\text{int},\epsilon}} \delta \pi \wedge \delta \phi d^3 \xi = \int_{\Gamma_\epsilon} \partial_\tau \delta f \wedge \delta f \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 d\tau d^2 \xi \\ &= \int_{S^2} \int_{\lambda \geq 0} \partial_\lambda \delta f_\epsilon^+ \wedge \delta f_\epsilon^+ d\lambda d^2 \sigma, \end{aligned} \quad (81)$$

where

$$f_\epsilon^+(\lambda, \dots) := \frac{1-\epsilon}{1+\epsilon} f(\lambda + \tau, \dots).$$

Formulae (76) and (81) prove that the global Cauchy data can be described by a single function (f_ϵ), equal to f_ϵ^- for $\lambda < 0$ and to f_ϵ^+ for $\lambda > 0$. Tailoring these two partial phase spaces into a single phase space, we have to impose compatibility condition (“corner condition”) at $\lambda = 0$, namely: the symplectic form:

$$\omega_\epsilon := \omega_{\text{ext},\epsilon} + \omega_{\text{int},\epsilon} = \int_{S^2} d^2 \sigma \int_{\lambda \in \mathbb{R}} \delta f'_\epsilon \wedge \delta f_\epsilon d\lambda \quad (82)$$

must be well defined. It means that the product of function which lives on a tube $\mathbb{R} \times S^2$ and its derivative along the tube must be integrable. Moreover, functions f_ϵ and $\partial_\lambda f_\epsilon$ have to belong to mutually dual spaces because they represent “positions” and “momenta” correspondingly. This implies that $f_\epsilon \in H^{\frac{1}{2}}(\mathbb{R}) \otimes L^2(S^2)$ and $\partial_\lambda f_\epsilon \in H^{-\frac{1}{2}}(\mathbb{R}) \otimes L^2(S^2)$.

Acknowledgments

One of the authors (J.K.) is grateful to the Mittag-Leffler Institute in Djursholm, Sweden, for providing support and an excellent research environment during the Fall 2008 program Geometry, Analysis, and General Relativity, when a portion of this research was carried out.

References

- [1] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner: *Proc. Roy. Soc. London A* **269**, 21-52 (1962).
- [2] P.R. Chernoff and J.E. Marsden: *Properties of Infinite Dimensional Hamiltonian Systems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin 1974.
- [3] P. Chruściel, J. Jezierski and J. Kijowski: *Hamiltonian Field Theory in the Radiating Regime*, Lecture Notes in Physics. Monographs, vol. 70, Springer, Berlin 2002.
- [4] P. Chruściel, J. Jezierski and Sz. Łęski: *Adv. Theor. Math. Phys.* **8**, 83-139 (2004).
- [5] P. Chruściel and O. Lengard: *Bull. Soc. Math. de France* **133**, 1-72 (2005).
- [6] P. Chruściel and Sz. Łęski: *J. Hyp. Diff. Eq.* **3**, 81-141 (2006).
- [7] B.A. Dubrovin and S.P. Novikov: *Dokl. Akad. Nauk SSSR* **270**:4, 781-785 (1983); English transl. *Soviet Math. Dokl.* **27**, 665-669 (1983).
- [8] B.A. Dubrovin: eprint arXiv:0804.3790 (2008).
- [9] L.D. Faddeev and L.A. Takhtajan: *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin 1987.
- [10] J. Jezierski: *Acta Phys. Polon. B* **29**, 667-743 (1998).
- [11] J. Jezierski, J. Kijowski and E. Czuchry: *Nuovo Cimento B* **119**, 733-748 (2004).
- [12] J. Kijowski and W.M. Tulczyjew: *A Symplectic Framework for Field Theories*, Lecture Notes in Physics, No. 107, Springer-Verlag, Berlin 1979.
- [13] S.P. Novikov: *Russ. Math. Surv.* **37**, 1-56 (1982).
- [14] A. Trautman: *Bull. Acad. Pol. Sci., Série sci. math., astr. et phys.* **VI**, 407-412 (1958).