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HIDDEN SYMMETRIES AND DECAY FOR THE WAVE EQUATION ON THE KERR SPACETIME

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ABSTRACT. Energy and decay estimates for the wave equation on the exterior region of slowly rotating Kerr spacetimes are proved. The method used is a generalization of the vector-field method, which allows the use of higher-order symmetry operators. In particular, our method makes use of the second-order Carter operator, which is a hidden symmetry in the sense that it does not correspond to a Killing symmetry of the spacetime.

The main result gives, in stationary regions, an almost inverse linear decay rate and the corresponding decay rate at the event horizon and null infinity. Except for the small loss in the decay rate, this generalizes the known decay results on the exterior region of the Schwarzschild spacetime.

CONTENTS

1. INTRODUCTION

In this paper we prove boundedness and decay for solutions of the covariant wave equation

$$\nabla^\alpha \nabla_\alpha \psi = 0$$

in the exterior region of the Kerr spacetime. In Boyer-Lindquist coordinates, the exterior region is given by $(t, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times S^2$ with the Lorentz metric

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mra\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\phi + \frac{\Pi\sin^2\theta}{\Sigma}\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \tag{1.1}$$

where $r_+ = M + \sqrt{M^2 - a^2}$ and

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad \Pi = (r^2 + a^2)^2 - a^2\sin^2\theta\Delta.$$

For $0 \le |a| \le M$, the Kerr metric describes a rotating black hole, with mass M and angular momentum Ma, and with horizon located at $r = r_+$. The Schwarzschild spacetime is the subcase with $a = 0$. The exterior region is globally hyperbolic, with the surfaces of constant t, Σ_t, as Cauchy surfaces. Thus, the wave equation is well posed in the exterior region, even though the Kerr spacetime can be extended. We consider initial data on the hypersurface Σ_0.

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The Kerr black-hole spacetime is expected to be the unique, stationary, asymptotically flat spacetime containing a nondegenerate Killing horizon [2]. Further, motivated by considerations including the weak cosmic censorship conjecture, the Kerr black hole is expected to be the asymptotic limit of the evolution of asymptotically flat, vacuum data in general relativity. An important step towards establishing the validity of this scenario is to prove the stability of the Kerr solution, i.e. to show that vacuum spacetimes evolving from data which represent a small perturbation of Kerr initial data asymptotically approach a Kerr solution. Decay for the scalar wave equation on the Kerr background is an important model problem for stability, and due to the importance of this application, the wave equation on black hole backgrounds has been actively studied in the last decade.

An essential tool in the analysis of both linear and nonlinear Lagrangian field equations is the use of Noetherian currents associated to Killing or conformal symmetries of the background spacetime. In the relativistic setting, we interpret these currents as momenta. A method, which may be referred to as the vector-field method, based on the systematic use of such currents, has been developed and has played an essential role in the proof of the nonlinear stability of Minkowski space [12], which built on earlier vector-field based estimates for the decay rates of solutions to linear and nonlinear wave equations [28] and to Maxwell's equations and the spin-2 field equations [11]. Generalizations of these ideas have played a central role in recent work concerning the wave and Maxwell equations on the Schwarzschild spacetime, and the wave equation on the Kerr spacetime. For these non-flat background spacetimes, the lack of symmetries presents an important new problem.

The 10 dimensional group of isometries of the Minkowski space is broken to a 4 dimensional group for the Schwarzschild spacetime, generated by ∂_t and the spatial rotations. Further, the Schwarzschild spacetime contains orbiting null geodesics, located at the photon spere, the hypersurface with $r = 3M$. As high frequency waves can track null geodesics for long times, an analysis of this feature is an essential step in a proof of decay for the wave equation.

In the general Kerr spacetime, with $a \neq 0$, which we consider in this paper, there are only two Killing fields, ∂_t and ∂_ϕ. In addition, in studying the wave equation on the Kerr spacetime, one encounters several new phenomena which are not present in the Schwarzschild case. There is an ergo-region outside the horizon, where the stationary Killing field ∂_t fails to be timelike. Thus, the Kerr spacetime admits no positive definite, conserved energy for the wave equation. Further, the orbiting null geodesics in Kerr fill an open region in spacetime. The lack of symmetries of the Kerr spacetime is compensated for by the presence of a fundamentally new feature, which we make essential use of in this paper, a hidden symmetry.

By a hidden symmetry we mean an operator which commutes with the wave operator, not associated to a Killing vector field, but rather to a second-rank Killing tensor. For the Kerr spacetime, the Killing tensor and the related conserved quantity found by Carter [8] provides, via the associated second-order Carter operator, a hidden symmetry. The existence of the two Killing vectors and the Killing tensor imply the separability of many important equations on the Kerr spacetime, including the wave equation.

One of the fundamentally new ideas introduced in this work is a generalization of the vector-field method which allows the use of not only Killing symmetries but also hidden symmetries in the construction of suitable Noetherian currents for the analysis of Lagrangian field equations. This allows us, in contrast to other, recent work on the wave equation on Kerr, to carry out our proof of uniform boundedness and decay results exclusively in physical space, using only the coordinate functions and differential operators. This technique almost eliminates the need for methods involving separation of variables or Fourier analysis.[1]

To state our main results, we use the tortoise coordinate r_*, defined by

$$\frac{dr}{dr_*} = (r^2 - 2Mr + a^2)(r^2 + a^2), \quad r_*(3M) = 0,$$

and the almost null coordinates $u_\pm$ given by

$$u_+ = t + r_*, \quad u_- = t - r_*.$$

Our main results are:

Theorem 1.1 (Uniformly bounded, positive energy)**.** *There are positive constants C_1 and $\bar{a}_1$, and a nonnegative quadratic form on each hypersurface of constant t, $E_{\mathbf{T}_\chi}[\psi](t)$, such that, if $|a| < \bar{a}_1$ and $\psi : \mathbb{R} \times (r_+, \infty) \times S^2 \to \mathbb{R}$ is a solution of the wave equation, $\nabla^\alpha \nabla_\alpha \psi = 0$, then $\forall t$*

$$E_{\mathbf{T}_\chi}[\psi](t) \leq C_1 E_{\mathbf{T}_\chi}[\psi](0).$$

Theorem 1.2 (Decay estimates)**.** *There are positive constants C_2, C_2', and $\bar{a}_2$, and there is a nonnegative quadratic form on each hypersurface of constant t, $\|\psi\|^2(t)$, such that, if $|a| < \bar{a}_2$ and $\psi : \mathbb{R} \times (r_+, \infty) \times S^2 \to \mathbb{R}$ is a solution of the wave equation, $\nabla^\alpha \nabla_\alpha \psi = 0$, then there are the following decay estimates $\forall t > 0, (\theta, \phi) \in S^2$:*

(1) *Decay in stationary regions: $\forall r \in (3M, 4M)$:*

$$|\psi(t, r, \theta, \phi)| \leq C_2 \max\{1, t\}^{-1+C_2'|a|} \|\psi\|(0).$$

(2) *Near decay: $\forall r < 3M$:*

$$|\psi(t, r, \theta, \phi)| \leq C_2 \max\{1, u_+\}^{-1+C_2'|a|} \|\psi\|(0).$$

(3) *Far decay: $\forall r$ with $r > 4M$ and $r < t$:*

$$|\psi(t, r, \theta, \phi)| \leq C_2 r^{-1} \max\{1, u_-\}^{C_2'|a|} \left(\frac{u_+ - u_-}{u_+ \max\{1, u_-\}} \right)^{1/2} \|\psi\|(0).$$

In particular, for $t/2 < r < t$:

$$|\psi(t, r, \theta, \phi)| \leq C_2 r^{-1} \max\{1, u_-\}^{-1/2+C_2'|a|} \|\psi\|(0).$$

(4) *Decay near spatial infinity: $\forall r$ with $r > t$:*

$$|\psi(t, r, \theta, \phi)| \leq C_2 r^{-1} \max\{1, -u_-\}^{-1/2} \|\psi\|(0).$$

Theorem 1.1 is the conclusion of section 3 and is given in theorem 3.13. Theorem 1.2 follows from the conclusions of theorems 5.1, 5.2, and 5.4. Except for the loss in the exponent of $C_2'|a|$, the estimates stated in theorem 1.2 are the same as the results proven using vector-field techniques in the Schwarzschild spacetime.[2] The decay along null infinity is the same (modulo the loss in the exponent) as can be obtained in Minkowski space from initial data on $t = 0$ which decays like $r^{-3/2}$. This is roughly the decay rate we require for the initial data in Kerr. Note that the Kerr spacetime has a discrete, time reversal symmetry $(t, \phi) \mapsto (-t, -\phi)$, so that the results of theorems 1.1 and 1.2 can also be reversed to $t < 0$.

The norm $\|\psi\|(0)$ in theorem 1.2 is bounded if ψ is smooth on the hypersurface Σ_0 and if ψ and its first nine derivatives (with respect to the Boyer-Lindquist

[1]We do not use separability or the Fourier transform (which are essentially equivalent to each other) in the t coordinate. In the proof of lemma 3.12, we need to use separability in the ϕ coordinate to control the axially symmetric component of the solution, but are otherwise able to avoid using separability (or the Fourier transform) in ϕ. We never use separability in the r and θ coordinates, since this is only possible once separation in the t and ϕ coordinates has already been performed.

[2]Recently, better decay estimates have been proven in the Schwarzschild case, giving, for any positive δ, decay rates of $t^{-3/2+\delta}$ and $u_+^{-3/2+\delta}$ in stationary and near regions respectively [32].

coordinates) decay like $r^{-3/2+\delta}$ for some positive δ as $r \to \infty$. By "smooth" we mean C^∞ with respect to local coordinates. As $r \to r_+$, this is not the same as being smooth with respect to the Boyer-Lindquist coordinates, since they degenerate here. The quadratic form $E_{\mathbf{T}_\chi}[\psi](0)$ in theorem 1.1 is bounded when $\|\psi\|^2(0)$ in theorem 1.2 is bounded.

We briefly comment on other related work. Estimates for the decay rate of solutions to the wave equation have been proven in the subcase of the Schwarzschild spacetime, where $a = 0$. Birkhoff's theorem states that the Schwarzschild spacetime is the unique spherically symmetric, vacuum spacetime solution of Einstein's equation. For the coupled Einstein and scalar wave system, a decay rate and non-linear stability of the Schwarzschild solution have been proven in the spherically symmetric setting [13].

As mentioned earlier, for the wave equation without a symmetry assumption but on a fixed background spacetime, the case of the linear wave equation on the Schwarzschild spacetime is significantly simpler than the corresponding case in the Kerr spacetime, since the ∂_t Killing vector is timelike in the entire exterior region and generates a conserved, positive energy, there is the full $SO(3)$ group of rotation symmetries available to generate higher energies, and the orbiting null geodesics are restricted to $r = 3M$. The first two of these properties imply that solutions remain bounded. Following the introduction of a Morawetz vector field and of the equivalent of an almost conformal vector field to the Schwarzschild spacetime [30], decay estimates for the wave equation were proven [7], proven with a weaker decay rate but less regularity loss [6], and proven separately with a stronger estimate near the event horizon [14]. These were extended to Strichartz estimates for the wave equation [33] and decay estimates for Maxwell's equation [4]. The Morawetz vector field which made these estimates possible was centred about the orbiting geodesics at $r = 3M$. This construction of a classical vector field fails when $a \neq 0$, since there are orbiting geodesics filling an open set in spacetime.

Recently, new Morawetz vector fields with coefficients that depend on both spacetime position and on Fourier operators have been introduced to construct a uniformly bounded, positive energy and to show that solutions to the Kerr wave equation remain uniformly bounded [17, 39]. These might reasonably be called Fourier-analytic, pseudodifferential, microlocal, or phase-space techniques, since the Fourier operators represent coordinates in momentum space in contrast to spacetime coordinates in physical or configuration space. These results include a form of weak decay, since the Morawetz estimate implies integrability of the local energy. In [16], it has been announced that these Fourier-analytic techniques can be extended to decay results very similar to ours in theorem 1.2. Like our work, these require that $|a|$ is very small relative to M. We compare this work with our own in more detail in section 1.3. Fourier-analytic vector fields were used previously to prove Mourre estimates, which are similar to Morawetz estimates, in the proof of scattering for the Klein-Gordon equation [25] and the Dirac equation [26].

Finally, we recall that decay for the wave equation has previously been obtained [21] from an explicit representation of solutions [20] using the complete separability of the Kerr wave equation. These decay results are of the form $\lim_{t\to\infty} |\psi(t, r, \theta, \phi)| \to 0$, where $\psi(t, r, \theta, \phi) = \psi_{L_z}(t, r, \theta)e^{iL_z\phi}$ or where ψ is made up of a finite number of azimuthal modes of this form. Decay rates have been obtained from this separability method for solutions to the Dirac equation [19] and spherically symmetric solutions to the wave equation when $a = 0$ [29]. The decay without rate results for the wave equation built on the earlier result that there are no exponentially growing modes [43] and applies for all $|a| \in [0, M]$.

1.1. **Hidden symmetries and the vector-field method.**

In the classical vector-field method, one seeks to control the solution of field equations by making use of energy fluxes and deformation terms calculated for suitably chosen vector fields. The vector fields used are often (approximate) conformal symmetries of the background spacetime. Higher-order estimates are achieved by Lie differentiating the field along further vector fields. An important advantage of the vector-field method is that one works entirely in terms of quantities in physical space.

A particularly clear example of the vector-field method was the use of the Lorentz group in the Minkowski spacetime, $\mathbb{R}^{1+3}$, to construct norms which could be used in the Klainerman-Sobolev inequality and to use this to prove the well-posedness of nonlinear wave equations [28]. The term "vector-field method" seems to have been introduced relatively recently, especially to describe generalisations of the work in $\mathbb{R}^{1+3}$ to situations where one lacks the full Lorentz group of symmetries. Applying this terminology retroactively, we would now describe the early uses of the radial Morawetz vector-field [35] and the conformal vector-field [24] as applications of the vector-field method. This may have previously been refered to as the method of multipliers or the Euler-Lagrange method, although these terms can also be applied to more general techniques.

A central result in mathematical relativity, and perhaps the most important application of the vector-field method, was the proof of the nonlinear stability of the Minkowski spacetime [12]. There had also been earlier work on the stability of the Minkowski spacetime, but this required hyperboloidal initial data [22]. The monumental proof of nonlinear stability built upon previous vector-field estimates for linear and nonlinear wave equations [28], and for the Maxwell and spin-2 field equations [11], which are better models for Einstein's equations. This partly motivates our work on the linear wave equation in the Kerr spacetime using generalisations of the vector-field method. Since the original proof of nonlinear stability for the Minkowski spacetime, a simpler proof has been developed, but this also makes use of the vector-field method [31].

As mentioned above, in the Kerr spacetime, the lack of symmetries, as well as the complicated nature of the orbiting null geodesics, makes it impossible to derive the required estimates using only classical vector fields. In this section, we outline a generalization of the vector-field method which allows us to take advantage of the presence of hidden symmetries in the Kerr spacetime. In particular, we consider energies based on operators of order greater than one, rather than just vector fields.

Let $\Box_g = \nabla^\alpha \nabla_\alpha$. In the discussion here, we consider the scalar wave equation $\Box_g \psi = 0$, but much of the discussion applies equally to general field equations derived from a quadratic action. We define a symmetry operator to be a differential operator S such that if $\Box_g \psi = 0$, then also $\Box_g S\psi = 0$. The set of symmetry operators is closed under scalar multiplication, addition, and composition, and each symmetry operator has a well-defined order as a differential operator. Thus, the set of symmetry operators forms a graded algebra. Given a set of generators of the set of symmetries, we can consider the subset consisting of generators of order n. We denote this subset of the generators of the symmetry operators by $\mathbb{S}_n$ and denote the elements of $\mathbb{S}_n$ with an underlined index, e.g. $S_{\underline{a}} \in \mathbb{S}_n$.

If $\mathbf{X}$ is a conformal Killing field, then the operator $\mathcal{L}_\mathbf{X}$ generated by Lie differentiation with respect to $\mathbf{X}$ is clearly a symmetry operator. We take a hidden symmetry to be a symmetry operator which is not in the algebra generated by the Killing vector fields. Since the Minkowski spacetime saturates the Delong-Takeuchi-Thompson inequality, there are no hidden symmetries [10]. In the Schwarzschild spacetime, there are no hidden symmetries [9]. In the Kerr spacetime, it is well-known that there is Carter's Killing 2 tensor and that this generates a hidden symmetry [8, 42].

The energy-momentum tensor for the wave equation is

$$T[\psi]_{\alpha\beta} = \nabla_\alpha \psi \nabla_\beta \psi - \frac{1}{2} g_{\alpha\beta} \left(\nabla_\gamma \psi \nabla^\gamma \psi \right). \tag{1.2}$$

The momentum associated with a vector field $\mathbf{X}$ and the energy associated with a vector field $\mathbf{X}$ and evaluated on a hypersurface Σ are

$$P_{\mathbf{X}}[\psi]_\alpha = T[\psi]_{\alpha\beta}\mathbf{X}^\beta,$$
$$E_{\mathbf{X}}[\psi](\Sigma) = \int_\Sigma P_{\mathbf{X}}[\psi]_\alpha \mathrm{d}\eta^\alpha,$$

where $\mathrm{d}\eta$ is integration with respect to the surface volume induced by g on Σ. In the following, unless there is room for confusion, we will drop reference to ψ in the notation for momentum and energy. When the spacetime is foliated by surfaces of constant time, we will denote these surfaces by Σ_t and typically denote the energy on such a surface by $E_{\mathbf{X}}(t) = E_{\mathbf{X}}(\Sigma_t)$.

The energy momentum tensor (1.2) satisfies the dominant energy condition, and hence for $\mathbf{X}$ timelike, the energy induced on a hypersurface with a timelike normal (i.e. a spacelike hypersurface) is positive definite. The energy conservation law takes the form

$$E_{\mathbf{X}}(\Sigma_2) - E_{\mathbf{X}}(\Sigma_1) = \int_\Omega (\nabla_\alpha P^\alpha_{\mathbf{X}}) \sqrt{-|g|}\mathrm{d}^4x,$$

where Ω is the region enclosed between Σ_1 and Σ_2. This is often referred to as the deformation formula. Energy estimates are often performed by controlling the bulk (also called deformation) terms $\nabla_\alpha P^\alpha_{\mathbf{X}}$. However, for the Morawetz estimate (e.g. inequality (3.6)), one makes use of the sign of the bulk term itself.

By estimating higher-order energies one may, via Sobolev estimates, get pointwise control of the fields. Higher-order energies may be defined by using symmetries. If for $0 \leq i \leq n$, there is a collection of order-n differential operators, $\mathbb{S}_i$, then we can define the higher-order energy (of order n) for a vector field $\mathbf{X}$ to be

$$E_{\mathbf{X},n}[\psi](\Sigma) = \sum_{i=0}^n \sum_{S\in\mathbb{S}_i} E_{\mathbf{X}}[S\psi](\Sigma).$$

Since the energy momentum tensor is quadratic in ψ, we can define a bilinear form of the energy momentum by

$$T[\psi_1,\psi_1]_{\alpha\beta} = \frac{1}{4}\left(T[\psi_1+\psi_2]_{\alpha\beta} - T[\psi_1-\psi_2]_{\alpha\beta}\right).$$

It is convenient to define an index version of the bilinear energy momentum, with respect to a set of symmetry operators $\{S_{\underline{a}}\}$ by

$$T[\psi]_{\underline{ab}\alpha\beta} = T[S_{\underline{a}}\psi, S_{\underline{b}}\psi]_{\alpha\beta}.$$

Given a double-indexed collection of vector fields, $\{\mathbf{X}^{\underline{ab}}\}$, we define the associated generalized momentum and energy to be

$$P_{\mathbf{X}^{\underline{ab}}}[\psi]_\alpha = T[\psi]_{\underline{ab}\alpha\beta}\mathbf{X}^{\underline{ab}\beta},$$
$$E_{\mathbf{X}^{\underline{ab}}}[\psi](\Sigma) = \int_\Sigma P_{\mathbf{X}^{\underline{ab}}}[\psi]_\alpha \mathrm{d}\eta^\alpha.$$

In practice it is convenient to consider momenta with lower-order terms, designed to improve certain deformation terms in $\nabla_\alpha P^\alpha_{\mathbf{X}}$. For a scalar function, q ([33], but previously appearing in [15]), or a double-indexed collection of functions, $q^{\underline{ab}}$, the associated momenta are defined to be

$$P_q[\psi]_\alpha = q(\nabla_\alpha\psi)\psi - \frac{1}{2}(\partial_\alpha q)\psi^2,$$
$$P_{q^{\underline{ab}}}[\psi]_\alpha = q^{\underline{ab}}(\nabla_\alpha S_{\underline{a}}\psi)S_{\underline{b}}\psi - \frac{1}{2}(\partial_\alpha q^{\underline{ab}})(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).$$

For a pair consisting of a vector field and a scalar function, $(\mathbf{X}, q)$, the associated momentum is defined to be the sum of the momenta associated with the vector field and the scalar. For a pair of collections, $(\mathbf{X}^{\underline{ab}}, q^{\underline{ab}})$, again the momentum is defined to be the sum of the momenta. In all cases, the energy on a hypersurface is given by the flux, defined with respect to the momentum vector, through the hypersurface.

It is important to point out, as we show in lemma 2.1, that the deformation terms for the generalized momenta are computationally not much more difficult to handle than the classical ones. As for the classical momenta and energies, in defining the generalized vector fields, momenta, and energies as outlined above, one is interested in getting positive definiteness of the energies or bulk terms. Here, an additional subtlety arises. Namely, in the Morawetz estimate presented in equation (3.6), one achieves positive definiteness only modulo boundary terms. We generate these boundary terms when we integrate by parts to use the formal self-adjointness of the second-order symmetry operators. These boundary terms can then be controlled by the energy. The presence of these boundary terms is a completely new feature compared to the classical energies and momenta.

1.2. **Symmetries and null geodesics of Kerr.** For any geodesic, the quantity $g_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta$ is a constant of the motion. Given a Killing field ξ, the quantity $p_\xi = g_{\alpha\beta}\dot\gamma^\alpha\xi^\beta$ is an additional conserved quantity. In Kerr, we have the Killing fields ∂_t and ∂_ϕ with the associated constants of motion p_t and p_ϕ. For a timelike or null geodesic, these correspond to the energy and the angular momentum of a particle or photon with world line γ and are denoted E and L_z.

More generally, if the spacetime admits a Killing k-tensor, i.e. a symmetric tensor $K_{\alpha_1\cdots\alpha_k}$ which solves the Killing equation $\nabla_{(\alpha_1}K_{\alpha_2\cdots\alpha_{k+1})} = 0$, then $K = K_{\alpha_1\cdots\alpha_k}\dot\gamma^{\alpha_1}\cdots\dot\gamma^{\alpha_k}$ is a conserved quantity. In the particular case of Killing 2-tensors, which is the only case we are interested in here, there is associated to the Killing tensor a symmetry operator $K = \nabla_\alpha K^{\alpha\beta}\nabla_\beta$, such that $[K, \Box_g] = 0$ [8, 42]. Since the commutator is zero, this operator is clearly a symmetry in the slightly weaker sense defined in the previous section.

In Kerr, Carter's Killing 2-tensor, provides a fourth constant of the motion $Q = Q_{\alpha\beta}\dot\gamma^\alpha\dot\gamma^\beta$. For a null geodesic, we have

$$Q = p_\theta^2 + \frac{\cos^2\theta}{\sin^2\theta}p_\phi^2 + a^2\sin^2\theta p_t^2.$$

A similar expression exists for timelike or spacelike geodesics. Any linear combination of E^2, EL_z, and L_z^2 can be added to Q to give an alternate choice for the fourth constant of the motion. The form we have chosen is uncommon, but useful for our purposes because it is nonnegative.

As was demonstrated by Carter, the presence of the extra conserved quantity allows one to separate the equations of geodesic motion. Of most interest to us is the equation for the r coordinate of null geodesics,

$$\Sigma^2\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = -\mathcal{R}(r; M, a; E, L_z, Q), \tag{1.3}$$

where

$$\mathcal{R}(r; M, a; E, L_z, Q) = -(r^2+a^2)^2E^2 - 4aMrEL_z + (\Delta - a^2)L_z^2 + \Delta Q. \tag{1.4}$$

One finds that orbiting null geodesics, i.e. ones which do not fall into the black hole or escape to infinity, must have orbits with constant r. The r values allowing orbiting null geodesics are solutions to the equations $\mathcal{R} = 0$, $\partial\mathcal{R}/\partial r = 0$. The solutions to this system in the exterior region turn out to be unstable, which corresponds with our conventions to $\partial^2\mathcal{R}/\partial r^2 < 0$.

In the Schwarzschild case, i.e. for $a = 0$, there are only orbits on the sphere at $r = 3M$, which is called the photon sphere. For nonzero a, the orbiting null geodesics fill up an open region in spacetime which we shall also refer to as the photon sphere in the Kerr case. As $a \to 0$, the photon sphere tends to $r = 3M$.

There are many descriptions of the Kerr spacetime and its geodesics, including [3, 23, 40].

In Boyer-Lindquist coordinates, the d'Alembertian $\Box_g = \nabla^\alpha\nabla_\alpha$ takes the form

$$\Box_g = \frac{1}{\Sigma}\left(\partial_r\Delta\partial_r + \frac{1}{\Delta}\mathcal{R}(r;M,a;\partial_t,\partial_\phi,Q)\right), \tag{1.5}$$

where $\mathcal{R}$ is given by (1.4) with the conserved quantities E, L_z, Q replaced by their corresponding operators $\partial_t, \partial_\phi$, and the second-order Carter operator[3] Q,

$$\begin{aligned} Q =& \frac{1}{\sin}\theta\partial_\theta\sin\theta\partial_\theta + \frac{\cos^2\theta}{\sin^2\theta}\partial_\phi^2 + a^2\sin^2\theta\partial_t^2, \\ \mathcal{R}(r;M,a;\partial_t,\partial_\phi,Q) =& -(r^2+a^2)^2\partial_t^2 - 4aMr\partial_t\partial_\phi + (\Delta - a^2)\partial_\phi^2 + \Delta Q. \end{aligned} \tag{1.6}$$

We have used some unusual sign conventions in defining $\mathcal{R}$ to avoid factors of i when replacing the constants of motion by differential operators.

It is clear from the above that ∂_t, ∂_ϕ, and Q are symmetry operators for the wave equation on Kerr. We denote the set of order-n generators of the symmetry algebra generated by these operators by

$$\mathbb{S}_n = \{\partial_t^{n_t}\partial_\phi^{n_\phi}Q^{n_Q} | n_t + n_\phi + 2n_Q = n; n_t, n_\phi, n_Q \in \mathbb{N}\}. \tag{1.7}$$

In particular,

$$\mathbb{S}_0 =\{\mathrm{Id}\}, \qquad \mathbb{S}_1 =\{\partial_t,\partial_\phi\}.$$

Of particular importance in our analysis will be the set of second-order symmetry operators,

$$\mathbb{S}_2 = \{\partial_t^2, \partial_t\partial_\phi, \partial_\phi^2, Q\} = \{S_{\underline{a}}\},$$

and underlined indices always refer to the index in this set.

The function $\mathcal{R}$ is polynomial in its last three variables, so $\mathcal{R}(r;M,a;\partial_t,\partial_\phi,Q)$ is well defined. Furthermore, it can be written as a linear combination of the second-order symmetries with coefficients which are rational in r, M, and a,

$$\mathcal{R}(r;M,a;\partial_t,\partial_\phi,Q) =\mathcal{R}^{\underline{a}}S_{\underline{a}}.$$

1.3. **Strategy of proof and further results.** Recall from earlier in the introduction that there are three major problems in the Kerr spacetime:

(1) No positive, conserved energy: There is no timelike, Killing vector. In particular, the vector field ∂_t, which is Killing, is only timelike outside the ergosphere, $r > M + \sqrt{M^2 - a^2\cos^2\theta}$.
(2) Lack of sufficient classical symmetries: The higher energies generated the Lie derivatives in the ∂_t and ∂_ϕ directions do not control enough directions to control Sobolev norms or the function.
(3) Complicated trapping: There are null geodesics which orbit the black hole, in the sense that they neither escape to null infinity nor enter the black hole. Since solutions to the wave equation can follow null geodesics for an arbitrarily long time, this presents an obstacle to decay. Furthermore, there are orbiting geodesics occuring over a range of r in the Kerr spacetime (with $|a| > 0$), which makes the situation more complicated than in the Schwarzschild spacetime ($a = 0$), where there are only orbiting geodesics at $r = 3M$.

To overcome the first problem, we first observe that the vector ∂_t is timelike for sufficiently large r; that, if

$$\omega_H = \frac{a}{r^2 + a^2}$$

[3]Since the Carter operator Q and the Carter constant are closely related, we use the same notation for both.

denotes the angular velocity of the horizon, then the vector $\partial_t + \omega_H \partial_\phi$ is null on the horizon and timelike for sufficiently small $r > r_+$; that the regions where ∂_t and $\partial_t + \omega_H \partial_\phi$ are timelike overlap when $|a|$ is sufficiently small; and that both ∂_t and $\partial_t + \omega_H \partial_\phi$ are Killing. Thus, if we let

$$\mathbf{T}_\chi = \partial_t + \chi \omega_H \partial_\phi, \tag{1.8}$$

where χ is identically 1 for $r < r_\chi$ for some constant r_χ, identically 0 for $r > r_\chi + M$, and smoothly decreases on $[r_\chi, r_\chi + M]$, then, for sufficiently small a, this vector-field will be timelike everywhere and will be Killing outside the fixed region $r \in [r_\chi, r_\chi + M]$. Thus, to prove the boundedness of this positive, it will be sufficient to control the behaviour of solutions in this fixed region through a Morawetz estimate.

To overcome problem (2), we note that the second-order operator Q is a symmetry and is a weakly elliptic operator. Using Q, ∂_ϕ^2, and ∂_t^2 as symmetries to generate higher energies, we can control energies of the the spherical Laplacian of ψ. These control Sobolev norms which are sufficiently strong to control $|\psi|^2$.

To handle the complicated trapping, we will use our extension of the vector-field method to include hidden symmetries. To construct a Morawetz multiplier, we would like to construct a vector field with a weight that changes sign at the orbiting geodesic, but this is not possible using a classical vector-field. If we introduce $\mathcal{L} = \mathcal{L}^{\underline{a}} S_{\underline{a}} = \partial_t^2 + \partial_\phi^2 + Q$ to give us an elliptic operator and an extra, free, underlined index, we can take as our collection of Morawetz vector fields

$$\begin{aligned}
\mathbf{A}^{\underline{ab}} =& - z w \tilde{\mathcal{R}}'^{(\underline{a}} \mathcal{L}^{\underline{b})} \partial_r, \\
q_{\mathbf{A}}^{\underline{ab}} =& - \frac{1}{2} z \left(\partial_r \left(w \tilde{\mathcal{R}}'^{(\underline{a}} \right) \right) \mathcal{L}^{\underline{b})}, \\
\tilde{\mathcal{R}}'^{\underline{a}} =& \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{\underline{a}} \right),
\end{aligned}$$

with z and w smooth, positive functions to be chosen. Applying the analogy of the deformation formula, the difference between the energies on one hypersurface and another is

$$E_{(\mathbf{A}^{\underline{ab}}, q_{\mathbf{A}}^{\underline{ab}})}(\Sigma_2) - E_{(\mathbf{A}^{\underline{ab}}, q_{\mathbf{A}}^{\underline{ab}})}(\Sigma_1) = \int \left(\nabla_\alpha P^\alpha_{(\mathbf{A}^{\underline{ab}}, q_{\mathbf{A}}^{\underline{ab}})} \right) \sqrt{-|g|} \mathrm{d}^4 x.$$

Ignoring several distracting details, the deformation is of the form

$$\begin{aligned}
&\frac{1}{2} z \tilde{\mathcal{R}}'^{\underline{a}} \tilde{\mathcal{R}}'^{\underline{b}} \mathcal{L}^{\alpha\beta} (\partial_\alpha S_{\underline{a}} \psi)(\partial_\beta S_{\underline{b}} \psi) \\
&+ z^{1/2} \Delta^{3/2} \left(-\partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}'^{\underline{a}} \right) \right) \mathcal{L}^{\underline{b}} (\partial_r S_{\underline{a}} \psi)(\partial_r S_{\underline{b}} \psi) \\
&+ \frac{1}{4} (\partial_r \Delta \partial_r z (\partial_r w \tilde{\mathcal{R}}'^{\underline{a}})) \mathcal{L}^{\underline{b}} (S_{\underline{a}} \psi)(S_{\underline{b}} \psi).
\end{aligned}$$

In the first line, one factor of $\tilde{\mathcal{R}}'$ arises from the wave equation, and the other from our choice of the Morawetz vector field $\mathbf{A}^{\underline{ab}}$, which allows us to construct a perfect square to obtain positivity. In the second line, the term involves two derivatives of $-\tilde{\mathcal{R}}$. Near the photon orbits, the convexity properties of $\mathcal{R}$, which ensured that the orbits are unstable, ensure that this term is positive. We are free to choose z and w to get positivity away from the photon orbits. The fourth term is lower-order, since it involves fewer derivatives.

For small a, with v denoting terms of the form $S_{\underline{a}} \psi$, and with our choices of z and w, the sum of the second and third terms is of the form

$$M \left(\frac{\Delta^2}{r^2 (r^2 + a^2)} (\partial_r v)^2 + \frac{9r^2 - 46Mr + 54M^2}{r^4} v^2 \right) \tag{1.9}$$

with small perturbations on the coefficients. The coefficient on v^2 is positive outside a compact interval in (r_+, ∞). As shown in [5], it is sufficient to prove a Hardy estimate which bounds the quadratic form in (1.9) from below by a sum of positive weights times $(\partial_r v)^2$ and v^2.

The positive terms arising from the deformation of $\mathbf{A}^{\underline{ab}}$ dominate the deformation terms (with extra derivatives) terms arising in the failure of $\mathbf{T}_\chi$ to be Killing. (In fact, the terms in the Morawetz estimate only control the second and third derivatives of ψ, where as the deformation terms from the third-order $\mathbf{T}_\chi$ energy also involve the first derivatives and undifferentiated factors of ψ. It is at this point, in the proof of lemma 3.12, where we are forced to make a decomposition in harmonics of ∂_ϕ, i.e. to separate variables in ϕ, to obtain additional control on the rotationally symmetric components.) On the other hand, the energy associated with $\mathbf{A}^{\underline{ab}}$ is dominated by the (third-order) energy associated with $\mathbf{T}_\chi$. Since there is a factor of a on the $\mathbf{T}_\chi$ deformation terms, we have a small parameter, which allows us to close the boot-strap argument in which the $\mathbf{T}_\chi$ energy is controlled by the integral of its deformation, which is controlled by the integral of the $\mathbf{A}^{\underline{ab}}$ deformation, which is controlled by the $\mathbf{A}^{\underline{ab}}$ energy, which is finally controlled by the $\mathbf{T}_\chi$ energy. This allows us to establish theorem 1.1.

To prove decay, we introduce the vector field

$$\mathbf{K} = (t^2 + r_*^2 + 1)\partial_t + 2tr_* \left(\frac{(r^2+a^2)^2}{\Pi} \right) \partial_{r_*}.$$

In the Minkowski spacetime, the corresponding vector field is a conformal Killing vector field and generates a positive, conserved energy, sometimes called the conformal energy or conformal charge. In the Schwarzschild spacetime, it is now well known that the corresponding vector field has a deformation tensor which is favourable outside a compact region in (r_+, ∞), and that growth in the $\mathbf{K}$ energy can only occur as a result of the wave remaining inside the compact interval for long periods of time. Since the Morawetz estimate rules this out, after another boot-strap argument, it is possible to bound the $\mathbf{K}$ energy. After changing variables, we are able to mimic most of the argument from the Schwarzschild spacetime. We need to introduce some lower-order terms, $q_{\mathbf{K}}$, to remove the worst terms in the deformation expression, but are still left with terms, in the $\mathbf{K}$ deformation, of the form

$$at^2(\partial_r\psi)(\partial_\phi\psi).$$

since these involve a quadratic expression in the derivatives times a factor of t^2, these are of the same order as the $\mathbf{K}$ energy itself. Even after applying the Morawetz estimate, one is left with an estimate of the form

$$E_{\mathbf{K}}(t_2) - E_{\mathbf{K}}(t_1) \le C|a| \int_{t_1}^{t_2} \frac{E_{\mathbf{K}}(t)}{t} \mathrm{d}t + \text{(more easily controlled terms)},$$

where $E_{\mathbf{K}}(t)$ denotes the energy evaluated on the hypersurface $\{t\} \times (r_+, \infty) \times S^2$. Although one cannot obtain a uniform bound on the $\mathbf{K}$ energy from this, if the surfaces are of constant t, then the growth cannot be faster than $t^{C|a|}$. Since, in regions of fixed r, the $\mathbf{K}$ energy is like the $\mathbf{T}_\chi$ energy times t^2, this means that the energy density in regions of fixed r (away from the horizon) decays like $t^{-2+C|a|}$. Since the energy density is quadratic, if we use the third-order $\mathbf{K}$ energy, we can control the solution ψ in regions of fixed r by $t^{-1+C|a|}$, which proves the first part of theorem 1.2.

We also prove decay near to and far from the black hole. In both cases, we drop a null geodesic (or almost-null curves) from the point q at (t, r_*, θ, ϕ) to a point p at $r = 3M$, where we already have decay by theorem 1.2, and then estimate $|\tilde{\psi}(q) - \tilde{\psi}(p)|$, where $\tilde{\psi} = (r^2+a^2)^{1/2}\psi$. The point p at $r = 3M$ has a t coordinate given by the u_+ or u_- coordinate of q in the near or far cases respectively. In the near case, following [4, 38], we consider the region in the (u_+, u_-) plane and enclosed by p, q, curves of constant u_+ and of constant u_-, and $t = 0$. We apply Stokes' theorem in this region to the one-form $(\partial_- \tilde{\psi})\mathrm{d}u_-$ and estimate the bulk term using the Morawetz and $\mathbf{K}$ estimates. This allows us to control $|\tilde{\psi}(q) - \tilde{\psi}(p)|$ as one piece of the boundary term. In the far case, we drop an almost-null curve from q to the surface $r = t/2$, and we call this intersection point p. Although p is not in a stationary region, we are still able to show sufficiently strong decay for $\tilde{\psi}(p)$ directly from the $\mathbf{K}$ energy bound. We evolve the solution from the hypersurface $t = 0$ to the hypersurface of constant t through p. Since the deformation of the $\mathbf{K}$ energy is easily controlled when $r > t/2$, we can deform the hypersurface so that it becomes almost null and passes through q. We then integrate the derivative of ψ along an almost null curve in this hypersurface to estimate $|\tilde{\psi}(q) - \tilde{\psi}(p)|$. By applying the Cauchy-Schwarz inequality, we are able to control the integral of this derivative by the product of the $\mathbf{K}$ energy on the hypersurface and by the desired decaying factor. The contribution from the endpoint p at $r = 3M$ decays much faster. These estimates give the remaining parts of theorem 1.2.

The small a condition which we impose is significantly stronger than the condition that $|a| \leq M$ which implies the existence of a black hole and which might be ideally imposed. There are several fundamental and technical reasons for this small a condition. Perhaps most importantly, the construction of $\mathbf{T}_\chi$ relies on there being a region where both $\partial_t + \chi\omega_H\partial_\phi$ and ∂_t are timelike in which to perform the blending. When a is sufficiently large, but still smaller than M, there is no such overlapping region, so this particular construction fails. In addition, we use the assumption on the smallness of a to close the bounded $\mathbf{T}_\chi$ energy argument and the $\mathbf{K}$ energy growth argument. If a is not small relative to the absolute constants appearing in those estimates, it would not be possible to close the boot strap. A clear technical obstacle is that, in the proof of the Morawetz estimate, we perturb the Hardy estimate in (1.9). If a were too large, the perturbation argument would fail, and our numerical investigation suggests that when a is larger than about $.9M$, there are no longer positive solutions of the associated ODE, which we use to prove the estimate. These obstacles are the most fundamental obstacles to extending the range of a, but there are also numerous other, technical estimates in which we have made use of the smallness of a.

Having summarized our method, we will now compare it with methods used in recent, related work. Recently, others have constructed a bounded energy [17, 39]. To make a comparison, we point to several features which they share but which are different from those in our approach.

Since our energy is based on $\mathbf{T}_\chi$, which becomes null on the event horizon, the energy we control has a weight which vanishes linearly at $r = r_+$. The other works make use of the horizon-penetrating vector field, first introduced in [14]. This is denoted Y [17] or X_2 [39]. By combining the horizon-penetrating vector field with the equivalent of $\mathbf{T}_\chi$, they are able to construct a timelike vector field and an energy which do not degenerate near the event horizon. This is clearly advantageous. Since the properties of the Y or X_2 vecter field are quite stable [16], it can be used in a separate step, which we omit, following our result.

Neither [17] nor [39] use Q to generate higher energies. Away from the event horizon, they use the symmetries ∂_t^2, $\partial_t\partial_\phi$, and ∂_ϕ^2 and the fact that ψ satisfies the wave equation. Near the event horizon, they generate higher energies using ∂_t and a horizon-penetrating, radial vector field (e.g. Y in [17]). This is possible because of a favourable sign in the error terms arising from the failure of the radial vector field to be a symmetry.

Both of [17, 39] perform a Fourier transform in the t and ϕ variables to construct a pseudodifferential Morawetz multiplier, which we have avoided in favour of differential operators.

Less importantly, both avoid surfaces of constant t in favour of surfaces and coordinates which go through the event horizon. Since vector-field arguments can be deformed from one surface to another, this is a minor difference, although, the presence of the functions, q, slightly complicates this. Although all known Morawetz arguments have, in some sense, a troublesome lower-order term, [17, 39] use a different construction so that they can use positivity arising from Y or X_2, instead of the Hardy estimate we use to control the negativity in (1.9).

In [16], a decay rate of $t^{-1+\delta(a)}$ is stated with $\delta(a) \to 0$ as $a \to 0$ and with the corresponding decay rates along the event horizon and null infinity. The detailed outline of a proof focuses on using a vector field $\mathbf{K}$, which is very similar to the one used previously in Schwarzschild and, hence, to the one we use in this paper.

The structure of this paper is as follows: In section 2, we provide some further notation which we use in this paper. In section 3, the main argument of this paper, we expand the energy associated with $\mathbf{T}_\chi$ and prove the Morawetz estimate using the symmetry-indexed vector fields. This is followed by the $\mathbf{K}$ argument to prove local energy decay in section 4, and then by the decay estimate for ψ itself in section 5.

2. Notation and preliminaries

In this section, we present some more notation and basic estimates which we will use through out the paper.

To begin, we note that, in estimates, C is used to denote an absolute constant or a constant which depends only on M. The notation $x \lesssim y$ means $x \leq Cy$, and the notation $x \approx y$ means $x \lesssim y$ and $y \lesssim x$. Further, it is convenient to introduce the following notation

$$\begin{aligned} d^2\mu =& \mu d\theta d\phi, & \mu =& \sin\theta, \\ d^3\mu =& d^2\mu dr, & d^3\mu_* =& d^2\mu dr_*, \\ d^4\mu =& d^2\mu dr dt, & d^4\mu_* =& d^2\mu dr_* dt. \end{aligned}$$

2.1. **Canonical analysis.**

The volume element for the Kerr metric in Boyer-Lindquist coordinates is

$$\sqrt{-|g|} = \Sigma \sin\theta.$$

It is convenient to consider instead of the covariant d'Alembertian $\Box_g$, the transformed d'Alembertian $\Box = \Sigma \Box_g$. Recalling (1.5), we can write $\Box$ in the form

$$\Box = \partial_r \Delta \partial_r + \frac{1}{\Delta} \mathcal{R}(r; M, a; \partial_t, \partial_\phi, Q).$$

This is the form of the d'Alembertian that we shall consider throughout this paper. Let

$$\mathcal{G}^{\alpha\beta} = \Sigma g^{\alpha\beta}, \quad \mu = \sin\theta.$$

Then $\Box\psi = 0$ is the Euler-Lagrange equation for the action

$$S = \int \mathcal{L} d^4\mu,$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2} \mathcal{G}^{\alpha\beta} (\partial_\alpha \psi)(\partial_\beta \psi).$$

The canonical energy-momentum tensor for S is

$$\mathcal{T}^\alpha{}_\beta = -\frac{\delta \mathcal{L}}{\delta \partial_\alpha \psi} (\partial_\beta \psi) + \delta^\alpha{}_\beta \mathcal{L},$$

and the momentum vector for a vector field $\mathbf{X}$ is given by

$$\mathcal{P}^{\alpha}_{\mathbf{X}} = \mathcal{T}^{\alpha}{}_{\beta}\mathbf{X}^{\beta}$$

or explicitely

$$\begin{aligned}\mathcal{P}^{\alpha}_{(\mathbf{X},q)}[\psi] =& \left(-\mathcal{G}^{\alpha\gamma}(\partial_\gamma\psi)(\partial_\beta\psi) + \frac{1}{2}\delta^{\alpha}{}_{\beta}\left(\mathcal{G}^{\rho\sigma}(\partial_\rho\psi)(\partial_\sigma\psi)\right)\right)\mathbf{X}^{\beta}\\ &+ \mathcal{G}^{\alpha\beta}\left(-q(\partial_\beta\psi)\psi + \frac{1}{2}(\partial_\beta q)\psi^2\right).\end{aligned}$$

We have the following relations with the quantities introduced in section 1.1,

$$\begin{aligned}\nabla_\alpha P^{\alpha}_{(\mathbf{X},q)}[\psi] =& -\frac{1}{\Sigma}\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}^{\alpha}_{(\mathbf{X},q)}[\psi]\right),\\ \int_\Omega \left(\nabla_\alpha P^{\alpha}_{\mathbf{X},q}\right)\sqrt{-|g|}\mathrm{d}^4x =& -\int_\Omega \left(\partial_\alpha\mu\mathcal{P}^{\alpha}_{(\mathbf{X},q)}\right)\mathrm{d}^4\mu,\\ E_{(\mathbf{X},q)}[\psi](t) =& \int_{\Sigma_t}\mathcal{P}^{t}_{(\mathbf{X},q)}[\psi]\mathrm{d}^3\mu.\end{aligned}$$

When it is clear from context what the arguments are, we will frequently write $E_{\mathbf{X}}[\psi]$, $E_{\mathbf{X}}(t)$, or $E_{\mathbf{X}}$. Similarly, we will typically write $P_{(\mathbf{X},q)}$ for $P_{(\mathbf{X},q)}[\psi]$. Towards the end of this paper, we also need to evaluate energies on other hypersurfaces. In this case, we generate a three form by contracting the vector field $P_{(\mathbf{X},q)}$ against the volume form generated by the coordinate one forms, and then integrate that 3-form over the corresponding three-dimensional hypersurface. This is explained further in section 5.3.

The advantage of the canonical formalism just introduced is that without computing covariant derivatives, one can calculate the divergence of the momentum.

Lemma 2.1. *If $\mathbf{X}$ is a vector field, q is a function, and if $P_{(\mathbf{X},q)}$ is the associated momentum relative to the Lagrangian and weight*

$$\bar{\mathcal{L}} = \frac{1}{2}\left(\bar{\mathcal{G}}^{\alpha\beta}(\partial_\alpha u)(\partial_\beta u) + \bar{V}\psi^2\right), \quad \bar{\mu}$$

then the divergence of the corresponding momentum is

$$\begin{aligned}&\frac{1}{\bar{\mu}}\partial_\alpha\left(\bar{\mu}\bar{\mathcal{P}}_{(\mathbf{X},q)}[\psi]^{\alpha}\right)\\ &\quad= \left(-\bar{\mathcal{G}}^{\alpha\gamma}(\partial_\gamma\mathbf{X}^{\beta}) + \frac{1}{2}(\partial_\gamma\mathbf{X}^{\gamma})\bar{\mathcal{G}}^{\alpha\beta} + \frac{1}{2}\frac{1}{\bar{\mu}}\mathbf{X}^{\gamma}(\partial_\gamma\bar{\mu}\bar{\mathcal{G}}^{\alpha\beta})\right)(\partial_\alpha\psi)(\partial_\beta\psi)\\ &\qquad+ \frac{1}{2}(\partial_\gamma\mathbf{X}^{\gamma})\bar{V}\psi^2 + \frac{1}{2}\mathbf{X}^{\gamma}(\partial_\gamma\bar{V})\psi^2\\ &\qquad- q(\bar{\mathcal{G}}^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + \bar{V}\psi^2) + \frac{1}{2}\left(\partial_\beta\bar{\mathcal{G}}^{\alpha\beta}\partial_\alpha q\right)\psi^2\\ &\qquad+ \mathbf{X}^{\gamma}(\partial_\gamma\psi + q\psi)\left(\left(\frac{1}{\bar{\mu}}\partial_\alpha\bar{\mu}\bar{\mathcal{G}}^{\alpha\beta}\partial_\beta - \bar{V}\right)\psi\right).\end{aligned}$$

If ψ is a solution of $\left(\bar{\mu}^{-1}\partial_\alpha\bar{\mu}\bar{\mathcal{G}}^{\alpha\beta}\partial_\beta - \bar{V}\right)\psi = 0$, then the last term in this formula vanishes.

Similarly, if $\mathbf{X}^{\underline{ab}}$ *and* $q^{\underline{ab}}$ *are symmetric collections of double indexed vectors and scalars respectively and* ψ *is a solution of* $\left(\bar{\mu}^{-1}\partial_\alpha\bar{\mu}\bar{\mathcal{G}}^{\alpha\beta}\partial_\beta - \bar{V}\right)\psi = 0$*, then*

$$\begin{aligned}
&\frac{1}{\bar{\mu}}\partial_\alpha\left(\bar{\mu}\bar{\mathcal{P}}^\alpha_{(\mathbf{X}^{\underline{ab}},q^{\underline{ab}})}\right)\\
&=\left(-\bar{\mathcal{G}}^{\alpha\gamma}(\partial_\gamma\mathbf{X}^{\underline{ab}\beta})+\frac{1}{2}(\partial_\gamma\mathbf{X}^{\underline{ab}\gamma})\bar{\mathcal{G}}^{\alpha\beta}+\frac{1}{2}\mathbf{X}^{\underline{ab}\gamma}(\partial_\gamma\bar{\mathcal{G}}^{\alpha\beta})-q^{\underline{ab}}\bar{\mathcal{G}}^{\alpha\beta}\right)(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&\quad+\frac{1}{2}(\partial_\gamma\mathbf{X}^{\underline{ab}\gamma})\bar{V}\psi^2+\frac{1}{2}\mathbf{X}^{\underline{ab}\gamma}(\partial_\gamma\bar{V})\psi^2\\
&\quad+\frac{1}{2}(\partial_\alpha\bar{\mathcal{G}}^{\alpha\beta}\partial_\beta q^{\underline{ab}})(S_{\underline{a}}\psi)(S_{\underline{b}}\psi). \qquad (2.1)
\end{aligned}$$

2.2. **The $3+1$ decomposition.** The normal to the surfaces of constant t is

$$n_{\Sigma_t} = \left(\frac{\Pi}{\Sigma\Delta}\right)^{1/2}\left(\partial_t + \omega_\perp\partial_\phi\right),$$

with $\omega_\perp$ defined below. The lapse function is defined solely in terms of the foliation and its normal and is

$$\begin{aligned}
N =&(n^\alpha_{\Sigma_t}\partial_\alpha t)^{-1}\\
=&\left(\frac{\Pi}{\Sigma\Delta}\right)^{-1/2}
\end{aligned}$$

Unfortunately, the t co-ordinates do not extend beyond the exterior of the black hole, so the vector field n_{Σ_t} does not extend continuously beyond the exterior.

We introduce the vector field

$$\begin{aligned}
\mathbf{T}_\perp =&\partial_t + \omega_\perp\partial_\phi,\\
\omega_\perp =&\frac{2aMr}{\Pi}.
\end{aligned}$$

This vector field has the properties that

$$\mathbf{T}_\perp = N n_{\Sigma_t},$$

that $\mathbf{T}_\perp$ is timelike in the exterior, and that it extends continuously to the event horizon and the bifurcation sphere. In fact, it extends smoothly through the event horizon and the bifurcation sphere.[4] This vector field extends to the null tangent vector on the event horizon and to axial rotation (with coefficient ω_H) on the bifurcation sphere.

2.3. **Norms.** Given a set of differential operators, $\mathbb{X}$, we use the notation

$$|\psi|^2_{\mathbb{X}} = |\mathbb{X}\psi|^2 = \sum_{X\in\mathbb{X}} |X\psi|^2.$$

If no set is specified, simply an index, we mean

$$|\psi|^2_n = \sum_{i=0}^{n} |\mathbb{S}_n\psi|^2,$$

where $\mathbb{S}_n$ is the set of generators of the order-n symmetries given in equation (1.7).

Lemma 2.2 (Spherical Sobolev estimate using symmetries)**.** *There is a constant,* C*, such that* $\forall(\theta,\phi)\in S^2$ *and all* $(t,r)\in\mathbb{R}\times(r_+,\infty)$*, if* ψ *is sufficiently smooth that the norm on the right is bounded, then*

$$\sup_{(t,r)\times S^2}|\psi|^2 \le C\int_{(t,r)\times S^2}|\psi|^2_2\, d^2\mu.$$

if u *is sufficiently smooth that the integral on the right is bounded.*

[4] The vector fields ∂_t and ∂_ϕ are known to extend smoothly through the bifurcation sphere [26].

Proof. We use $\Delta\!\!\!/$ to denote the spherical Laplacian

$$\Delta\!\!\!/ = \frac{1}{\mu}\partial_\theta \mu \partial_\theta + \frac{1}{\mu^2}\partial_\phi^2.$$

The absolute value of the spherical Laplacian of u can be estimated by

$$\begin{aligned}
|\Delta\!\!\!/\psi| =& \left| \left(\frac{1}{\mu}\partial_\theta\mu\partial_\theta + \cot^2\theta\partial_\phi^2 + \partial_\phi^2 \right) \psi \right| \\
\leq& \left| \left(\frac{1}{\mu}\partial_\theta\mu\partial_\theta + \cot^2\theta\partial_\phi^2 \right) \psi \right| + |\partial_\phi^2\psi| \\
\leq& |Q\psi| + a^2\sin^2\theta|\partial_t^2\psi| + |\partial_\phi^2\psi| \\
\lesssim& |\mathbb{S}_2\psi|.
\end{aligned}$$

By a standard, spherical, Sobolev estimate,

$$|\psi|^2_{L^\infty(S^2)} \lesssim \int_{S^2} \left(|\Delta\!\!\!/\psi|^2 + |\psi|^2\right) \mathrm{d}^2\mu.$$

Since the integrand on the right is bounded by $|\psi|_2$, the desired estimate holds with a uniform constant in (t, r). □

In subsection 3.4, we also require the following operator and the associated weaker norms.

Definition 2.3. *Let*

$$\begin{aligned}
\mathcal{L} =& \partial_t^2 + Q + \partial_\phi^2, \\
\mathcal{L}_\epsilon =& \epsilon\partial_t^2 + Q + \partial_\phi^2,
\end{aligned}$$

and

$$\begin{aligned}
|\psi|^2_{2,\epsilon} =& \epsilon|\partial_t^2\psi|^2 + |\partial_t\partial_\theta\psi|^2 + \frac{1}{\mu^2}|\partial_t\partial_\phi^2\psi|^2 + |\Delta\!\!\!/\psi|^2, \\
|\psi|^2_{3,\epsilon} =& \epsilon^2|\partial_t^3\psi|^2 + \epsilon|\partial_t^2\nabla\!\!\!\!/\psi|^2 + |\partial_t\Delta\!\!\!/\psi|^2 + |\nabla\!\!\!\!/\Delta\!\!\!/\psi|^2.
\end{aligned}$$

We also introduce the homogeneous norms, generated from the previous norm by taking $\epsilon = 1$,

$$|\psi|_{n,1}.$$

Lemma 2.4 (The $\mathcal{L}_\epsilon\mathcal{L}$ estimate)**.** *There is a positive constant C such that, for positive values of the parameter ϵ, if $|a| \leq C\epsilon$ and if ψ smooth, then*

$$\begin{aligned}
(\mathcal{L}_\epsilon\psi)(\mathcal{L}\psi) \geq& |\psi|^2_{2,\epsilon} \\
&+ \frac{1}{\mu}\partial_t(\mu(\partial_t\psi)(\Delta\!\!\!/\psi))(1+\epsilon) \\
&- \frac{1}{\mu}\nabla\!\!\!\!/\cdot(\mu(\partial_t\psi)(\nabla\!\!\!\!/\partial_t\psi)) \\
&+ \frac{1}{\mu}2a^2\sin^2\theta(\partial_t(\mu(\partial_t\psi)(\partial_\phi^2\psi)) - \partial_\phi(\mu(\partial_t\psi)(\partial_\phi\partial_t\psi))).
\end{aligned} \tag{2.2}$$

Proof. This follows by direct computation, but it is important to integrate by parts in t first.

$$\begin{aligned}
(\mathcal{L}_\epsilon\psi)(\mathcal{L}\psi) =& \left((\epsilon\partial_t^2 + Q + \partial_\phi^2)\psi\right)\left((\partial_t^2 + Q + \partial_\phi^2)\psi\right) \\
=& \epsilon(\partial_t^2\psi)^2 + (Q\psi)^2 + (\partial_\phi^2\psi)^2 \\
&+ 2(Q\psi)(\partial_\phi^2\psi) \\
&+ (1+\epsilon)(\partial_t^2\psi)(Q\psi) + (1+\epsilon)(\partial_t^2\psi)(\partial_\phi^2\psi).
\end{aligned}$$

In the last line, the contribution from $\partial_t^2\psi$ in $Q\psi$ gives a strictly positive term, so it can be dropped, and integration by parts can be applied to the remainder

$$\begin{aligned}\mu(\partial_t^2\psi)((Q+\partial_\phi^2)\psi) \geq&\mu(\partial_t^2\psi)(\not\Delta\psi)\\ \geq&-\mu(\partial_t\psi)(\not\Delta\partial_t\psi)+\partial_t(\mu(\partial_t\psi)(\not\Delta\psi))\\ \geq&\mu|\not\nabla\partial_t\psi|^2+\partial_t(\mu(\partial_t\psi)(\not\Delta\psi))-\not\nabla\cdot(\mu(\partial_t\psi)(\not\nabla\partial_t\psi)).\end{aligned}$$

Finally, it remains to show that

$$\begin{aligned}\epsilon|dt^2\psi|^2+|Q\psi|^2+|\partial_\phi^2\psi|^2+2(Q\psi)(\partial_\phi^2\psi)=&\epsilon|\partial_t^2\psi|^2+|(Q+\partial_\phi^2)\psi|^2\\ \gtrsim&\epsilon|dt^2\psi|^2+|\not\Delta\psi|^2.\end{aligned}$$

This follows from expanding the left-hand side using $Q+\partial_\phi^2=\not\Delta+a^2\sin^2\theta\partial_t^2$, estimating the mixed term $2a^2\sin^2\theta(\partial_t^2\psi)(\not\Delta\psi)$ by the Cauchy-Schwarz inequality, and using the fact that $|a|\lesssim\epsilon$. $\square$

It is sometimes useful to have an estimate which gives equality, except for some small error terms, instead of an inequality. In such cases, we relate $Q+\partial_\phi^2$ to $\not\Delta$ to observe that

$$\begin{aligned}(\mathcal{L}_\epsilon\psi)(\mathcal{L}\psi)=&\left((\epsilon\partial_t^2+\not\Delta)\psi\right)\left((\partial_t^2+\not\Delta)\psi\right)\\ &+a^2\sin^2\theta\left((\partial_t^2\psi)(\mathcal{L}\psi)+((\epsilon\partial_t^2+\not\Delta)\psi)(\partial_t^2\psi)\right).\end{aligned}$$

Thus,

$$\left|(\mathcal{L}_\epsilon\psi)(\mathcal{L}\psi)-\left((\epsilon\partial_t^2+\not\Delta)\psi\right)\left((\partial_t^2+\not\Delta)\psi\right)\right|\leq a^2|\psi|_2^2.$$

Expanding $\left((\epsilon\partial_t^2+\not\Delta)\psi\right)\left((\partial_t^2+\not\Delta)\psi\right)$ in the same way as in lemma 2.4, we find

$$\left|(\mathcal{L}_\epsilon\psi)(\mathcal{L}\psi)-|\psi|_{2,\epsilon}^2+(\text{time and angular derivatives})\right|\leq a^2|\psi|_{2,1}^2, \tag{2.3}$$

where the time-derivatives terms are time derivatives of terms dominated by $|\partial_t\psi||\mathbb{S}_2\psi|$. Similarly,

$$\left|(\mathcal{L}_\epsilon\psi)^2-|\psi|_{3,\epsilon}^2+(\text{time and angular derivatives})\right|\leq a^2|\psi|_{3,1}^2, \tag{2.4}$$

where the time derivative terms are derivatives of quantities bounded by $|\partial_t\mathbb{T}_1\psi||\mathbb{S}_2\mathbb{T}_1\psi|$ and where the angular derivatives are derivatives of smooth terms.

2.4. **Further notation.** It is convenient to write the second-order symmetry operators with respect to coordinate partial derivatives

$$S_{\underline{a}}=\frac{1}{\mu}\partial_\alpha\mu S_{\underline{a}}^{\alpha\beta}\partial_\beta.$$

All other operator built from these, such as $\mathcal{L}$, $\mathcal{L}_\epsilon$, and $\mathcal{R}$, can be similarly expanded. For example,

$$\mathcal{R}=\frac{1}{\mu}\partial_\alpha\mu\mathcal{R}^{\alpha\beta}\partial_\beta.$$

We use the notation

$$f=O(r^p)$$

to mean that there is a constant, uniformly in a in some small interval of a values containing 0, such that $\forall r>r_+$, $f(r)<Cr^p$. Introduce also the notation

$$f=O\left(\left(\frac{\Delta}{r^2}\right)^q,r^p\right)$$

to mean that there is a constant, uniformly in a in some small interval of $a$values containing 0, such that $\forall r>r_+$,

$$f(r)<C\left(\frac{\Delta}{r^2}\right)^q r^p.$$

This measures the decay rate at r_+ and ∞. If f is continuous, this is all the information that is required to bound the function.

We use $\nabla\!\!\!/$ to denote angular partial derivatives and $\Delta\!\!\!/$ for the spherical Laplacian in (θ, ϕ) coordinates. We use Θ_i for the rotation vector-fields about the coordinate axes. With the exception of $\Theta_3 = \partial_\phi$, these are not symmetries in Kerr. We use $\mathbb{O}_1 = \{\Theta_i\}$ to denote the set of these rotations, and we use $\mathbb{T}_1$ for $\{\partial_t, \Theta_i\}$.

We use $\mathbb{1}_X$ to denote the indicator function, which is identically one on X and zero elsewhere.

We define a function to be smooth on a closed interval if it is smooth on the interior and if all the derivatives are continuous up to the boundary.

3. The bounded-energy argument

In this section, we construct a bounded energy by first constructing an almost conserved energy and then proving a Morawetz estimate to control the growth of the energy.

3.1. **The blended energy.**

Recall from (1.8) that for $|a|$ sufficiently small, the vector field

$$\mathbf{T}_\chi = \partial_t + \chi\omega_H\partial_\phi$$

is timelike in the exterior and Killing outside the region $[r_\chi, r_\chi + M]$, since χ is constant outside this region and decreases from one to zero inside this region. If we choose r_χ sufficiently large so that it corresponds to a larger value of r than any photon orbit for our initial choice of small $|a|$, this property will remain true for any subsequent decrease in the range of $|a|$ we allow. For specificity, we take $r_\chi = 10M$, which is beyond the range of photon orbits for any Kerr black hole.

The vector field $\mathbf{T}_\chi$ becomes null on the horizon, so the associated energy degenerates there. We compare this with the energy associated with $\mathbf{T}_\perp = (\Delta\Sigma/\Pi)^{1/2} n_{\Sigma_t}$ to make clear that the rate of degeneration with respect to the normal is roughtly $(\Delta/(r^2+a^2))^{1/2}$. We also provide a coordinate expression which is useful for making estimates. The apparently singular contribution to the energy from $\Delta^{-1}(\mathbf{T}_\perp\psi)^2$ is in fact vanishing, since the vector-field $\mathbf{T}_\perp$ vanishes on the bifurcation sphere at such a rate to exactly compensate for the factor of Δ^{-1}, and then the form $\mathrm{d}r$ is degenerating at a rate of $(\Delta/(r^2+a^2))^{1/2}$ near the bifurcation sphere.

Lemma 3.1. *There is a positive $\bar{a}$ such that for $|a| \le \bar{a}$ and any smooth function ψ, $\mathbf{T}_\chi$ is timelike and*

$$\begin{aligned}\mathcal{P}^t_{\mathbf{T}_\chi} \approx& \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\perp\psi)^2 + \Delta(\partial_r\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi)\\ \approx& \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\perp\psi) + \Delta(\partial_r\psi) + \Delta(\partial_t\psi)^2 + \sum_i |\Theta_i\psi|^2 \end{aligned} \tag{3.1}$$

$$\begin{aligned}\int_{\Sigma_t} \mathcal{P}^t_{\mathbf{T}_\chi} d^3\mu =& \int_{\Sigma_t} \mathcal{P}^\alpha_{\mathbf{T}_\chi} d\eta_\alpha\\ \approx& \int_{\Sigma_t} \mathcal{P}^\alpha_{\mathbf{T}_\perp} d\eta_\alpha. \end{aligned} \tag{3.2}$$

Furthermore, if ψ is a solution of the wave equation $\Box\psi = 0$, then

$$\left|\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}_{\mathbf{T}_\chi}[\psi]^\alpha\right)\right| = \Delta\omega_H|\partial_r\chi||\partial_\phi\psi||\partial_r\psi|. \tag{3.3}$$

Proof. We first expand

$$\begin{aligned}\mathcal{G}^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) =&\Delta(\partial_r\psi)^2 - \frac{(r^2+a^2)^2}{\Delta}(\partial_t\psi)^2 - \frac{4aMr}{\Delta}(\partial_t\psi)(\partial_\phi\psi)\\ &+ Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + \frac{\Delta - a^2}{\Delta^2}(\partial_\phi\psi)^2.\end{aligned}$$

We now substitute $\partial_t = \mathbf{T}_\chi + \chi\omega_H\partial_\phi$ and estimate the terms arising from the difference. We find

$$\begin{aligned}\mathcal{G}^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) =&\Delta(\partial_r\psi)^2 - \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\chi\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2\\ &+\left(\frac{4aMr}{\Delta} - \chi\omega_H\frac{(r^2+a^2)^2}{\Delta}\right)(\partial_t\psi)(\partial_\phi\psi)\\ &+\frac{1}{\Delta}\left(-a^2 + (r^2+a^2)^2\omega_H^2\right)(\partial_\phi^2)\\ =&\Delta(\partial_r\psi)^2 - \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\chi\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2\\ &+ aO(1,r^{-1})(\partial_t\psi)(\partial_\phi\psi) + a^2O(1,r^{-2})(\partial_\phi)^2.\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{G}^{t\beta}(\partial_\beta\psi) =& -\frac{\Pi}{\Delta}\partial_t\psi - \frac{2aMr}{\Delta}\psi\\ =& -(r^2+a^2)^2\mathbf{T}_\chi\psi\\ &+ a^2O(1,r^{-2})\partial_t\psi + aO(1,r^{-1})\partial_\phi\psi.\end{aligned}$$

The t component of the momentum associated with $\mathbf{T}_\chi$ is

$$\begin{aligned}\mathcal{P}^t_{\mathbf{T}_\chi} =& -\mathcal{G}^{t\beta}(\partial_\beta\psi)(\mathbf{T}_\chi\psi) + \frac{1}{2}\mathcal{G}^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi)\\ =&\frac{1}{2}\left(\Delta(\partial_r\psi)^2 + \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\chi\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2\right)\\ &+ a^2O(1,r^{-2})\partial_t\psi + aO(1,r^{-1})(\partial_t\psi)(\partial_\phi\psi) + a^2O(1,r^{-2})(\partial_\phi\psi)^2.\end{aligned}$$

Since the asymptotics of the coefficients on the last line grow no faster than the coefficients of the terms in the first line, we can take a sufficiently small that the first line easily dominates the last, and we can conclude

$$\mathcal{P}^t_{\mathbf{T}_\chi} \gtrsim \Delta(\partial_r\psi)^2 + \frac{(r^2+a^2)^2}{\Delta}(\mathbf{T}_\chi\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2.$$

Since the ∂_t^2 term in Q has a bounded factor times a^2,

$$\begin{aligned}\frac{(r^2+a^2)^2}{\Delta}(\partial_t\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2 \gtrsim& (\partial_t\psi)^2 + Q^{\alpha\beta}(\partial_\alpha\psi)(\partial_\beta\psi) + (\partial_\phi\psi)^2\\ \gtrsim& \sum_i |\Theta_i u|^2.\end{aligned}$$

Since

$$\mathbf{T}_\perp = \mathbf{T}_\chi + aO(\Delta, r^{-3})\partial_\phi,$$

the difference between the two associated energies is

$$\begin{aligned}aO(1,r^{-3})\mathcal{P}^t_{\partial_\phi} =& aO(\Delta,r^{-3})\left(\frac{2aMr}{\Delta}(\partial_\phi\psi)(\partial_t\psi)\right)\\ =& aO(1,r^{-4})(\partial_\phi\psi)(\partial_t\psi),\end{aligned}$$

which is easily dominated by $\mathcal{P}^t_{\mathbf{T}_\chi}$, and the two momenta are equivalent. Hence, their integrals are equivalent.

We compute the divergence of the momentum using equation (2.1) and find it be given by

$$-\mathcal{G}^{rr}(\partial_r \mathbf{T}^{\phi}_{\chi})(\partial_\phi\psi)(\partial_r\psi) = -\Delta(\partial_r\chi\omega_H)(\partial_\phi\psi)(\partial_r\psi),$$

which we estimate in absolute value. □

Recall that we defined higher-order energies by

$$E_{\mathbf{T}_\chi,n}[\psi] = \sum_{i=0}^{n} E_{\mathbf{T}_\chi}[\mathbb{S}_i\psi],$$

where $\mathbb{S}_i$ is the set of order-i symmetries from (1.7).

Corollary 3.2. *If ψ is a solution of the wave equation $\Box\psi = 0$,*

$$\frac{d}{dt}E_{\mathbf{T}_\chi,n}[\psi] \leq C\int_{r_+}^{\infty}\int_{S^2} \mathbb{1}_{supp\chi'}|\partial_r\psi|_n|\partial_\phi\psi|_n r^2 dr\sin\theta d\theta d\phi, \tag{3.4}$$

where the norms on the right are defined in subsection 2.3.

Proof. This follows from the previous lemma applied to the functions obtained from applying the symmetry operators to the solution ψ and then summing over the operators. □

3.2. **Set-up for radial vector fields and their fifth-order analogues.** If z and w are smooth functions of r and the parameters M and a, then we can define the following single- and double-indexed quantities

$$\begin{aligned}
\tilde{\mathcal{R}}^{\underline{a}} =& \frac{z}{\Delta}\mathcal{R}^{\underline{a}},\\
\tilde{\mathcal{R}}'^{\underline{a}} =& \partial_r\left(\frac{z}{\Delta}\mathcal{R}^{\underline{a}}\right),\\
\tilde{\tilde{\mathcal{R}}}'^{\underline{a}} =& w\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'^{\underline{a}},\\
\tilde{\tilde{\mathcal{R}}}''^{\underline{a}} =& \partial_r\left(w\frac{z^{1/2}}{\Delta^{1/2}}\tilde{\mathcal{R}}'^{\underline{a}}\right).
\end{aligned}$$

These can be used to define a double-indexed family of vectors and scalars which we will use to prove a Morawetz estimate.

Definition 3.3. *The Morawetz vector fields and scalar functions are defined to be*

$$\begin{aligned}
& & \mathbf{A}^{\underline{ab}} =& \mathcal{F}^{\underline{ab}}\partial_r,\\
q^{\underline{a}}_{\mathbf{A}} =& \frac{1}{2}(\partial_\gamma\mathbf{A}^{\underline{a}\gamma}) - q^{\underline{a}}_{\mathbf{A}'}, & q^{\underline{ab}}_{\mathbf{A}} =& \frac{1}{2}(\partial_\gamma\mathbf{A}^{\underline{ab}\gamma}) - q^{\underline{ab}}_{\mathbf{A}'},\\
\mathcal{F}^{\underline{a}} =& zw\tilde{\mathcal{R}}'^{\underline{a}}, & \mathcal{F}^{\underline{ab}} =& zw\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})},\\
q^{\underline{a}}_{\mathbf{A}'} =& \frac{1}{2}(\partial_r z)w\tilde{\mathcal{R}}'^{\underline{a}}, & q^{\underline{ab}}_{\mathbf{A}'} =& \frac{1}{2}(\partial_r z)w\tilde{\mathcal{R}}'^{(\underline{a}}\mathcal{L}^{\underline{b})}.
\end{aligned}$$

For simplicity, we introduce the following notation for the pair consisting of the previous vector field and function,

$$A = (\mathbf{A}^{\underline{ab}}, q^{\underline{ab}}_{\mathbf{A}}).$$

Lemma 3.4. *If ψ is a solution to the wave equation $\Box\psi = 0$, then the divergence of the momentum associated with these quantities is given by*

$$\begin{aligned}
\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}_A[\psi]^\alpha\right) =& \mathcal{A}^{\underline{ab}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)\\
&+ \mathcal{U}^{\underline{ab}\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&+ \mathcal{V}^{\underline{ab}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi),
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{A}^{\underline{ab}} =& \mathcal{A}^{(\underline{a}}\mathcal{L}^{\underline{b})},\\
\mathcal{U}^{\underline{ab}} =& \frac{1}{2}w\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{b}},\\
\mathcal{V}^{\underline{ab}} =& \mathcal{V}^{(\underline{a}}\mathcal{L}^{\underline{b})},\\
\mathcal{A}^{\underline{a}} =& z^{1/2}\Delta^{3/2}(-\tilde{\tilde{\mathcal{R}}}''^{\underline{a}}),\\
\mathcal{V}^{\underline{a}} =& \frac{1}{4}(\partial_r\Delta\partial_r z(\partial_r w\tilde{\mathcal{R}}'^{\underline{a}})),\\
\mathcal{U}^{\underline{ab}\alpha\beta} =& \mathcal{U}^{\underline{c}(\underline{a}}\mathcal{L}^{\underline{b})}S_{\underline{c}}^{\alpha\beta}.
\end{aligned}$$

Proof. In the formula for the divergence of the momentum, (2.1), the terms involving $\mathcal{G}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}u)(\partial_\beta S_{\underline{b}}u)$ are refered to as the Lagrangian contributions. The Lagrangian contribution from $(1/2)(\partial_\gamma \mathbf{A}^{\underline{ab}\gamma})$ in $q_{\mathbf{A}}^{\underline{ab}}$ exactly cancels the Lagrangian contribution from the vector field.

The divergence of the momentum is given by

$$\begin{aligned}
-\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}_A^\alpha\right) =& \left(\Delta(\partial_r\mathcal{F}^{\underline{ab}}) - \frac{1}{2}\mathcal{F}^{\underline{ab}}(\partial_r\Delta)\right)(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)\\
&- \frac{1}{2}\mathcal{F}^{\underline{ab}}\left(\partial_r\left(\frac{\mathcal{R}^{\alpha\beta}}{\Delta}\right)\right)(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&- q_{\mathbf{A}'}^{\underline{ab}}\Delta(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi) - q_{\mathbf{A}'}^{\underline{ab}}\frac{\mathcal{R}^{\alpha\beta}}{\Delta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi).\\
&- \frac{1}{2}(\partial_\beta\mathcal{G}^{\alpha\beta}\partial_\alpha q^{\underline{ab}})(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).
\end{aligned}$$

In the coefficient of the radial derivative terms, the part coming from the vector field can be rewritten as

$$\left(\Delta(\partial_r\mathcal{F}^{\underline{ab}}) - \frac{1}{2}\mathcal{F}^{\underline{ab}}(\partial_r\Delta)\right) = \left(\partial_r\left(\frac{\mathcal{F}^{\underline{ab}}}{\Delta^{1/2}}\right)\right)\Delta^{3/2}.$$

Expanding using the definitions of z, w, and $\tilde{\mathcal{R}}'$, we have

$$\begin{aligned}
\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}_A^\alpha\right) =& - z^{1/2}\Delta^{3/2}\left(\partial_r\left(\frac{z^{1/2}}{\Delta^{1/2}}w\tilde{\mathcal{R}}'^{\underline{a}}\right)\right)\mathcal{L}^{\underline{b}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)\\
&+ \frac{1}{2}w\tilde{\mathcal{R}}'^{\underline{a}}\mathcal{L}^{\underline{b}}\left(\partial_r\left(\frac{z\mathcal{R}^{\alpha\beta}}{\Delta}\right)\right)(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&+ \frac{1}{2}(\partial_\beta\mathcal{G}^{\alpha\beta}\partial_\alpha(z(\partial_r w\tilde{\mathcal{R}}'^{(\underline{a}}))\mathcal{L}^{\underline{b})}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).
\end{aligned}$$

The expression $\tilde{\mathcal{R}}'$ was chosen so that it is exactly the derivative in the second term. Similarly, the quantity $\tilde{\tilde{\mathcal{R}}}''$ was chosen so that it is the derivative in the first term. Thus, the total bulk term is

$$\begin{aligned}
\frac{1}{\mu}\partial_\alpha\left(\mu\mathcal{P}_A^\alpha\right) =& - z^{1/2}\Delta^{3/2}\tilde{\tilde{\mathcal{R}}}''^{\underline{a}}\mathcal{L}^{\underline{b}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)\\
&+ \frac{1}{2}(\mathcal{L}^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{b}})\tilde{\mathcal{R}}'^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&+ \frac{1}{4}(\partial_r\Delta\partial_r z(\partial_r w\tilde{\mathcal{R}}'^{(\underline{a}}))\mathcal{L}^{\underline{b})}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi).
\end{aligned}$$

Since $\tilde{\tilde{\mathcal{R}}}''^{\underline{a}}\mathcal{L}^{\underline{b}}$ is contracted against a quantity which is symmetric in $\underline{ab}$, it is not necessary to distinguish between $\tilde{\tilde{\mathcal{R}}}''^{\underline{a}}\mathcal{L}^{\underline{b}}$ and $\tilde{\tilde{\mathcal{R}}}''^{(\underline{a}}\mathcal{L}^{\underline{b})}$. Substituting the definitions of $\mathcal{A}^{\underline{ab}}$, $\mathcal{U}^{\underline{ab}\alpha\beta}$, and $\mathcal{V}^{\underline{ab}}$ gives the desired result. □

3.3. **Rearrangements.** We rearrange the terms related to $\mathcal{U}$ to get a strictly positive contribution to the divergence.

Lemma 3.5. *If ψ is a solution to the wave equation $\Box\psi = 0$, then*

$$\begin{aligned}\frac{1}{\mu}\partial_\alpha\left(\mu(\mathcal{P}_A^\alpha + \mathcal{B}_{A;I}^\alpha)\right) =&\mathcal{A}^{\underline{ab}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi)\\ &+\mathcal{U}^{\underline{ab}}\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\ &+\mathcal{V}^{\underline{ab}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi),\end{aligned}$$

where $\mathcal{A}$, $\mathcal{U}$, and $\mathcal{V}$ are defined in lemma 3.4 and

$$\mathcal{B}_{A;I}[\psi]^\alpha = \left(\mathcal{U}^{\underline{ab}}\mathcal{L}^{\alpha\beta} - \mathcal{U}^{\underline{ab}\alpha\beta}\right)(S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi).$$

We will refer to $\mathcal{B}_{A;I}$ as the first boundary term.

Proof. Starting from (3.5), it is only the second term on the right side that needs to be manipulated. First, we rearrange the derivative term to get

$$\begin{aligned}\mu\mathcal{U}^{\underline{ab}\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi) =&\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\ =&-\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(\partial_\alpha \mu S_{\underline{c}}^{\alpha\beta}\partial_\beta S_{\underline{b}}\psi)\\ &+\partial_\alpha(\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}(S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)).\end{aligned}$$

The first term on the right can be rewritten in terms of $S_{\underline{c}}$, which can be commuted with $S_{\underline{b}}$, which in turn can be expanded in partial derivatives:

$$\begin{aligned}-\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(\partial_\alpha\mu S_{\underline{c}}^{\alpha\beta}\partial_\beta S_{\underline{b}}\psi) =&-\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{c}}S_{\underline{b}}\psi)\\ =&-\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(S_{\underline{b}}S_{\underline{c}}\psi)\\ =&-\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}(S_{\underline{a}}\psi)(\partial_\alpha\mu S_{\underline{b}}^{\alpha\beta}\partial_\beta S_{\underline{c}}\psi).\end{aligned}$$

We can substitute this into the previous calculation, rearrange a derivative in the new expression, reindex, and use the symmetry of $\mathcal{U}^{\underline{ab}}$ to conclude that

$$\begin{aligned}\mu\mathcal{U}^{\underline{ab}\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi) =&\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}S_{\underline{b}}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{c}}\psi)\\ &-\partial_\alpha(\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}S_{\underline{b}}^{\alpha\beta}(S_{\underline{a}}\psi)(\partial_\beta S_{\underline{c}}\psi))\\ &+\partial_\alpha(\mu\mathcal{U}^{\underline{ca}}\mathcal{L}^{\underline{b}}S_{\underline{c}}^{\alpha\beta}(S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi))\\ =&\mu\mathcal{U}^{\underline{ab}}\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\ &-\partial_\alpha\left(\mu(\mathcal{U}^{\underline{ab}}\mathcal{L}^{\alpha\beta} - \mathcal{U}^{\underline{ab}\alpha\beta})(S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\right).\end{aligned}$$

Applying the definition $\mathcal{B}_{A;\mathrm{I}}$ gives the desired result. ☐

3.4. **Choosing the weights.** In this section, we choose the weights z and w to ensure the positivity of the highest order terms in the right-hand side of the estimate in the previous lemma, lemma 3.5.

Definition 3.6. *Given a positive value for the parameter $\epsilon_{\partial_t^2}$, we use the following weights to define the Morawetz vector field,*

$$\begin{aligned}z =&z_1 z_2, & w =&w_1 w_2,\\ z_1 =&\frac{\Delta}{(r^2+a^2)^2}, & w_1 =&\frac{(r^2+a^2)^4}{3r^2-a^2},\\ z_2 =&1-\epsilon_{\partial_t^2}\left(\frac{\Delta}{(r^2+a^2)^2}\right), & w_2 =&\frac{1}{2r}.\end{aligned}$$

This choice of weights generate a momentum which has a positive divergence, and for which this positive divergence dominates the square of third derivatives of ψ. The statement and proof of the following lemma make use of the norms given in subsection 2.3

Remark 3.7. *A simpler version of this argument can be run taking $z = z_1$ and $w = w_1$. These weights are chosen so that the coefficient of ∂_t^2 in $\tilde{\mathcal{R}}'$ and of $\partial_\phi\partial_t$ in $\tilde{\tilde{\mathcal{R}}}''$ vanish respectively. Eliminating the ∂_t^2 term in $\tilde{\mathcal{R}}'$ is a natural first step, since, in essence, this is what has been done in all previous vector-field arguments in the Schwarzschild spacetime. Eliminating the $\partial_\phi\partial_t$ term in $\tilde{\tilde{\mathcal{R}}}''$ is also natural, since this leaves only Q and ∂_ϕ^2 terms, which have a natural ellipticity. Taking these choices gives a collection of vector fields for which the divergence is positive. However, these simpler choices generate a divergence which fails to dominate third derivatives which involve two t derivatives, and they generate an energy which is not dominated by the $\mathbf{T}_\chi$ energy. We have further introduced the weights z_2 to obtain better control over the ∂_t^2 derivatives and w_2 to "temper" the vector fields, so that the associated energy can be controlled by the energies associated with $\mathbf{T}_\chi$.*

Lemma 3.8. *There are positive constants $\bar{a}$, $\overline{\epsilon_{\partial_t^2}}$, and C such that if $|a| \leq \bar{a}$ and $\epsilon_{\partial_t^2} < \overline{\epsilon_{\partial_t^2}}$ and ψ is a solution to the wave equation $\Box\psi = 0$ then*

$$
\begin{aligned}
&\frac{1}{\mu}\partial_\alpha\left(\mu\left(\mathcal{P}_A[u]^\alpha + \mathcal{B}^\alpha_{A;I} + \mathcal{B}^\alpha_{A;II}\right)\right) \qquad (3.6)\\
\geq& M\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi|^2_{2,\epsilon_{\partial_t^2}} + \frac{1}{6}\frac{9Mr^2 - 46M^2r + 54M^3}{r^4}|\psi|^2_{2,\epsilon_{\partial_t^2}}\\
&+ \frac{1}{4r}\frac{(r^2+a^2)^4}{3r^2-a^2}\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi)\\
&- C\frac{\Delta^2}{r^2(r^2+a^2)}(a|\partial_r\psi|^2_{2,1} + \epsilon_{\partial_t^2}|\partial_r\psi|^2_{2,a^2})\\
&- C\frac{1}{r^2}(a|\psi|^2_{2,1} + \epsilon_{\partial_t^2}|\psi|^2_{2,a^2}).
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{R}}' =& -2(r-3M)r^{-4}\mathcal{L}_{\epsilon_{\partial_t^2}}\\
&+ aO(r^{-4})\partial_\phi\partial_t + a^2O(r^{-5})Q + a^2O(r^{-5})\partial_\phi^2\\
&+ \epsilon_{\partial_t^2}a^2\partial_t^2 + \epsilon_{\partial_t^2}O(r^{-5})Q + \epsilon_{\partial_t^2}O(r^{-5})\partial_\phi^2
\end{aligned}
$$

and where the $\mathcal{B}^\alpha_{A;II}$ satisfy

$$
\begin{aligned}
|\mathcal{B}^t_{A;II}| \lesssim& \frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\partial_t\psi|\sum_{\underline{a}}|\partial_r S_{\underline{a}}\psi| + \frac{1}{r^2}|\partial_t\psi|\sum_{\underline{a}}|S_{\underline{a}}\psi|,\\
\mathcal{B}^r_{A;II} =& 0,
\end{aligned}
$$

and the angular components are smooth functions.

Proof. From the lemma 3.5, there are three terms to control, the $\mathcal{U}$, $\mathcal{A}$, and $\mathcal{V}$ terms.

Step 1: The $\mathcal{U}$ term. The $\mathcal{U}$ term can be expanded using the definition in lemma 3.4 as

$$
\frac{1}{2}\mathcal{U}^{\underline{ab}}\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi) = \frac{1}{4}w\tilde{\mathcal{R}}'^{\underline{a}}\tilde{\mathcal{R}}'^{\underline{b}}\mathcal{L}^{\alpha\beta}(\partial_\alpha S_{\underline{a}}\psi)(\partial_\beta S_{\underline{b}}\psi),
$$

so it is sufficient to calculate $\tilde{\mathcal{R}}'$. With our choice of z and w, this is

$$
\begin{aligned}
\tilde{\mathcal{R}}' =& -\epsilon_{\partial_t^2}(2(r-3M)r^{-4} + a^2O(r^{-5}))\partial_t^2\\
&+ aO(r^{-4})\partial_\phi\partial_t\\
&- (2(r-3M)r^{-4} + a^2O(r^{-5}) + \epsilon_{\partial_t^2}O(r^{-5}))Q\\
&- (2(r-3M)r^{-4} + a^2O(r^{-5}) + \epsilon_{\partial_t^2}O(r^{-5}))\partial_\phi^2.
\end{aligned}
$$

Step 2: The $\mathcal{A}$ term. The $\mathcal{A}$ term is

$$\mathcal{A}^{\underline{a}}\mathcal{L}^{\underline{b}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi) = \frac{\Delta^2}{r^2+a^2}(-\tilde{\tilde{\mathcal{R}}}''^{\underline{a}})\mathcal{L}^{\underline{b}}(\partial_r S_{\underline{a}}\psi)(\partial_r S_{\underline{b}}\psi).$$

With our choices of z and w, we find:

$$\begin{aligned}-\tilde{\tilde{\mathcal{R}}}'' =& M\epsilon_{\partial_t^2}(r^{-2}+a^2O(r^{-3}))\partial_t^2\\ &- aMO(r^{-2})\partial_\phi\partial_t\\ &+ M(r^{-2}+a^2O(r^{-3})+\epsilon_{\partial_t^2}O(r^{-3}))Q\\ &+ M(r^{-2}+a^2O(r^{-3})+\epsilon_{\partial_t^2}O(r^{-3}))\partial_\phi^2.\end{aligned}$$

We are interested in this because the operator $\tilde{\tilde{\mathcal{R}}}''$ is very close to $\mathcal{L}_{\epsilon_{\partial_t^2}}$ in the sense that

$$\begin{aligned}\left|\left((-\tilde{\tilde{\mathcal{R}}}'')-\frac{M}{r^2}\mathcal{L}_{\epsilon_{\partial_t^2}}\right)\partial_r\psi\right| =& aO(r^{-3})|\mathbb{T}_1^2\partial_r\psi|\\ &+ aO(r^{-2})|\mathbb{T}_1^2\partial_r\psi|\\ &+ aO(r^{-3})|\mathbb{T}_1^2\partial_r\psi|\\ &+ \epsilon_{\partial_t^2}O(r^{-3})|\mathbb{O}_1^2\partial_r\psi| + \epsilon_{\partial_t^2}aO(r^{-3})|\sin^2\theta\partial_t^2\partial_r\psi|\\ =& aO(1)|\partial_r\psi|_{2,1} + \epsilon_{\partial_t^2}O(1)|\partial_r\psi|_{2,a^2}.\end{aligned}$$

Since $\mathcal{L}$ and $\mathcal{L}_{\epsilon_{\partial_t^2}}$ commute with functions of r, we can apply lemma 2.4 to $\partial_r\psi$, to get

$$\begin{aligned}M\frac{\Delta^2}{r^2(r^2+a^2)}(\mathcal{L}_{\epsilon_{\partial_t^2}}\partial_r\psi)(\mathcal{L}\partial_r\psi) \geq& M\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi|^2_{2,\epsilon_{\partial_t^2}}\\ &+ \text{time and angular derivatives.}\end{aligned}$$

The time and angular derivatives are exactly those coming from lemma 2.4. The terms from the angular derivatives are smooth, and the terms from the time derivative are of the form

$$M\frac{\Delta^2}{r^2(r^2+a^2)}\partial_t((\partial_t\partial_r\psi)((\not\Delta+a^2\sin^2\theta\partial_\phi^2)\partial_r\psi).$$

Thus, we only need to control contributions from these terms when they appear as boundary terms on hypersurfaces of constant t. They are controlled by

$$M\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_t\partial_r\psi||(\not\Delta+a^2\sin^2\theta\partial_\phi^2)\partial_r\psi)| \lesssim M\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_t\partial_r\psi|\sum_{\underline{a}}|S_{\underline{a}}\partial_r\psi|. \tag{3.7}$$

Thus,

$$\begin{aligned}\mathcal{A}^{\underline{ab}}(S_{\underline{a}}\partial_r\psi)(S_{\underline{b}}\partial_r\psi) \geq& M\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi|^2_{2,\epsilon_{\partial_t^2}}\\ &- C\frac{\Delta^2}{r^2(r^2+a^2)}(a|\partial_r\psi|^2_{2,1}+\epsilon_{\partial_t^2}|\partial_r\psi|^2_{2,a^2})\\ &+ \text{time and angular derivatives}\end{aligned}$$

with the time and angular derivatives satisfying the bound given in the statement of this lemma.

Step 3: The $\mathcal{V}$ term. By direct computation, the $\mathcal{V}$ term is given by

$$\begin{aligned}
\mathcal{V}^{\underline{a}}S_{\underline{a}} =&\frac{1}{4}\partial_r\Delta\partial_r q^{\underline{a}}_{\mathbf{A}}S_{\underline{a}}\\
=&\left(\epsilon_{\partial_t^2}\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4}+aO(r^{-4}))\right)\partial_t^2\\
&+aO(r^{-4})\partial_\phi\partial_t\\
&+\left(\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4}+aO(r^{-4})+\epsilon_{\partial_t^2}O(r^{-4}))\right)Q\\
&+\left(\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4}+aO(r^{-4})+\epsilon_{\partial_t^2}O(r^{-4}))\right)\partial_\phi^2,\\
=&\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4})\mathcal{L}_{\epsilon_{\partial_t^2}}\\
&+(a+\epsilon_{\partial_t^2})O(r^{-4})\mathbb{S}_2,
\end{aligned}$$

where we have used $O(r^{-4})\mathbb{S}_2$ to denote terms of the form $O(r^{-4})S_{\underline{a}}$ with $S_{\underline{a}}\in\mathbb{S}_2$.

Applying the estimate in (2.3), we find

$$\begin{aligned}
\mathcal{V}^{\underline{ab}}(S_{\underline{a}}\psi)(S_{\underline{b}}\psi) =&\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4})(\mathcal{L}_{\epsilon_{\partial_t^2}}\psi)(\mathcal{L}\psi)\\
&+O(r^{-2})(a|\psi|^2_{2,1}+\epsilon_{\partial_t^2}|\psi|^2_{2,a})\\
\geq&\frac{1}{6}(9Mr^{-2}-46M^2r^{-3}+54M^3r^{-4})|\psi|^2_{2,\epsilon_{\partial_t^2}}\\
&+O(r^{-2})(a|\psi|^2_{2,1}+\epsilon_{\partial_t^2}|\psi|^2_{2,a})\\
&+\text{time and angular derivatives.}
\end{aligned}$$

Again, the time and angular derivatives come from the application of lemma 2.4, so that the terms from the angular derivatives are smooth, and the terms from the time derivative give a contribution of the form

$$\frac{C}{r^2}|\partial_t\psi||(\not{\Delta}+a^2\sin^2\theta\partial_\phi^2)\psi| \leq\frac{C}{r^2}|\partial_t\psi|\sum_{\underline{a}}|S_{\underline{a}}\psi|. \tag{3.8}$$

The time and angular derivative terms arising in this step and the previous one are combined into $\mathcal{B}_{A;\mathrm{II}}$ and are controlled by and (3.7)-(3.8). □

Lemma 3.9 (Controlling the boundary terms)**.** *If ψ is sufficiently smooth, satisfies $\Box\psi=0$, and has initial data which decays sufficiently rapidly at infinity, then*

$$\begin{aligned}
|\mathcal{P}_A[\psi]^t| \leq&C(\mathcal{P}_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]^t+|\mathbb{S}_2\psi|^2)\\
|\mathcal{B}^t_{A;I}| \leq&C(\mathcal{P}_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]^t+|\mathbb{S}_2\psi|^2)\\
|\mathcal{B}^t_{A;II}| \leq&C(\mathcal{P}_{\mathbf{T}_\chi}[\partial_t\psi]^t+\mathcal{P}_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]^t+|\partial_t\psi|^2+|\mathbb{S}_2\psi|^2),\\
\lim_{r\to r_+}\mathcal{P}^r_A =&0\\
\lim_{r\to r_+}\mathcal{B}^r_{A;I} =&0\\
\lim_{r\to\infty}\mathcal{P}^r_A =&0\\
\lim_{r\to\infty}\mathcal{B}^r_{A;I} =&0.
\end{aligned}$$

Here, by "sufficiently rapidly", we mean that $\lim_{r\to\infty}\psi=0$, $\lim_{r\to\infty}r\partial_r\psi=0$, and the same estimates hold for $\mathbb{S}_1\psi$ and $\mathbb{S}_2\psi$.

Proof. By direct computation,

$$\begin{aligned}
\mathcal{P}^t_A =&\left(-\left(\left((r^2+a^2)^2-a^2\Delta\sin^2\theta\right)(\partial_t\mathbb{S}_2\psi)-2aMr(\partial_\phi\mathbb{S}_2\psi)\right)\left(O(1)(\partial_r\mathbb{S}_2\psi)+O(r^{-2})(\mathbb{S}_2\psi)\right)\right.\\
\mathcal{B}^t_{A;\mathrm{I}} =&O(r^{-1})((\mathbb{S}_2\psi)(\mathbb{T}_1\mathbb{S}_2\psi))\\
\mathcal{P}^r_A =&O((\Delta/r^2)^2,r^2)(\partial_r\mathbb{S}_2\psi)(\partial_r\mathbb{S}_2\psi)\\
&+O(\Delta/r^2,r)(\partial_r\mathbb{S}_2\psi)(\mathbb{S}_2\psi)+O(\Delta/r^2,1)(\mathbb{S}_2\psi)(\mathbb{S}_2\psi)\\
\mathcal{B}^r_{A;\mathrm{I}} =&0.
\end{aligned}$$

Here we have used $\mathbb{S}_2\psi$ to denote a term which is bounded in absolute value by $|\mathbb{S}_2\psi|$. First, the t component of the momentum can be dominated by

$$\begin{aligned}
&\frac{1}{\Delta}\left(\left((r^2+a^2)^2-a^2\Delta\sin^2\theta\right)(\partial_t\mathbb{S}_2\psi)+(2aMr)(\partial_\phi\mathbb{S}_2\psi)\right)^2\\
&+O((\Delta/r^2),r^2)(\partial_r\mathbb{S}_2\psi)^2+O(\Delta/r^2,1)(\mathbb{S}_2\psi)^2
\end{aligned}$$

From the expression for the energy, (3.1), the second term is dominated by $\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]$. The third is clearly dominated by ψ^2. This leaves the first. Since the coefficients in the first term are polynomial, and the ratio of the coefficients of the $\partial_t\psi$ and $\partial_\phi\psi$ term in the first expression is

$$\begin{aligned}
\frac{(r^2+a^2)^2-a^2\Delta\sin^2\theta}{2aMr} =&\frac{(r_+^2+a^2)^2}{2aMr_+}+O(\Delta/r^2,r^4)\\
=&\frac{(r_+^2+a^2)}{a}+O(\Delta/r^2,r^4)\\
=&\omega_H^{-1}+O(\Delta/r^2,r^4),
\end{aligned}$$

the first term can be estimated by

$$C\frac{\chi}{\Delta}(\partial_t\mathbb{S}_2\psi+\omega_H\partial_\phi\mathbb{S}_2\psi)^2+O(r^2)(\partial_t\mathbb{S}_2\psi)^2+O(1)(\partial_\phi\mathbb{S}_2\psi)^2.$$

Thus, the expression for the energy in estimate (3.1) shows that this is bounded by $\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]$. This controls the t component of the momentum.

The t-component of the first boundary term is controlled by

$$\begin{aligned}
\mathcal{B}^t_{A;\mathrm{I}} =&O(r^{-1})((\mathbb{S}_2\psi)(\partial_t\mathbb{S}_2\psi))\\
\leq&O(r^2)(\partial_t\mathbb{S}_2\psi)^2+O(1)(\mathbb{S}_2\psi)^2\\
\leq&\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]+|\mathbb{S}_2\psi|^2.
\end{aligned}$$

The t component of the second boundary term was partially estimated in the statement of lemma 3.8, so that

$$\begin{aligned}
|\mathcal{B}^t_{A;\mathrm{II}}| \leq&O((\Delta/r^2)^2,1)|\partial_r\partial_t\psi||\partial_r\mathbb{S}_2\psi|\\
&+O(r^{-2})|\partial_t\psi||\mathbb{S}_2\psi|\\
\leq&O((\Delta/r^2)^2,1)|\partial_r\partial_t\psi|^2+O((\Delta/r^2)^2,1)|\partial_r\mathbb{S}_2\psi|^2\\
&+|\partial_t\psi|^2+|\mathbb{S}_2\psi|^2\\
\leq&C(\mathcal{P}^t_{\mathbf{T}_\chi}[\partial_t\psi]+\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]+|\partial_t\psi|^2+|\mathbb{S}_2\psi|^2).
\end{aligned}$$

The limits at r_+ and ∞ are easily evaluated. The radial component of the momentum and the first boundary terms are bounded functions times a power of Δ, so they vanish at r_+. If the initial data decays sufficiently rapidly towards infinity, by finite speed of propogation, the same will be true for the solution at any time, so that in the limit as $r\to\infty$, at fixed t, the momentum and first boundary term will go to zero. The radial term of the second boundary term is identically zero, so all its limits vanish. □

3.5. **The Hardy estimate.** In (3.6), the coefficient of $|\psi|^2_{2,\epsilon_{\partial_t^2}}$ is positive except in a compact range of r values. The purpose of this section is to prove a Hardy estimate which allows us to get a globally positive coefficient for $|\psi|^2_{2,\epsilon_{\partial_t^2}}$ by using the positivity of the term involving $|\partial_r\psi|^2_{2,\epsilon_{\partial_t^2}}$. The proof is a bit technical, and the reader can omit it on a first reading, since the proof is independent of the rest of the Morawetz estimate.

Lemma 3.10. *There exist positive $\bar{a}$ and ϵ_{Hardy} such that if $|a| < \bar{a}$, then for any smooth function ϕ on $[r_+, \infty) \times S^2$ which decays sufficiently rapidly at ∞,*

$$\int_{r_+}^{\infty} \left(\frac{\Delta^2}{r^2(r^2+a^2)} (\partial_r\phi)^2 + \frac{1}{6}\frac{9r^2 - 46Mr + 54M^2}{r^4}\phi^2 \right) dr \geq \epsilon_{Hardy} \int_{r_+}^{\infty} \frac{\Delta^2}{r^2(r^2+a^2)} (\partial_r\phi)^2 + \frac{1}{r^2}\phi^2 \, dr. \tag{3.9}$$

Proof. The proof consists of several parts. The early parts of this proof follow the method of [5]. First, we will demonstrate that it is sufficient to find a positive solution to an associated ODE (ordinary differential equation). Second, we rewrite the estimate and ODE in terms of a new function, φ. Third, we will construct an explicit solution for the new ODE when $a = 0$ and $\epsilon_{\text{Hardy}} = 0$. Fourth, we will argue that the construction of the explicit solution can be perturbed to cover nonzero a and ϵ_{Hardy}, which will give a perturbed estimate for φ. Fifth, we will show that this gives the estimate for the original function ϕ. Finally, we will show that boundary conditions for the ODE do not place restrictions on the function ϕ.

Step 1: Find a positive solution to the associated ODE. We wish to show that if the ODE

$$-\partial_r A \partial_r u + V u = 0,$$

has a smooth, positive solution u on $[r_0, \infty]$, then for any smooth function ϕ on $[r_0, \infty]$, there is the estimate

$$\int_{r_0}^{\infty} A(\partial_r\phi)^2 + V\phi^2 dr \geq 0,$$

as long as

$$\phi^2 A \frac{\partial_r u}{u} \tag{3.10}$$

vanishes at r_0 and ∞. Recall that we define a function to be smooth on a closed interval if it is smooth on the interior and all derivatives have a limit at the boundary.

Since u is positive, for any smooth ϕ, we can define $f = \phi/u$. From integration by parts,

$$\begin{aligned}\int_{r_0}^{\infty} A(\partial_r\phi)^2 + V\phi^2 dr - [Auf(\partial_r uf)]_{r_0}^{\infty} = &\int_{r_0}^{\infty} uf^2(-\partial_r A\partial_r u + Vu)dr \\ &+ \int_{r_0}^{\infty} u^2 A(\partial_r f)^2 dr \\ &- [u^2 A f(\partial_r f)]_{r_0}^{\infty}.\end{aligned}$$

Since u satisfies the ODE $-\partial_r A\partial_r u + Vu = 0$, the first term on the right is positive. Cancelling the boundary terms on the right from those on the left leaves the estimate

$$\int_{r_0}^{\infty} A(\partial_r\phi)^2 + V\phi^2 dr = \int_{r_0}^{\infty} u^2 A(\partial_r f)^2 dr + [f^2 Au(\partial_r u)]_{r_0}^{\infty}.$$

The boundary term vanishes under condition (3.10), and the integrand on the right is non-negative, since $\phi = fu$. Therefore,

$$\int_{r_0}^{\infty} A(\partial_r \phi)^2 + V\phi^2 dx \geq 0.$$

Step 2: Simplify the estimate to eliminate one of the coefficients. We will first introduce the notation

$$A = \frac{\Delta^2}{r^2(r^2+a^2)}.$$

We will consider the function

$$\varphi = A^{1/2}\phi.$$

Since $A^{1/2}$ is smooth on $[r_+, \infty)$ and vanishes linearly at r_+, the new function φ is also smooth and vanishes at least linearly at r_+. Its derivative satisfies

$$\partial_r \phi = \frac{1}{A^{1/2}}(\partial_r \varphi) - \frac{1}{2}\frac{\partial_r A}{A^{3/2}}\varphi.$$

Therefore, the right-hand side of (3.9) is given by

$$\begin{aligned}
&\int_{r_+}^{\infty} (\partial_r \varphi)^2 + \left(\frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A^2} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}\right)\varphi^2 \mathrm{d}r \\
&\quad = \int_{r_+}^{\infty} (\partial_r \varphi)^2 - \frac{\partial_r A}{A}\varphi(\partial_r \varphi) + \left(\frac{1}{4}\frac{(\partial_r A)^2}{A^2} + \frac{V}{A}\right)\varphi^2 \mathrm{d}r + \left[\frac{1}{2}\frac{\partial_r A}{A}\varphi^2\right]_{r_+}^{\infty}
\end{aligned}$$

If this satisfies

$$\int_{r_+}^{\infty} (\partial_r \varphi)^2 + \left(\frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A^2} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}\right)\varphi^2 \mathrm{d}r \geq \epsilon_{\mathrm{Hardy},2}\int_{r_+}^{\infty} \frac{1}{Ar^2}\varphi^2 \mathrm{d}r$$

then by multiplying this estimate by $1 - \epsilon_{\mathrm{Hardy},3}$

$$\begin{aligned}
&\int_{r_+}^{\infty} (\partial_r \varphi)^2 + \left(\frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A^2} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}\right)\varphi^2 \mathrm{d}r \\
&\quad \geq \int_{r_+}^{\infty} \epsilon_{\mathrm{Hardy},3}(\partial_r \varphi)^2 + \left(\epsilon_{\mathrm{Hardy},3}\left(\frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A^2} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}\right) + \epsilon_{\mathrm{Hardy},2}\frac{1}{Ar^2}\varphi^2\right)\mathrm{d}r.
\end{aligned}$$

By taking $\epsilon_{\mathrm{Hardy},3}$ sufficiently small, we can conclude inequality (3.9) holds.

Step 3: Construction of the explicit solution for $a = 0$ and $\epsilon_{\mathbf{Hardy}} = 0$. Following the arguments in the first section, we could prove the desired estimate (for $a = 0$ and $\epsilon_{\mathrm{Hardy}} = 0$) by finding a positive solution to

$$-\partial_r A \partial_r u + Vu = 0 \tag{3.11}$$

with

$$\begin{aligned}
A &= \frac{(r^2 - 2Mr)^2}{r^4}, \\
V &= \frac{1}{6}\frac{9r^2 - 46Mr + 54M^2}{r^4}
\end{aligned}$$

on the interval $[2M, \infty)$. However, by using the argument in the previous section, it is easier to use the transformed function

$$v = A^{1/2}u = \left(\frac{(r^2 - 2r)^2}{r^4}\right)^{1/2} u, \tag{3.12}$$

$$x = r - 2M, \tag{3.13}$$

and to solve the ODE (3.11)

$$-\partial_x^2 v + W v =0, \tag{3.14}$$

$$\begin{aligned} W =&\frac{V}{A} + \frac{1}{2}\frac{\partial_x^2 A}{A} - \frac{1}{4}\frac{(\partial_x A)^2}{A^2} \\ =&\frac{9x^2 - 34Mx - 2M^2}{6x^2(x+2M)^2} \end{aligned} \tag{3.15}$$

on the interval $x \in [0, \infty)$.

We first note the following properties of hypergeometric functions [1, 18]. The hypergeometric function is typically written with parameters $F(a, b; c; z)$. It should be clear in all cases whether a refers to the first parameter of the hypergeometric function or to the angular momentum parameter of the Kerr spacetime. The hypergeometric function $F(a, b; c; z)$ has the following integral representation for $a < 0 < b < c$ and $z \notin [1, \infty)$

$$F(a, b; c; z) =\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt. \tag{3.16}$$

It is not obvious from this representation, but it is true, that F is symmetric in its first two arguments, $F(a, b; c; z) = F(b, a; c; z)$. There are a vast number of further relations. The hypergeometric differential equation is

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0. \tag{3.17}$$

A pair of solutions to this equation is

$$\begin{aligned} &F(a, b; c; z), \\ z^{1-c}&F(a - c + 1, b - c + 1; 2 - c; z). \end{aligned}$$

Returning to the ODE arising from the Hardy estimate, we introduce the further substitution

$$v =x^\alpha (x+d)^\beta \tilde{v}. \tag{3.18}$$

The ODE now becomes

$$\begin{aligned} v'' =& \left(\alpha(\alpha-1)x^{\alpha-2}(x+2)^\beta + 2\alpha\beta x^{\alpha-1}(x+d)^{\beta-1} + \beta(\beta-1)x^\alpha(x+d)^{\beta-2}\right)\tilde{v} \\ &+ 2\left(\alpha x^{\alpha-1}(x+d)^\beta + \beta x^\alpha(x+d)^{\beta-1}\right)\tilde{v}' \\ &+ x^\alpha(x+d)^\beta\tilde{v}'', \\ 0 =& - v'' + Wv \\ =&x^{\alpha-2}(x+d)^{\beta-2}P, \end{aligned} \tag{3.19}$$

$$\begin{aligned} P =&x^2(x+d)^2\tilde{v}'' \\ &- 2x(x+d)((\alpha+\beta)x + \alpha d)\tilde{v}' \\ &+ \Bigg(- (\alpha(\alpha-1)(x+d)^2 + 2\alpha\beta x(x+d) + \beta(\beta-1)x^2) \\ &\qquad + \frac{9x^2 - 34Mx - 2M^2}{6x^2(x+2M)^2}x^2(x+d)^2\Bigg)\tilde{v}. \end{aligned} \tag{3.20}$$

The prefactor of $x^{\alpha-2}(x+d)^{\beta-2}$ in the ODE (3.19) can be ignored. If we choose

$$d =2M, \tag{3.21}$$

then the rational function in the last term on the right reduces to a polynomial.

The coefficient of $\tilde{v}''$ is $x^2(x+d)^2$, of $\tilde{v}'$ is $x(x+d)$ times a linear function, and of $\tilde{v}$ a quadratic. If we choose the parameters α and β so that the coefficient of $\tilde{v}$ is a constant multiple of $x(x+d)$, then an over all factor of $x(x+d)$ can be dropped, leaving the coffecients of $\tilde{v}''$, $\tilde{v}'$, and $\tilde{v}$ as $x(x+d)$, a linear function, and a constant respectively. The substitution $z = -x/d$, then transforms the equation to the hypergeometric differential equation. Our goal is to show such choices of α and β can be made.

It is now merely a matter of checking by direct calculation that this can be done. The coefficient of $\tilde{v}$ is

$$\begin{aligned} &-\alpha(\alpha-1)(x^2+2xd+d^2) - 2\alpha\beta(x^2+dx) - \beta(\beta-1)x^2 \\ &+ (3/2)x^2 - (17/3)x - 1/3. \end{aligned} \tag{3.22}$$

In this coefficient, we set the constant order term to zero

$$\begin{aligned} -\alpha(\alpha-1)d^2 - 1/3 =&0, \\ \alpha =&\frac{1}{2} \pm \frac{\sqrt{6}}{6}. \end{aligned}$$

Fortunately, the term $\alpha\beta(x^2+dx)$ is already a multiple of x^2+dx, so we may ignore it when trying to get the coefficient of $\tilde{v}$ to be a multiple of x^2+dx. We set the ratios of the remaining coefficients of x^2 and of x to be d:

$$d\left((3/2) - \alpha(\alpha-1) - \beta(\beta-1)\right) = -2d\alpha(\alpha-1) - (17/3).$$

We can substitute $-\alpha(\alpha-1) = 1/12$ to get

$$\begin{aligned} 2\left((3/2) + (1/12) - \beta(\beta-1)\right) =&(1/3) - (17/3), \\ \beta =&\frac{1}{2} \pm \frac{3\sqrt{2}}{2}. \end{aligned}$$

The four choices of sign provide four choices of simplified equations to study. For simplicity, we will consider only the equation arising from taking the + sign in α and the − sign in β.[5]

We are left with the differential equation for $\tilde{v}$

$$\begin{aligned} x(x+2)\tilde{v}'' - 2((1+\sqrt{6}/6+3\sqrt{2}/2)x + 1 + \sqrt{6}/3)\tilde{v}' \\ +(19/6 - 3\sqrt{2}/2 + \sqrt{6}/6 - \sqrt{3})\tilde{v} = 0, \end{aligned}$$

Making the substitutions $z = -x/d$ and $\tilde{\psi}(z) = \tilde{v}(x)$ gives

$$\begin{aligned} z(1-z)\tilde{\psi}'' + \Big((1+\sqrt{6}/3) - (2 - 3\sqrt{2} + \sqrt{6}/3)z\Big)\tilde{\psi}' \\ + \Big(-19/6 + 3\sqrt{2}/2 + \sqrt{3} - \sqrt{6}/6\Big)\tilde{\psi} = 0. \end{aligned} \tag{3.23}$$

Thus we have a hypergeometric differential equation, with solution $\tilde{v} = F(a,b;c,-x/d)$. We can immediately read off some quantites in terms of the hypergeometric parameters

$$c = 1 + \sqrt{6}/3, \tag{3.24}$$

$$\begin{aligned} -a-b-1 =& -2 + 3\sqrt{2} - \sqrt{6}/3, \\ -ab =& -19/6 + 3\sqrt{2}/2 + \sqrt{3} - \sqrt{6}/6. \end{aligned}$$

We can now solve for the remaining two parameters

$$\{a,b\} = \frac{1}{2} - \frac{3}{2}\sqrt{2} + \frac{\sqrt{6}}{6} \pm \frac{1}{2}\sqrt{7}. \tag{3.25}$$

We will make the choice $a < b$ so that

$$a < -2.5 < 0 < .1 < b < .2 < 1.8 < c.$$

In particular

$$a < 0 < b < c.$$

[5] This choice simplifies some expressions in the rest of this argument.

Thus, the integral representation (3.16) holds. Multiplying by $(-1)^{-a}\Gamma(c)/(\Gamma(a)\Gamma(b))$, we find that $\tilde{\psi}(z)$ is positive when $z \leq 0$. This means that $\tilde{v}$ is positive when $x \geq 0$, v is also positive when $x \geq 0$, and u is positive when $r > 2M$.

Step 4: The perturbed estimate for v. In this step, we will prove that there are $0 < \bar{a}_{\text{Hardy},4}$ and $0 < \epsilon_{\text{Hardy},4}$ such that for $|a| < \bar{a}_{\text{Hardy},4}$ and all suitable φ,

$$\int_0^\infty |\partial_r\varphi|^2 + \bar{W}\varphi^2 \mathrm{d}x \geq 0,$$

for

$$\begin{aligned}\bar{W} =& \frac{9x^2 - 34Mx - 2M^2}{6x^2(x+d)^2} - \epsilon_{\text{Hardy},4}\frac{(M+x)^2}{x^2(x+d)^2},\\ d =& r_+ - r_-\\ r_- =& M - \sqrt{M^2 - a^2}.\end{aligned}$$

This potential is of the form

$$\bar{W} = \frac{C_1x^2 + C_2x + C_3}{C_4x^2(x+d)^2}, \tag{3.26}$$

with the coefficients $C_1, \ldots, C_4$, and d perturbed from their original values in equation (3.15).

From the argument in step 1, it is sufficient to find a positive solution to the associated ODE (3.14),

$$-\partial_x^2 v + \bar{W}v = 0,$$

with the perturbed potential $\bar{W}$. The analysis in step 3 found an explicit, positive solution for $x \in [0, \infty)$ for the parameter values dictated by the potential in equation (3.15). This step shows that the previous analysis also applies when the coefficients are perturbed.

The previous analysis began by making the definition of $\tilde{v}$ in equation (3.18), in terms of the parameters α and β. The analysis then proceeded by choosing values for α and β by solving quadratic equations coming from the coefficient in formula (3.22), which lead to the new ODE (3.23). This ODE could be solved explicitely in terms of a hypergeometric function by solving linear and quadratic equations for the non-zero quantities a, b, and c. Since the coefficients in formula (3.22) depend continuously on the parameters C_1, C_2, C_3, C_4, and d in the potential; since the coefficients in the ODE (3.23) depend continuously on α, β, and the coefficients in the potential; since all the quadratic equations involved had distinct, real roots; and since solutions to linear and quadratic equations depend continuously on the coefficients; it follows that positive solutions to the ODEs (3.14) and (3.23) can be found explicitely in terms of hypergeometric functions with parameters a, b, and c depending continuously on the parameters in $\bar{W}$, at least when those parameter values are sufficiently close to the values given in equation (3.15). Similarly, when the perturbation of the parameter values in the potential $\bar{W}$ is sufficiently small, then the hypergeometric parameters maintain their order $a < 0 < b < c$. This gives the existence of positive $\bar{a}_{\text{Hardy},4}$ and $\epsilon_{\text{Hardy},4}$ which give the desired estimate for this step.

Step 5: The perturbed estimate for the original function ϕ. From the argument in step 2, we wish to prove that there exist $0 < \bar{a}$ and $0 < \epsilon_{\text{Hardy}}$ such that for $0 \leq |a| < \bar{a}$ and suitable φ

$$\int_{r_+}^\infty (\partial_r\varphi)^2 + \left(\frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A^2} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}\right)\varphi^2 \mathrm{d}r \geq \epsilon_{\text{Hardy},2}\int_{r_+}^\infty \frac{1}{r^2}\frac{1}{Ar^2}\varphi^2 \mathrm{d}r, \tag{3.27}$$

with

$$A = \frac{\Delta^2}{r^2(r^2+a^2)},$$
$$V = \frac{1}{6}\frac{9r^2 - 46Mr + 54M^2}{r^4}.$$

With the benefit of foresight, we introduce a new rotation parameter

$$\tilde{a} = M - \sqrt{M^2 - a^2}.$$

When $\tilde{a}$ is treated as a function of $|a|$ with M fixed, this is a continuous, increasing function on the interval $[0, M]$, which maps the interval $[0, M]$ to $[0, M]$. In addition, since the quantities which appear in our estimate (such as Δ and r^2+a^2) only have a quadratic dependence on a, and since a^2 can be solved for as a quadratic expression in $\tilde{a}$, it follows that the quantities A and V are rational functions in $(r, M, \tilde{a})$.

The new radial coordinate, analagous to the one defined in (3.13), is now defined to be

$$x = r - r_+ = r - (2M - 2\tilde{a}).$$

Since r can be solved for linearly in terms of $(x, M, \tilde{a})$, the quantities A and V are rational functions in $(r, M, \tilde{a})$.

The quantity

$$W = \frac{V}{A} + \frac{1}{2}\frac{\partial_r^2 A}{A} - \frac{1}{4}\frac{(\partial_r A)^2}{A^2}$$

is rational in $(x, M, \tilde{a})$; has degree, with respect to x, two lower in the numerator than in the denominator; has singularities in $x \in [-d, \infty)$ only at $x \in \{0, -d\}$ for fixed M and $\tilde{a}$; these are of order at most two; and, for sufficiently small $\tilde{a}$, has no singularities in $\tilde{a}$ for fixed $x > 0$ and M. Thus, we may expand it as

$$W = \frac{1}{\Delta^2}\frac{P_0 + \tilde{a}P_>}{Q_0 + \tilde{a}Q_>},$$

where the functions P_0 and Q_0 are polynomials in (x, M), the functions $P_>$ and $Q_>$ are polynomials in $(x, M, \tilde{a})$, and Q_0 and $Q_>$ have no roots in $x \in [-d, \infty)$. Since P_0/Q_0 is determined explicitly by equation (3.15), it follows that

$$W - \frac{1}{\Delta^2}\frac{P_0}{Q_0} = \frac{\tilde{a}}{\Delta^2}\frac{P_> Q_0 - P_0 Q_>}{Q_0(Q_0 + \tilde{a}Q_>)}$$

must decay like r^{-2} as $r \to \infty$ for fixed $\tilde{a}$ and M and has no singularities in $[-d, \infty)$. Since this is a rational function, there is a constant C such that

$$\left| W - \frac{1}{\Delta^2}\frac{P_0}{Q_0} \right| \leq \tilde{a} C \frac{(M+x)^2}{\Delta^2}.$$

Thus, there are sufficiently small $\bar{a}$ and $\epsilon_{\text{Hardy},2}$ such that for $0 \leq |a| < \bar{a}$

$$W - \epsilon_{\text{Hardy},2}\frac{1}{Ar^2} > \bar{W},$$

with $\bar{W}$ as in equation (3.26) The smallness of $\bar{a}$ and $\epsilon_{\text{Hardy},2}$ is determined by the smallness of $\bar{a}_{\text{Hardy},4}$ and $\epsilon_{\text{Hardy},4}$. These then give $\bar{a}$ and ϵ_{Hardy} for which the desired estimate holds.

Step 6: Controlling the boundary terms. Since the argument from step 1 was applied to the function φ, the boundary condition which must be imposed for this argument to hold is that

$$\varphi^2 \frac{\partial_r v}{v}$$

vanishes at r_+ and at ∞. Since the positive solution to the ODE is given by

$$\begin{aligned}v(r) =&x^\alpha(x+d)^\beta\tilde{v} = x^\alpha(x+d)^\beta F(a,b;c;-z/d)\\ =&(r-r_+)^\alpha(r-r_-)^\beta F\left(a,b;c;-\frac{r-r_+}{r_+-r_-}\right),\end{aligned}$$

and the hypergeometric function is analytic (in its fourth argument) near zero, the ratio $\partial_r v/v$ will diverge at most inverse linearly at $r=r_+$. Thus, it is sufficient that φ vanish linearly at $r=r_+$. Since $\varphi = \Delta/(r(r^2+a^2)^{1/2})\phi$, it is sufficient that ϕ be smooth near r_+.

To show the decay near ∞, we first note that from the form of the potential $\bar{W}$ in the ODE, the solution $v(r)$ will behave like a polynomial as $r\to\infty$, so that $\partial_r v/v$ will decay like a constant times $1/r$. Thus, it is sufficient that φ remains bounded at infinity. Thus, to get decay at both r_+ and ∞, it is sufficient that ϕ be smooth and bounded on $[r_+,\infty)$. □

3.6. **Integrating the Morawetz estimate.**

Lemma 3.11. *For positive parameters $\bar{a}$ and $\epsilon_{\partial_t^2}$, there is a positive constant C such that, for all $|a|<\bar{a}$ and all smooth ψ solving the wave equation $\Box\psi=0$, the estimate*

$$\begin{aligned}&C(E_{\mathbf{T}_\chi}[\mathbb{S}_2\psi](T_2)+E_{\mathbf{T}_\chi}[\mathbb{S}_1\psi](T_2)+E_{\mathbf{T}_\chi}[\mathbb{S}_2\psi](T_1)+E_{\mathbf{T}_\chi}[\mathbb{S}_1\psi](T_1))\\ &\geq \int_{T_1}^{T_2}\int_{r_+}^\infty\int_{S^2}\left(\frac{\Delta^2}{r^4}\right)|\partial_r\psi|_{2,1}^2 + r^{-2}|\psi|_{2,1}^2 + \mathbb{1}_{r\not\approx 3M}\frac{1}{r}|\psi|_{3,1}^2\, d^4\mu, \qquad (3.28)\end{aligned}$$

holds, where $\mathbb{1}_{r\not\approx 3M}$ is identically one, except in an open neighbourhood of the values of r for which there are orbiting geodesics. In this neighbourhood, we take $\mathbb{1}_{r\not\approx 3M}$ to be indentically zero.

Proof. We integrate the result of lemma 3.8 over the coordinate slab $(t,r,\theta,\phi)\in[T_1,T_2]\times(r_+,\infty)\times S^2$, from which we get the integral of the right-hand side of estimate (3.6). From the Hardy estimate (3.9), the integral of the first two terms on the right-hand side of (3.6) dominates an absolute constant times

$$\int_{T_1}^{T_2}\int_{S^2}\int_{r_+}^\infty\left(\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi|_{2,\epsilon_{\partial_t^2}}^2+\frac{1}{r^2}|\psi|_{2,\epsilon_{\partial_t^2}}^2\right)\mathrm{d}^4\mu.$$

By taking $|a|\lesssim\epsilon_{\partial_t^2}\lesssim 1$, these terms will also dominate the fourth and fifth terms, with a constant factor left over. Since $\epsilon_{\partial_t^2}$ can be chosen independently of a, the norms $|\psi|_{2,\epsilon_{\partial_t^2}}$ can be replaced by $|\psi|_{2,1}$ at the price of a fixed constant. The same is true for the norms of $\partial_r\psi$.

The only term which we still need to estimate is the third,

$$\int_{T_1}^{T_2}\int_{r_+}^\infty\int_{S^2}\frac{(r^2+a^2)^4}{4r(3r^2-a^2)}\mathcal{L}^{\alpha\beta}(\partial_\alpha\tilde{\mathcal{R}}'\psi)(\partial_\beta\tilde{\mathcal{R}}'\psi)\mathrm{d}^4\mu.$$

The integrand can be estimated by

$$\begin{aligned}\frac{(r^2+a^2)^4}{4r(3r^2-a^2)}\mathcal{L}^{\alpha\beta}(\partial_\alpha\tilde{\mathcal{R}}'\psi)(\partial_\beta\tilde{\mathcal{R}}'\psi) \geq&\frac{(r^2+a^2)^4}{4r(3r^2-a^2)}|\mathbb{T}_1\tilde{\mathcal{R}}'\psi|^2\\ \geq&\mathbb{1}_{r\not\approx 3M}\frac{(r^2+a^2)^4}{4r(3r^2-a^2)}|\mathbb{T}_1\tilde{\mathcal{R}}'\psi|^2\\ \geq&\mathbb{1}_{r\not\approx 3M}O(r^{-1})|\mathbb{T}_1\mathcal{L}_{\epsilon_{\partial_t^2}}\psi|^2\\ &+\mathbb{1}_{r\not\approx 3M}O(r^{-5})\epsilon_{\partial_t^2}|\mathbb{T}_1\mathcal{L}_{a^2}\psi|^2\\ &+\mathbb{1}_{r\not\approx 3M}O(r^{-3})a|\mathbb{T}_1\mathbb{S}_2\psi|^2.\end{aligned}$$

The first term in this expression can be bounded from below by inequality (2.4), so that

$$|\mathbb{T}_1\mathcal{L}_{\epsilon_{\partial_t^2}}\psi|^2 \gtrsim |\psi|^2_{3,\epsilon_{\partial_t^2}} + a^2|\psi|^2_{3,1} + \text{(time and angular derivatives)}.$$

To control the remaining terms, we note that

$$\begin{aligned}\epsilon_{\partial_t^2}|\mathbb{T}_1\mathcal{L}_{a^2}\psi|^2 + a|\mathbb{T}_1\mathbb{S}_2\psi|^2 \lesssim &(\epsilon_{\partial_t^2}a^2+a)|\partial_t^3\psi|^2 + (\epsilon_{\partial_t^2}a^2+a)|\partial_t^2\nabla\!\!\!/\psi|^2\\ &+ (\epsilon_{\partial_t^2}+a)|\partial_t\Delta\!\!\!\!/\psi|^2 + (\epsilon_{\partial_t^2}+a)|\nabla\!\!\!/\Delta\!\!\!\!/\psi|^2.\end{aligned}$$

If we impose the conditions that $|a| \lesssim \epsilon^2_{\partial_t^2} \lesssim 1$, then these terms are controlled by

$$|\psi|^2_{3,\epsilon_{\partial_t^2}} \leq |\mathbb{T}_1\mathcal{L}_{\epsilon_{\partial_t^2}}\psi|^2 + \text{time and angular derivatives},$$

so that

$$\begin{aligned}\frac{(r^2+a^2)^4}{2r(3r^2-a^2)}\mathcal{L}^{\alpha\beta}(\partial_\alpha\tilde{\mathcal{R}}'\psi)(\partial_\beta\tilde{\mathcal{R}}'\psi) \gtrsim &\mathbb{1}_{r\not\approx 3M}r^{-1}|\psi|^2_{3,\epsilon_{\partial_t^2}}\\ &+ \mathbb{1}_{r\not\approx 3M}O(r^{-1})\text{(time and angular derivatives)}.\end{aligned}$$

Thus, if we fix a sufficiently small $\epsilon_{\partial_t^2}$, and then require $|a| < \epsilon^2_{\partial_t^2}$, we have control of $|u|_3$ with the weight $\mathbb{1}_{r\not\approx 3M}r^{-1}$.

The time derivative generated in this part of the argument is

$$\partial_t(\mathbb{1}_{r\not\approx 3M}O(r^{-1})(\partial_t\mathbb{T}_1\psi)(\Delta\!\!\!\!/\mathbb{T}_1\psi)),$$

where $\mathbb{T}_1$ is the set defined in section 2.4 to contain ∂_t and the rotations around the coordinate axes. Thus, the contribution of this time derivative on the boundary of the region of integration is bounded by $P^t_{\mathbf{T}_\chi}[\mathbb{S}_1\psi] + P^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]$.

We must now control the integral of the momentum and the boundary terms over the boundary of the slab. All the angular derivate terms vanish, since S^2 has no boundary. Similarly, the boundary contributions along $r = r_+$ and $r \to \infty$ are zero by lemma 3.9. (Geometrically, one would expect this, since $r = r_+$ is actually a two-dimensional surface, the bifurcation sphere, and not a three-dimensional hypersurface, so it should not contribute any boundary terms.)

We are left to control the integral of the momentum and the boundary terms over the hypersurfaces $t = T_1$ and $t = T_2$. From lemma 3.9, these are controlled, at fixed t, by

$$\begin{aligned}&|\int \mathcal{P}^t_A + \mathcal{B}^t_{A;\mathrm{I}} + \mathcal{B}^t_{A;\mathrm{II}}\mathrm{d}^3\mu|\\ &\qquad\leq C\int\left(\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2u] + |\mathbb{S}_2u|^2 + \mathcal{P}^t_{\mathbf{T}_\chi}[\partial_tu] + |\partial_tu|^2\right)\mathrm{d}^3\mu\\ &\qquad\leq E_{\mathbf{T}_\chi}[\mathbb{S}_2u] + E_{\mathbf{T}_\chi}[\partial_tu].\end{aligned}$$

In this, we have used the 1-dimensional Hardy estimate:

$$\int_0^\infty |\psi|^2dx \leq \int_0^\infty x^2|\partial_x\psi|^2\mathrm{d}x \tag{3.29}$$

with $x = r - r_+$. □

We are unable to make use of the previous lemma since it controls only third derivatives, but the boundary terms involve both the second- and third-order energies. (Certain second-derivative terms are controlled, but these are not the important ones.) In the following lemma, we control the lower-order derivatives. This is the only place where we use any separability or spectral information.

Lemma 3.12. *For the positive parameters $\bar{a}$ and $\epsilon_{\partial_t^2}$, there is a positive constant C such that for all $|a| < \bar{a}$ and all smooth ψ solving the wave equation $\Box\psi = 0$, the estimate*

$$C(E_{\mathbf{T}_\chi,3}[\psi](T_2) + E_{\mathbf{T}_\chi,3}[\psi](T_1)) \geq \int_{T_1}^{T_2}\int_{r_+}^{\infty}\int_{S^2}\left(\left(\frac{\Delta^2}{r^4}\right)|\partial_r\psi|_2^2 + r^{-2}|\psi|_2^2 + \mathbb{1}_{r\not\approx 3M}\frac{1}{r}\left(|\partial_t\psi|_2^2 + |\nabla\!\!\!/\psi|_2^2\right)\right)d^4\mu. \tag{3.30}$$

holds, where $\mathbb{1}_{r\not\approx 3M}$ is identically one, except in an open neighbourhood of the values of r for which there are orbiting geodesics.

Proof. We must control the weighted integral of the first, second, and third derivatives. The Morawetz estimate, (3.28), controls the third derivatives, so we only need to control the lower-order derivatives.

Since ∂_ϕ commutes with the d'Alembertian, it is possible to apply separate out the zero eigenmode and write ψ in the form

$$\psi = \psi_{L_z=0} + \psi_{L_z\neq 0}.$$

From the Morawetz estimate, (3.6), we control the integral of the third derivatives. If ψ is real valued, then $\psi_{L_z=0}$ and $\psi_{L_z\neq 0}$ are also real valued.

Since the ∂_ϕ eigenvalue, L_z, satisfies $|L_z| \geq 1$, for $\psi_{L_z\neq 0}$, the integral over the sphere of the third derivatives controls all lower derivatives, i.e. the homogeneous norm controls the inhomogeneous norm,

$$\int_{S^2}|\psi_{L_z\neq 0}|_n^2 d^2\mu \leq C\int_{S^2}|\psi_{L_z\neq 0}|_{n,1}^2 d^2\mu.$$

It remains to control the lower derivatives of $\psi_{L_z=0}$. To do this, we prove a Morawetz estimate using a classical, first-order vector-field. Consider the momentum associated with

$$\begin{aligned}
\mathbf{A} =& zwf\partial_r,\\
q_{\mathbf{A}} =& \frac{1}{2}\left(\partial_r\mathbf{A}^r\right) - q'_{\mathbf{A}},\\
q'_{\mathbf{A}} =& \frac{1}{2}\left(\partial_r z\right)wf,\\
f =& \partial_r\left(\frac{z^{1/2}}{\Delta^{1/2}}\frac{\Delta}{(r^2+a^2)}\right).
\end{aligned}$$

Using the same sort of calculations as before, we can obtain the analogue of (3.5)

$$\frac{1}{\mu}\partial_\alpha\left(\mu P^\alpha_{(\mathbf{A},q_{\mathbf{A}})}\right) = \mathcal{A}(\partial_r\psi_{L_z=0})^2 + \mathcal{U}^{\alpha\beta}(\partial_\alpha\psi_{L_z=0})(\partial_\beta\psi_{L_z=0}) + \mathcal{V}|\psi_{L_z=0}|^2,$$

with

$$\begin{aligned}
\mathcal{A} =& \frac{1}{2}z^{1/2}\Delta^{3/2}\partial_r\left(w\frac{w^{1/2}}{\Delta^{1/2}}f\right),\\
\mathcal{U}^{\alpha\beta} =& \frac{1}{2}w(\partial_r f)\tilde{\mathcal{R}}'^{\alpha\beta},\\
\mathcal{V} =& -\frac{1}{4}\partial_r\Delta\partial_r z(\partial_r wf).
\end{aligned}$$

Taking the same choices of z and w as before, we find

$$\begin{aligned}\mathcal{A} =&\frac{1}{2}\frac{\Delta^2}{r^2+a^2}\left(\frac{1}{r^2}+(a+\epsilon_{\partial_t^2})O(r^{-3})\right),\\ \mathcal{U}^{\alpha\beta}(\partial_\alpha\psi_{L_z=0})(\partial_\beta\psi_{L_z=0}) =&\frac{1}{2}(\partial_r f)^2 Q^{\alpha\beta}(\partial_\alpha\psi_{L_z=0})(\partial_\beta\psi_{L_z=0})\\ &+\frac{1}{2}\epsilon_{\partial_t^2}\left(\partial_r\frac{\Delta}{(r^2+a^2)^2}\right)\left(1-2\epsilon_{\partial_t^2}\frac{\Delta}{(r^2+a^2)^2}\right)(\partial_t\psi_{L_z=0})^2\\ \mathcal{V} =&\frac{1}{6}\frac{9Mr^2-46M^2r+54M^3}{r^4}+(a+\epsilon_{\partial_t^2})O(r^{-4}).\end{aligned}$$

Note that there are no $\partial_t\partial_\phi$ or ∂_ϕ^2 arising from $\tilde{\mathcal{R}}'$ when acting on $\psi_{L_z=0}$. From the Hardy estimate, (3.9), it follows taht

$$\mathcal{A}(\partial_r\psi_{L_z=0})^2+\mathcal{V}|\psi_{L_z=0}|^2 \gtrsim \frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi_{L_z=0}|^2+\frac{1}{r^2}|\psi_{L_z=0}|^2.$$

Thus,

$$\begin{aligned}\frac{1}{\mu}\partial_\alpha\left(\mu P^\alpha_{(\mathbf{A},q_\mathbf{A})}\right) \gtrsim&\frac{\Delta^2}{r^2(r^2+a^2)}|\partial_r\psi_{L_z=0}|^2+\frac{1}{r^2}|\psi_{L_z=0}|^2\\ &+\mathbb{1}_{r\not\approx 3M}(|\partial_t\psi_{L_z=0}|^2+|\nabla\!\!\!/\psi_{L_z=0}|^2).\end{aligned}$$

In analogy with the previous results in lemma 3.9, there is a constant and an upper bound on a such that

$$|E_{(\mathbf{A},q_\mathbf{A})}[\psi_{L_z=0}]| \le CE_{\mathbf{T}_\chi}[\psi_{L_z=0}].$$

Thus, we have the Morawetz estimate

$$C(E_{\mathbf{T}_\chi}(T_2)+E_{\mathbf{T}_\chi}(T_1)) \tag{3.31}$$

$$\begin{aligned}\geq\int_{T_1}^{T_2}\int_{r_+}^{\infty}\int_{S^2}\Bigg(\left(\frac{\Delta^2}{r^4}\right)|\partial_r\psi_{L_z=0}|^2+r^{-2}|\psi_{L_z=0}|^2\\ +\mathbb{1}_{r\not\approx 3M}\frac{1}{r}\left(|\partial_t\psi_{L_z=0}|^2+|\nabla\!\!\!/\psi_{L_z=0}|^2\right)\Bigg)\mathrm{d}^4\mu,\end{aligned} \tag{3.32}$$

which controls the first derivatives.

Second derivatives have either two, one, or zero time derivatives. Any term containing one or more time derivative can be controlled by applying (3.32) to $\partial_t\psi_{L_z=0}$. Terms which contain no time derivatives contain at least one angular derivative. Thus, by integrating by parts in the angular derivative and applying the Cauchy-Schwarz inequality, we can control such terms by the first and third derivatives, which are controlled by (3.32) and (3.6).

By combining the results for $\psi_{L_z=0}$ and $\psi_{L_z\neq 0}$, we have the desired result. □

3.7. **Closing the argument.** We are now able to show that the energy associated with $\mathbf{T}_\chi$ is uniformly bounded by its value on the initial hypersurface. When $a=0$, the energy is conserved. When $a\neq 0$, the energy is no longer conserved, but the factor by which it can change vanishes linearly in $|a|$.

Theorem 3.13. *There are positive constants $\bar{a}$ and C such that if $|a|<\bar{a}$ and ψ is a solution to the wave equation $\Box\psi=0$, then for all $t_2\geq t_1\geq 0$:*

$$E_{\mathbf{T}_\chi,3}[\psi](t_2)\leq(1+C|a|)E_{\mathbf{T}_\chi,3}[\psi](t_1).$$

Proof. By lemma 3.2

$$\begin{aligned}&E_{\mathbf{T}_\chi,3}[\psi](t_2)+E_{\mathbf{T}_\chi,3}[\psi](t_1)\\ &\qquad\leq|a|C\int_{[t_1,t_2]\times(r_+,\infty)\times S^2}\mathbb{1}_{\mathrm{supp}\chi'}\left(|\partial_r\psi|_2^2+|\psi|_3^2\right)\mathrm{d}^4\mu.\end{aligned}$$

By the Morawetz estimate, lemma (3.12), for sufficiently small a, there is a constant C' such that the integral of the third derivatives is controlled by the energies. Thus,

$$E_{\mathbf{T}_\chi,3}[\psi](t_2) - E_{\mathbf{T}_\chi,3}[\psi](t_1) \le |a|C' \left(E_{\mathbf{T}_\chi,3}[\psi](t_2) + E_{\mathbf{T}_\chi,3}[\psi](t_1)\right).$$

Thus, for sufficiently small a,

$$(1-|a|C')E_{\mathbf{T}_\chi,3}[\psi](t_2) \le (1+|a|C')E_{\mathbf{T}_\chi,3}[\psi](t_1),$$

$$E_{\mathbf{T}_\chi,3}[\psi](t_2) \le \frac{1+|a|C'}{1-|a|C'}E_{\mathbf{T}_\chi,3}[\psi](t_1).$$

Since, for sufficiently small $|a|$, the rational function $(1+|a|C')/(1-|a|C')$ is bounded above by $1+C|a|$ for some C, the desired result holds. □

4. Decay estimates for the local energy

In this section, we prove decay using the $\mathbf{K}$ vector field. To prove decay, we find it convenient to work in a different set of coordinates and to work with a transformed function. This allows us to write the wave equation as the Euler-Lagrange equation for a different Lagrangian. A single equation can be the Euler-Lagrange equation for more than one Lagrangian. The advantage of this Lagrangian is that it allows us to control terms involving $|\psi|_2$ with the energy.

4.1. Reformulation of the problem.

Recall that the r_* coordinate is defined by

$$\frac{dr}{dr_*} = \frac{\Delta}{r^2+a^2},$$

and by the choice of initial condition that $r_*=0$ at $r=3M$. In the Schwarzschild case, this choice of initial condition is convenient since it fixes the origin of the r_* coordinates at the photon orbits, and, in the Kerr case, no other choice seems more convenient. In this section, we will work with the (t,r_*,θ,ϕ) coordinates and treat r as a function of r_*. All indices will refer to these coordinates.

Lemma 4.1. *Let*

$$\tilde\psi = \sqrt{r^2+a^2}\,\psi.$$

The wave equation $\Box\psi=0$ *is equivalent to*

$$\mu^{-1}\partial_\alpha\left(\mu\tilde{\mathcal{G}}^{\alpha\beta}\right)\partial_\beta\tilde\psi - V\tilde\psi = 0, \tag{4.1}$$

where

$$\frac{1}{\mu}\partial_\alpha\left(\mu\tilde{\mathcal{G}}^{\alpha\beta}\right)\partial_\beta = \partial_{r_*}^2 + \frac{1}{(r^2+a^2)^2}\mathcal{R},$$

$$V = \frac{\Delta}{(r^2+a^2)^4}(2Mr^3+r^2a^2-4Mra^2+a^4).$$

Here, it is understood that $\mathcal{R}$ *is the operator defined in* (1.6).

Proof. Direct computation. □

The matrix $\tilde{\mathcal{G}}^{\alpha\beta}$ is obtained from multiplying the inverse metric $g^{\alpha\beta}$, evaluated in the (t,r_*,θ,ϕ) coordinate system by the function $\Sigma\Delta(r^2+a^2)^{-2}$. The transformed wave equation in (4.1) can be written as the Euler-Lagrange equation for

$$S = \int \tilde{\mathcal{L}}\,\mathrm{d}^4\mu_*,$$

$$\tilde{\mathcal{L}} = \frac{1}{2}\left(\tilde{\mathcal{G}}^{\alpha\beta}(\partial_\alpha\tilde\psi)(\partial_\beta\tilde\psi) + V\tilde\psi^2\right), \quad \mu=\sin\theta.$$

Section 2.1 defines the canonical energy-momentum tensor, the momentum vector, and the energy for this Lagrangian. We denote these

$$\tilde{\mathcal{T}}[\tilde\psi], \qquad \tilde{\mathcal{P}}_{\mathbf{X}}[\tilde\psi], \qquad \tilde E_{\mathbf{X}}[\tilde\psi].$$

The related bilinear terms are defined similarly.

In the r_* coordinates, we have

$$\tilde{\mathcal{G}}^{\alpha\beta} = -\tilde{N}^{-2}\mathbf{T}_\perp^\alpha \mathbf{T}_\perp^\beta + \partial_*^\alpha \partial_*^\beta + \tilde{h}^{\alpha\beta}, \tag{4.2}$$

with

$$\begin{aligned}\tilde{V}_Q =& \frac{\Delta}{(r^2+a^2)^2},\\ \tilde{N}^{-2} =& 1 - a^2\sin^2\theta \tilde{V}_Q, \qquad (4.3)\\ \tilde{h}^{\alpha\beta} =& \tilde{V}_Q \partial_\theta^\alpha \partial_\theta^\beta\\ &+ \tilde{V}_Q\left(1 - a^2\sin^2\theta \frac{r^2+2Mr+a^2\cos^2\theta}{\Pi}\right)\frac{1}{\sin^2\theta}\partial_\phi^\alpha\partial_\phi^\beta. \qquad (4.4)\end{aligned}$$

4.2. **Energies in the reformulation.**

Lemma 4.2 (Dominant energy condition for $\tilde{\mathcal{T}}$)**.** *There are positive constant $\bar{a}$ and C, such that, if $|a| < \bar{a}$, and if $\mathbf{X}$ and $\mathbf{Y}$ are future-directed, non-spacelike vectors (with respect to ∂_t and g), then*

$$-\tilde{\mathcal{T}}^\alpha{}_\beta g_{\alpha\gamma}\mathbf{X}^\beta \mathbf{Y}^\gamma \geq 0 \tag{4.5}$$

In particular,

$$C\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi} \geq (\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2 + \tilde{V}_Q|\nabla\!\!\!/\tilde{\psi}|^2 + V|\tilde{\psi}|^2. \tag{4.6}$$

Proof. The energy-momentum tensor can be written as

$$\begin{aligned}-\tilde{\mathcal{T}}[\tilde{\psi}]^\alpha{}_\beta =& \tilde{\mathcal{G}}^{\alpha\gamma}(\partial_\gamma\tilde{\psi})(\partial_\beta\tilde{\psi}) - \frac{1}{2}\delta^\alpha{}_\beta \tilde{\mathcal{G}}^{\delta\epsilon}(\partial_\epsilon\tilde{\psi})(\partial_\delta\tilde{\psi}) - \frac{1}{2}\delta^\alpha{}_\beta V\tilde{\psi}^2. \qquad (4.7)\\ =& \frac{\Delta\Sigma}{(r^2+a^2)^2}\left(g^{\alpha\gamma}(\partial_\gamma\tilde{\psi})(\partial_\beta\tilde{\psi}) - \frac{1}{2}g^\alpha{}_\beta g^{\delta\epsilon}(\partial_\epsilon\tilde{\psi})(\partial_\delta\tilde{\psi})\right) - \frac{1}{2}g^\alpha{}_\beta V\tilde{\psi}^2\\ =& \frac{\Delta\Sigma}{(r^2+a^2)^2}T[\tilde{\psi}]^\alpha{}_\beta - \frac{1}{2}g^\alpha{}_\beta V\tilde{\psi}^2.\end{aligned}$$

Thus, the reformulated energy-momentum tensor can be written as the sum of two terms. The first satisfies the dominant energy condition because it is a positive multiple of the standard energy-momentum tensor evaluated on $\tilde{\psi}$. The second term is a negative factor multiplied by the metric, so it also satisfies the dominant energy condition. Thus, $\tilde{\mathcal{T}}[\tilde{\psi}]^\alpha{}_\beta g_{\alpha\gamma}$ is the sum of two terms which each satisfy the dominant energy condition, so it satisfies the dominant energy condition.

We now turn to the second part of the lemma. From equation (4.7),

$$\tilde{\mathcal{T}}^\alpha{}_\beta \mathbf{T}_\chi^\beta(\mathrm{d}t)_\alpha = -(\mathbf{T}_\chi\tilde{\psi})\tilde{\mathcal{G}}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\tilde{\mathcal{G}}^{\gamma\delta}(\partial_\gamma\tilde{\psi})(\partial_\delta\tilde{\psi}) + \frac{1}{2}V\tilde{\psi}^2$$

From the fact that $\mathbf{T}_\chi^\alpha\partial_\alpha = -\tilde{\mathcal{G}}^{t\alpha}\partial_\alpha$, $\mathbf{T}_\chi = \mathbf{T}_\perp + aO((\Delta/r^2), r^{-2})\partial_\phi$, and the expansion of $\tilde{\mathcal{G}}$ in (4.2) and (4.4),

$$\begin{aligned}\tilde{\mathcal{T}}^\alpha{}_\beta \mathbf{T}_\chi^\beta(\mathrm{d}t)_\alpha =& \tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})(\mathbf{T}_\perp\tilde{\psi}) + aO((\Delta/r^2), r^{-2})(\mathbf{T}_\perp\tilde{\psi})(\partial_\phi\tilde{\psi})\\ &+ \frac{1}{2}\left(-\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_*\tilde{\psi})^2 + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi})\right) + \frac{1}{2}V\tilde{\psi}^2.\end{aligned}$$

Since we assume that a is small, and since $O((\Delta/r^2)^2, r^{-4})$ decays more rapidly than the product of $\tilde{N}^{-2} \approx 1$ and $O((\Delta/r^2), r^{-2})$, which is the decay rate of the leading-order piece of $\tilde{h}$, the term $aO((\Delta/r^2), r^{-2})(\mathbf{T}_\perp\tilde{\psi})(\partial_\phi\tilde{\psi})$ is controlled by the $(1/2)(\mathbf{T}_\chi\tilde{\psi})^2 + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}))$ terms with only a small loss. This proves the desired result. ☐

Lemma 4.3 (Relation between energy for u and $\tilde{\psi}$)**.** *There are positive constants $\bar{a}$ and C, such that, if $|a| \leq \bar{a}$, and if $\tilde{\psi} = (r^2+a^2)^{1/2}\psi$, then*

(1) *(Global identity)*

$$E_{\mathbf{T}_\chi}[\psi](t) = \tilde{E}_{\mathbf{T}_\chi}[\tilde{\psi}](t). \tag{4.8}$$

(2) *(Local estimate) Given $r_1 < r_2$ and $r_{*1} < 0 < r_{*2}$ such that $r_* = r_{*1}$ corresponds to $r = r_1$ and $r_* = r_{*2}$ corresponds to $r = r_2$,*

$$\int_{r_1}^{r_2} \int_{S^2} |\mathcal{P}^t_{\mathbf{T}_\chi}[\psi]| + |\psi|^2 d^3\mu \lesssim \int_{r_{*1}}^{r_{*2}} \int_{S^2} \tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}[\tilde{\psi}] d^3\mu_*.$$

Proof. This argument follows by substituting $\tilde{\psi} = (r^2+a^2)^{-1/2}\psi$ into the energy $E_{\mathbf{T}_\chi}$, changing from (t, r, θ, ϕ) to (t, r_*, θ, ϕ) variables, and then integrating by parts in the r_* variable.

$$\begin{aligned}
&E_{\mathbf{T}_\chi}[\psi](t) \\
&= \int \mathcal{T}[\psi]^t{}_\beta \mathbf{T}_\chi^\beta \mathrm{d}^3\mu \\
&= \int -(\mathbf{T}_\chi\psi)\mathcal{G}^{t\gamma}(\partial_\gamma\psi) + \frac{1}{2}(1)\mathcal{G}^{\beta\gamma}(\partial_\beta\psi)(\partial_\gamma\psi)\mathrm{d}^3\mu \\
&= \int -(\mathbf{T}_\chi\tilde{\psi})\frac{1}{r^2+a^2}\mathcal{G}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\frac{1}{r^2+a^2}\mathcal{G}^{\beta\gamma}(\partial_\beta\tilde{\psi})(\partial_\gamma\tilde{\psi})\mathrm{d}^3\mu \\
&\quad + \int \frac{\Delta}{\sqrt{r^2+a^2}}\left(\partial_r \frac{1}{\sqrt{r^2+a^2}}\right)\tilde{\psi}(\partial_r\tilde{\psi}) + \frac{1}{2}\Delta\left(\partial_r\frac{1}{\sqrt{r^2+a^2}}\right)^2\tilde{\psi}^2\mathrm{d}^3\mu.
\end{aligned} \tag{4.9}$$

We can simplify the final term using integration by parts to obtain

$$\begin{aligned}
&\int \frac{\Delta}{\sqrt{r^2+a^2}}\left(\partial_r \frac{1}{\sqrt{r^2+a^2}}\right)\tilde{\psi}(\partial_r\tilde{\psi}) + \frac{1}{2}\Delta\left(\partial_r\frac{1}{\sqrt{r^2+a^2}}\right)^2\tilde{\psi}^2\mathrm{d}^3\mu \\
&= \int \frac{1}{2}\left(\partial_r\left(\frac{\Delta}{\sqrt{r^2+a^2}}\partial_r\frac{1}{\sqrt{r^2+a^2}}\right) + \Delta\left(\partial_r\frac{1}{\sqrt{r^2+a^2}}\right)^2\right)\tilde{\psi}^2\mathrm{d}^3\mu \\
&= \int \frac{1}{2}\frac{\Delta}{r^2+a^2}\left(\partial_r\left(\frac{\Delta}{\sqrt{r^2+a^2}}\partial_r\frac{1}{\sqrt{r^2+a^2}}\right) + \Delta\left(\partial_r\frac{1}{\sqrt{r^2+a^2}}\right)^2\right)\tilde{\psi}^2\mathrm{d}^3\mu \\
&= \int \frac{1}{2}V\tilde{\psi}^2\mathrm{d}^3\mu.
\end{aligned}$$

We were able to drop all boundary terms in the integration by parts, since the boundary terms involve bounded factors times Δ, which vanishes at r_+, or involve factors of $\tilde{\psi}$ or its derivative which are assumed to decay rapidly as $r \to \infty$ on surfaces of fixed t. (Merely decay such that $r^{-1/2}\tilde{\psi} \to 0$ or $r^{1/2}\psi \to 0$ is sufficient here.) Thus, we find

$$E_{\mathbf{T}_\chi}[\psi](t) = \tilde{E}_{\mathbf{T}_\chi}[\tilde{\psi}](t).$$

By a similar argument, using the Cauchy-Schwarz inequality on (4.9)

$$\begin{aligned}
&\int_{r_1}^{r_2}\int_{S^2} |\mathcal{P}_{\mathbf{T}_\chi}[\psi]^t| \mathrm{d}^3\mu \\
&\quad= \int_{r_1}^{r_2}\int_{S^2} -(\mathbf{T}_\chi\tilde{\psi})\frac{1}{r^2+a^2}\mathcal{G}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\frac{1}{r^2+a^2}\mathcal{G}^{\beta\gamma}(\partial_\beta\tilde{\psi})(\partial_\gamma\tilde{\psi})\mathrm{d}^3\mu \\
&\qquad+ \int_{r_1}^{r_2}\int_{S^2} \frac{\Delta}{(r^2+a^2)^2}|\tilde{\psi}||\partial_r\tilde{\psi}| + \frac{1}{2}\frac{\Delta}{(r^2+a^2)^3}|\tilde{\psi}|^2\mathrm{d}^3\mu \\
&\quad\lesssim \int_{r_1}^{r_2}\int_{S^2} -(\mathbf{T}_\chi\tilde{\psi})\frac{1}{r^2+a^2}\mathcal{G}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\frac{1}{r^2+a^2}\mathcal{G}^{\beta\gamma}(\partial_\beta\tilde{\psi})(\partial_\gamma\tilde{\psi})\mathrm{d}^3\mu \\
&\qquad+ \int_{r_1}^{r_2}\int_{S^2} \frac{\Delta}{(r^2+a^2)^3}|\tilde{\psi}^2|\mathrm{d}^3\mu \\
&\quad\lesssim \int_{r_{*1}}^{r_{*2}}\int_{S^2} -(\mathbf{T}_\chi\tilde{\psi})\tilde{\mathcal{G}}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\tilde{\mathcal{G}}^{\beta\gamma}(\partial_\beta\tilde{\psi})(\partial_\gamma\tilde{\psi})\mathrm{d}^3\mu_* \\
&\qquad+ \int_{r_{*1}}^{r_{*2}}\int_{S^2} \frac{\Delta^2}{(r^2+a^2)^4}|\tilde{\psi}^2|\mathrm{d}^3\mu_*.
\end{aligned}$$

As a consequence of [7, equation (36)], there is the following Hardy estimate for any non-negative, continuous weight χ which is positive at zero

$$\int_{r_{*1}}^{r_{*2}} \frac{1}{1+x^2}|\phi|^2\mathrm{d}x \lesssim \int_{r_{*1}}^{r_{*2}} |\partial_x\phi|^2 + \chi|\phi|^2\mathrm{d}x, \tag{4.10}$$

with the implicit constant in $\lesssim$ depending on χ, but not on r_{*1} and r_{*2}, as long as $r_{*1} < -1$ and $1 < r_{*2}$. Since $\tilde{\mathcal{G}}^{**} = 1$, we can apply this result to the energy for $\tilde{\psi}$, so that

$$\begin{aligned}
&\int_{r_{*1}}^{r_{*2}}\int_{S^2} \tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^3\mu_* \\
&\quad\gtrsim \int_{r_{*1}}^{r_{*2}}\int_{S^2} -(\mathbf{T}_\chi\tilde{\psi})\tilde{\mathcal{G}}^{t\gamma}(\partial_\gamma\tilde{\psi}) + \frac{1}{2}\tilde{\mathcal{G}}^{\beta\gamma}(\partial_\beta\tilde{\psi})(\partial_\gamma\tilde{\psi})\mathrm{d}^3\mu_* \\
&\qquad+ \int_{r_{*1}}^{r_{*2}}\int_{S^2} \frac{1}{1+r_*^2}|\tilde{\psi}|^2\mathrm{d}^3\mu_*.
\end{aligned}$$

Since $\Delta^2(r^2+a^2)^{-4} \lesssim (1+r_*^2)^{-1}$,

$$\int_{r_1}^{r_2}\int_{S^2} |\mathcal{P}_{\mathbf{T}_\chi}[\psi]^t|\mathrm{d}^3\mu \lesssim \int_{r_{*1}}^{r_{*2}}\int_{S^2} \tilde{\mathcal{P}}_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^3\mu_*.$$

By the same argument

$$\begin{aligned}
\int_{r_1}^{r_2}\int_{S^2} |\psi|^2\mathrm{d}^3\mu &= \int_{r_{*1}}^{r_{*2}}\int_{S^2} \frac{\Delta}{(r^2+a^2)^2}|\tilde{\psi}|^2\mathrm{d}^3\mu_* \\
&\lesssim \int_{r_{*1}}^{r_{*2}}\int_{S^2} \tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^3\mu_*.
\end{aligned}$$

□

Corollary 4.4 (Local decay for $\tilde{\psi}$)**.** *If ψ is a solution to the wave equation $\Box\psi = 0$, then the right-hand side of* (3.30) *is greater than*

$$\int \left(\frac{\Delta}{(r^2+a^2)}\frac{1}{r^2}|\partial_{r_*}\tilde{\psi}|_2^2 + \frac{\Delta}{r^2+a^2}\frac{1}{r^4}|\tilde{\psi}|_2^2 + \frac{\Delta}{r^2+a^2}\frac{\mathbb{1}_{r\not\approx 3M}}{r^3}(|\partial_t\tilde{\psi}|_2^2 + |\nabla\!\!\!/\tilde{\psi}|_2^2) \right) d^4\mu_*. \tag{4.11}$$

Proof. The right-hand side of (3.30) is

$$\int_{T_1}^{T_2}\int_{r_+}^{\infty}\int_{S^2}\left(\left(\frac{\Delta^2}{r^4}\right)|\partial_r\psi|_2^2+\frac{1}{r^2}|\psi|_2^2+\frac{\mathbb{1}_{r\not\approx 3M}}{r}\left(|\partial_t\psi|_2^2+|\nabla\!\!\!/\psi|_2^2\right)\right)\mathrm{d}^4\mu.$$

Making the substitution $\psi=\tilde{\psi}(r^2+a^2)^{-1/2}$ introduces an extra factor of $(r^2+a^2)^{-1}$, except in the factor with the radial derivatives, where the situation is a little more complicated. The norm $|u|_2$ only introduces derivatives which commute with the radial derivative, so we may ignore them. We will use v to denote $\tilde{\psi}$ with possibly a derivative operator acting on it.

The sum of the radial derivative and lower-order terms has the form

$$\begin{aligned}\int\frac{\Delta^2}{r^4}\left|\partial_r\frac{v}{\sqrt{r^2+a^2}}\right|^2+\frac{1}{r^2}\frac{1}{r^2+a^2}|v|^2\mathrm{d}r\\ =\int\frac{\Delta^2}{r^4(r^2+a^2)}\left|\partial_r v\right|^2-2\frac{r\Delta^2}{r^4(r^2+a^2)^2}|\partial_r v||v|\\ +\left(\frac{\Delta^2}{(r^2+a^2)^2}\frac{r^2}{(r^2+a^2)^3}+\frac{1}{r^2}\frac{1}{r^2+a^2}\right)|v|^2\mathrm{d}r.\end{aligned}$$

Applying Cauchy-Schwarz to the term involving $v\partial_r v$ and using the extra positivity from the original lower-order term, we have that

$$\begin{aligned}\int\frac{\Delta^2}{r^4}\left|\partial_r\frac{v}{\sqrt{r^2+a^2}}\right|^2+\frac{1}{r^2}\frac{1}{r^2+a^2}|v|^2\mathrm{d}r&\gtrsim\int\frac{\Delta^2}{(r^2+a^2)^3}\left|\partial_r v\right|^2+\frac{1}{r^4}|v|^2\mathrm{d}r\\&\gtrsim\int\frac{1}{r^2+a^2}\left|\partial_{r_*}v\right|^2+\frac{1}{r^4}|v|^2\mathrm{d}r.\end{aligned}$$

Making the change of variables from r to r_*, introduces an extra factor of $\Delta/(r^2+a^2)$ in the measure. Thus, the right-hand side of (3.30)dominates

$$\int\left(\frac{\Delta}{(r^2+a^2)^2}|\partial_{r_*}\tilde{\psi}|_2^2+\frac{\Delta}{r^6}|\tilde{\psi}|_2^2+\mathbb{1}_{r\not\approx 3M}\frac{\Delta}{r^5}|\tilde{\psi}|_3^2\right)\mathrm{d}^4\mu_*.$$

These coefficients dominate the coefficients given in the statement of this theorem are dominated, so the desired result holds. □

Because we will need to work with this later to integrate by parts in the angular variables, we introduce the following Morawetz density with additional angular derivatives. Note that since the angular derivatives do not commute with $\mathbb{S}_2$, this is not quite the same as the Morawetz density evaluated on angular derivatives of u.

Definition 4.5. *We define the higher-order angular Morawetz densities to*

$$\begin{aligned}(\delta\mathcal{P}_A)[\mathbb{O}_1^n,\psi]=&\frac{\Delta}{(r^2+a^2)}\frac{1}{r^2}|\mathbb{O}_1^n\partial_r\mathbb{S}_2\tilde{\psi}|^2\\&+\frac{\Delta}{(r^2+a^2)}\frac{1}{r^3}(|\mathbb{O}_1^n\partial_t\mathbb{S}_2\tilde{\psi}|^2+|\mathbb{O}_1^n\nabla\!\!\!/\mathbb{S}_2\tilde{\psi}|^2)\\&+\frac{\Delta}{(r^2+a^2)}\frac{1}{r^4}|\mathbb{O}_1^n\mathbb{S}_2\tilde{\psi}|^2.\end{aligned}$$

(Recall $\mathbb{O}_1=\{\Theta_i\}$ was defined to be the set of rotations about the coordinate axes in section 2.4.)

4.3. **K in Kerr.** We now define the **K** vector field in analogy with the corresponding vector field in the Schwarzschild or Minkowski spacestimes. It is well known that this type of vector field generates a stronger energy which can be used to prove decay.

Definition 4.6. *We define*

$$\begin{aligned}\mathbf{K} =&(t^2+r_*^2+1)\mathbf{T}_\chi + 2tr_*\tilde{N}^2\partial_{r_*},\\ q_{\mathbf{K}} =&t(\tilde{N}^2-1),\end{aligned}$$

where $\tilde{N}$ *is defined in* (4.3)*.*

We now prove $\mathbf{K}$ is timelike. The essence of this argument is the same as in the Minkowski or Schwarzschild spacetime. The length of $\mathbf{T}_\chi$ is very close to that of ∂_*, so $(t^2+r_*^2)\mathbf{T}_\chi + 2tr\partial_{r_*}$ is timelike when $t^2+r_*^2$ dominates $2tr_*$. When $t^2+r_*^2$ is comparable to $2tr_*$, $(t^2+r_*^2)\mathbf{T}_\chi+2tr_*\partial_{r_*}$ can become null or even slightly spacelike. By adding a little more $\mathbf{T}_\chi$, we have built the always timelike vector-field $\mathbf{K} = (t^2+r_*^2+1)\mathbf{T}_\chi + 2tr_*\partial_{r_*}$. This trick of adding a little more of the timelike vector-field to get a globally timelike $\mathbf{K}$ is common in the literature for the Minkowski or Schwarschild spacetime.

Lemma 4.7. *There is a positive constant* $\bar{a}$*, such that if* $|a|<\bar{a}$*, then* $\mathbf{K}$ *is timelike with respect to* g*.*

Proof. Since $\mathbf{T}_\chi$ is timelike and orthogonal to ∂_{r_*}, it is sufficient to show that the absolute value of the norm of $(t^2+r_*^2+1)\mathbf{T}_\chi$ dominates the norm of $2tr_*\partial_{r_*}$.

For (t,r_*) close to $(0,0)$, the term $\mathbf{T}_\chi$ dominates the contribution from all the other pieces of $\mathbf{K}$, so that $\mathbf{K}$ is timelike in this region. We can therefore assume that at least one of t or $|r_*|$ is bigger than some fixed constant in the rest of this proof. By taking a sufficiently small, we may assume χ is constant for r_* bigger than this constant. (Alternatively, we could take $\mathbf{K} = (t^2+r_*^2+C)\mathbf{T}_\chi + 2tr\tilde{N}^2\partial_r$ with C sufficiently large, so that this step in the argument would not impose any further restriction on the size of a.)

To prove the lemma, we break the spacetime into three different regions. In the limit $a\to 0$, the surfaces $t=r_*$ describe light cones. For $a\neq 0$, the surfaces $t=r_*$ are no longer light cones, although give a rough approximation of the location of the light cones. In particular, for a small $\epsilon>0$, if a is sufficiently small, the regions $|r_*|<(1-\epsilon)t$ remain inside the light cone.

Before considering different regions, we note that

$$\begin{aligned}-g(\mathbf{T}_\perp,\mathbf{T}_\perp) =&\frac{\Delta\Sigma}{(r^2+a^2)^2}\tilde{N}^{-2} = \frac{\Delta\Sigma}{(r^2+a^2)^2}\left(1-a^2\sin^2\theta\frac{\Delta}{(r^2+a^2)^2}\right)\\ -g(\partial_t,\partial_t) =&1-\frac{2Mr}{\Sigma}\\ g(\partial_{r_*},\partial_{r_*}) =&\frac{\Sigma}{\Delta}\left(\frac{\Delta}{r^2+a^2}\right)^2 = \frac{\Delta\Sigma}{(r^2+a^2)^2}.\end{aligned}$$

First, we consider the region $r_*\le-(1-\epsilon)t$. In this region, the difference in norm between $\mathbf{T}_\chi$ and $\mathbf{T}_\perp$ is $Ca\Delta^2$, so the absolute value of the norm of $\mathbf{T}_\chi$ is bounded below by $\Delta(1-Ca^2\Delta)$, and the norm of ∂_{r_*} is bounded above by $\Delta(1+Ca^2\Delta)$. The coefficient $(t^2+r_*^2+1)^2$ dominates $(2tr_*)^2+t^2$. For a sufficiently small

$$\begin{aligned}|(t^2+r_*^2+1)^2 g(\mathbf{T}_\chi,\mathbf{T}_\chi)| \ge&(t^2+r_*^2+1)^2\Delta(1-Ca^2\Delta)\\ \ge&((2tr_*)^2+t^2)\Delta(1-Ca^2\Delta)\\ \ge&(2tr_*)^2\Delta+(1/2)t^2\Delta\\ \ge&(2tr_*)^2\Delta(1+Ca^2\Delta)\\ \ge&(2tr_*)^2|g(\partial_{r_*},\partial_{r_*})|^2.\end{aligned}$$

Second, we consider the region $-(1-\epsilon)t\le r_*\le(1-\epsilon)t$. From the estimates on the norms of $\mathbf{T}_\perp$ and ∂_t, it follows that, uniformly in the region currently under consideration,

$$-g(\mathbf{T}_\chi, \mathbf{T}_\chi) \geq \frac{\Delta\Sigma}{(r^2+a^2)^2}\left(1 - Ca^2\frac{\Delta}{(r^2+a^2)^2}\right).$$

In the region under consideration, we also have

$$\begin{aligned}
|t \pm r_*| \geq& \epsilon t \\
t^2 + r_*^2 \geq& |2tr_*| + \epsilon^2 t^2 \\
(t^2+r_*^2)^2 \geq& (2tr_*)^2 + \epsilon^4 t^4 \\
\geq& (1+\epsilon^4/8)(2tr_*)^2 + (\epsilon^4/2)t^4.
\end{aligned}$$

Thus, for sufficiently small a based on ϵ

$$\begin{aligned}
|(t^2+r_*^2+1)^2 g(\mathbf{T}_\chi,\mathbf{T}_\chi)| \geq& (1+\epsilon^4/8)(2tr_*)^2 g(\partial_{r_*},\partial_{r_*}) \\
&+ (\epsilon^4/2)t^4|g(\mathbf{T}_\chi,\mathbf{T}_\chi)|.
\end{aligned}$$

Thus, $\mathbf{K}$ is timelike.

Finally, we turn to the region $(1-\epsilon)t \leq r_*$. (We may also assume we're sufficiently far from $(t, r_*) = (0, 0)$ that $\mathbf{T}_\chi = \partial_t$.) Here, $-g(\mathbf{T}_\chi,\mathbf{T}_\chi) = -g(\partial_t,\partial_t) = 1 - 2Mr/\Sigma \geq (1-2Mr^{-1})(1-Ca^2r^{-2})$, and $g(\partial_{r_*},\partial_{r_*}) = \Delta\Sigma(r^2+a^2)^{-2} \leq (1 - 2Mr^{-1})(1+Ca^2r^{-2})$. Following the same type of argument as we used in the first region, we have

$$\begin{aligned}
&(t^2+r_*^2+1)^2|g(\mathbf{T}_\chi,\mathbf{T}_\chi)| \\
&\quad\geq(1-2Mr^{-1})\left((t^2+r_*^2)^2(1-Ca^2r^{-2}) + (t^2+r_*^2)(1-Ca^2r^{-2})\right) \\
&\quad\geq(1-2Mr^{-1})\left((2tr_*)^2 + t^2(1-Ca^2)\right) \\
&\quad\geq(1-2Mr^{-1})(2tr_*)^2(1+Ca^2r^{-2}) \\
&\quad\geq(2tr_*)^2 g(\partial_{r_*},\partial_{r_*}).
\end{aligned}$$

□

We now show that the $\mathbf{K}$ momentum dominates the $\mathbf{T}_\chi$ momentum and that, on hypersurfaces of constant t, the energy assciated with $q_{\mathbf{K}}$ is small relative to the $\mathbf{K}$ energy, so that we can freely add or drop the energy associated with $q_{\mathbf{K}}$ when we find it convenient to do so.

Lemma 4.8 ($\mathbf{K}$ energy)**.** *There are positive constants $\bar{a}$, C_{KE}, and $C_{KE,2}$, such that if $|a| \leq \bar{a}$, and if ψ is smooth, then :*

(1) *(estimate on the energy)*

$$\begin{aligned}
C_{KE}\tilde{\mathcal{P}}^t_{\mathbf{K}} \geq& \left|(t+r_*)(\mathbf{T}_\perp + \partial_{r_*})\tilde{\psi}\right|^2 + \left|(t-r_*)(\mathbf{T}_\perp - \partial_{r_*})\tilde{\psi}\right|^2 \\
&+ (t^2+r_*^2+1)(\tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2).
\end{aligned}$$

As a consequence, for $|r_*|^2 \leq (3/4)t$,

$$C_{KE}\tilde{\mathcal{P}}^t_{\mathbf{K}} \geq t^2\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}.$$

(2) *The contribution to the energy from $q_{\mathbf{K}}$ is small in the sense that*

$$|\tilde{E}_{q_{\mathbf{K}}}(t)| \leq C_{KE}\int_{\Sigma_t} |\tilde{\mathcal{G}}^{t\beta}q_{\mathbf{K}}(\partial_\beta\tilde{\psi})\tilde{\psi} - \frac{1}{2}\tilde{\mathcal{G}}^{t\beta}(\partial_\beta q_{\mathbf{K}})\tilde{\psi}^2| d^3\mu_* \leq a^2 C_{KE,2}\tilde{E}_{\mathbf{K}}(t).$$

Proof. This follows the same ideas as the previous lemma. We start by computing the energy,

$$\begin{aligned}
\tilde{\mathcal{P}}^t_{\mathbf{K}} =& -\tilde{\mathcal{G}}^{t\alpha}(\partial_\alpha\tilde{\psi})\mathbf{K}^\beta(\partial_\beta) + \frac{1}{2}\mathbf{K}^t\tilde{\mathcal{G}}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) \\
=&\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})(t^2+r_*^2+1)(\mathbf{T}_\perp\tilde{\psi}) + (\mathbf{T}_\perp\tilde{\psi})2tr_*\tilde{N}^2(\partial_{r_*}\tilde{N}) \\
&-\frac{1}{2}(t^2+r_*^2+1)(\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})^2 - (\partial_{r_*}\tilde{\psi})^2 - \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) - V\tilde{\psi}^2) \\
=&\frac{1}{2}(t^2+r_*^2+1)(\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2 + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2) \qquad (4.12a) \\
&+(t^2+r_*^2+1)\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})(\omega_H\chi - \omega_\perp)(\partial_\phi\tilde{\psi}) \qquad (4.12b) \\
&+2tr_*\tilde{N}^2(\mathbf{T}_\perp\tilde{\psi})(\partial_{r_*}\tilde{\psi}). \qquad (4.12c)
\end{aligned}$$

Since $\omega_H\chi - \omega_\perp \lesssim a\Delta/(r^2+a^2)^2$, the term (4.12b) is dominated by a small multiple of (4.12a). In fact, because of the weights, we are left with

$$\frac{1}{2}(t^2+r_*^2+\frac{1}{2})(\tilde{N}^{-2}(\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2 + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2)$$

from dominating (4.12a) and (4.12b) together. We now seek to control (4.12c) using this remaining term.

First we consider the region $||r_*| - t| < t/2$. In this region, $|tr_*(\tilde{N}^2-1)| \lesssim a$ and $|(t^2+r_*^2+1)(\tilde{N}^2-1)| \lesssim a$, so that the term $(\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2$ dominates any terms arising from replacing all the factors of $\tilde{N}$ by 1. Thus, in this region,

$$\begin{aligned}
\tilde{\mathcal{P}}^t_{\mathbf{K}} \geq& \frac{1}{2}\left(t^2+r_*^2+\frac{1}{2}\right)\left((\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2 + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2\right) \\
&+2tr_*(\mathbf{T}_\perp\tilde{\psi})(\partial_{r_*}\tilde{\psi}) \\
\gtrsim& \left|(t+r_*)(\mathbf{T}_\perp + \partial_{r_*})\tilde{\psi}\right|^2 + \left|(t-r_*)(\mathbf{T}_\perp - \partial_{r_*})\tilde{\psi}\right|^2 \\
&+(t^2+r_*^2+1)(\tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2).
\end{aligned}$$

In the region $||r_*| - t| \geq t/2$, the dominant term $(t^2+r_*^2)((\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2)$ dominates $(t^2+r_*^2)(\tilde{N}^{-2}-1)((\mathbf{T}_\perp\tilde{\psi})^2 + (\partial_{r_*}\tilde{\psi})^2)$ and the terms in (4.12b) and (4.12c) by at least a factor of 2, so that the desired estimate holds. The control over $\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}$ follows from this.

We now turn to the terms arising from $q_{\mathbf{K}}$. These terms are

$$\begin{aligned}
&|\tilde{\mathcal{G}}^{t\beta}q_{\mathbf{K}}(\partial_\beta\tilde{\psi})\tilde{\psi} - \frac{1}{2}\tilde{\mathcal{G}}^{t\beta}(\partial_\beta q_{\mathbf{K}})\tilde{\psi}^2| \\
&\qquad\lesssim t|\tilde{N}^2-1||\mathbf{T}_\perp\tilde{\psi}||\tilde{\psi}| + \frac{1}{2}|\tilde{N}^2-1|\tilde{\psi}^2 \\
&\qquad\lesssim a^2t^2\frac{\Delta^2}{(r^2+a^2)^4}\tilde{\psi}^2 + a^2(\mathbf{T}_\perp\tilde{\psi})^2 + \frac{a^2\Delta}{(r^2+a^2)^2}\tilde{\psi}^2.
\end{aligned}$$

The first term is controlled by $\tilde{\mathcal{P}}^t_{\mathbf{K}}$, and the second, by $\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}$. The decay of the final term is too slow to be directly estimated by the energy, but after integrating along a hypersurface of constant t and applying the Hardy estimate (4.10), the third term can be estimated by $\tilde{E}_{\mathbf{T}_\chi}$. Thus,

$$\int_{\Sigma_t}|\tilde{\mathcal{G}}^{t\beta}q_{\mathbf{K}}(\partial_\beta\tilde{\psi})\tilde{\psi} - \frac{1}{2}\tilde{\mathcal{G}}^{t\beta}(\partial_\beta q_{\mathbf{K}})\tilde{\psi}^2|\mathrm{d}^3\mu_* \lesssim a^2\tilde{E}^t_{\mathbf{K}}.$$

□

Lemma 4.9 (Bulk term for $\mathbf{K}$). *If $\tilde\psi$ is a solution to the wave equation* (4.1) *and* $\psi = \sqrt{r^2+a^2}\tilde\psi$*, then*

$$\frac{1}{\mu}\partial_\alpha\left(\mu\tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_\mathbf{K}),3}\right) \lesssim \begin{cases} t^{-2}\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi,3} & \text{for } r_* \leq -t/2 \\ (t^{-2}\log t)\tilde{\mathcal{P}}^t_{\mathbf{K},3} & \text{for } r_* \geq t/2 \end{cases},$$

and, in the region $-t/2 \leq r_* \leq t/2$,

$$\begin{aligned}\frac{1}{\mu}\partial_\alpha\left(\mu\tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_\mathbf{K}),3}\right) \lesssim & t(\delta\mathcal{P}_A)[\psi] \\ &+ t(\delta\mathcal{P}_A)[\mathbb{O}^1_1,\psi] \\ &+ |a|t^2(\delta\mathcal{P}_A)[\psi] \\ &+ |a|t^{-1}\tilde{\mathcal{P}}^t_{\mathbf{K},3}.\end{aligned}$$

Proof. We begin by considering

$$\begin{aligned}\mathbf{K} =&(t^2+r_*^2+1)T_\mathbf{K} + 2tr_*\tilde{N}^2\partial_{r_*},\\ T_\mathbf{K} =&\partial_t + \omega_\mathbf{K}\partial_\phi,\end{aligned}$$

and derive conditions on $\omega_\mathbf{K}$. Near the end of this proof we will conclude that $T_\mathbf{K} = \mathbf{T}_\chi$ is permissible. For simiplicity, for most of this proof, we use $\tilde v$ to denote a second-order symmetry operator acting on $\tilde\psi$, and consider the deformation of $\tilde{\mathcal{P}}_{(\mathbf{K},q_\mathbf{K})}[\tilde v]$. Similarly, unless otherwise specified, $\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}$ and $\tilde{\mathcal{P}}^t_\mathbf{K}$ are understood to be evaluated on $\tilde v$.

The bulk term we want to control is

$$\frac{1}{\mu}\partial_\alpha\mu\mathcal{P}^\alpha_{(\mathbf{K},q_\mathbf{K})} = \left(-\tilde{\mathcal{G}}^{\alpha\gamma}\partial_\gamma\mathbf{K}^\beta + \frac12(\partial_\gamma\mathbf{K}^\gamma)\tilde{\mathcal{G}}^{\alpha\beta} + \frac12\mathbf{K}^\gamma(\partial_\gamma\tilde{\mathcal{G}}^{\alpha\beta})\right)(\partial_\alpha\tilde v)(\partial_\beta\tilde v) \tag{4.13a}$$

$$- q_\mathbf{K}\tilde{\mathcal{G}}^{\alpha\beta}(\partial_\alpha\tilde v)(\partial_\beta\tilde v) \tag{4.13b}$$

$$+\left(\frac12(\partial_\gamma\mathbf{K}^\gamma)V + \frac12\mathbf{K}^\gamma(\partial_\gamma V) - q_\mathbf{K}V\right)\tilde v^2, \tag{4.13c}$$

$$+\frac12(\partial_\beta\tilde{\mathcal{G}}^{\alpha\beta}\partial_\alpha q_\mathbf{K})\tilde v^2. \tag{4.13d}$$

With an eye to computing the derivative terms, we first compute

$$\begin{aligned}\partial_\gamma\mathbf{K}^\gamma =&2t + 2t\tilde N^2 + 2tr_*(\partial_{r_*}\tilde N^2)\\ =&4t\tilde N^2 + 2t(1-\tilde N^2 + r_*\partial_{r_*}\tilde N^2).\end{aligned}$$

The terms involving the derivatives and arising from the vector field are

$$\begin{aligned}&(\partial_\alpha\tilde v)(\partial_\beta\tilde v)\left(-\tilde{\mathcal{G}}^{\alpha\gamma}\partial_\gamma\mathbf{K}^\beta + \frac12(\partial_\gamma\mathbf{K}^\gamma)\tilde{\mathcal{G}}^{\alpha\beta} + \frac12\mathbf{K}^\gamma(\partial_\gamma\tilde{\mathcal{G}}^{\alpha\beta})\right)\\
&=\tilde N^{-2}(\mathbf{T}_\perp\tilde v)2t(T_\mathbf{K}\tilde v) + (\mathbf{T}_\perp\tilde v)2r_*(\partial_*\tilde v)\\
&\quad-(\partial_*\tilde v)2r_*(T_\mathbf{K}\tilde v) - (\partial_*\tilde v)(t^2+r_*^2)(\partial_{r_*}\omega_\mathbf{K})(\partial_\phi\tilde v)\\
&\quad-(\partial_*\tilde v)2t\tilde N^2(\partial_*\tilde v) - (\partial_*\tilde v)^2 2tr_*(\partial_{r_*}\tilde N^2)\\
&\quad-\tilde h^{\alpha\beta}(\partial_\alpha\tilde v)(t^2+r_*^2+1)(\partial_\beta\omega_\mathbf{K})(\partial_\phi\tilde v) - \tilde h^{\alpha\beta}2tr_*(\partial_\beta\tilde N^2)(\partial_{r_*}\tilde v)\\
&\quad-(2t\tilde N^2 + t(1-\tilde N^2 + r_*\partial_{r_*}\tilde N^2))\tilde N^{-2}(\mathbf{T}_\perp\tilde v)^2\\
&\quad+(2t\tilde N^2 + t(1-\tilde N^2 + r_*\partial_{r_*}\tilde N^2))(\partial_*\tilde v)^2\\
&\quad+(2t\tilde N^2 + t(1-\tilde N^2 + r_*\partial_{r_*}\tilde N^2))\tilde h^{\alpha\beta}(\partial_\alpha\tilde v)(\partial_\beta\tilde v)\\
&\quad-tr_*(\partial_{r_*}\tilde N^{-2})(\mathbf{T}_\perp\tilde v)^2\\
&\quad+tr_*(\partial_{r_*}\tilde h^{\alpha\beta})(\partial_\alpha\tilde v)(\partial_\beta\tilde v).\end{aligned}$$

Expanding $T_{\mathbf{K}}$ as $\mathbf{T}_\perp + aO(\Delta, r^{-3})$ and grouping terms, we find

$$
\begin{aligned}
&(\partial_\alpha\tilde{v})(\partial_\beta\tilde{v})\left(-\tilde{\mathcal{G}}^{\alpha\gamma}\partial_\gamma\mathbf{K}^\beta + \frac{1}{2}(\partial_\gamma\mathbf{K}^\gamma)\tilde{\mathcal{G}}^{\alpha\beta} + \frac{1}{2}\mathbf{K}^\gamma(\partial_\gamma\tilde{\mathcal{G}}^{\alpha\beta})\right)\\
&\quad= -(\mathbf{T}_\perp\tilde{v})^2 t(2(1-\tilde{N}^{-2}) + \tilde{N}^{-2}(1-\tilde{N}^2 + r_*\partial_{r_*}\tilde{N}^2) + r_*\partial_{r_*}\tilde{N}^{-2}) && \text{(4.14a)}\\
&\qquad + (\mathbf{T}_\perp\tilde{v})(\partial_\phi\tilde{v})(2t)(\omega_{\mathbf{K}} - \omega_\perp) && \text{(4.14b)}\\
&\qquad - (\partial_*\tilde{v})^2 t(2r_*(\partial_{r_*}\tilde{N}^{-2}) - (1-\tilde{N}^2 + r_*\partial_{r_*}\tilde{N}^2)) && \text{(4.14c)}\\
&\qquad - (\partial_*\tilde{v})(\partial_\phi\tilde{v})((t^2+r_*^2+1)(\partial_{r_*}\omega_{\mathbf{K}}) + 2r_*(\omega_{\mathbf{K}} - \omega_\perp)) && \text{(4.14d)}\\
&\qquad - (\partial_*\tilde{v})(\partial_\beta\tilde{v})\tilde{h}^{\alpha\beta}2tr_*(\partial_\beta\tilde{N}^2) && \text{(4.14e)}\\
&\qquad - (\partial_\alpha\tilde{v})(\partial_\beta\tilde{v})t(-2\tilde{N}^2\tilde{h}^{\alpha\beta} - r_*(\partial_{r_*}\tilde{h}^{\alpha\beta}) - (1-\tilde{N}^2 + r_*\partial_{r_*}\tilde{N}^2)\tilde{h}^{\alpha\beta}) && \text{(4.14f)}\\
&\qquad - (\partial_\alpha\tilde{v})(\partial_\phi\tilde{v})\tilde{h}^{\alpha\beta}(t^2+r_*^2)(\partial_\beta\omega_{\mathbf{K}}). && \text{(4.14g)}
\end{aligned}
$$

In considering these calculations, it is important to note that the worst terms, involving $(\mathbf{T}_\perp\tilde{v})(\partial_*\tilde{v})$ are exactly cancelled. To force this cancellation, we have included $\tilde{N}^2$ in $\mathbf{K} = (t^2+r_*^2+1)\mathbf{T}_\chi + 2\tilde{N}^2 tr_*\partial_{r_*}$, instead of using $(t^2+r_*^2+1)\partial_t + 2tr_*\partial_{r_*}$, which is a more straight-forward analogue of the vector field used in the Minkowski and Schwarzschild spacetimes. We now analyse the remaining terms.

For the terms (4.14a), (4.14c), and (4.14f), which we consider to be like derivatives squared times t times coefficients, we look at just the coefficients. We also include the contribution from $q_{\mathbf{K}}\tilde{\mathcal{G}}^{\alpha\beta}$. The coefficients of $(\mathbf{T}_\perp\tilde{\psi})^2$ and $(\partial_{r_*}\tilde{\psi})^2$ are

$$
\begin{aligned}
\text{Coefficient from (4.14a)} &= -\tilde{N}^{-2}(\tilde{N}^2 - 1 + r_*\partial_{r_*}\tilde{N}^2) - \tilde{N}^2 r_*\partial_{r_*}\tilde{N}^{-2} + q_{\mathbf{K}}\tilde{N}^{-2}\\
&= -\tilde{N}^{-2}(\tilde{N}^2 - 1 + r_*\partial_{r_*}\tilde{N}^2) + \tilde{N}^{-2} r_*\partial_{r_*}\tilde{N}^{2} + q_{\mathbf{K}}\tilde{N}^{-2}\\
&= -\tilde{N}^{-2}(\tilde{N}^2 - 1 - (q_{\mathbf{K}}/t))\\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\text{Coefficient from (4.14c)} &= -2r_*\partial_{r_*}\tilde{N}^2 + (1-\tilde{N}^2 + r_*\partial_{r_*}\tilde{N}^2) - (q_{\mathbf{K}}/t)\\
&= -\tilde{N}^2 + 1 - r_*\partial_{r_*}\tilde{N}^2 - (q_{\mathbf{K}}/t)\\
&= -2(\tilde{N}^2 - 1) + \tilde{N}^4 r_*\partial_{r_*}\tilde{N}^{-2}\\
&= a^2\sin^2\theta O(r_*\Delta, r^{-3}\log r).
\end{aligned}
$$

Thus, for $r_* < -t/2$, with the $t(\partial_{r_*}\tilde{v})^2$ factor, the term is of the form $a^2tr_*\Delta(\partial_{r_*}\tilde{v})^2 \lesssim a^2t^{-2}(\partial_{r_*}\tilde{v})^2 \lesssim t^{-2}\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}$. For $-t/2 \le r_* \le t/2$, the term is bounded by $a^2t^{-1}\tilde{\mathcal{P}}^t_{\mathbf{K}}$. For $r_* \le t/2$, the term is bounded by $a^2(t^{-2}\log t)\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi} \lesssim a^2(t^{-2}\log t)\tilde{\mathcal{P}}^t_{\mathbf{K}}$.

The angular derivatives are given by

$$
\begin{aligned}
\text{Coefficient from (4.14f)} =& 2\tilde{h}^{\alpha\beta} + r_*(\partial_{r_*}\tilde{h}^{\alpha\beta})\\
&+ (2(\tilde{N}^2-1) + (1-\tilde{N}^2) - r_*(\partial_{r_*}\tilde{N}^2) - (q_{\mathbf{K}}/t))\tilde{h}^{\alpha\beta}.
\end{aligned}
$$

the first two terms of the coefficient of (4.14f) are of a familiar form from the Schwarzschild spacetime. Outside a compact set near the orbiting null-geodesics, they have a good sign, i.e. are negative, and decay like Δ near the event horizon and not slower than $r^{-3}\log r$ as $r \to \infty$. The remaining terms involve only factors involving either $(\tilde{N}^2 - 1)$ or $r_*\partial_{r_*}\tilde{N}^{-2}$ multiplied by the potential $\tilde{V}_Q$ in $\tilde{h}$. Thus, all the terms should decay at least like $\log(\Delta)\Delta^2$ near the horizon and at least as fast as r^{-4} as $r \to \infty$. In addition, these all have a factor of a on them. Thus, outside a slightly larger compact set, the negativity of the first two terms is sufficient to control the rest. Inside that compact set, the sum can be controlled by

$$(\delta\mathcal{P}_A)[\mathbb{O}^1_1,\psi].$$

We now turn to showing the remaining terms are small. The terms we wish to control all involve mixed derivatives. We first estimate these away from photon orbits. We break the remainder of the spacetime into the three regions I, IIa, and III, where $r_* < -t/2$, where $|r_*| \leq t/2$ but $|r_*| > C$ to avoid the photon sphere, and where $|r_*| > t/2$ respectively. Region IIb will refer to the region $|r_*| \leq C$. To estimate the remaining terms, we will apply the Cauchy-Schwarz inequality and then use the following estimates in the regions I, IIa, and III,

$$\begin{aligned}
\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi} \gtrsim& (\mathbf{T}_\perp\tilde{v})^2 + (\partial_{r_*}\tilde{v}) + \tilde{V}_Q|\nabla\!\!\!/\tilde{v}|^2 + V\tilde{v}^2,\\
t(\delta\mathcal{P}_A)[\psi] \gtrsim& t\frac{\Delta}{(r^2+a^2)}\frac{1}{r^2}|\partial_{r_*}\tilde{v}|^2 + t\frac{\Delta}{(r^2+a^2)}\frac{1}{r^3}|\tilde{v}|_1 + t\frac{\Delta}{(r^2+a^2)}\frac{1}{r^4}\tilde{v}^2,\\
\tilde{\mathcal{P}}^t_{\mathbf{K}} \gtrsim& (\mathbf{T}_\perp\tilde{v})^2 + (\partial_{r_*}\tilde{v}) + t^2\tilde{V}_Q|\nabla\!\!\!/\tilde{v}|^2 + t^2V\tilde{v}^2
\end{aligned}$$

respectively. Since we are applying the Cauchy-Schwarz inequality, these estimates limit the asymptotic behaviour of the coefficients of the mixed derivatives. This imposes restrictions on the behaviour of $\omega_{\mathbf{K}}$ and its derivatives. Thus, in these calculations, we find ourselves having certain coefficients, needing them to satisfy certain asymptotics so that we can apply the relevant estimates, and then requiring certain conditions on $\omega_{\mathbf{K}}$ and its derivatives. We call this "need-have-require" analysis. When there is no condition which we require, we simply say the required condition is "nothing".

The first term, coming from (4.14b), is

$$|(\mathbf{T}_\perp\tilde{v})(\partial_\phi\tilde{v})(-2t\tilde{N}^2(\omega_{\mathbf{K}}-\omega_\perp))| \lesssim |\mathbf{T}_\perp\tilde{v}||\partial_\phi\tilde{v}|t|\omega_{\mathbf{K}}-\omega_\perp|.$$

In the three regions of interest, the asymptotics we have, those we need, and the conditions we require are

Region	Have	Need	Require
I	$t(\omega_{\mathbf{K}}-\omega_\perp)$	$\Delta^{1/2+\epsilon}$	$\omega_{\mathbf{K}}-\omega_\perp \lesssim \Delta^{1/2+\epsilon}$
IIa	$t(\omega_{\mathbf{K}}-\omega_\perp)$	$tO(\Delta, r^{-3})$	$\omega_{\mathbf{K}}-\omega_\perp = O(\Delta, r^{-3})$
III	$t(\omega_{\mathbf{K}}-\omega_\perp)$	r^{-2}	$\omega_{\mathbf{K}}-\omega_\perp \lesssim r^{-3}$

The second term, from (4.14d), is bounded by the sum of two terms

$$\begin{aligned}
&|(\partial_*\tilde{v})(\partial_\phi\tilde{v})((t^2+r_*^2)(\partial_{r_*}\omega_{\mathbf{K}}) + 2r_*(\omega_{\mathbf{K}}-\omega_\perp))|\\
&\qquad\lesssim |\partial_*\tilde{v}||\partial_\phi\tilde{v}|\left((t^2+r_*^2)|\partial_{r_*}\omega_{\mathbf{K}}| + 2|r_*(\omega_{\mathbf{K}}-\omega_\perp)|\right)
\end{aligned}$$

We perform a have-need-require analysis on each of these. For the first of these terms, in region IIa, we will not estimate the term by $t(\delta\mathcal{P}_A)[\psi]$ but by $at^2(\delta\mathcal{P}_A)[\psi]$, which means we "have" an extra factor of t.

Region	Have	Need	Require
I	$(t^2+r_*^2+1)\partial_{r_*}\omega_{\mathbf{K}}$	$\Delta^{1/2+\epsilon}$	$\|\partial_{r_*}\omega_{\mathbf{K}}\| \lesssim \Delta^{1/2+\epsilon}$
IIa	$(t^2+r_*^2+1)\partial_{r_*}\omega_{\mathbf{K}}$	$at^2O(\Delta, r^{-5/2})$	$\partial_{r_*}\omega_{\mathbf{K}} = O(\Delta, r^{-5/2})$
III	$(t^2+r_*^2+1)\partial_{r_*}\omega_{\mathbf{K}}$	r^{-2}	$\|\partial_{r_*}\omega_{\mathbf{K}}\| \lesssim r^{-4}$

For the second, we again use the standard estimates.

Region	Have	Need	Require
I	$r_*(\omega_{\mathbf{K}} - \omega_\perp)$	$\Delta^{1/2+\epsilon}$	$\lvert\omega_{\mathbf{K}} - \omega_\perp\rvert \lesssim \Delta^{1/2+\epsilon}$
IIa	$r_*(\omega_{\mathbf{K}} - \omega_\perp)$	$a^2 t O(\Delta, r^{-5/2})$	$\omega_{\mathbf{K}} - \omega_\perp = O(\Delta, r^{-5/2})$
III	$r_*(\omega_{\mathbf{K}} - \omega_\perp)$	r^{-2}	$\lvert\omega_{\mathbf{K}} - \omega_\perp\rvert \lesssim r^{-3}$

The remaining derivative terms are estimated similarly. These are

$$|(\partial_*\tilde{v})(\partial_\beta\tilde{v})\tilde{h}^{\alpha\beta}2tr_*(\partial_\beta\tilde{N}^2)| \lesssim |\partial_*\tilde{v}||\not\nabla\tilde{v}|\tilde{V}_Q^2 t|r_*|a^2,$$

and we require

Region	Have	Need	Require
I	$\tilde{V}_Q^2 t\lvert r_*\rvert a^2$	$\Delta^{1/2+\epsilon}$	nothing
IIa	$\tilde{V}_Q^2 t\lvert r_*\rvert a^2$	$tO(\Delta, r^{-5/2})$	nothing
III	$\tilde{V}_Q^2 t\lvert r_*\rvert a^2$	r^{-2}	nothing

For the final term, from (4.14g), we again need to use $at^2(\delta\mathcal{P}_A)[\psi]$ to dominate

$$|(\partial_\alpha\tilde{v})(\partial_\phi\tilde{v})\tilde{h}^{\alpha\beta}(t^2+r_*^2)(\partial_\beta\omega_{\mathbf{K}})| \lesssim |\not\nabla\tilde{v}|^2\tilde{V}_Q(t^2+r_*^2+1)|\not\nabla\omega_{\mathbf{K}}|.$$

Region	Have	Need	Require
I	$\tilde{V}_Q(t^2+r_*^2+1)\lvert\not\nabla\omega_{\mathbf{K}}\rvert$	$\Delta^{1+\epsilon}$	$\lvert\not\nabla\omega_{\mathbf{K}}\rvert \lesssim \Delta^\epsilon$
IIa	$\tilde{V}_Q(t^2+r_*^2+1)\lvert\not\nabla\omega_{\mathbf{K}}\rvert$	$at^2 O(\Delta, r^{-3})$	$\not\nabla\omega_{\mathbf{K}} = O(1, r^{-3})$
III	$\tilde{V}_Q(t^2+r_*^2+1)\lvert\not\nabla\omega_{\mathbf{K}}\rvert$	r^{-2}	$\lvert\not\nabla\omega_{\mathbf{K}}\rvert \lesssim r^{-2}$

We turn to controlling these terms in region IIb, near $r = 3M$. If we assume that $\omega_{\mathbf{K}}$ is constant in a neighbourhood of the photon orbits, then the worst terms involving t^2, coming from (4.14d) and (4.14g), vanish in this region. We will further assume that in this region $\omega_{\mathbf{K}} - \omega_\perp$ is bounded by a constant times a, which bounds the term coming from (4.14b). Applying Cauhcy-Schwartz, all the other terms are bounded by

$$\begin{aligned} t\mathbb{1}_{r\approx 3M}|\partial_{r_*}\tilde{v}|^2 + t\mathbb{1}_{r\approx 3M}|\not\nabla\tilde{v}|^2 + at\mathbb{1}_{r\approx 3M}|\partial_t\tilde{v}|^2 \\ \leq &(\delta\mathcal{P}_A)[\psi] \\ &+ (\delta\mathcal{P}_A)[\mathbb{O}_1^1, \psi] \\ &+ at^{-1}\tilde{\mathcal{P}}_{\mathbf{K}}[\mathbb{S}_1\tilde{v}]^t. \end{aligned}$$

By taking the most restrictive conditions, we are left with the following requirements:

$$\begin{aligned} \omega_{\mathbf{K}} - \omega_\perp =& O(\Delta, r^{-3}), \\ \partial_{r_*}\omega_{\mathbf{K}} =& aO(\Delta, r^{-4}), \\ \not\nabla\omega_{\mathbf{K}} =& a^2 O(\Delta^\epsilon, r^{-3}), \end{aligned}$$

and that $\omega_{\mathbf{K}}$ is constant in a neighbourhood of the photon orbits, and that $\omega_{\mathbf{K}} - \omega_\perp$ is bounded by a constant times a in that neighbourhood.

There is a wide range of possibilities for choice of $\omega_{\mathbf{K}}$. In particular, our choice

$$\omega_{\mathbf{K}} = \omega_H\chi$$

satisfies all the necessary conditions.

The remaining terms to be estimated, which come from (4.13c)-(4.13d), are given by $\tilde{v}^2$ times the coefficient

$$
\begin{aligned}
+\frac{1}{2}(\partial_\gamma \mathbf{K}^\gamma)V + \frac{1}{2}\mathbf{K}^\gamma\partial_\gamma V - q_{\mathbf{K}}V + \frac{1}{2}(\partial_\alpha \tilde{\mathcal{G}}^{\alpha\beta}\partial_\beta q_{\mathbf{K}}) =& + t\tilde{N}^2(2V + r_*\partial_{r_*}V)\\
&+ t(\tilde{N}^2 - 1 + r_*\partial_{r_*}\tilde{N})V\\
&- (\tilde{N}^2-1)q_{\mathbf{K}} - \frac{1}{2}(\partial_{r_*}^2 q_{\mathbf{K}}).
\end{aligned}
$$

As with the angular derivative terms in (4.14f), the first term is familiar from the Scwarzschild case and gives a weight which is positive in a compact region, but which is negative, and hence trivially bounded, outside this region. The behaviour outside this region is $\approx -t\Delta\log\Delta$ approaching the event horizon and $\approx -tr^{-3}\log r$ approaching the horizon. The other terms are all $a^2O(\Delta, r^{-4})$ at worst, so they are easily dominated by the first term outside a slightly larger compact region. Thus, the contribution involving $\tilde{v}^2$ without derivatves is easily controlled by $t(\delta\mathcal{P}_A)[\psi]$. □

4.4. **Stronger, light-cone localised Morawetz estimates.** In this section, we prove a stronger Morawetz estimate to gain factors of t inside the light cone, $|r_*| < t$. We do this by using a Morawetz vector field with additional factors of t and localising inside the light cone.

Definition 4.10. *Let $\chi_{LC}: \mathbb{R}\to\mathbb{R}$ be a smooth, even function which is identically 1 for $|x| < 1/2$, identically 0 for[6] $|x| > 3/4$, which is weakly decreasing for $x > 0$, and which is weakly increasing for $x < 0$. If χ_{LC} is written without an argument, it understood to mean*

$$\chi_{LC} = \chi_{LC}\left(\frac{r_*}{t}\right).$$

The stronger, light-cone localised Morawetz vector field (with strength p) is defined to be

$$
\begin{aligned}
\bar{\mathbf{A}}_p^{\underline{ab}} =& t^p\chi_{LC}\mathbf{A}^{\underline{ab}},\\
\bar{q}_{A,p}^{\underline{ab}} =& t^p\chi_{LC}q_{\mathbf{A}}^{\underline{ab}},\\
A_p =& (\bar{\mathbf{A}}_p^{\underline{ab}}, \bar{q}_{A,p}^{\underline{ab}}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\bar{\mathbf{A}}_0^{\underline{ab}} =& \mathbf{A}^{\underline{ab}}\chi_{\mathrm{LC}},\\
\bar{q}_{A,0}^{\underline{ab}} =& q_{\mathbf{A}}^{\underline{ab}}\chi_{\mathrm{LC}}.
\end{aligned}
$$

Lemma 4.11. *There are positive constant $\bar{a}$ and C, such that, if $|a|\le\bar{a}$, if ψ is a solution to the wave equation $\Box\psi = 0$, and if $\tilde{\psi} = (r^2+a^2)^{1/2}\psi$, then for $t_2\ge t_1\ge 0$ and $p\in\mathbb{N}^+$,*

$$
\begin{aligned}
C^{-1}\int_{t_1}^{t_2}\int t^p\chi_{LC}(\delta\mathcal{P}_A)[\psi]\,d^4\mu \le& (1+t_2)^{p-2}\tilde{E}_{\mathbf{K},3}[\tilde{\psi}](t_2) + (1+t_1)^{p-2}\tilde{E}_{\mathbf{K},3}[\tilde{\psi}](t_1)\\
&+ \int_{t_1}^{t_2}(1+t)^{p-3}\tilde{E}_{\mathbf{K},3}[\tilde{\psi}](t)\,dt.
\end{aligned}
$$

Proof. We begin by relating the divergence of the momentum of the stronger light-cone localised Morawetz vector field to that of the original Morawetz vector-field.

[6]The parameters 1/2 and 3/4 are not important. The important thing is the ordering $0 < 1/2 < 3/4 < 1$.

We denote the right-hand side of (3.6) by $\mathcal{W}[\psi]$. The divergence is

$$\begin{aligned}
\frac{1}{\mu}\partial_\alpha(\mu\mathcal{P}_{A_p}) =&\frac{1}{\mu}\partial_\alpha(\mu t^p\chi_{\mathrm{LC}}\mathcal{P}_A)\\
=&t^p\chi_{\mathrm{LC}}\frac{1}{\mu}\partial_\alpha(\mu\mathcal{P}_A^\alpha)+\mathcal{P}_A^\alpha(\partial_\alpha\chi_{\mathrm{LC}}t^p),\\
t^p\chi_{\mathrm{LC}}\mathcal{W}[\psi] =&\frac{1}{\mu}\partial_\alpha(\mu\mathcal{P}_{A_p})\\
&-P_A^\alpha(\partial_\alpha t^p\chi_{\mathrm{LC}})-t^p\chi_{\mathrm{LC}}\frac{1}{\mu}\partial_\alpha(\mu(\mathcal{B}_{A;\mathrm{I}}+\mathcal{B}_{A;\mathrm{II}})).
\end{aligned}$$

We now move the localisation onto the solution itself to find

$$\begin{aligned}
&t^p\mathcal{W}[\chi_{\mathrm{LC}}^{1/2}\psi]\\
&=\frac{1}{\mu}\partial_\alpha(\mu\mathcal{P}_{A_p})\\
&\quad-P_A^\alpha(\partial_\alpha t^p\chi_{\mathrm{LC}})-t^p\chi_{\mathrm{LC}}\frac{1}{\mu}\partial_\alpha(\mu(\mathcal{B}_{A;\mathrm{I}}+\mathcal{B}_{A;\mathrm{II}}))\\
&\quad-t^p\chi_{\mathrm{LC}}\frac{M\Delta^2}{r^2(r^2+a^2)}\sum_{\underline{a}}(2(\partial_r S_{\underline{a}}\psi)(\partial_r\chi_{\mathrm{LC}})S_{\underline{a}}\psi+(\partial_r\chi_{\mathrm{LC}})^2(S_{\underline{a}}\psi^2)),\\
&=\frac{1}{\mu}\partial_\alpha(\mu t^p\chi_{\mathrm{LC}}(\mathcal{P}_A^\alpha+\mathcal{B}_{A;\mathrm{I}}^\alpha+\mathcal{B}_{A;\mathrm{II}}^\alpha))\\
&\quad-(P_A^\alpha+\mathcal{B}_{A;\mathrm{I}}^\alpha+\mathcal{B}_{A;\mathrm{II}}^\alpha)(\partial_\alpha t^p\chi_{\mathrm{LC}})\\
&\quad-t^p\chi_{\mathrm{LC}}\frac{M\Delta^2}{r^2(r^2+a^2)}\sum_{\underline{a}}(2(\partial_r S_{\underline{a}}\psi)(\partial_r\chi_{\mathrm{LC}})S_{\underline{a}}\psi+(\partial_r\chi_{\mathrm{LC}})^2(S_{\underline{a}}\psi^2)).
\end{aligned}$$

We now estimate the last term, using χ_2 as a smooth localisation which is 1 on the support of χ_{LC} but still supported well inside the light cone, and using ϕ to denote $S_{\underline{a}}\psi$,

$$\begin{aligned}
t^p\chi_{\mathrm{LC}}&\frac{M\Delta^2}{r^2(r^2+a^2)}(2(\partial_r\phi)(\partial_r\chi_{\mathrm{LC}})\phi+(\partial_r\chi_{\mathrm{LC}})^2\phi^2)\\
&\lesssim t^p\chi_{\mathrm{LC}}\frac{M\Delta^2}{r^2(r^2+a^2)}((\partial_r\phi)^2+(\partial_r\chi_{\mathrm{LC}})^2\phi^2)\\
&\lesssim t^p\chi_{\mathrm{LC}}\frac{M\Delta^2}{r^2(r^2+a^2)}(\partial_r\phi)^2+t^{p-2}\chi_2\phi^2\\
&\lesssim t^p\chi_2\left(\Delta(\partial_r\phi)^2+\frac{1}{1+r_*^2}\phi^2\right).
\end{aligned}$$

Integrating this over a hypersurface of constant t,

$$\begin{aligned}
\int t^p\chi_2&\frac{\Delta^2}{(r^2+a^2)^2}((\partial_r\psi)(\partial_r\chi_{\mathrm{LC}})\psi+(\partial_r\chi_{\mathrm{LC}})^2\psi^2)\mathrm{d}^3\mu\\
&\leq\int t^p\chi_2\left(\mathcal{P}^t_{\mathbf{T}_\chi}+\frac{1}{1+r_*^2}\psi^2\right)\mathrm{d}^3\mu\\
&\lesssim\int t^{p-2}\tilde{\mathcal{P}}^t_{\mathbf{K}}\mathrm{d}^3\mu_*.
\end{aligned}$$

Since $\mathcal{P}_A^t$, $\mathcal{P}_A^r$, $\mathcal{B}_{A;\mathrm{I}}^t$, $\mathcal{B}_{A;\mathrm{I}}^r$, $\mathcal{B}_{A;\mathrm{II}}^t$, and $\mathcal{B}_{A;\mathrm{II}}^r$ are all bounded by $\mathcal{P}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\psi]+|\mathbb{S}_2\psi|^2$, which is bounded by $\tilde{\mathcal{P}}^t_{\mathbf{T}_\chi}[\mathbb{S}_2\tilde{\psi}]$, at least in an integrated sense over a portion of the hypersurface inside the light cone, it follows that

$$\int t^p \mathcal{W}[\chi_{\mathrm{LC}}^{1/2}\psi]\mathrm{d}^3\mu \lesssim \int \frac{1}{\mu}\partial_\alpha(\mu t^p \chi_{\mathrm{LC}}(\mathcal{P}_A^\alpha + \mathcal{B}_{A;\mathrm{I}}^\alpha + \mathcal{B}_{A;\mathrm{II}}^\alpha))\mathrm{d}^3\mu + t^{p-1}\int_{-(3/4)t}^{(3/4)t} \tilde{\mathcal{P}}_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^3\mu_*.$$

Now applying the Hardy estimate and an argument similar to the one earlier in this proof, it follows that

$$\int \chi_{\mathrm{LC}} t^p (\delta\mathcal{P}_A)[\psi]\mathrm{d}^3\mu \lesssim \int \frac{1}{\mu}\partial_\alpha(\mu t^p \chi_{\mathrm{LC}}(\mathcal{P}_A^\alpha + \mathcal{B}_{A;\mathrm{I}}^\alpha + \mathcal{B}_{A;\mathrm{II}}^\alpha))\mathrm{d}^3\mu + t^{p-1}\int_{-(3/4)t}^{(3/4)t} \tilde{\mathcal{P}}_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^3\mu_*.$$

Finally, integrating in time, we get

$$\begin{aligned}
\int_{t_1}^{t_2}\int \chi_{\mathrm{LC}} t^p (\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu \lesssim & t_2^p \int_{\Sigma_{t_2}} \chi_{\mathrm{LC}}(\mathcal{P}_A^t + \mathcal{B}_{A;\mathrm{I}}^t + \mathcal{B}_{A;\mathrm{II}}^t)\mathrm{d}^3\mu \\
& - t_1^p \int_{\Sigma_{t_1}} \chi_{\mathrm{LC}}(\mathcal{P}_A^t + \mathcal{B}_{A;\mathrm{I}}^t + \mathcal{B}_{A;\mathrm{II}}^t)\mathrm{d}^3\mu \\
& + \int_{t_1}^{t_2} t^{p-1}\int_{-(3/4)t}^{(3/4)t} \tilde{\mathcal{P}}_{\mathbf{T}_\chi}[\tilde{\psi}]\mathrm{d}^4\mu_* \\
\lesssim & t_2^p \int_{\Sigma_{t_2}} \chi_{\mathrm{LC}}\mathcal{P}_A^t[\mathbb{S}_2\psi]\mathrm{d}^3\mu \\
& + t_1^p \int_{\Sigma_{t_1}} \chi_{\mathrm{LC}}\mathcal{P}_A^t[\mathbb{S}_2\psi]\mathrm{d}^3\mu \\
& + \int_{t_1}^{t_2} t^{p-1}\int_{-(3/4)t}^{(3/4)t} \tilde{\mathcal{P}}_{\mathbf{T}_\chi}^t[\mathbb{S}_2\tilde{\psi}]\mathrm{d}^4\mu_* \\
\lesssim & t_2^{p-2}\tilde{E}_{\mathbf{K}}[\mathbb{S}_2\psi](t_2) + t_1^{p-2}\tilde{E}_{\mathbf{K}}[\mathbb{S}_2\psi](t_1) \\
& + \int_{t_1}^{t_2} t^{p-3}\tilde{E}_{\mathbf{K}}[\mathbb{S}_2\tilde{\psi}]\mathrm{d}t.
\end{aligned}$$

□

4.5. **Closing the K estimate.** In this section, we use the stronger, light-cone localised Morawetz estimate to control the growth of the **K** energy.

Theorem 4.12. *There are positive constants $\bar{a}$, C, and C' such that, if $|a| < \bar{a}$ and if $\tilde{\psi}$ is a solution of the transformed wave equation* (4.1)*, then* $\forall t \geq 0$

$$\tilde{E}_{\mathbf{K},3}(t) \leq C(1+t)^{C'|a|}\left(\tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,5}(0)\right).$$

Proof. Essentially, the stronger Morawetz-estimate, lemma 4.11, controls the growth of the **K** energy, and the **K** energy controls the boundary terms in the stronger Morawetz estimate. Thus, a standard Gronwall's argument closes the estimate. In this proof, an integral without limits typically refers to integration over $(r_*, \theta, \phi) \in \mathbb{R} \times S^2$.

First, we can bound the growth of the $\mathbf{K}$ energy, using lemma 4.9, by

$$\tilde{E}_{(\mathbf{K},q_{\mathbf{K}}),3}(t_2) - \tilde{E}_{(\mathbf{K},q_{\mathbf{K}}),3}(t_1) \leq \int_{t_1}^{t_2} \tilde{E}_{\mathbf{T}_\chi,3}(t)(1+t)^{-2}\mathrm{d}t + \int_{t_1}^{t_2}\int t(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* + \int_{t_1}^{t_2}\int t(\delta\mathcal{P}_A)[\mathbb{O}_1^1,\psi]\mathrm{d}^4\mu_* + a\int_{t_1}^{t_2}\int t^2(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* + \int_{t_1}^{t_2} \tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t. \tag{4.15}$$

Since $\tilde{E}_{\mathbf{T}_\chi,3} \lesssim \tilde{E}_{\mathbf{K},3}$ and $(1+t)^{-2} \lesssim (1+t)^{-2}\log(2+t)$, the first term can be absorbed into the last.

Before applying the Morawetz estimate, we apply integration by parts in the angular derivatives to the term in (4.15) and apply the Cauchy-Schwarz inequality to find

$$\begin{aligned}\int_{S_2}\int(\delta\mathcal{P}_A)[\mathbb{O}_1^1,\psi]\mathrm{d}^2\mu &\leq \int_{S_2}(\delta\mathcal{P}_A)[\psi]^{1/2}(\delta\mathcal{P}_A)[\mathbb{O}_1^2,\psi]^{1/2}\mathrm{d}^2\mu\\ &\leq \left(\int_{S_2}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^2\mu\right)^{1/2}\left(\int_{S_2}(\delta\mathcal{P}_A)[\mathbb{O}_1^2,\psi]\mathrm{d}^2\mu\right)^{1/2}\\ &\leq \left(\int_{S_2}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^2\mu\right)^{3/4}\left(\int_{S_2}(\delta\mathcal{P}_A)[\mathbb{O}_1^4,\psi]\mathrm{d}^2\mu\right)^{1/4}.\end{aligned}$$

Now applying the integration in the remaining variables and the Cauchy-Schwarz inequality one more time, we find

$$\int_{t_1}^{t_2}\int t(\delta\mathcal{P}_A)[\mathbb{O}_1^1,\psi]\mathrm{d}^4\mu_* \lesssim \int_{t_1}^{t_2}\int t^{4/3}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* + \int_{t_1}^{t_2}\int(\delta\mathcal{P}_A)[\mathbb{O}_1^4,\psi]\mathrm{d}^4\mu_*.$$

From this, we can bound the growth of $\tilde{E}_{\mathbf{K},3}$ by

$$\begin{aligned}\tilde{E}_{\mathbf{K},3}(t_2) - \tilde{E}_{\mathbf{K},3}(t_1) \lesssim{}& \int_{t_1}^{t_2}\int t^{4/3}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_*\\ &+ \int_{t_1}^{t_2}\int(\delta\mathcal{P}_A)[\mathbb{O}_1^4,\psi]\mathrm{d}^4\mu_*\\ &+ a\int_{t_1}^{t_2}\int t^2(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_*\\ &+ \int_{t_1}^{t_2}\tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t.\end{aligned}$$

In the first integral, with the aim of applying lemma 4.11, we use the convergence of $\int_1^\infty t^{4/3-3}\mathrm{d}t$ to get a bounded piece plus what we hope to show is a small piece. To do this, we choose a time T such that $\int_T^\infty (1+t)^{-5/3}\mathrm{d}t$ is small relative to the absolute constants implicit in $\lesssim$. The first integral we break into an integral from t_1 to T and then from T to t_2. In the first part, we replace $(1+t)^{4/3}$ by $(1+T)^{4/3}$, which is also an absolute constant. Thus, the first integral is bounded by

$$\int_{t_1}^{t_2}\int t^{4/3}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* \lesssim \tilde{E}_{3,\mathbf{T}_\chi} + \int_T^\infty\int t^{4/3}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_*.$$

Similarly, in the last integral, we want to use the integrability of $\int t^{-2}\log t\mathrm{d}t$ and the bound $\tilde{E}_{\mathbf{K}} \lesssim (1+t)^2\tilde{E}_{\mathbf{T}_\chi}$, so that we can take T sufficiently large that $\int_T^\infty t^{-2}\log t\mathrm{d}t$ is small, and obtain the bound

$$\int_{t_1}^{t_2} \tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t \lesssim \tilde{E}_{\mathbf{T}_\chi,3} + \int_T^{t_2} \tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t.$$

We can also apply the original Morawetz estimate, lemma 3.12, with an extra four derivatives, to control the integral of $(\delta\mathcal{P}_A)[\mathbb{O}_1^4,\psi]$. This leaves us,

$$\begin{aligned}\tilde{E}_{\mathbf{K},3}(t_2) - \tilde{E}_{\mathbf{K},3}(t_1) \lesssim & \int_T^{t_2} t^{4/3}(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* \\ &+ \tilde{E}_{\mathbf{T}_\chi,7} \\ &+ |a|\int t^2(\delta\mathcal{P}_A)[\psi]\mathrm{d}^4\mu_* \\ &+ \int_T^{t_2} \tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t.\end{aligned}$$

Using the stronger Morawetz lemma 4.11, we can bound the integral of the Morawetz terms by,

$$\begin{aligned}\tilde{E}_{\mathbf{K},3}(t_2) - \tilde{E}_{\mathbf{K},3}(t_1) \lesssim &(1+t_2)^{-2/3}\tilde{E}_{\mathbf{K},3}(t_2) + (1+T)^{-2/3}\tilde{E}_{\mathbf{K},3}(T) && (4.16a)\\ &+ \int_T^{t_2}(1+t)^{-5/3}\tilde{E}_{\mathbf{K},3}(t)\mathrm{d}t && (4.16b)\\ &+ \tilde{E}_{\mathbf{T}_\chi,7} && (4.16c)\\ &+ |a|\tilde{E}_{\mathbf{K},3}(t_2) + |a|\tilde{E}_{\mathbf{K},3}(t_1) && (4.16d)\\ &+ |a|\int_{t_1}^{t_2}(1+t)^{-1}\tilde{E}_{\mathbf{K},3}(t)\mathrm{d}t && (4.16e)\\ &+ \int_T^{t_2} \tilde{E}_{\mathbf{K},3}(1+t)^{-2}\log(2+t)\mathrm{d}t., && (4.16f)\end{aligned}$$

Let

$$\begin{aligned}f(t) =& \sup_{\tau\in[t_1,t]} \tilde{E}_{\mathbf{K},3}(\tau),\\ F(t) =& \int_{t^1}^t f(\tau)(1+\tau)^{-1}\mathrm{d}\tau,\\ K_0 =& \tilde{E}_{\mathbf{K},3}(t_1) + \tilde{E}_{\mathbf{T}_\chi,7}(t_1).\end{aligned}$$

We can replace $\tilde{E}_{\mathbf{K},3}$ on the left of (4.16a) by f. We can make a similar substitution in (4.16a), (4.16b), (4.16d), and (4.16f), and then use the smallness of $t_2^{-2/3}$, $T^{-2/3}$, $\int_T^\infty t^{-5/3}$, a, and $\int_T^\infty t^{-2}\log t$ to absorb all those terms into the left-hand side, with only a small loss. Note that by the bounded energy result, one has that $\tilde{E}_{\mathbf{T}_\chi,7}(t) \lesssim K_0$ uniformly in t. Thus, we are left with

$$f(t_2) \lesssim K_0 + a\int_{t_1}^{t_2}(1+t)^{-1}f(t)\mathrm{d}t.$$

We now write this as an integral inequality to apply the standard Gronwall's inequality techniques

$$
\begin{aligned}
f(t) - C|a| \int_{t^1}^{t} (1+\tau)^{-1} f(\tau) \mathrm{d}\tau \lesssim & K_0, \\
(1+t)F'(t) - CaF(t) \lesssim & K_0, \\
\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{-C|a|} F(t) \right) \lesssim & K_0 t^{-1-C|a|}, \\
F(t) \lesssim & \frac{1}{a} K_0 (1+t)^{C|a|} \\
f(t) \lesssim & K_0 (1+t)^{C|a|}.
\end{aligned}
$$

In terms of the original quantities of interest, this gives

$$
\tilde{E}_{\mathbf{K},3}(t) \lesssim (1+t)^{C|a|} \left(\tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,7}(0) \right).
$$

□

We also want to evaluate the **K** energy on surfaces other than those of constant t. The following lemma allows us to control the **K** energy on such surfaces. Although this lemma is valid for any surface, it is only interesting on time-like or null hypersurfaces on which the energy should be positive.

Theorem 4.13 (**K** bounds for other surfaces)**.** *Consider the almost-null cones given by*

$$
\mathcal{C}_\tau = \{t - |r_*| = \tau\}.
$$

There are positive constants $\bar{a}$, C, and C', such that, if $|a| \leq \bar{a}$, and if $\tilde{\psi}$ is a solution of the transformed wave equation (4.1)2 then for any hypersurface Σ, which need not be in the futute of Σ_0, if Σ can be represented as a graph over $(r_*, \theta, \phi) \in \mathbb{R} \times S^2$ which lies beneath $\mathcal{C}_\tau$ and above $-\mathcal{C}_\tau$, then

$$
\tilde{E}_{(\mathbf{K},q_{\mathbf{K}}),3}(\Sigma) \leq C \max\{1, \tau\}^{C'|a|} \left(\tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,7}(0) \right)
$$

Proof. In this proof we will exploit the time symmetry of the previous results. For simplicity let $K_0 = \tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,7}(0)$. Lemma 4.9 allows us to estimate the change in $\tilde{E}_{(\mathbf{K},q_{\mathbf{K}})}$ from $\{t=0\}$ to Σ by the four-dimensional integral of $\mu^{-1}\partial_\alpha(\mu \tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_{\mathbf{K}})})$. This integral can be broken into three pieces: where $r_* < -|t|/2$, where $-|t|/2 \leq r_* \leq |t|/2$, and where $|t|/2 < r_*$.

We first consider the middle region, where $-|t|/2 \leq r_* \leq |t|/2$. The four-dimensional volume inside this region and beneath Σ is a subset of the volume inside this region where $|t| \leq 2\tau$. Thus, from the proof of lemma 4.5, the integral of the absolute value of $\mu^{-1}\partial_\alpha(\mu \tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_{\mathbf{K}})})$ is bounded by $C(1+\tau)^{Ca} K_0$.

We now consider the other two regions. First, consider the far region, $|t|/2 < r_*$. In this region, $|\mu^{-1}\partial_\alpha(\mu \tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_{\mathbf{K}})})|$ is bounded by $\log(2+|t|)(1+|t|)^{-2}\tilde{\mathcal{P}}^t_{(\mathbf{K},q_{\mathbf{K}})}$. If we now foliate the region beneath Σ by hypersurfaces of constant t, we can bound the four-dimensional integral of $\mu^{-1}\partial_\alpha(\mu \tilde{\mathcal{P}}^\alpha_{(\mathbf{K},q_{\mathbf{K}})})$ by the integral in time of the integral over each of the leaves of the foliation. On each leaf, the integral is bounded by $\log(2+|t|)(1+|t|)^{-2}\tilde{E}_{(\mathbf{K},q_{\mathbf{K}})}(t) \lesssim \log(2+|t|)(1+|t|)^{Ca-2}K_0$. Thus, the four-dimensional integral is bounded by $K_0 \int \log(2+|t|)(1+|t|)^{Ca-2}\mathrm{d}t \lesssim K_0$. The near region, $r_* < -|t|/2$ can be handled similarly.

Thus, the four-dimensional integral is bounded by $C\tau^{Ca}K_0$. Hence, so is $\tilde{E}_{(\mathbf{K},q_{\mathbf{K}})}(\Sigma)$.

□

5. Pointwise decay estimates

5.1. **Decay in stationary regions.** In this subsection, we prove decay in regions of fixed r bounded away from the event horizon, $r_+ < r_1 < r < r_2$. Since these are preserved by the flow of the stationary, Killing field, ∂_t, we refer to these as stationary regions.

The essence of the proof of this decay result is that the bound on $\tilde{E}_{\mathbf{K},3}$ gives a bound on t^2 times the square of a local H^3 norm, which controls the square of the field, $|\psi|^2$, through a Sobolev estimate. Thus, $|\psi|$ decays like t^{-1} with a small loss from the growth of the $\mathbf{K}$ energy.

Theorem 5.1. *There are positive constants $\bar{a}$ and C' such that, given $r_+ < r_1 < r_2$, there is a constant, C_{r_1,r_2}, such that, if $|a| \leq \bar{a}$, and if ψ is a solution to the wave equation $\Box\psi = 0$, then $\forall t \geq 0$, $r \in [r_1, r_2]$, $(\theta, \phi) \in S^2$, there is the estimate*

$$|\psi(t, r, \theta, \phi)| \leq C_{r_1,r_2} \max\{1, t\}^{-1+C'|a|} \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right).$$

Proof. In this proof, a constant of the form C'_{r_1,r_2} refers to a constant which depends only on r_1 and r_2. As is common in analysis, the value of this constant may vary from line to line. Since $r_+ < r_1 < r_2 < \infty$, there are corresponding r_* values $-\infty < r_{*1} < r_{*2} < \infty$.

From lemma 2.2, for fixed (t, r_*) and all $(\theta, \phi) \in S^2$

$$|\tilde{\psi}(t, r_*, \theta, \phi)|^2 \lesssim \int_{S^2} |\tilde{\psi}|_2 \mathrm{d}^2\mu.$$

Given any smooth $v : \mathbb{R} \to \mathbb{R}$, by a one-dimensional Sobolev estimate (or simply through the fundamental theorem of calculus, localisation, and Cauchy-Schwarz), for $r_{*1} \leq r_* \leq r_{*2}$

$$|v(r_*)|^2 \leq C'_{r_1,r_2} \int_{r_{*1}}^{r_{*2}} |\partial_{r_*} v|^2 + |v|^2 \mathrm{d}r_*.$$

Applying this result with v being a symmetry operator applied to $\tilde{\psi}$, then summing over the symmetry operators, and then integrating over S^2, we find

$$\int_{S^2} |\tilde{\psi}|_2 \mathrm{d}^2\mu \leq C'_{r_1,r_2} \int_{S^2} \int_{r_{*1}}^{r_{*2}} |\partial_{r_*} \tilde{\psi}|_2^2 + |\tilde{\psi}|_2^2 \mathrm{d}^3\mu_*.$$

The left-hand side dominates $|\tilde{\psi}|^2$. Since the range of r is fixed, there is a lower bound on V, so that

$$|\tilde{\psi}(t, r_*, \theta, \phi)|^2 \leq C'_{r_1,r_2} \int_{S^2} \int_{r_{*1}}^{r_{*2}} |\partial_{r_*} \tilde{\psi}|_2^2 + V|\tilde{\psi}|_2^2 \mathrm{d}^3\mu_*.$$

For sufficiently large t, the region $[r_{*1}, r_{*2}] \times S^2$ is inside $|r_*| \leq t/2$, so that the right-hand side can be bounded by the $\mathbf{K}$ energy, and

$$\begin{aligned} |\tilde{\psi}(t, r_*, \theta, \phi)|^2 &\leq C'_{r_1,r_2} t^{-2} \tilde{E}_{\mathbf{K},3}(t) \\ &\leq C'_{r_1,r_2} t^{-2-Ca} \left(\tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,7}(0)\right). \end{aligned}$$

Since $(r^2 + a^2)^{1/2}$ is uniformly bounded above and below on (r_1, r_2) we may replace $\tilde{\psi}(t, r_*, \theta, \phi)$ by $\psi(t, r_*, \theta, \phi)$ to get the desired result.

For small t, the result holds from a similar argument using the energy bound. □

5.2. **Near Decay.** To study the wave equation near the event horizon, it is convenient to use u_+ andu_-, which were defined in the introduction, and to use

$$\phi_* = \phi - \omega_H t.$$

The functions (u_+, r, θ, ϕ_*) are known to form a coordinate system which can be extended to an open set containing the future event horizon.

Theorem 5.2 (Near Decay). *There are positive constants $\bar{a}$, C and C', such that if $|a| < \bar{a}$ and if $\tilde{\psi}$ is a solution of the wave equation (4.1), then for all $t > r_*$, $r < 3M$, $(\theta, \phi) \in S^2$,*

$$|\psi(t, r, \theta, \phi)| < C \max\{1, u_+\}^{-1+C'a} \left(\tilde{E}_{\mathbf{K},5}(0)^{1/2} + E_{\mathbf{T}_\chi,9}(0)^{1/2} + \tilde{E}_{n_{\Sigma_t},3}(0)^{1/2} \right).$$

Proof. Near the horizon, it is convenient to rewrite the wave equation, (4.1), for $\tilde{\psi}$ as

$$\begin{aligned}
0 =& -(\partial_t + \omega_H \partial_\phi)^2 \tilde{\psi} + \partial_{r_*}^2 \tilde{\psi} \\
&+ 2\left(\omega_H - \frac{2aMr}{(r^2+a^2)^2}\right) \partial_t \partial_\phi \tilde{\psi} \\
&+ \Delta(Q + \partial_\phi^2)\tilde{\psi} \\
&+ \left(\omega_H^2 - \frac{a^2}{(r^2+a^2)^2}\right) \tilde{\psi} \\
&- V\tilde{\psi} \\
=& -(\partial_t + \omega_H \partial_\phi)^2 \tilde{\psi} + \partial_{r_*}^2 \tilde{\psi} + \mathcal{Z}\tilde{\psi} - V\tilde{\psi},
\end{aligned}$$

where $\mathcal{Z}$ can be expanded as $\mathcal{Z} = \mathcal{Z}^{\underline{a}} S_{\underline{a}}$ and the $\mathcal{Z}^{\underline{a}}$ vanish linearly in $r - r_+$. Equivalently, the functions $\mathcal{Z}^{\underline{a}}/\Delta$ are bounded for $r \in [r_+, 3M]$.

To prove decay, it is convenient to use $(u_+, u_-, \theta, \phi_*)$ as a coordinate system in the exterior region. The coordinate derivatives in this coordinate system can be related to the earlier (t, r_*, θ, ϕ) coordinate derivatives. The relations that we are interested in are

$$\begin{aligned}
\partial_\pm = \partial_{u_\pm} =& \frac{1}{2}\left(\partial_t + \omega_H \partial_\phi \pm \partial_{r_*}\right), \\
\partial_\theta =& \partial_\theta, \\
\partial_{\phi_*} =& \partial_\phi.
\end{aligned}$$

The relation $\partial_\theta = \partial_\theta$ is understood to mean that the θ-coordinate derivatives in the two systems generate the same vector field. We can, without ambiguity, write formulas involving coordinate derivatives from both systems. In particular, we can write the wave equation, (4.1), as

$$0 = -4\partial_+ \partial_- \tilde{\psi} + \mathcal{Z}\tilde{\psi} - V\tilde{\psi}$$

with $\mathcal{Z}$ defined in terms of ∂_t, ∂_θ, and ∂_ϕ as above.

To prove decay, we use the one-form

$$\xi = \left(\partial_- \tilde{\psi}\right) \mathrm{d}u_-.$$

For a given (θ, ϕ_*), let

$$\begin{aligned}
\Omega =& \{(u'_+, u'_-, \theta, \phi_*) | u'_- \in [u_+, u_-], u'_+ \in [-u'_-, u_+]\} \\
=& \{(t' r'_*, \theta', \phi') | t' \geq 0, t + r_* \leq t' - r'_* \leq t - r_*, t' - r'_* \leq t + r_*\}.
\end{aligned}$$

It is convenient to first consider $u_+ \geq 1$. We apply Stokes' theorem to ξ on Ω and in $(u_+, u_- \theta, \phi_*)$ coordinates to get

$$\begin{aligned}
&\int_{\partial\Omega} \xi = \int_\Omega \mathrm{d}\xi, \\
&\tilde{\psi}(t, r_*, \theta, \phi - \omega_H t) - \tilde{\psi}(t + r_*, 0, \theta, \phi - \omega_H(t + r_*)) \\
&\qquad + \int_{\mathcal{C}} (\partial_- \tilde{\psi}) \mathrm{d}r_* = \int_\Omega (\partial_+ \partial_- \tilde{\psi}) \mathrm{d}u_+ \mathrm{d}u_-,
\end{aligned} \tag{5.1}$$

where $\mathcal{C}$ is the curve with $t' = 0$, $r'_* \in [-u_-, u_+]$, and $(\theta', \phi') = (\theta, \phi)$.

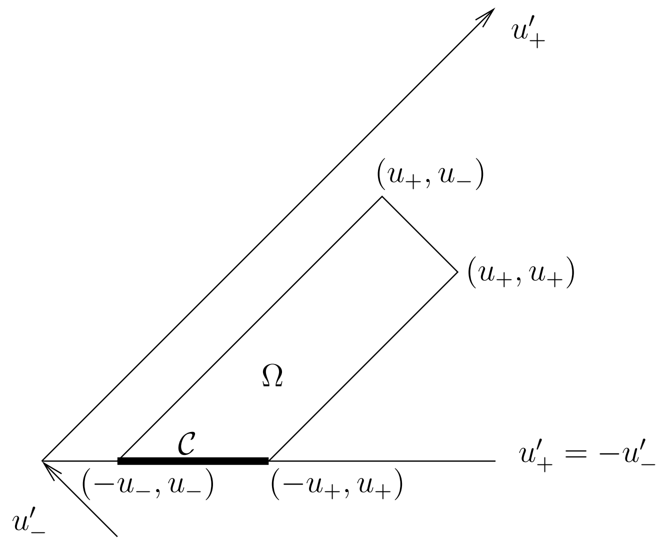


FIGURE 1. The region Ω in the (u'_+, u'_-) plane with vertices at (u_+, u_-), $(-u_-, u_-)$, $(-u_+, u_+)$, and (u_+, u_+). In the (t, r_*) plane, these vertives correspond to (t, r_*), $(0, -t + r_*)$, $(0, t + r_*)$, and $(t + r_*, 0)$.

The function $\tilde{\psi}$ evaluated at $(u_+, u_+, \theta, \phi_*)$ corresponds to the value at $(t', r'_*, \theta', \phi') = (u_+, 0, \theta, \phi_* - \omega_H u_+)$, so it can be estimated by

$$|\tilde{\psi}(t + r_*, 0, \theta, \phi - \omega_H(t + r_*))| \lesssim u_+^{-1+Ca} \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2} \right).$$

The integral over $\mathcal{C}$ corresponds to an integral in the hypersurface $t = 0$ with $r'_* \in [-u_-, -u_+] \subset (-\infty, -u_+]$. Since ∂_- is in the span of $\mathbf{T}_\perp$, ∂_{r_*}, and $\Delta\partial_\phi$ with smooth and uniformally bounded coefficients when $r \in [r_+, 3M]$, it follows that

$$|(\Delta^{-1/2}\partial_-)\tilde{\psi}|^2 = \Delta^{-1}|\partial_-\tilde{\psi}|^2 \lesssim \Delta^{-1}\tilde{\mathcal{P}}^t_{\mathbf{T}_\perp}.$$

Since $\mathbf{T}_\perp$ is parallel to the normal, n_{Σ_t}, it follows that

$$|(\Delta^{-1/2}\partial_-)\tilde{\psi}|^2 \lesssim \Delta^{-1/2}\tilde{\mathcal{P}}^t_{n_{\Sigma_t}}.$$

Therefore,

$$\int_{\mathcal{C}} (\partial_-\tilde{\psi}) \mathrm{d}r_* \le \left(\int_{-\infty}^{-u_+} \Delta^{1/2} \mathrm{d}r_* \right)^{1/2} \left(\int_{-\infty}^{-u_+} |(\Delta^{-1/2}\partial_-)\tilde{\psi}|^2 \Delta^{1/2} \mathrm{d}r_* \right)^{1/2}.$$

The second integral can be bounded by

$$\begin{aligned} \int_{-\infty}^{-u_+} |(\Delta^{-1/2}\partial_-)\tilde{\psi}|^2 \Delta^{1/2} \mathrm{d}r_* &\lesssim \int_{S^2} \int_{-\infty}^{-u_+} |(\Delta^{-1/2}\partial_-)\tilde{\psi}|_2^2 \Delta^{1/2} \mathrm{d}^3\mu_* \\ &\lesssim \int_{S^2} \int_{-\infty}^{-u_+} \tilde{\mathcal{P}}^t_{n_{\Sigma_t},3} \mathrm{d}^3\mu_* \\ &\lesssim \tilde{E}_{n_{\Sigma_t},3}(0). \end{aligned}$$

Since Δ decays exponentially in r_*, which is faster than any polynomial

$$\begin{aligned} \int_{\mathcal{C}} (\partial_-\tilde{\psi}) \mathrm{d}r_* &\le \left(\int_{-\infty}^{-u_+} \Delta^{1/2} \mathrm{d}r_* \right)^{1/2} \tilde{E}_{n_{\Sigma_t},3}(0)^{1/2} \\ &\le u_+^{-1+Ca} \tilde{E}_{n_{\Sigma_t},3}(0)^{1/2}. \end{aligned}$$

Finally, to estimate the integral over Ω, we break the integral into two regions

$$A = \Omega \cap \{r_* > -t/2\},$$
$$B = \Omega \cap \{r_* \leq -t/2\}.$$

We first prove a preliminary estimate in region A. For $-t/2 \leq r_* \leq 0$, since $\Delta/(r^2+a^2)$ is increasing, it follows that, in (t, r_*, θ, ϕ) coordinates

$$\partial_{r_*}\left(\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+^2\tilde{\psi}^2\right) \geq 2\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+\tilde{\psi}^2 + 2\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+^2\tilde{\psi}(\partial_{r_*}\tilde{\psi}).$$

Integrating along a curve of constant (t, θ, ϕ), we have

$$\begin{aligned}
&\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+^2\tilde{\psi}(t, r_*, \theta, \phi)^2 \\
&\quad = -\int_{r_*}^{0} \partial_{r_*}\left(\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+^2\tilde{\psi}^2\right) \mathrm{d}r_* + Ct^2\tilde{\psi}(t, 0, \theta, \phi)^2 \\
&\quad \lesssim -\int_{r_*}^{0} \left(2\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+\tilde{\psi}^2 + 2\left(\frac{\Delta}{r^2+a^2}\right)^{1/2} u_+^2\tilde{\psi}(\partial_{r_*}\tilde{\psi})\right) \mathrm{d}r_* \\
&\qquad + t^2\tilde{\psi}(t, 0, \theta, \phi)^2 \\
&\quad \lesssim \int_{r_*}^{0} \frac{1}{1+r_*^2} u_+^2\tilde{\psi}^2 \mathrm{d}r_* + \int_{r_*}^{0} u_+^2(\partial_{r_*}\tilde{\psi})^2 \mathrm{d}r_* + t^2\tilde{\psi}(t, 0, \theta, \phi)^2 \\
&\quad \lesssim \tilde{E}_{\mathbf{K},3}(t) \\
&\quad \lesssim t^{C|a|}\left(\tilde{E}_{\mathbf{K},3}(0) + \tilde{E}_{\mathbf{T}_\chi,7}(0)\right).
\end{aligned}$$

Therefore,

$$|\tilde{\psi}(t, r_*, \theta, \phi)| \lesssim \left(\frac{\Delta}{r^2+a^2}\right)^{-1/4} t^{-1+C|a|}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + \tilde{E}_{\mathbf{T}_\chi,7}(0)^{1/2}\right).$$

Returning to the argument involving Stokes' theorem, the integral over Ω can be estimated by

$$\begin{aligned}
\int_\Omega (\partial_+\partial_-\tilde{\psi})\mathrm{d}u_+\mathrm{d}u_- &= \int_\Omega (\mathcal{Z}\tilde{\psi} - V\tilde{\psi})\mathrm{d}u_+\mathrm{d}u_- \\
&= 2\int_\Omega (\mathcal{Z}\tilde{\psi} - V\tilde{\psi})\mathrm{d}t\mathrm{d}r_* \\
&\leq 2\sup_A |\Delta^{1/4}\tilde{\psi}|_2 \int_A \Delta^{3/4}\mathrm{d}t\mathrm{d}r_* \\
&\quad + 2\left(\int_B \Delta \mathrm{d}t\mathrm{d}r_*\right)^{1/2}\left(\int_B \frac{1}{\Delta}\left((\mathcal{Z}\tilde{\psi})^2 + (V\tilde{\psi})^2\right)\mathrm{d}t\mathrm{d}r_*\right)^{1/2}.
\end{aligned}$$

We now control each of these terms. We start with the first. By the preliminary estimate in region A, there is the estimate $\Delta^{1/4}|\tilde{\psi}| \lesssim t^{-1+C|a|}(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2})$. After applying up to two more derivatives, and noting $t' \approx u_+'$ in A, we have that the supremum term decays like

$$\sup_A |\Delta^{1/4}\tilde{\psi}|_2 \lesssim u_+^{-1+C|a|}(\tilde{E}_{\mathbf{K},5}(0)^{1/2} + E_{\mathbf{T}_\chi,9}(0)^{1/2}).$$

Since $A \subset \Omega$ has $|t' - 2u_+| < |r_*'|$, the integral in $\int_A \Delta^{3/4}\mathrm{d}t\mathrm{d}r_*$ is bounded by $\int_A 2\Delta^{3/4}r_*\mathrm{d}r_*$, which is bounded, since Δ decays exponentially.

We now turn to the integrals over B. Since $\mathcal{Z}$ and V decay linearly in $r - r_+$, the integral $\int \Delta^{-1}((\mathcal{Z}\tilde{\psi})^2 + (V\tilde{\psi})^2)\mathrm{d}t\mathrm{d}r_*$ is bounded by a multiple of Δ times the square of up to two derivatives of $\tilde{\psi}$. If there were no derivatives on $\tilde{\psi}$, this integral would be bounded by the Morawetz estimate. Therefore, after applying two more angular derivatives and a spherical Sobolev estimate, this integral can be controlled by the Morawez estimate

$$\int_B \frac{1}{\Delta}\left((\mathcal{Z}\tilde{\psi})^2+(V\tilde{\psi})^2\right)\mathrm{d}t\mathrm{d}r_* \lesssim E_{\mathbf{T}_\chi,5}(0).$$

Finally, the integral $\int_B \Delta \mathrm{d}t\mathrm{d}r_*$ is bounded by an integral of the form $\int \Delta r_* \mathrm{d}r_*$, but now the upper bound on r_* is $\approx u_+$, since we are restricted to the region B. Thus, $\int_B \Delta\mathrm{d}t\mathrm{d}r_*$ decays faster than any polynomial in u_+. Combining the estimates in regions A and B, we have

$$\int_\Omega (\partial_+\partial_-\tilde{\psi})\mathrm{d}u_+\mathrm{d}u_- \lesssim u_+^{-1+C|a|}\left(E_{\mathbf{T}_\chi,5}(0)^{1/2}+E_{\mathbf{T}_\chi,9}(0)^{1/2}\right).$$

Thus, from (5.1), the solution $\tilde{\psi}$ decays like

$$|\tilde{\psi}(t,r_*,\theta,\phi-\omega_H t)| \lesssim u_+^{-1+C|a|}\left(\tilde{E}_{\mathbf{K},5}(0)^{1/2}+E_{\mathbf{T}_\chi,9}(0)^{1/2}+\tilde{E}_{n_{\Sigma_t},3}(0)^{1/2}\right).$$

This gives a bound on $\tilde{\psi}$ at $(t,r_*,\theta,\phi-\omega_H t)$. Since the same decay rate holds at all points on the sphere, the desired result holds.

For $u_+\in[0,1]$, the same argument holds to give the boundedness of $\tilde{\psi}$. For $u_+\le 0$, a similar argument holds, except that Ω is a triangle in the (u_+,u_-) plane, with both endpoints on the surface $t=0$. The boundedness of the initial data on $t=0$ in the region $r<3M$, then gives boundedness for $u_+<0$.

Since $\tilde{\psi}$ and ψ are uniformly equivalent for $r<3M$, this gives the desired result. □

5.3. **Far decay.** We now prove decay in the far region, for $r\ge r_\chi$, with particular attention to the behaviour as $r\approx t$ and $r\to\infty$.

To study the far region, we introduce the radial coordinate, y, and almost null coordinates, $v_\pm$,

$$\begin{aligned} y =& \int_0^{r_*} h\mathrm{d}r_*', \\ h =& \sqrt{1-2a^2\frac{\Delta}{(r^2+a^2)^2}} \\ =& \sqrt{1-2a^2\tilde{V}_Q}, \\ v_\pm =& t\pm y. \end{aligned}$$

Since, for large r_*, $h\approx 1-a^2Cr^{-2}$, the coordinate y differs from r_* by at most a constant. Similarly $v_\pm$ differs from $u_\pm = t\pm r_*$ by at most a fixed constant. By direct computation, one finds that the length of the one-form

$$\mathrm{d}v_\pm = \mathrm{d}t \pm h\mathrm{d}r_*$$

is given by

$$\begin{aligned} \frac{\Sigma\Delta}{(r^2+a^2)^2} g(\mathrm{d}v_\pm,\mathrm{d}v_\pm) =& \tilde{\mathcal{G}}^{\alpha\beta}(\mathrm{d}v_\pm)_\alpha(\mathrm{d}v_\pm)_\beta, \\ \tilde{\mathcal{G}}^{\alpha\beta}(\mathrm{d}v_\pm)_\alpha(\mathrm{d}v_\pm)_\beta =& a^2(\sin^2\theta-2)\tilde{V}_Q \le -a^2\tilde{V}_Q < 0. \end{aligned}$$

Thus, the surfaces of constant v_- (or v_+) are timelike. Although we have chosen a particularly convenient form for the factor h, the only property that we use is that $h\approx 1-Cr_*^{-2}$, so that the surfaces are timelike, with a fixed rate at which they degenerate with respect to the $\mathbf{T}_\perp$ and ∂_{r_*} basis, and that the constant in this rate is sufficiently large that $a^2\lesssim C$. We denote the surfaces of constant v_- by

$$\Sigma^+_{v_-}.$$

Unless otherwise stated, we will restrict these to the region where $t > \max(2y, 0)$. Outside this region, we extend the surface as a surface of constant t. It is also convenient to introduce the almost null vectors

$$\xi_\pm = \mathbf{T}_\perp \pm h^{-1}\partial_{r_*}.$$

These satisfy $\xi_+ v_- = 0 = \xi_- v_+$ and $\xi_+ v_+ = 1 = \xi_- v_-$.

Before proving decay in the far region, we control the $\mathbf{K}$ energy on almost null hypersurfaces crossing null infinity.

Lemma 5.3. *There are positive constants $\bar{a}$, C, and C' such that, if $|a| < \bar{a}$, and if $\tilde{\psi}$ is sufficiently smooth, then*

$$C\tilde{E}_{\mathbf{K}}(\Sigma^+_{v_-}) \geq \int_{\Sigma^+_{v_-}} (v_+^2 + 1)\left((\xi_+\tilde{\psi})^2 \frac{|\nabla\!\!\!\!/\tilde{\psi}|^2}{r^2} + V\tilde{\psi}^2\right) dv_+ d^2\mu.$$

Furthermore, if $\tilde{\psi}$ is a solution to the wave equation, (4.1), then

$$\tilde{E}_{\mathbf{K}}(\Sigma^+_{v_-}) \leq C v_-^{C'|a|}\left(\tilde{E}_{\mathbf{K},3}(\Sigma_0) + \tilde{E}_{\mathbf{T}_\chi,7}(\Sigma_0)\right).$$

Proof. It is convenient to introduce the pair of null vectors

$$L_\pm = \tilde{N}^{-1}\mathbf{T}_\perp \pm \partial_{r_*}.$$

The wave equation can be written as

$$(\frac{1}{\mu}\partial_\alpha \mu \tilde{\mathcal{G}}^{\alpha\beta}\partial_\beta - V)\tilde{\psi} = \left(-\frac{1}{2}(L_+L_- + L_-L_+) + \frac{1}{\mu}\partial_\alpha\mu\tilde{h}^{\alpha\beta}\partial_\beta - V\right)\tilde{\psi} = 0.$$

For any vector in the span of $\mathbf{T}_\perp$ and ∂_{r_*},

$$\mathbf{X} = \mathbf{X}^\perp \mathbf{T}_\perp + \mathbf{X}^*\partial_{r_*},$$

the vector can be written as

$$\mathbf{X} = \mathbf{X}^+ L_+ + \mathbf{X}^- L_-,$$
$$\mathbf{X}^\pm = \frac{1}{2}\left(\tilde{N}\mathbf{X}^\perp \pm \mathbf{X}^*\right).$$

The vectors

$$B^\alpha_\pm = -\tilde{\mathcal{G}}^{\alpha\beta}(\mathrm{d}v_\pm)_\beta,$$
$$B_\pm = \tilde{N}^{-2}\mathbf{T}_\perp \mp h\partial_{r_*}$$

have null components (ie, components with respect to the L_+ and L_-)

$$B^\pm_- = \frac{1}{2}\left(\tilde{N} \pm h\right),$$
$$B^\pm_+ = \frac{1}{2}\left(\tilde{N} \mp h\right).$$

We're particularly interested in the components of B_-, which can be estimated by

$$B^+_- \approx 1$$
$$B^-_- \geq a^2\tilde{V}_Q \gtrsim a^2 r^{-2}.$$

Similarly, there is a decomposition of $\mathbf{K}$ as

$$\mathbf{K} = \mathbf{K}^+ L_+ + \mathbf{K}^- L_- - (t^2 + r_*^2 + 1)\omega_\perp\partial_\phi,$$
$$\mathbf{K}^\pm = \frac{1}{2}\left((t^2 + r_*^2 + 1)\tilde{N} \pm 2tr_*\tilde{N}^2\right)$$
$$= \frac{\tilde{N}}{2}(t \pm r_*)^2 + \frac{\tilde{N}}{2} \pm tr_*\tilde{N}(\tilde{N} - 1).$$

For $r_* \geq t/2$ and a small

$$|(t^2+r_*^2+1)\omega_\perp| \lesssim \frac{|a|}{r},$$
$$\mathbf{K}^\pm \gtrsim v_\pm^2+1.$$

The functions (v_-, r_*, θ, ϕ) can be used as a coordinate system. (In fact, the coordinates (v_-, r, θ, ϕ_*) can be used as a coordinate system extending through the (future) event horizon.) Since the new coordinates satisfy

$$\mathrm{d}v_- \wedge \mathrm{d}r_* \wedge \mathrm{d}\mu\theta \wedge \mathrm{d}\phi = \mathrm{d}t \wedge \mathrm{d}r_* \wedge \mathrm{d}\mu\theta \wedge \mathrm{d}\phi,$$

the energy associated with any vector-field $\mathbf{X}$ is the integral over the 3-surface of the contraction of the associated momentum (vector) against the 4-dimensional measure

$$\int_{\Sigma^+_{v_-}} \tilde{\mathcal{P}}_{\mathbf{X}}(\mathrm{d}t \wedge \mathrm{d}r_* \wedge \mathrm{d}\mu\theta \wedge \mathrm{d}\phi) = \int_{\Sigma^+_{v_-}} \tilde{\mathcal{P}}_{\mathbf{X}}(\mathrm{d}v_- \wedge \mathrm{d}r_* \wedge \mu\mathrm{d}\theta \wedge \mathrm{d}\phi)$$
$$= \int_{\Sigma^+_{v_-}} \tilde{\mathcal{P}}_{\mathbf{X}}^{v_-} \mathrm{d}^3\mu_*.$$

In (v_-, r_*, θ, ϕ) coordinates, the energy associated with $\mathbf{K}$ on a surface of constant v_- is the integral of

$$\begin{aligned}
\tilde{\mathcal{P}}_{\mathbf{K}}^{v_-} =&\tilde{\mathcal{P}}_{\mathbf{K}}^{\beta}(\mathrm{d}v_-)_\beta \\
=&(B_-)^\alpha(\mathbf{K}^\beta)(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) \\
&+ \frac{1}{2}\delta_\alpha^\beta(\mathrm{d}v_-)_\beta\mathbf{K}^\alpha\left(-(L_+\tilde{\psi})(L_-\tilde{\psi}) + \tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi})\right) \\
=&(B_-^+)(\mathbf{K}^+)(L_+\tilde{\psi})^2 \\
&+ \left((B_-^+)(\mathbf{K}^-) + (B_-^-)(\mathbf{K}^+) - \frac{1}{2}(\mathrm{d}v_-)_\alpha\mathbf{K}^\alpha\right)(L_+\tilde{\psi})(L_-\tilde{\psi}) \\
&+ (B_-^-)(\mathbf{K}^-)(L_-\tilde{\psi})^2 \\
&+ \frac{1}{2}(\mathrm{d}v_-)_\alpha\mathbf{K}^\alpha\left(\tilde{h}^{\alpha\beta}(\partial_\alpha\tilde{\psi})(\partial_\beta\tilde{\psi}) + V\tilde{\psi}^2\right).
\end{aligned}$$

By direct computation,

$$(\mathrm{d}v_-)_\alpha\mathbf{K}^\alpha =2\left((B_-^+)(\mathbf{K}^-) + (B_-^-)(\mathbf{K}^+)\right),$$

so that

$$\begin{aligned}
\tilde{\mathcal{P}}_{\mathbf{K}}^{v_-} \gtrsim&(v_+^2+1)(L_+\tilde{\psi})^2 \\
&+ (v_-^2+1)\frac{a^2}{r^2}(L_-\tilde{\psi})^2 \\
&+ (v_+^2+v_-^2+1)\left(\frac{|\nabla\!\!\!/\tilde{\psi}|^2}{r^2} + V\tilde{\psi}^2\right).
\end{aligned}$$

Since

$$L_\pm =\tilde{N}^{-1}\partial_t + \tilde{N}\omega_\perp\partial_\phi \pm \partial_{r_*},$$

it follows that

$$|\partial_t\tilde{\psi}|^2 + |\partial_{r_*}\tilde{\psi}|^2 \lesssim |L_+\tilde{\psi}|^2 + |L_-\tilde{\psi}|^2 + \frac{a^2}{r^6}|\nabla\!\!\!/\tilde{\psi}|^2.$$

The difference between ξ_+ and L_+ can be estimated by

$$\begin{aligned}
\xi_+ - L_+ =&(1-\tilde{N}^{-1})\partial_t + (1-h^{-1})\partial_{r_*} + \tilde{N}^{-1}\omega_\perp\partial_\phi,\\
|(\xi_+ - L_+)\tilde\psi|^2 \lesssim&\frac{a^4}{r^4}\left(|\partial_t\tilde\psi|^2 + |\partial_{r_*}\tilde\psi|^2\right) + \frac{a^2}{r^6}|\partial_\phi\tilde\psi|^2\\
\lesssim&\frac{a^4}{r^4}\left(|L_+\tilde\psi|^2 + |L_-\tilde\psi|^2\right) + \frac{a^2}{r^6}|\partial_\phi\tilde\psi|^2.
\end{aligned}$$

A similar estimate holds for ξ_- and L_-. For $r_* \geq t/2$, the desired derivative can be controlled by

$$\begin{aligned}
(v_+^2+1)(\xi_+\tilde\psi)^2 =&(v_+^2+1)(L_+\tilde\psi)^2\\
&+(v_+^2+1)(-2(L_+\tilde\psi)((\xi_+ - L_+)\tilde\psi) + (\xi_+ - L_+)\tilde\psi)^2\\
\lesssim&(v_+^2+1)(L_+\tilde\psi)^2\\
&+(v_+^2+1)\left((\xi_+ - Lp)\tilde\psi\right)^2\\
\lesssim&(v_+^2+1)(L_+\tilde\psi)^2\\
&+\frac{a^4}{r^2}(L_-\tilde\psi)^2\\
&+\frac{a^2}{r^4}|\nabla\!\!\!/\tilde\psi|^2\\
\lesssim&\tilde{\mathcal{P}}_{\mathbf{K}}^{v_-}.
\end{aligned}$$

Thus, the energy on a surface of constant v_-, $\Sigma^+_{v_-}$, associated with the vector $\mathbf{K}$ is

$$\begin{aligned}
\tilde{E}_{\mathbf{K}}(\Sigma^+_{v_-}) =& \int_{\Sigma^+_{v_-}} \tilde{\mathcal{P}}_{\mathbf{K}}^{v_-}\mathrm{d}^3\mu_*\\
\gtrsim& \int_{\Sigma^+_{v_-}} (v_+^2+1)\left((\xi_+\tilde\psi)^2 + \frac{|\nabla\!\!\!/\tilde\psi|^2}{r^2} + V\tilde\psi^2\right)\mathrm{d}^3\mu_*.
\end{aligned}$$

At this point in the argument, we want to show that the energy associated with $q_{\mathbf{K}}$ is negligible so that the slow growth of the $(\mathbf{K}, q_{\mathbf{K}})$ energy implies that the $\mathbf{K}$ energy grows (at most) slowly. The energy associated with the scalar $q_{\mathbf{K}}$ is

$$\begin{aligned}
\tilde{E}_{q_{\mathbf{K}}}(\Sigma^+_{v_-}) =& \int_{\Sigma^+_{v_-}} \tilde{\mathcal{P}}_{q_{\mathbf{K}}}^{v_-}\mathrm{d}^3\mu_*\\
=& \int_{\Sigma^+_{v_-}} \left(\tilde{\mathcal{G}}^{v_-\beta}(\partial_\beta\tilde\psi)q_{\mathbf{K}}\tilde\psi - \frac12\left(\tilde{\mathcal{G}}^{v_-\beta}\partial_\beta q_{\mathbf{K}}\right)\tilde\psi^2\right)\mathrm{d}^3\mu_*.
\end{aligned}$$

If $\Sigma^+_{v_-}$ is understood to be restricted to the region $r_* > t/2$, then

$$\begin{aligned}
\tilde{E}_{q_{\mathbf{K}}}(\Sigma^+_{v_-}) \lesssim& \int_{\Sigma^+_{v_-}} \left(|B_-^\beta\partial_\beta\tilde\psi| a^2 r^{-1}|\tilde\psi| + a^2r^{-2}\tilde\psi^2\right)\mathrm{d}^3\mu_*\\
\lesssim& \int_{\Sigma^+_{v_-}} a^2|B_-^\beta\partial_\beta\tilde\psi|^2\mathrm{d}^3\mu_*\\
&+\int_{\Sigma^+_{v_-}} a^2r^{-2}\tilde\psi^2\mathrm{d}^3\mu_*\\
\lesssim& \int_{\Sigma^+_{v_-}} \left(a^2|L_+\tilde\psi|^2 + \frac{a^2}{r^2}|L_-\tilde\psi|^2\right)\mathrm{d}^3\mu_*\\
&+\int_{\Sigma^+_{v_-}} a^2r^{-2}\tilde\psi^2\mathrm{d}^3\mu_*.
\end{aligned}$$

The integrand in the first integral is controlled by $\tilde{\mathcal{P}}^{v_-}_{\mathbf{K}}$, so that the integral is controlled by $\tilde{E}_{\mathbf{K}}(\Sigma^+_{v_-})$. By a Hardy estimate, we have

$$\begin{aligned}\int_{r_*}^\infty a^2 r^{-2}\tilde{\psi}^2 \mathrm{d}r_* \lesssim& \int_{r_*}^\infty a^2\tilde{\psi}^2\mathrm{d}r_* + \tilde{\psi}(v_-,r_*,\theta,\phi)\\ \lesssim& \int_{r_*}^\infty a^2(\xi_+\tilde{\psi})^2\mathrm{d}r_* + \tilde{\psi}(v_-,r_*,\theta,\phi).\end{aligned}$$

The energy $\tilde{E}_{\mathbf{K},3}$ controls the first, integral term on the right.

To control the second, we prove a preliminary decay result for $0 \leq r_* \leq t/2$,

$$\begin{aligned}|\tilde{\psi}(t,r_*,\theta,\phi)| =& \int_0^{r_*} |\partial_{r_*}\psi|\mathrm{d}r_* + \psi(t,0,\theta,\phi)\\ \lesssim& \left(\int_0^{r_*}\mathrm{d}r_*\right)^{1/2}\left(\int_0^{r_*}|\partial_{r_*}\tilde{\psi}|^2\mathrm{d}r_*\right)^{1/2}\\ &+ t^{-1+C|a|}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right)\\ \lesssim& r_*^{1/2}t^{-1}\tilde{E}_{\mathbf{K},3}(t) + t^{-1+C|a|}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right)\\ \lesssim& t^{-1/2+C|a|}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right).\end{aligned} \tag{5.2}$$

In this calculation, we've used the spherical Sobolev estimate to relate the integral along the curve of constant (t,θ,ϕ) to one over the hypersurface of constant t on which $\tilde{E}_{\mathbf{K},3}(t)$ is evaluated. This provides uniform control on the end point. Since we were interested in the integral over the sphere, it was not necessary to introduce the two, additional derivatives, so that

$$\begin{aligned}&\int_{S^2}\int_{r_*}^\infty a^2 r^{-2}|\tilde{\psi}|_2^2\mathrm{d}^3\mu_*\\ &\leq \int_{S^2}\int_{r_*}^\infty a^2|\xi_+\tilde{\psi}|_2^2\mathrm{d}^3\mu_* + \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right).\end{aligned}$$

Thus, for sufficiently small a,

$$\tilde{E}_{q_{\mathbf{K}}}(\Sigma^+_{v_-}) \lesssim a^2(\tilde{E}_{\mathbf{K}}(\Sigma^+_{v_-}) + \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right)).$$

Theorem 4.13 gives us a bound on the $(\mathbf{K}, q_{\mathbf{K}})$ energy,

$$\tilde{E}_{(\mathbf{K},q_{\mathbf{K}})}(\Sigma^+_{v_-}) \lesssim v_-^{C|a|}\tilde{E}_{(\mathbf{K},q_{\mathbf{K}})}(\Sigma_0).$$

Since the $q_{\mathbf{K}}$ energy is small relative to the $\mathbf{K}$ energy and the initial data, we have the same estimate on the $\mathbf{K}$ energy. □

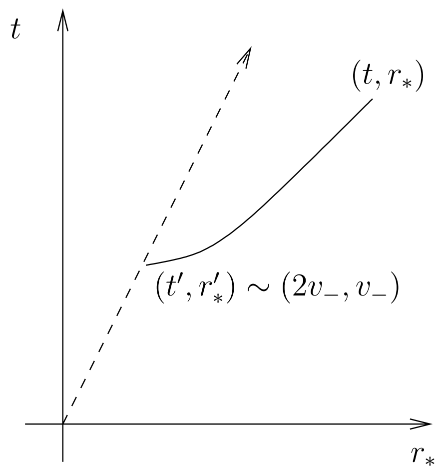
This control of the energy allows us to integrate along almost null surfaces to control the solution ψ near null infinity.

Theorem 5.4. *There are constants $\bar{a}$, C, and C', such that if $|a| \leq \bar{a}$ and ψ is a solution to the wave equation $\Box\psi = 0$, then for all $t > 0$, $r > r_\chi$, $(\theta,\phi) \in S^2$, if $u_- \geq 0$,*

$$\begin{aligned}&|\psi(t,r,\theta,\phi)|\\ &\leq C\max\{1,u_-\}^{C'|a|}\left(\frac{u_+ - u_-}{u_+(\max\{1,u_-\})}\right)^{1/2} r^{-1}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right),\end{aligned}$$

and if $u_- < 0$,

$$\begin{aligned}&|\psi(t,r,\theta,\phi)|\\ &\leq C\max\{1,-u_-\}^{-1/2}r^{-1}\left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2} + sup_{\Sigma_0}\left(r^{3/2}\psi\right)\right).\end{aligned}$$



Proof. As usual, we will start by working with the transformed function $\tilde{\psi}$ which satisfies (4.1).

With the control from the previous lemma, we can integrate along null curves. Consider the curve, $\mathcal{C}$, from (t, r_*, θ, ϕ) with tangent ξ_+ and parameter λ. Along this curve, since the ∂_ϕ component of ξ_- vanishes so rapidly, λ will be uniformally equivalent to t, to r_*, and to v_+.

First, we consider when $u_- \geq 0$. When the curve reaches the surface $r'_* = t'/2$, since $v_- = t - r_* + O(1)$, it will reach a point $(t', r'_*, \theta', \phi') = ((2/3)(t - r_*) + O(1), (1/3)(t-r_*)+O(1), \theta, \phi+O(1))$, at which u'_- will be $t-r_*+O(1) = v_-+O(1)$.

Integrating along the curve,

$$\begin{aligned}
\tilde{\psi}(t, r_*, \theta, \phi) &= \int_{\mathcal{C}} \xi_+ \tilde{\psi} \mathrm{d}\lambda + \tilde{\psi}\left(\frac{2}{3}u_- + O(1), \frac{1}{3}u_- + O(1), \theta, \phi + O(1)\right), \\
|\tilde{\psi}(t, r_*, \theta, \phi)| &\leq \left(\int_{\mathcal{C}} v_+^{-2} \mathrm{d}v_+\right)^{1/2} \left(\int_{\mathcal{C}} v_+^2 (\xi_+ \tilde{\psi})^2 \mathrm{d}r_*\right)^{1/2} \\
&\quad + \left|\tilde{\psi}\left(\frac{2}{3}u_- + O(1), \frac{1}{3}u_- + O(1), \theta, \phi + O(1)\right)\right| \\
&\lesssim \left(-v_+^{-1}|_{u_-+O(1)}^{u_++O(1)}\right)^{1/2} \left(\int_C v_+^2 (\xi_+ \tilde{\psi})^2 \mathrm{d}\lambda\right)^{1/2} \\
&\quad + |u_- + O(1)|^{-1+C|a|} \tilde{E}_{\mathbf{K},3}(0)^{1/2}.
\end{aligned}$$

Applying second-order symmetries, integrating in the angular variables, and applying a spherical Sobolev estimate in the usual way allows us to replace the remaining integral in this expression by the $\mathbf{K}$ energy. The end point is controlled by the preliminary estimate (5.2) in the previous lemma. Combining these we have

$$\begin{aligned}
|\tilde{\psi}(t, r_*, \theta, \phi)| &\lesssim \left(-(u_+ + O(1))^{-1} + (u_- + O(1))^{-1}\right)^{1/2} \tilde{E}_{\mathbf{K},3}(\Sigma^+_{v_-})^{1/2} \\
&\quad + v_+^{-1+C|a|} \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right) \\
&\lesssim \max\{1, u_-\}^{C|a|} \left(\frac{u_+ - u_-}{u_+(\max\{1, u_-\})}\right)^{-1/2} \left(\tilde{E}_{\mathbf{K},3}(0)^{1/2} + E_{\mathbf{T}_\chi,7}(0)^{1/2}\right).
\end{aligned}$$

To obtain the desired estimate, we note that $|\psi| \lesssim r^{-1}|\tilde{\psi}|$.

When $u_- < 0$, a similar argument applies, except that there is no $|u_-|^{C|a|}$ loss arising since the hypersurface of integration lies under the hypersurface $u_- = 0$ and except that the endpoint in the integration is at $(0, r_* - t + O(1), \theta, \phi)$, so that the endpoint is estimated by

$$|\tilde{\psi}(0, t - r_* + O(1), \theta, \phi)| \leq (r_* - t + O(1))^{-1/2} \sup(r^{-3/2}\psi).$$

This dictates the decay rate, since it is slower than the decay rate from $(u_+ - u_-)/(u_+(\max\{1, -u_-\}))$. Thus, the second estimate of this theorem holds. □

Appendix A. Relevance and global structure of Kerr

In this section, we review the physical relevance and global causal structure of the Kerr spacetime. Most of this description can be found in most introductory relativity texts (ie [34, 27, 41]).

The Lorentz metric for the Kerr spacetime is most simply given in terms of the Boyer-Lindquist coordinates (t, r, θ, ϕ) by (1.1). For a fixed constant $R \gg r_+$, one might imagine the region $(t, r, \theta, \phi) \in \mathbb{R} \times (R, \infty) \times S^2$ to represent the region of a spacetime which is a vacuum outside some astrophysical object.[7] The set $(t, r, \theta, \phi) \in \mathbb{R} \times (R, \infty) \times S^2$ can be endowed both with the Kerr metric in Boyer-Lindquist coordinates (1.1) and with the flat Minkowski metric in spherical coordinates $\eta = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$. In this coordinate patch, as $r \to \infty$, all components of g approach those of η, which suggests that the Kerr metric should be seen as asymptotically flat. [8]

At this point, it is useful to review the structure of infinity for the Minkowski spacetime. The Minkowski spacetime can be rewritten in terms of the coordinates (U_+, U_-, θ, ϕ) where $U_\pm = \arctan(t \pm r)$. If one then multiplies the Minkowski metric by the conformal factor $\cos^2 U_+ \cos^2 U_-$, one finds that the Minkowski spacetime can be conformally embedded into a compact subset of a larger spacetime [27]. The boundary of the embedded region is called the conformal boundary or points at infinity. Since conformal transformations preserve null geodesics, it is relatively easy to see that all geodesics terminate (as $t \to \infty$) on the hypersurface $U_+ = \pi/2$ and originate (as $t \to -\infty$) on the hypersurface $U_- = -\pi/2$. These surfaces are referred to as future and past null infinity, $\mathcal{I}^+$ and $\mathcal{I}^-$. All timelike geodesics terminate (as $t \to \infty$) at future timelike infinity i^+, they originiate (as $t \to -\infty$) at past timelike infinity i^-, and all spacelike geodesics originate and terminate (both as $r \to \infty$) at a single point, spacelike infinity, i_0. Since the Kerr spacetime is asymptotically flat, it is possible to construct the hypersurfaces $\mathcal{I}^+$ and $\mathcal{I}^-$ on which out-going (with $t \to \infty$ and $r \to \infty$) and in-going (with $t \to -\infty$ and $r \to \infty$) geodesics terminate and originate respectively.

The Kerr spacetime, and especially the subcase of the Schwarzschild spacetime where $a = 0$, provides the most important illustrations of the concepts of asymptotic flatness, black holes, and event horizons, which are central to general relativity. In an asymptotically flat spacetime, a black hole is the complement of the past of future null infinity, and its boundary is the future part of the event horizon. The event horizon must be a null hypersurface. The domain of outer communication, or more simply the exterior region, is the intersection of the past of future null infinity and the future of past null infinity. Ignoring the physical interpretation given above, one can continue the Kerr spacetime as a vacuum solution of Einstein's equations by continuing it to $r < R$ and even to $r < r_+$. When $|a| \leq M$, the maximally extended Kerr spacetime contains a black hole, the exterior region is given by $r > r_+$, and the event horizon is given by $r = r_+$.

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[7] In spherical symmetry, $a = 0$, it is easy to find explicit, globally smooth solutions of Einstein's equation coupled to matter models which can be written in spherical coordinates, in which there is a spherical material body from $r = 0$ to $r = R$, which contain vacuum for $r > R$, and in which the metric exactly coincides with (1.1) for $r > R$. We are not aware of such solutions for $a \neq 0$.

[8] We omit the precise definitions of asymptotically flat[41] and asymptotically simple[27], since they are quite technical.

The parameters M and a can be understood by comparing the Kerr spacetime in Boyer-Lindquist coordinates to the Minkowski metric in spherical coordinates. Timelike geodesics, representing the paths of small physical bodies, in the Kerr spacetime do not follow the same trajectories as geodesics in the Minkowski metric. The deviation between these geodesics can be treated as a relative acceleration, and, as $r \to \infty$, this acceleration corresponds to that generated by the gravitational force from Newtonian mechanics (also in spherical coordinates) generated by a central object with mass M. This is one reason why the parameter M is interpreted as the mass of the black hole, even though the Kerr spacetime is a solution of Einstein's equation in vacuum.

Alternatively, Einstein's equation can be treated as a Lagrangian theory[36], and the parameters M and a can be understood as energies associated with a vector fields. Normally, an energy is computed on a hypersurface, but, under certain conditions, one can apply integration by parts so that the energy can be computed as an integral over the boundary of the hypersurface. For any asymptotically flat spacetime, this allows energies to be defined with respect to vector fields defined near infinity in Minkowski space. In Minkowski space, the energy associated with ∂_t is referred to as simply the energy. In relativity, one expects that, with the correct choice of units, the energy and mass coincide. In Minkowski space, the energy associated with a rotation about an axis is called the angular momentum about that axis. Computing these quantities on two-surfaces very close to i_0, one finds that the mass of the Kerr black hole is M, that the angular momentum about the axis of symmetry is Ma, and that the angular momentum about orthogonal axes is zero.

The metric (1.1) is clearly smooth in the exterior region, since neither Δ nor Σ vanish. The roots of Δ are at

$$r_\pm = M \pm \sqrt{M^2 - a^2},$$

and the roots of Σ occur when both r and $a\cos\theta$ vanish. Thus, when $0 < |a| < M$, the metric is clearly smooth where $r > r_+$, where $r_- < r < r_+$, and where $r < r_-$ except at $(r,\theta) = (0,\pi/2)$. We shall refer to these regions as the exterior, intermediate, and deep-interior regions respectively and as block I, block II, and block III respectively. Using the coordinates (u_-, r, θ, ϕ_*) (with u_- defined in the introduction, r and θ the standard Boyer-Lindquist coordinates, and $\phi_* = \phi - \omega_H t$ given in section 5.2), it is possible to construct a smooth coordinate chart which covers a copy of both block I and block II.

Through similar choices of changes of coordinates, various copies of the blocks can be be smoothly joined together to form the maximal extension of the Kerr manifold as illustrated in figure A. The copies of block II and block III meet where $r = r_-$. The two-dimensional surfaces where two copies of block I and two copies of block II meet is also a regular point of the maximal extension and is called the bifurcation surface. The diagram in figure A is also a conformal diagram, which includes the conformal boundaries of the exterior regions and also of the deep-interior regions. The deep-interior regions contain singularities, in the sense that there are affinely parameterised geodesics along which $(r,\theta) \to (0,\pi/2)$, on which the affine parameter remains bounded, but which cannot be extended, and it is is not possible to smoothly extend the spacetime so that the geodesics can be extended to $(r,\theta) = (0,\pi/2)$. The diagram does not reveal other patholgical behaviour in the deep interior, in particular, the presence of closed timelike geodesics.

The subcritical ($|a| < M$) Kerr spacetime is expected to have a crucial physical role. From the singularity theorems, it is known that under a wide range of circumstances, spacetimes will develop singularities. The weak cosmic censorship conjecture asserts that, in asymptotically flat spacetimes, future null infinity is separated from any (future) singularity by an event horizon, as long as some

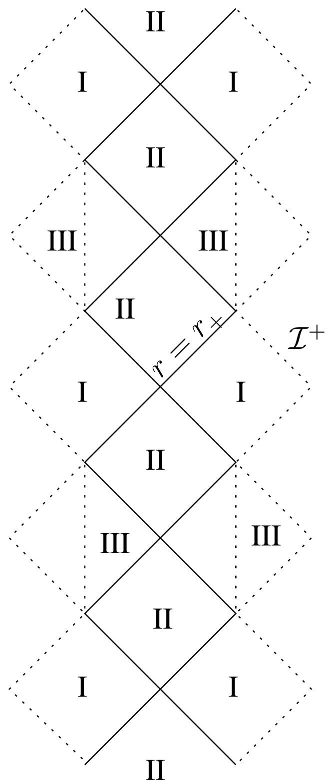

FIGURE 2. The conformal diagram for the Kerr spacetime: Angular variables have been suppressed. The boundaries of blocks are shown in solid lines. Boundaries at infinity and singularities are indicated by dotted lines. The event horizon at $r = r_+$ and $\mathcal{I}^+$ has been indicated for one exterior region. The maximal extension continues with this pattern infinitely forward and backward in time.

as-yet-to-be-determined genericity condition holds. The no hair theorem, or Kerr uniqueness theorem, states that the Kerr family of spacetimes (including Minkowski as $M = 0$) is the unique class of stationary spacetimes satisfying Einstein's equation in vacuum (see [37] for early work, and [2] for recent progress). It is common in physics that systems will relax towards stationary configurations, and, in particular, it is widely expected that solutions to Einstein's equation containing a single black hole will asymptotically approach a Kerr solution plus gravitational radiation going to $\mathcal{I}^+$, at least in the exterior region. A preliminary step in showing that this is true would be to show that the Kerr spacetimes are dynamically stable, i.e. that a small perturbation of the initial data which leads to a Kerr spacetime will generate a solution to Einstein's equation which asymptotically approaches a Kerr spacetime plus radiation.

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