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# Triangle-duality and equality of infinitely many bond percolation thresholds

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## Abstract

Recent research on percolation has led to the construction of an infinite class of lattices for which the percolation thresholds can be determined exactly. We discuss the mathematical basis for the solutions of bond percolation models, and, in particular, attempt to address some misconceptions that might arise from the seminal articles by Ziff and Scullard. As one consequence, we show that there exist infinitely many real numbers  $a \in [0, 1]$  for which there are infinitely many lattices that have bond percolation threshold equal to  $a$ .

## 1 Introduction

### 1.1 Bond Percolation

Percolation is an infinite random lattice graph model, which serves as the simplest example of a process exhibiting a phase transition. Even so, it provides some extremely challenging problems. Its study provides intuition for more elaborate statistical mechanics models. Due to its focus on clustering and connectivity phenomena, it is applied widely to problems such as magnetism and conductivity in materials, the spread of epidemics, fluid flow in a random porous medium, and gelation in polymer systems. Percolation models are studied extensively in both the mathematical and scientific literature. See Bollobás and Riordan [1], Grimmett [2], Hughes [3], and Kesten [4] for a comprehensive discussion of mathematical percolation theory, Stauffer and Aharony [5] for a physical science perspective, and Sahimi [6] for engineering science applications.

The bond percolation model may be described as follows. Consider an infinite connected graph  $G$ . Each edge of  $G$  is randomly declared to be open (respectively, closed) with probability  $p$  (respectively,  $1 - p$ ) independently of all other edges, where  $0 \leq p \leq 1$ . The corresponding parameterized family of product measures on configurations of edges is denoted by  $P_p$ . For each vertex  $v \in G$ , let  $C(v)$  be the open cluster containing  $v$ , i.e. the connected component of the subgraph of open edges in  $G$  containing  $v$ . Let  $|C(v)|$  denote the number of vertices in  $C(v)$ . The percolation threshold of the bond percolation model on  $G$ , denoted  $p_c(G \text{ bond})$ , is the unique real number such that

$$p > p_c(G \text{ bond}) \implies P_p(\exists v \text{ such that } |C(v)| = \infty) > 0 \quad (1)$$

and

$$p < p_c(G \text{ bond}) \implies P_p(\exists v \text{ such that } |C(v)| = \infty) = 0. \quad (2)$$

For over fifty years since the origins of percolation theory by Broadbent and Hammersley [7], the determination of critical probabilities has been a challenging problem. Until recently, exact solutions had been proved only for arbitrary trees [8] and a small number of periodic two-dimensional graphs [9, 4, 10, 11]. These results were obtained using graph duality and a star-triangle transformation. Scullard [12] introduced a generalized star-triangle transformation which allowed prediction of the exact stie percolation threshold for the so-called “martini” lattice. A triangle-triangle transformation and concept of triangle-duality was introduced by Ziff [13] and further developed by Ziff and Scullard [14, 15] and Chayes and Lei [16]. Triangle-duality allows calculation of exact thresholds for a collection of “martini-like” lattices.

## 1.2 The Triangle-Duality Construction

We first briefly and loosely describe the triangle-duality approach, in the context of bond percolation, with a slightly different perspective: We consider constructing a lattice graph rather than decomposing one. Consider an arrangement of non-overlapping triangular regions in the plane, with triangles touching only at their vertices. For convenience, it is sometimes desirable to represent the triangles as slightly concave, as illustrated in Figure 2.1. Such an arrangement may be transformed into another arrangement via the “triangle-triangle transformation,” in which each triangle is replaced by a “reversed triangle” as shown in Figure 3. If the resulting (dual) triangular arrangement is equivalent to the original arrangement, the arrangement is called “self-dual under the triangle-triangle transformation” by Ziff and Scullard. If the triangular arrangement is self-dual, then a lattice may be constructed by replacing each triangular region by a network of bonds which has vertices at all three vertices of the triangle. Such a network will be called the *generator* of the lattice. From such a generator, it may be possible to construct a dual generator, which creates another lattice when replacing the triangles in the dual triangular arrangement. By solving an equation derived from the connection probabilities in the generator and dual generator, a solution for the percolation threshold is obtained.

There are several issues in the description above which are ambiguous. Some important questions are: In what way must the reversed triangles be connected? What is meant by “equivalent” arrangements? What conditions must the generator satisfy? How must the dual generator be constructed? What connection probability equations are involved? Unfortunately, in the Ziff and Scullard papers, some of these issues are not explicitly addressed or statements are made which might lead to misinterpretation and misapplications of the approach. One goal of this article is to

make explicit some assumptions which may have been implicit in the previous papers, in order to prevent future misconceptions regarding these issues.

In the remainder of this article, we provide a careful discussion and justification of conditions which provide a valid exact solution for the bond percolation threshold, in the natural framework of 3-regular hypergraphs. Here we only note some of the most important remarks relating to the issues above.

(1) For bond percolation exact threshold determinations, it is crucial that the generator be a planar graph. While the examples and results given by Ziff and Scullard are for planar generators, the papers contain some statements that suggest much broader applicability: On page 2 of [15] it is said that the generator “can represent any network of bonds, correlated bonds, and sites, as long as they are contained between three vertices.” Page 1 of [13], states that “triangles represent any network of bonds, perhaps with correlations, that connect the three end points.” All exact percolation threshold solutions for periodic lattices depend on duality, which requires planarity of the graph. The given construction of a dual generator uses duality, and there is no proposed method for constructing a dual generator if the original generator is non-planar.

(2) Care must be taken when constructing the dual triangular arrangement, since the reversed triangles must be connected in a precise manner in order to create a proper dual structure. The way the reversed triangles are connected in the empty faces of the original structure is important. The reversed triangles do not need to be the same size or shape as the original triangles, but may need to be distorted instead of simply reversed. The example of the Kagomé lattice in Figure 2 of [15] illustrates this issue, but the text did not emphasize this crucial fact.

(3) In order to prove that the solution is valid, the resulting lattice graph must be periodic and have at least one axis of symmetry. While it is certainly plausible that the method may be valid for models that do not satisfy these conditions, there is currently no mathematical proof that such solutions are correct.

### 1.3 Equality of Percolation Thresholds

Note that, for lattices constructed by this method, the value of the bond percolation threshold is determined by equations describing the connectivity within the generator. Therefore, using the same generator in multiple self-dual triangular arrangements produces multiple lattices with equal bond percolation thresholds. Ziff and Scullard [15](Figures 1 and 6) illustrate three different self-dual arrangements, and remark that “many other self-dual arrangements can be constructed.” In section 7, we show that there are infinitely many self-dual arrangements, so each generator satisfying the appropriate conditions will generate an infinite set of lattices with equal percolation thresholds. Previously, it was only known that there were infinitely many lattices with bond percolation threshold equal to one-half, since Wierman [17] provided a construction for infinitely many periodic self-dual lattices. Since we also construct a sequence of nested generators which must give unequal percolation thresholds, we show that there are infinitely many values  $a$  for which there are infinitely many lattices with bond percolation threshold equal to  $a$ . The result also holds for site percolation thresholds, due to the bond-to-site transformation.

## 2 Background and Definitions

### 2.1 3-Uniform Hypergraphs

Given a set  $V$  of vertices, a hyperedge  $H$  is a subset of  $V$ . A hyperedge  $H$  is said to be incident to each of its vertices. A  $k$ -hyperedge is hyperedge containing exactly  $k$  vertices. In order to neglect the detailed structure of our generators, at times we will view a generator as a 3-hyperedge, and will represent it in the plane as a shaded triangular region bounded by a slightly concave triangular boundary.

A hypergraph is a vertex set  $V$  together with a set of hyperedges of vertices in  $V$ . A hypergraph containing only  $k$ -hyperedges is a  $k$ -uniform hypergraph. A hypergraph is planar if it can be embedded in the plane with each hyperedge represented by a bounded region enclosed by a simple closed curve with its vertices on the boundary, such that the intersection of two hyperedges is a set of vertices in  $V$ .

In order to construct exactly-solvable lattice graphs for bond percolation models, we will consider infinite connected planar periodic 3-uniform hypergraphs. A planar hypergraph  $H$  is periodic if there exists an embedding with a pair of basis vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $H$  is invariant under translation by any integer linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ , and such that every compact set of the plane is intersected by only finitely many hyperedges.

If a hypergraph  $H$  is planar, we may construct a dual hypergraph  $H^*$  as follows. Place a vertex of  $H^*$  in each face of  $H$ . For each hyperedge  $e$  of  $H$ , construct a hyperedge  $e^*$  of  $H^*$  consisting of the vertices in the faces surrounding  $e$ . Note that if the hyperedge is a 3-hyperedge represented by a triangular region, and each of the boundary vertices is in at least two hyperedges, then the dual hyperedge is a 3-hyperedge also, represented by a “reversed triangle.”

Two hypergraphs are isomorphic if there is a one-to-one correspondence between their vertex sets which preserves all hyperedges. A hypergraph is self-dual if its dual is isomorphic to its dual. If the hypergraph is 3-uniform, this corresponds to the term triangle-dual used by Ziff and Scullard. To illustrate, in Figure 1 we provide three examples of infinite connected planar periodic self-dual 3-uniform hypergraphs mentioned by Ziff and Scullard.

As a particular caution, note that the dual hypergraph is not obtained by simply reversing or flipping the triangles in the original hypergraph. The reversed triangles must be connected in a specific manner in order to create a proper dual structure. The way the reversed triangles are connected in the empty faces of the original structure is important. The reversed triangles do not need to be the same size or shape as the original triangles, but may need to be distorted instead of simply reversed. An example of a hypergraph which appears to be self-dual if one reverses each triangle, but is actually not self-dual, is given in Figure 2.

## 3 Generators and Duality

A planar graph can be embedded in the plane so that edges meet only at their endpoints, which divides the plane into regions bounded by edges, called “faces.” If the planar graph is finite and connected, one of these faces is unbounded. A generator is a finite connected planar graph embedded in the plane so that three vertices on the unbounded face are designated as boundary vertices A, B, and C.

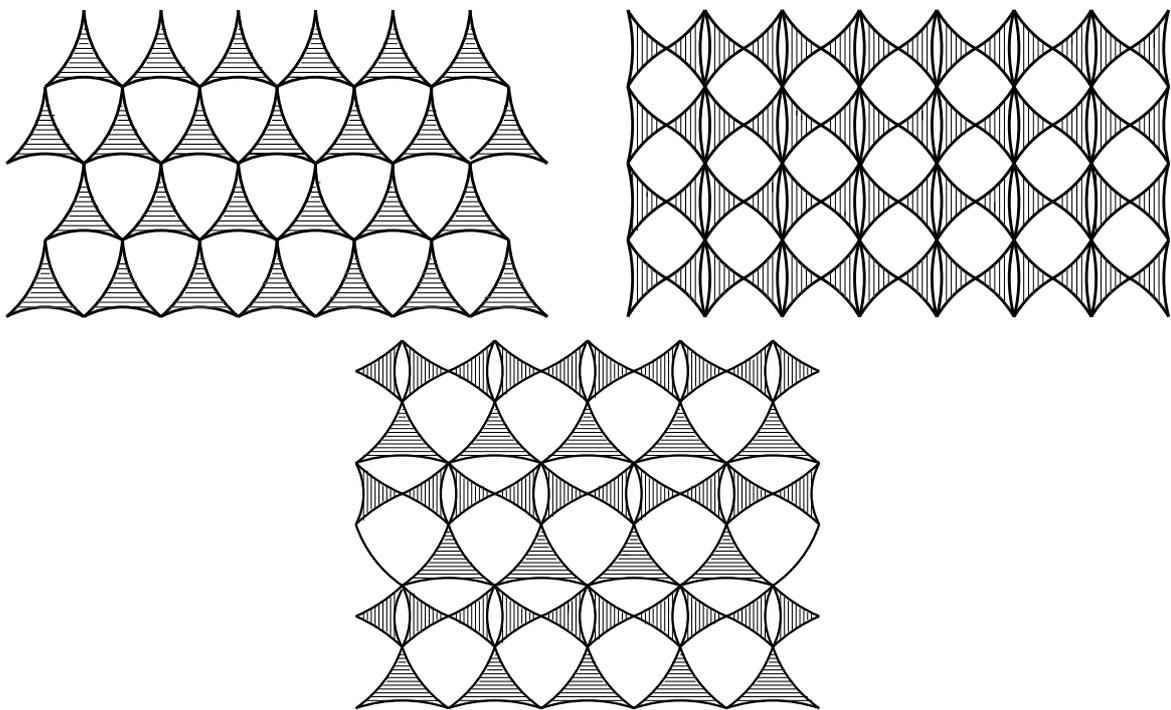


Figure 1: Self-dual hypergraph arrangements illustrated in Ziff and Scullard. In the top row, we refer to the left as the triangular arrangement, the right as the bow-tie arrangement. The third example alternates rows of triangles and bow-ties.

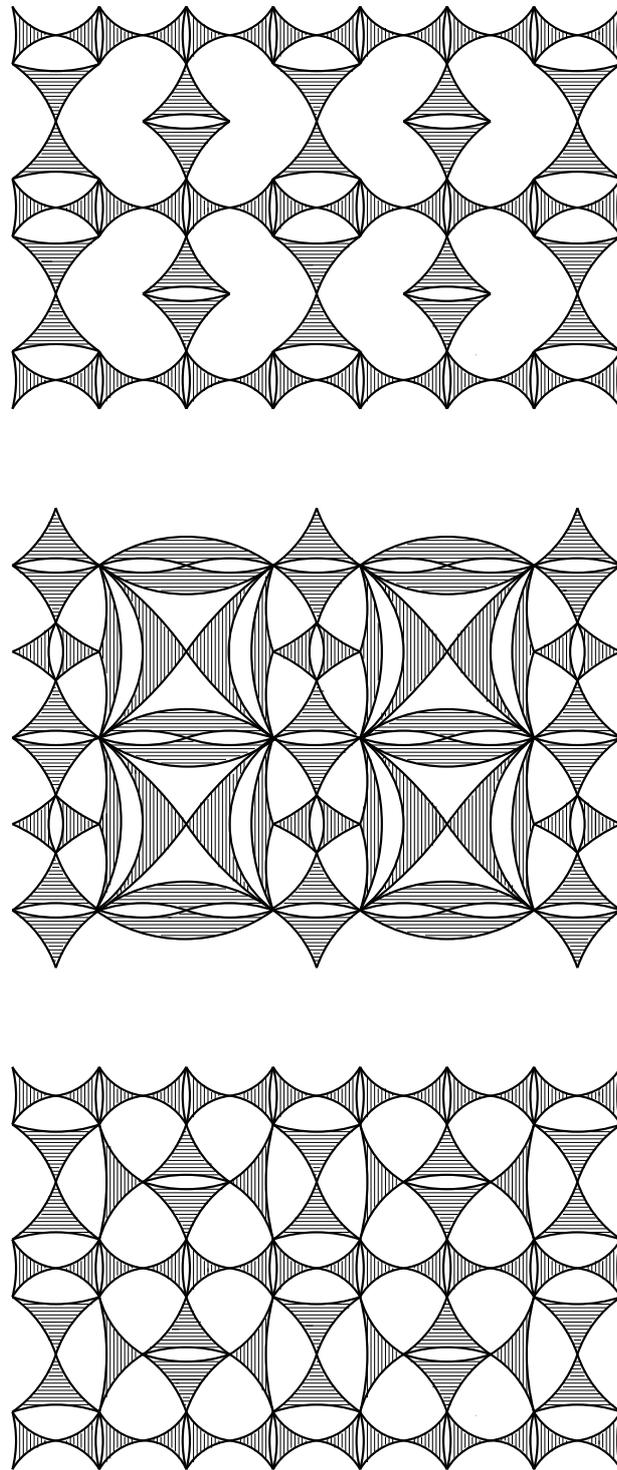


Figure 2: Top: A example of a hypergraph which is not self-dual, but appears to be if one simply reverses each triangle. Middle: The dual of the hypergraph above. Bottom: A self-dual hypergraph constructed by inserting additional 3-hyperedges in the top arrangement.

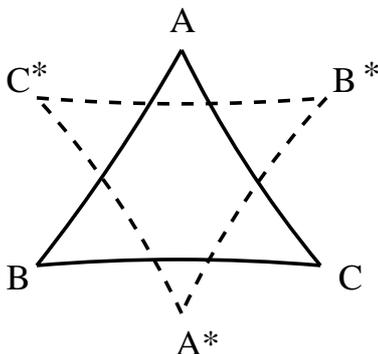


Figure 3: Solid lines represent a 3-hyperedge with boundary vertices  $A$ ,  $B$ , and  $C$ . Dashed lines represent the “reversed” or dual hyperedge, with its boundary vertices  $A^*$ ,  $B^*$ , and  $C^*$  labeled in the proper positions.

Given a generator  $G$ , we construct a dual generator  $G^*$  by placing a vertex in each bounded face of  $G$ , and three vertices  $A^*$ ,  $B^*$ , and  $C^*$  of  $G^*$  in the infinite face of  $G$ , as follows: The boundary of the infinite face can be decomposed into three (possibly intersecting) paths, from  $A$  to  $B$ ,  $B$  to  $C$ , and  $C$  to  $A$ . The infinite face may be partitioned into three infinite regions by three non-intersecting polygonal lines starting from  $A$ ,  $B$ , and  $C$ . Place  $A^*$  in the region containing the boundary path connecting  $B$  and  $C$ ,  $B^*$  in the region containing the boundary path connecting  $A$  and  $C$ , and  $C^*$  in the region containing the boundary path connecting  $A$  and  $B$ .  $A^*$ ,  $B^*$ , and  $C^*$  are the boundary vertices of  $G^*$ .

For each edge  $e$  of  $G$ , construct an edge  $e^*$  of  $G^*$  which crosses  $e$  and connects the vertices in the faces on opposite sides of  $e$ . If  $e$  is on the boundary of the infinite face, connect it to  $A^*$  if  $e$  is on the boundary path between  $B$  and  $C$ , to  $B^*$  if  $e$  is between  $A$  and  $C$ , and connect it to  $C^*$  if  $e$  is between  $A$  and  $B$ . (Note that it is possible for  $e^*$  to connect more than one of  $A^*$ ,  $B^*$ , and  $C^*$ , for example, if there is a single edge incident to  $A$  in  $G$ , so its dual edge connects  $B^*$  and  $C^*$ .)

Note that  $G^*$  is not the dual graph of  $G$ , which would have only one vertex in the infinite face. The three vertices  $A^*$ ,  $B^*$ , and  $C^*$  will correspond to separate faces of the lattice  $L_G$  generated from  $G$ .

Given a planar generator  $G$  and a connected periodic self-dual 3-regular hypergraph  $\mathcal{H}$ , a dual pair of periodic lattices may be constructed as follows: Construct a lattice graph  $L_{G,\mathcal{H}}$  by replacing each hyperedge of  $\mathcal{H}$  by a copy of the generator  $G$ , with the boundary vertices of the generator corresponding to the vertices of the hyperedge, in such a manner that the resulting lattice is periodic. This is always possible, by choosing the orientations of the generator in one period of the hypergraph, and extending the choice periodically. (However, for a generator without sufficient symmetry, it may be possible to replace hyperedges in a way that produces a non-periodic lattice, so some care is needed.)

We now construct a lattice  $L_{G^*,\mathcal{H}^*}$  as follows: Construct the embedding of the dual hypergraph  $\mathcal{H}^*$  in the plane, in which every hyperedge of  $\mathcal{H}$  is reversed. Replace each hyperedge of  $\mathcal{H}^*$  by a copy of the dual generator  $G^*$ , embedded so that it is consistent with the embedding of  $G$ , that is, in all hyperedges boundary vertex  $A'$  in  $G^*$  is opposite vertex  $A$  in  $G$ ,  $B'$  is opposite  $B$ , and  $C'$  is opposite  $C$ , and each edge of  $G^*$  crosses the appropriate edge of  $G$ . This results in a simultaneous embedding of  $L_{G^*,\mathcal{H}^*}$  and  $L_{G,\mathcal{H}}$ . An example of the construction for a particular generator is illustrated in Figure

3.

The constructions of the two lattices both produce a planar representation of the resulting lattice. From the simultaneous embeddings of the two lattices, it is seen that  $L_{G^*,\mathcal{H}^*}$  is the dual lattice of  $L_{G,\mathcal{H}}$ , since there is a one-to-one correspondence between vertices of one and faces of the other, and a one-to-one correspondence between edges, which are paired by crossing. (Note that the position of the boundary vertices in  $L_{G^*,\mathcal{H}^*}$  is completely determined by the positions of boundary vertices in  $L_{G,\mathcal{H}}$ . Rotations or reflections of the generator  $G^*$  for any hyperedge may not produce a dual pair of lattices.)

## 4 Reduction to a Single Equation

Consider a generator  $G$  and its dual generator  $G^*$ . In each case, denote the three boundary vertices by  $A$ ,  $B$ , and  $C$  listed counterclockwise around the triangle from the initial vertex. Any configuration (i.e., designation of edges or vertices as open or closed) on  $G$  determines a partition of the boundary vertices into clusters of vertices that are connected by open edges. Each such “boundary partition” may be denoted by a sequence of vertices and vertical bars, where vertices are in distinct open clusters if and only if they are separated by a vertical bar. For example,  $AB|C$  indicates that, within  $G$ , the vertices  $A$  and  $B$  are in the same open cluster, but  $C$  is in a separate cluster.

Given a planar embedding of the lattice  $L_{G,\mathcal{H}}$  and a planar embedding of  $L_{G^*,\mathcal{H}^*}$  with each edge crossing its dual edge, we may define coupled percolation models. Let each edge of  $L_{G,\mathcal{H}}$  be open with probability  $p$  independently of all other edges, and define each edge of  $L_{G^*,\mathcal{H}^*}$  to be open if and only if its dual edge is open.

Suppose we have two bond percolation models on  $L_{G,\mathcal{H}}$  and  $L_{G^*,\mathcal{H}^*}$ , perhaps with different edge probability parameters  $p$  and  $q$ , each assigning probability to configurations on  $G$  and  $G^*$ , respectively. The probability, denoted  $P_p^G(\pi)$  or  $P_q^{G^*}(\pi)$  respectively, for the partition  $\pi$  is determined by summing the probabilities of all configurations that produce the partition  $\pi$  of the boundary vertices.

The set of boundary partitions is a partially ordered set (poset). A partition  $\pi$  is a *refinement* of  $\sigma$ , denoted  $\pi \leq \sigma$ , if every cluster of  $\pi$  is contained entirely in a cluster of  $\sigma$ . The set of boundary partitions ordered by refinement is a combinatorial lattice, called the partition lattice.

Thus, we have two probability measures,  $P_p^G$  and  $P_q^{G^*}$  on the partition lattice, which summarize connections between the boundary vertices without explicitly referring to the detailed structure of the generator and its dual. The remarkable fact that allows exact bond percolation threshold values to be obtained is that it is possible to choose the parameters  $p$  and  $q$  so that the two probability measures are exactly equal. (Note that in cases with more boundary vertices, where the probability measures cannot be made equal, the concept of stochastic ordering of probability measures may be used to determine mathematically rigorous bounds for percolation thresholds, using the substitution method [18, 19, 20, 21, 22, 23, 24].)

By the duality relationship between  $G$  and  $G^*$ , we have that for each configuration of open and closed edges, the following five statements hold:

1.  $A$ ,  $B$ , and  $C$  are connected by open paths if and only if  $A^*$ ,  $B^*$ , and  $C^*$  are in separate closed components.

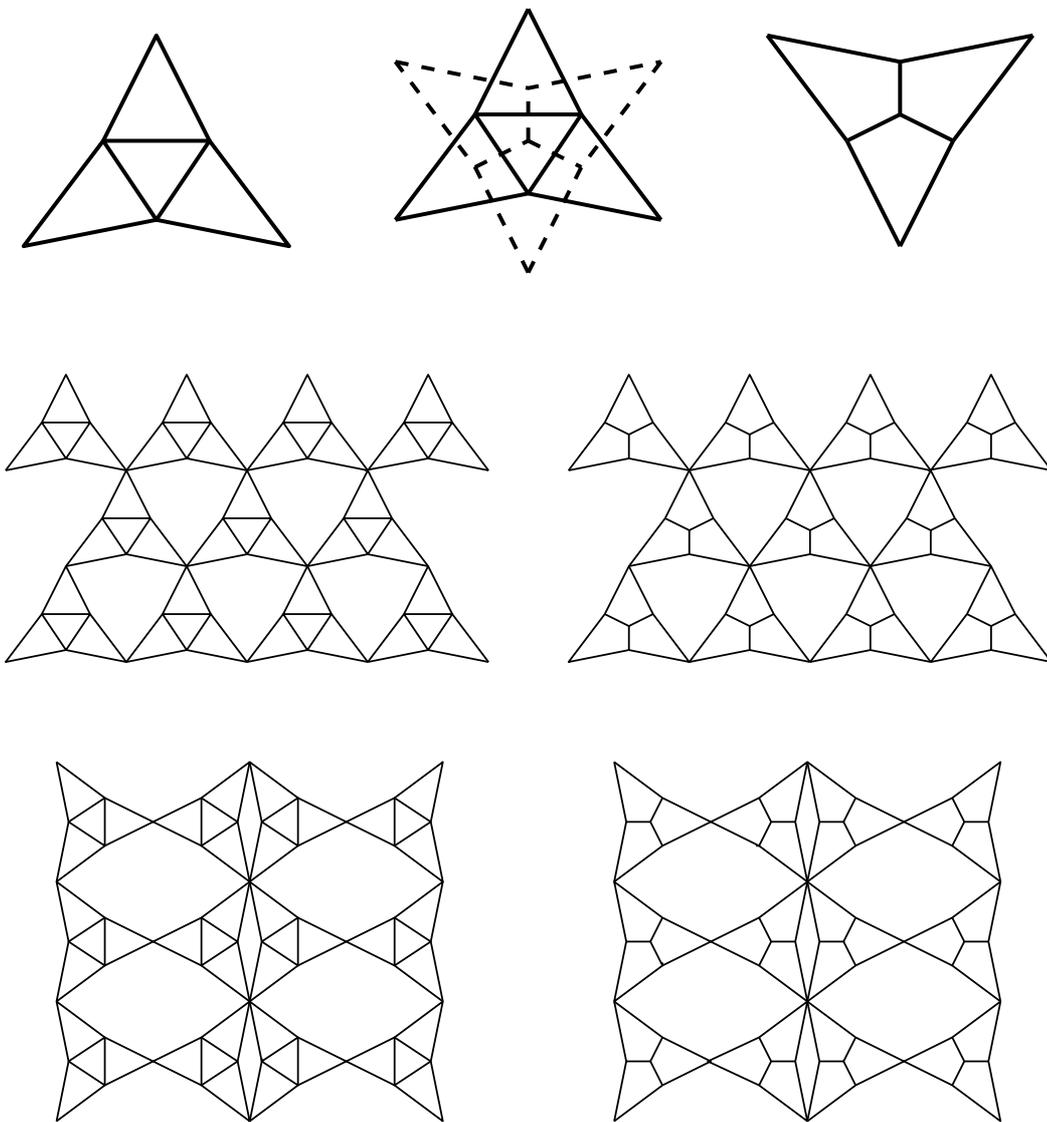


Figure 4: The construction of lattices based on a specific generator. Top: The generator, the duality relationship, and the dual generator. Middle: The lattices based on the generator and the triangular hypergraph arrangement. Bottom: The lattices based on the generator and the bow-tie hypergraph arrangement.

2.  $A$  and  $B$  are connected by an open path, but  $C$  is in a separate open component if and only if  $A^*$  and  $B^*$  are connected by a closed path, but  $C^*$  is in a separate closed component.
3.  $A$  and  $C$  are connected by an open path, but  $B$  is in a separate closed component, if and only if  $A^*$  and  $C^*$  are connected by a closed path, but  $B^*$  is in a separate closed component.
4.  $B$  and  $C$  are connected by an open path, but  $A$  is in a separate closed component, if and only if  $B^*$  and  $C^*$  are connected by a closed path, but  $A^*$  is in a separate closed component.
5.  $A$ ,  $B$ , and  $C$  are in separate open components if and only if  $A^*$ ,  $B^*$ , and  $C^*$  are connected by closed paths.

While these statements are intuitively clear by drawing diagrams, the proofs of these statements rely on duality. However, since the dual generator is not the dual graph of the generator, some additional vertices and edges must be added to apply duality results. An example of such reasoning is given in Smythe and Wierman [25, pp. 8-9].

When considering the random configurations induced by a percolation model, the five statements become statements of equality of events, which then have equal probabilities, yielding

$$P_p^G[ABC] = P_{1-p}^{G^*}[A^*|B^*|C^*], \quad (3)$$

$$P_p^G[AB|C] = P_{1-p}^{G^*}[A^*B^*|C^*], \quad (4)$$

$$P_p^G[AC|B] = P_{1-p}^{G^*}[A^*C^*|B^*], \quad (5)$$

$$P_p^G[A|BC] = P_{1-p}^{G^*}[A^*|B^*C^*], \quad (6)$$

$$P_p^G[A|B|C] = P_{1-p}^{G^*}[A^*B^*C^*]. \quad (7)$$

Since  $p$  is still a free parameter, we may choose it to satisfy

$$P_p^G[ABC] = P_{1-p}^{G^*}[A^*B^*C^*]. \quad (8)$$

This equation always has a solution in  $[0,1]$  since the left side is an increasing polynomial function of  $p$  while the right side is decreasing polynomial, both with values varying between 0 and 1. With this choice of  $p$ , the four probabilities in the first and last equations are equal, and, in fact, the two probability measures are equal. Thus, it is equivalent to solve the *connectivity equations*

$$P_p^G[ABC] = P_p^G[A|B|C], \quad (9)$$

equating the probabilities of all boundary vertices connected and no boundary vertices connected in the generator of the lattice. In the following, we denote the solution to this polynomial equation by  $p_0$ .

## 5 Proof of Exact Solution

To verify that the solution of the equation provides the exact value of the percolation threshold, we may rely on relatively standard results in mathematical percolation theory. Kesten [4] proved that for a dual pair of periodic planar lattices,  $L$  and  $L^*$ , with at least one axis of symmetry,

$$p_c(L) + p_c(L^*) = 1 \quad (10)$$

and that various definitions of the percolation threshold, denoted by  $p_H$ ,  $p_T$ , and  $p_S$  are all equal for each lattice. (The result was actually stated for the more general setting of site models, and applies to bond models by first applying the bond-to-site transformation to obtain an equivalent site model.) To apply Kesten's result to lattices constructed by the method in this article, we use the  $p_S$  concept of the percolation threshold.

For a periodic graph, the percolation threshold  $p_S$  is defined in terms of the asymptotic behavior of probabilities that open clusters connect opposite sides of rectangles in sequences of similar rectangles whose areas are increasing to infinity. In the following, we give a brief general description which can be adapted to each of the lattices in our construction, but avoiding some technicalities that are specific to each lattice.

Now let  $L$  be a lattice constructed in this article, and let  $L^*$  denote its dual lattice.  $L$  may be embedded periodically in the plane such that the unit vectors in the  $x$ - and  $y$ -axis directions are the basis vectors and the  $x$ -axis is an axis of symmetry. For each pair of positive integers,  $m$  and  $n$ , let  $L_{m \times n}$  denote the graph containing all edges of hyperedges of  $L$  that intersect the region  $[0, m] \times [0, n]$ . We can specify sets of vertices of  $L_{m \times n}$  which we call its top, bottom, left, and right sides. We are interested in the event that there is an open path which crosses  $L_{m \times n}$  from left to right, or top to bottom, which we denote by  $\{open \leftrightarrow L_{m \times n}\}$ , or  $\{open \downarrow L_{m \times n}\}$ , respectively. Similarly, we consider also closed crossings. The percolation threshold  $p_S$  is defined as

$$p_S = \sup\{p : \lim_{n \rightarrow \infty} P_p[open \downarrow L_{3n \times n}] = 0 \text{ and } \lim_{n \rightarrow \infty} P_p[open \leftrightarrow L_{n \times 3n}] = 0\}. \quad (11)$$

In the following, we will use square regions, noting that if  $\limsup_{n \rightarrow \infty} P_p[open \leftrightarrow L_{n \times n}] > 0$ , then  $\limsup_{n \rightarrow \infty} P_p[open \leftrightarrow L_{n \times 3n}] > 0$ , and similarly for vertical crossings. In either case,  $p > p_S$ .

Now let  $L_{n \times m}^*$  denote the graph corresponding to the dual hyperedges of the hyperedges in  $L_{n \times m}$ . For  $L_{n \times m}^*$ , we can also specify sets of vertices of the lattices which we can call the top, bottom, left, and right sides of these regions, and define similar crossing events, in such a manner that duality yields

$$P_{p_0}[open \leftrightarrow \text{ in } L_{n \times n}] + P_{p_0}[closed \uparrow \text{ in } L_{n \times n}^*] = 1, \quad (12)$$

which implies that

$$P_{p_0}[open \leftrightarrow \text{ in } L_{n \times n}] + P_{1-p_0}[open \uparrow \text{ in } L_{n \times n}^*] = 1. \quad (13)$$

We now replace generators of  $L^*$  by generators of  $L$ . In the duality relationship, each hyperedge in  $L^*$  is obtained by rotating the corresponding hyperedge of  $L$  by  $180^\circ$  degrees, so this replacement results in a copy of  $L$  reflected through some horizontal line. However, the subgraph of  $L$  corresponding to the replacement of hyperedges in  $L_{n \times n}^*$  may not be isomorphic to  $L_{n \times n}$ . For convenience, the following argument is written as if it is isomorphic. (If not, by monotonicity, we may consider a rectangle which is slightly smaller vertically and slightly larger horizontally, which is sufficient to obtain the conclusion.)

While replacing the hyperedges, by using the equality of connectivities at parameters  $p_0$  in  $L$  and  $1 - p_0$  in  $L^*$ , we have

$$P_{p_0}[\textit{open} \leftrightarrow \textit{ in } L_{n \times n}] + P_{p_0}[\textit{open} \updownarrow \textit{ in } L_{n \times n}] = 1. \quad (14)$$

As a consequence, either

$$\limsup_{n \rightarrow \infty} P_{p_0}[\textit{open} \leftrightarrow \textit{ in } L_{n \times n}] \geq \frac{1}{2} \quad (15)$$

or

$$\limsup_{n \rightarrow \infty} P_{p_0}[\textit{open} \updownarrow \textit{ in } L_{n \times n}] \geq \frac{1}{2}, \quad (16)$$

(or both) so

$$p_S(L) \leq p_0. \quad (17)$$

Reversing the roles of  $L$  and  $L^*$ , we also obtain

$$p_S(L^*) \leq 1 - p_0. \quad (18)$$

Therefore,

$$1 \leq p_c(L) + p_c(L^*) = p_S(L) + p_S(L^*) \leq p_0 + (1 - p_0) = 1, \quad (19)$$

so the percolation thresholds must equal their upper bounds, yielding

$$p_c(L) = p_0 \quad (20)$$

and

$$p_c(L^*) = 1 - p_0. \quad (21)$$

For lattices that do not have an axis of symmetry, there is no general proof that  $p_c(L) + p_c(L^*) = 1$ , in which case the reasoning above is not valid. However, Grimmett [2] (Section 11.9) provides a proof for a multi-parameter bond model on the triangle lattice which is asymmetric, which presents special difficulties. It is plausible that his reasoning may be adapted to the lattices constructed in this article, and research is in progress on this issue.

## 6 Subdivided and Split Edges

There are two situations which can produce lattices which some people may consider “non-physical.” If the generator has a boundary vertex with only one incident edge, it is possible for the resulting lattice to have two edges in series. If the generator has an edge which is incident to two boundary vertices, it is possible for the resulting lattice to have double edges, i.e., two edges between the same pair of vertices.

These problems cannot occur for some self-dual hypergraphs: In any hypergraph for which each boundary vertex is shared by three hyperedges, such as the triangular lattice arrangement, each boundary vertex has at least three incident edges, and thus there is no edges in series. In any hypergraph in which any two hyperedges intersect in at most one boundary vertex, such as the triangular arrangement, there cannot be parallel edges.

The generators illustrated in Figures 3 and 8 do not produce subdivided or split edges in any self-dual hypergraph arrangement.

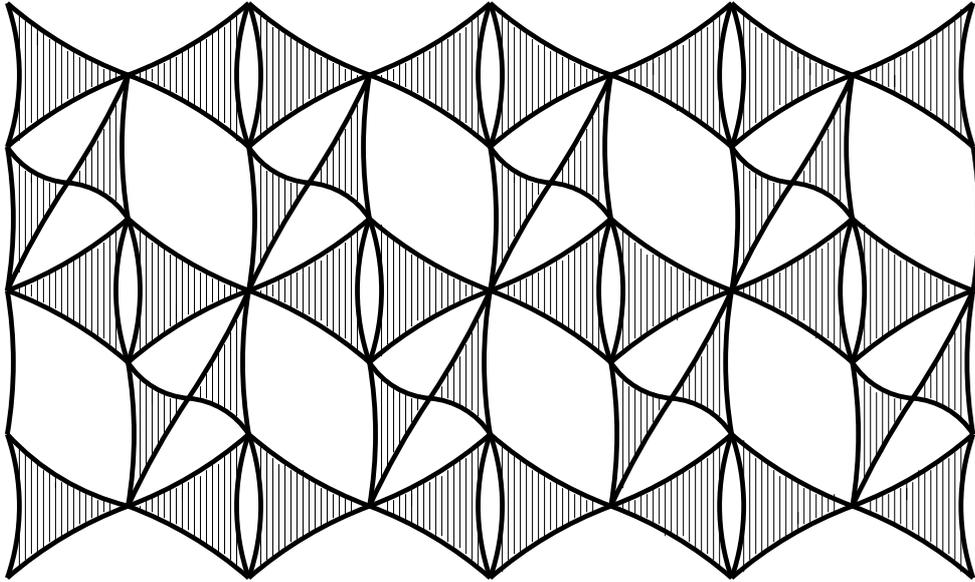


Figure 5: A self-dual hypergraph.

However, for other hypergraph arrangements, such as the bow-tie structure, there can be subdivided edges and split edges. If all hyperedges have the same number of pendant boundary vertices involved in series edges and the same number of adjacent boundary vertices involved in parallel edges, in a periodic pattern, then the lattice can be transformed into a lattice without series or parallel edges for which the bond percolation threshold can be exactly determined: (1) If two hyperedges both have pendant vertices at the same boundary vertex, then consider each to have probability  $\sqrt{p}$  of being open, which is equivalent to one edge with probability  $p$ . (2) If two hyperedges have two boundary vertices in common and in both hyperedges these boundary vertices are adjacent, then to get one edge with probability  $p$ , consider each edge to be open with probability  $s$  which satisfies  $p = s^2 + 2s(1 - s) = 1 - (1 - s)^2$ , which implies that  $s = 1 - \sqrt{1 - p}$ . Note that in both cases these edge probability functions are increasing functions of the parameter  $p$ .

## 7 Infinitely Many Structures

In section 4, Ziff and Scullard [15] illustrate a third triangle-dual structure, in addition to the triangular lattice structure and bow-tie lattice structure. We provide another self-dual hypergraph, which has not appeared previously, in Figure 7.

Ziff and Scullard mention that “Many other self-dual arrangements can be constructed.” In this section, we make this comment more precise, by showing that there are in fact infinitely many such structures.

Ziff and Scullard [15] remark that “we have shown that a great variety of lattices with exactly determined percolation thresholds can now be found. We also see that there are many variations of a given lattice (...) that share the same threshold, even though the bonds are in different arrangements.”

Note that the bond percolation threshold of a graph constructed from a generator and a self-dual

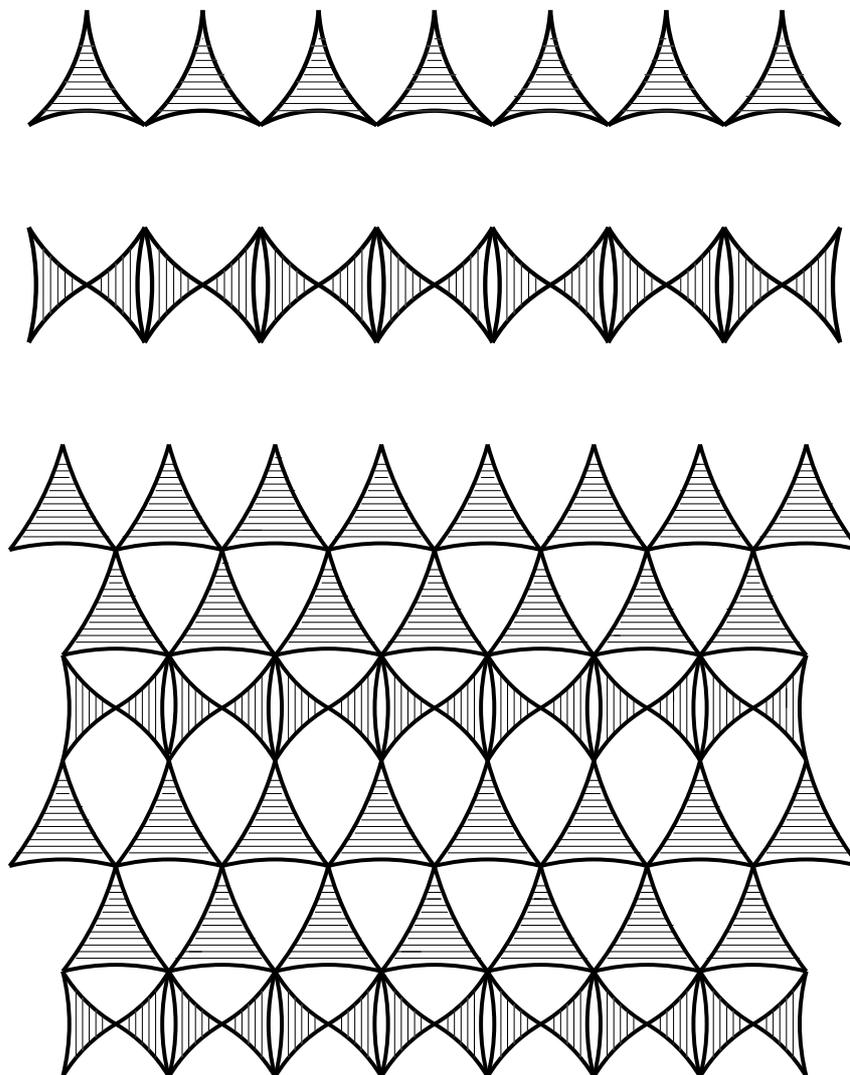


Figure 6: Top: A row of upward-pointing triangles. Middle: A row of bow-ties. Bottom: A periodic self-dual 3-regular hypergraph which alternates one row of bow-ties with two rows of triangles.

arrangement of triangles is uniquely determined by the connectivity equation. By filling all such structures with the same generator, we construct infinitely many planar lattices which have the same value for the bond percolation threshold.

We may construct triangle-dual arrangements using rows of similarly-oriented triangles and rows of similarly-oriented bow-ties, as shown in Figure 7. To form an infinite periodic triangle-dual arrangement, we combine these into a periodic pattern.

To see that there are infinitely many triangle-dual arrangements, consider forming a sequence of rows which alternates between  $m$  rows of triangles and  $n$  rows of bow-ties, for any  $m \geq 1$  and  $n \geq 1$ . When constructing the dual arrangement of triangles, each triangle is reflected 180 degrees, while the row of bow-ties is translated so the boundary vertices match up with the triangle vertices. As a result, the dual arrangement is isomorphic to the original arrangement reflected and translated. See Figure 7 for an illustration.

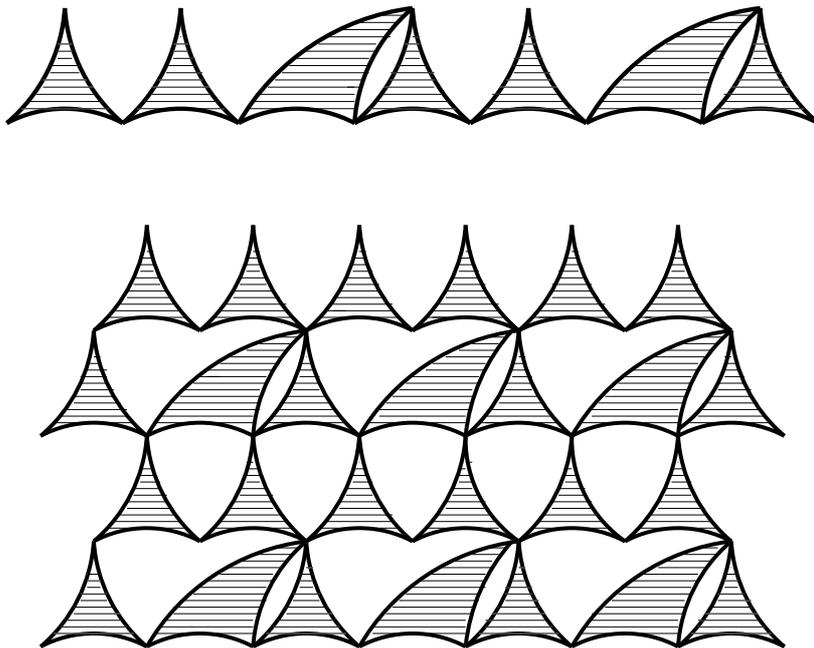


Figure 7: Top: A row of triangles with every third top moved to the right. Bottom: A self-dual hypergraph constructed by alternating rows of triangles and rows of triangles with every other top moved to the right.

We may construct another infinite collection of self-dual hypergraphs similarly. In Figure /refrow2, we show a row of triangles with the top of every third triangle moved to the right. Each time a top vertex is moved to the right, it produces two triangles which share two boundary vertices, as occurs in the bow-tie lattice structure. Consider forming a sequence of rows which alternates between  $m$  rows of triangles and  $n$  rows of triangles with tops moved, in a periodic manner. Again, the dual hypergraph is isomorphic to the original arrangement reflected and translated. See Figure 7, which illustrates this construction. Note that an arrangement with all rows have every other top moved, the hypergraph is isomorphic to the bow-tie hypergraph.

## 8 Infinitely-Many Threshold Values for Triangle-Dual Structures

Figure 8 shows a sequence of generator graphs that are nested so that each graph contains every previous graph as a subgraph. Each generator can be put into a infinitely many triangle-dual arrangements, producing infinitely many lattices which have the same bond percolation threshold. (Note that the construction does not have single edges connecting the boundary vertices, in order to avoid complications such as the lattice or its dual lattice having split edges or subdivided edges.) By the strict subgraph relationships between the generators, a result of Aizenmann and Grimmett [26] shows that the percolation thresholds of the families of graphs for different generators are not equal. Therefore, we have established that for infinitely many real numbers  $a$  there are infinitely many periodic lattice graphs with bond percolation threshold equal to  $a$ .

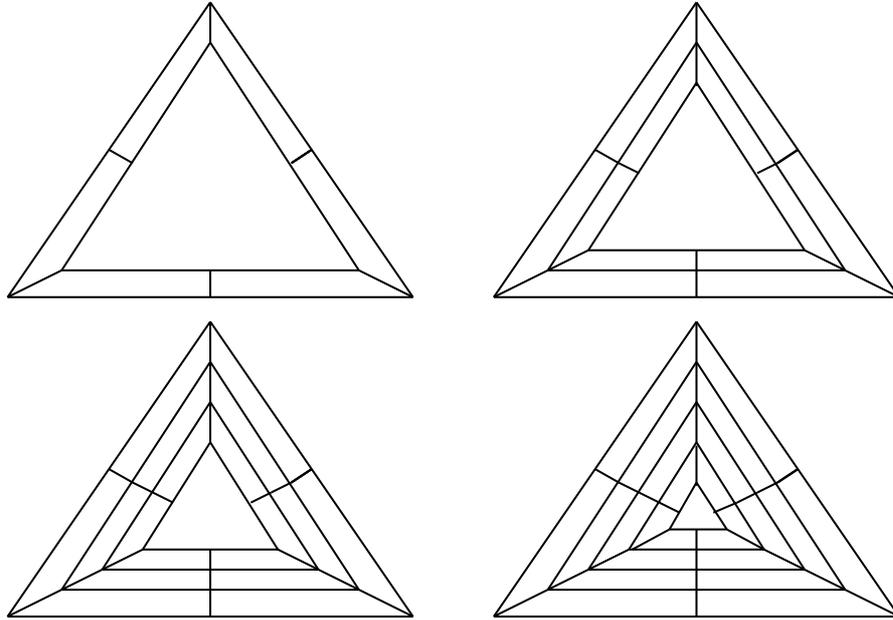


Figure 8: The first four graphs in an infinitely sequence of nested generators. Each contains all previous graphs in the sequence as subgraphs.

Recall that by Fisher’s [27] bond-to-site transformation, the bond percolation model on a lattice  $L$  is equivalent to the site percolation model on the covering lattice (in the physics literature) or line graph (in the mathematical literature), so that their percolation thresholds are equal. Thus, we have also shown that for infinitely many values  $a$  there are infinitely many site percolation models with percolation threshold equal to  $a$ .

## 9 Concluding Remarks

We have provided sufficient conditions on the underlying lattice for correct application of the methods of Ziff and Scullard for determining exact bond percolation thresholds. The method applies if the lattice is constructed from an infinite connected planar periodic 3-regular hypergraph with one axis of symmetry, using a generator which is a finite connected planar graph with three boundary vertices. These conditions are naturally stated in terms of planar hypergraphs, reflect the need for planarity of the lattice in order to apply graph duality results, and reflect the need for technical conditions for validity of the mathematical proofs.

In particular, note that sufficient conditions reflect the current state of knowledge in mathematical percolation theory, requiring periodicity and an axis of symmetry. It is plausible that the results may be valid in a larger context, where these conditions are relaxed. As evidence for this, in a recent breakthrough, Bollobás and Riordan [1, 28] exactly determined the site percolation threshold in a certain continuum percolation model which is not periodic or symmetric.

Ziff and Scullard also apply their approach to site percolation models and correlated percolation models (either all open or all closed). The formalism allows correlated bonds within each generator. As in the case of bond percolation, care must be taken to understand the conditions on the generator

and the connections between generators that produce a valid solution. Research on these issues is in progress.

Wierman [29] used the so-called “substitution method” with the star-triangle transformation to show that, if they exist, the critical exponents are equal for the triangular and hexagonal lattice bond percolation models, and that they are equal for the bond percolation models on the bow-tie and its dual lattice. These results are being generalized to the class of lattices identified in this article.

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