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Random graph asymptotics on high-dimensional tori II. Volume, diameter and mixing time

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Abstract: For critical (bond-) percolation on general high-dimensional torus, this paper answers the following questions: What is the diameter of the largest cluster? What is the mixing time of simple random walk on the largest cluster? The answer is the same as for critical Erdős-Rényi random graphs, and extends an earlier result by Nachmias and Peres [35].

We further improve our bound on the size of the largest cluster in [24], and extend the results on the largest clusters in [9, 10] to any *finite* number of the largest clusters. Finally, we show that any weak limit of the largest connected component is non-degenerate, which can be viewed as a significant sign of critical behavior. This result further justifies that the critical value defined in [9, 10] is appropriate in our rather general setting of random subgraphs of high-dimensional tori.

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1 Introduction

1.1 The model

For bond percolation on a graph $\mathbb{G}$ we make any edge (or 'bond') *occupied* with probability p, independently of each other, and otherwise leave it *vacant*. The connected components of the random subgraph of occupied edges are called *clusters*. For a vertex v we denote by $\mathcal{C}(v)$ the unique cluster containing v, and by $|\mathcal{C}(v)|$ the number of vertices in that cluster. For our purposes it is important to consider clusters as subgraphs (thus not only as a set of vertices). Our main interest is bond percolation on high-dimensional tori, but our techniques are based on a comparison with $\mathbb{Z}^d$ results. We describe the $\mathbb{Z}^d$-setting first.

Bond percolation on $\mathbb{Z}^d$. For $\mathbb{G} = \mathbb{Z}^d$, we consider two sets of edges. In the *nearest-neighbor model*, two vertices x and y are linked by an edge whenever $|x - y| = 1$, whereas in the *spread-out model*, they are linked whenever $0 < \|x - y\|_\infty \leq L$. Here, and throughout the paper, we write $\|\cdot\|_\infty$ for the supremum norm, and $|\cdot|$ for the Euclidean norm. The integer parameter L is typically chosen large.

The resulting product measure for percolation with parameter $p \in [0, 1]$ is denoted by $\mathbb{P}_{\mathbb{Z},p}$, and the corresponding expectation $\mathbb{E}_{\mathbb{Z},p}$. We write $\{0 \leftrightarrow x\}$ for the event that there exists a path of occupied edges from the origin 0 to the lattice site x (alternatively, 0 and x are in the same cluster), and define

$$\tau_{\mathbb{Z},p}(x) := \mathbb{P}_{\mathbb{Z},p}(0 \leftrightarrow x) \tag{1.1}$$

to be the *two-point* function. By

$$\chi_{\mathbb{Z}}(p) := \sum_{x \in \mathbb{Z}^d} \tau_{\mathbb{Z},p}(x) = \mathbb{E}_{\mathbb{Z},p}|\mathcal{C}(0)|$$

we denote the expected cluster size on $\mathbb{Z}^d$. The degree of the graph, which we denote by Ω, is $\Omega = 2d$ in the nearest-neighbor case and $\Omega = (2L + 1)^d - 1$ in the spread-out case.

Percolation on $\mathbb{Z}^d$ undergoes a phase transition as p varies, and it is well known that there exists a critical value

$$p_c(\mathbb{Z}^d) = \inf\{p \colon \mathbb{P}_{\mathbb{Z},p}(|\mathcal{C}(0)| = \infty) > 0\} = \sup\{p \colon \chi_{\mathbb{Z}}(p) < \infty\}, \tag{1.2}$$

where the last equality is due to Aizenman and Barsky [2] and Menshikov [31].

Bond percolation on the torus. By $\mathbb{T}_{r,d}$ we denote a graph with vertex set $\{-\lfloor r/2\rfloor, \dots, \lceil r/2\rceil - 1\}^d$ and two related set of edges:

(i) The nearest-neighbor torus: an edge joins vertices that differ by 1 (modulo r) in exactly one component. For d fixed and r large, this is a periodic approximation to $\mathbb{Z}^d$. Here $\Omega = 2d$ for $r \geq 3$. We study the limit in which $r \to \infty$ with $d > 6$ fixed, but large.

(ii) The spread-out torus: an edge joins vertices $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ if $0 < \max_{i=1,\dots,d} |x_i - y_i|_r \leq L$ (with $|\cdot|_r$ the metric on $\mathbb{Z}_r$). We study the limit $r \to \infty$, with $d > 6$ fixed and L large (depending on d) and fixed. This gives a periodic approximation to range-L percolation on $\mathbb{Z}^d$. Here $\Omega = (2L+1)^d - 1$ provided that $r \geq 2L+1$, which we will always assume.

We write $V = r^d$ for the number of vertices in the torus. We consider bond percolation on these tori with edge occupation probability p and write $\mathbb{P}_{\mathbb{T},p}$ and $\mathbb{E}_{\mathbb{T},p}$ for the product measure and corresponding expectation, respectively. We use notation analogously to $\mathbb{Z}^d$-quantities, e.g.

$$\chi_{\mathbb{T}}(p) := \sum_{x \in \mathbb{T}_{r,d}} \mathbb{P}_{\mathbb{T},p}(0 \leftrightarrow x) = \mathbb{E}_{\mathbb{T},p}|\mathcal{C}(0)|$$

for the expected cluster size on the torus.

Mean-field behavior in high dimensions. In the past decades, there has been substantial progress in the understanding of percolation in high-dimensions (see e.g., [3, 6, 18, 19, 20, 21, 22, 23, 37] for detailed results on high-dimensional percolation), and the results show that percolation on high-dimensional infinite lattices is similar to percolation on infinite trees (see e.g., [17, Section 10.1] for a discussion of percolation on a tree). Thus, informally speaking, the mean-field model for percolation on $\mathbb{Z}^d$ is percolation on the tree.

More recently, the question has been addressed what the mean-field model is of percolation on *finite* subsets of $\mathbb{Z}^d$, such as the torus. Aizenman [1] conjectured that critical percolation on high-dimensional tori behaves similarly to critical Erdős-Rényi random graphs, thus suggesting that the mean-field model for percolation on a torus is the Erdős-Rényi random graph. In the past years, substantial progress was made in this direction, see in particular [9, 10, 24]. In this paper, we bring this discussion to the next level, by showing that large critical clusters on various high-dimensional tori share many features of the Erdős-Rényi random graph.

1.2 Random graph asymptotics on high-dimensional tori

We investigate the size of the maximal cluster on the torus $\mathbb{T}_{r,d}$, i.e.,

$$|\mathcal{C}_{\max}| := \max_{x\in\mathbb{T}_{r,d}} |\mathcal{C}(x)|, \tag{1.3}$$

at the critical percolation threshold $p_c(\mathbb{Z}^d)$. We start by improving the asymptotics of the largest connected component as proved in [24]:

Theorem 1.1 (Random graph asymptotics of the largest cluster size)**.** *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then there exists a constant $b > 0$, such that for all $\omega \geq 1$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega} \qquad \text{as } r\to\infty. \tag{1.4}$$

The constant b can be chosen equal to b_6 in [9, Theorem 1.3]. Furthermore, there are positive constants c_1 and c_2 such that

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{\max}| > \omega V^{2/3}\Big) \geq \frac{c_1}{\omega^{3/2}}\,\mathrm{e}^{-c_2\,\omega} \qquad \text{as } r\to\infty. \tag{1.5}$$

The upper bound in (1.4) in Theorem 1.1 is already proved in [24, Theorem 1.1], whereas the lower bound in [24, Theorem 1.1] contains a logarithmic correction, which we remove here by a more careful analysis.

We next extend the above result to the other large clusters. For this, we write $\mathcal{C}_{(i)}$ for the i^{th} largest cluster for percolation on $\mathbb{T}_{r,d}$, so that $\mathcal{C}_{(1)} = \mathcal{C}_{\max}$ and $|\mathcal{C}_{(2)}| \leq |\mathcal{C}_{(1)}|$ is the size of the second largest component; etc.

Theorem 1.2 (Random graph asymptotics of the ordered cluster sizes)**.** *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. For every $m = 1,2,\dots$ there exist constants $b_1,\dots,b_m > 0$, such that for all $\omega \geq 1$, and all $i = 1,\dots,m$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{(i)}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b_i}{\omega} \qquad \text{as } r\to\infty. \tag{1.6}$$

Consequently, the expected cluster sizes satisfy $\mathbb{E}_{\mathbb{T},p_c(\mathbb{Z}^d)}|\mathcal{C}_{(i)}| \geq b_i' V^{2/3}$ for certain constants $b_i' > 0$. Moreover, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated.

By the tightness of $|\mathcal{C}_{\max}|V^{-2/3}$ proved in Theorem 1.1, $|\mathcal{C}_{\max}|V^{-2/3}$ not being concentrated is equivalent to the statement that any weak limit of $|\mathcal{C}_{\max}|V^{-2/3}$ is non-degenerate.

In the next result, we investigate the *diameter* of the large critical clusters. In its statement, we write $\mathrm{diam}(\mathcal{C})$ for the diameter of the cluster $\mathcal{C}$.

Theorem 1.3 (The diameter of large critical clusters)**.** *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then, for every $m = 1,2,\dots,$ there exist constants $c_1,\dots,c_m > 0$, such that for all $\omega \geq 1$, and all $i = 1,\dots,m$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{1/3} \leq \mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}\Big) \geq 1 - \frac{c_i}{\omega^{1/3}} \qquad \text{as } r\to\infty. \tag{1.7}$$

We now state a result on the *mixing time of lazy simple random walk* on the percolation clusters. To this end, we call a *lazy simple random walk* on a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a Markov chain on the vertices $\mathcal{V}$ with transition probabilities

$$\mathrm{p}(x,y) = \begin{cases} 1/2 & \text{if } x = y; \\ \frac{1}{2\deg(x)} & \text{if } (x,y) \in \mathcal{E}; \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

where $\deg(x)$ denotes the degree of a vertex $x \in \mathcal{V}$. The stationary distribution of this Markov chain π is given by $\pi(x) = \deg(x)/(2|\mathcal{E}|)$. The *mixing time* of lazy simple random walk is defined as

$$T_{\mathrm{mix}}(G) = \min\big\{n \colon \|\mathrm{p}^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq 1/4 \text{ for all } x \in V\big\}, \tag{1.9}$$

with p^n being the distribution after n steps (i.e., the n-fold convolution of p), and $\|\cdot\|_{\mathrm{TV}}$ denoting the total variation distance.

Theorem 1.4 (The mixing time on large critical clusters)**.** *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then, for every $m = 1, 2, \ldots$, there exist constants $c_1, \ldots, c_m > 0$, such that for all $\omega \geq 1$, and all $i = 1, \ldots, m$,*

$$\mathbb{P}_{\mathbb{T}, p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V \leq T_{\mathrm{mix}}(\mathcal{C}_{(i)}) \leq \omega V\Big) \geq 1 - \frac{c_i}{\omega^{1/34}} \qquad \textit{as } r \to \infty. \tag{1.10}$$

1.3 Random graph asymptotics on general high-dimensional tori

In [9, 10], random graph asymptotics was proved for random subgraphs of rather general high-dimensional tori, such as the n-cube $\{-1,+1\}^n$, the complete graph, and the Hamming graph, as well as the finite, but high-dimensional tori studied in Section 1.2. We shall now prove that the results in the previous section apply also to this case, thereby further explaining the role of the critical value chosen in [9, 10].

An alternative definition for the critical percolation threshold on a general high-dimensional torus, denoted by $p_c(\mathbb{T}_{r,d})$, was given in [9, (1.7)] as the solution to

$$\chi_{\mathbb{T}}(p_c(\mathbb{T}_{r,d})) = \lambda V^{1/3}, \tag{1.11}$$

where λ is a sufficiently small constant (recall that $V = |\mathbb{T}_{r,d}| = r^d$ denotes the volume of the torus). The definition of $p_c(\mathbb{T}_{r,d})$ in (1.11) is an *internal* definition only, due to the fact that [10] deals with rather general tori, for which an external definition (such as $p_c(\mathbb{Z}^d)$ as in (1.2)) does not always exist. On the high-dimensional torus $\mathbb{T}_{r,d}$, we therefore have *two* sensible critical values, the externally defined $p_c(\mathbb{Z}^d)$, and the internally defined $p_c(\mathbb{T}_{r,d})$ in (1.11). One way of formulating Theorem 1.1 is to say that $p_c(\mathbb{T}_{r,d})$ and $p_c(\mathbb{Z}^d)$, under the assumptions of Theorem 1.1, are asymptotically equivalent.

We shall next extend the results in Theorems 1.2–1.3 to the generality of [9, 10]. We let the vertex set of $\mathbb{T}_{r,d}$ be given by $\mathbb{V} = \{0, 1, \ldots, r-1\}^d$, and the edge sets shall be denoted by $\mathbb{B}$. For $x, y \in \mathbb{T}_{r,d}$, let

$$D(x,y) = D(y-x) = \frac{1}{\Omega} \mathbb{1}_{\{\{x,y\} \in \mathbb{B}\}}, \tag{1.12}$$

where $\mathbb{B}$ denotes a particular choice of edge set for the torus. Thus, $D(x)$ represents the 1-step transition probability for a random walk to step from 0 to a neighbor x. We assume that $\mathbb{B}$ is symmetric in the sense that $\{0, x\} \in \mathbb{B}$ if and only if $\{y, y \pm x\} \in \mathbb{B}$ for every vertex y.

We denote the Fourier dual of the torus $\mathbb{T}_{r,d}$ by $\mathbb{T}^*_{r,d} = \frac{2\pi}{r}\mathbb{T}_{r,d}$. We will identify the dual torus as $\mathbb{T}^*_{r,d} = \frac{2\pi}{r}\{-\lfloor\frac{r-1}{2}\rfloor, \ldots, \lceil\frac{r-1}{2}\rceil\}^n$, so that each component of $k \in \mathbb{T}^*_{r,d}$ is between $-\pi$ and π. Let

$k \cdot x = \sum_{j=1}^{n} k_j x_j$ denote the dot product of $k \in \mathbb{T}^*_{r,d}$ with $x \in \mathbb{T}_{r,d}$. The Fourier transform of $f\colon \mathbb{T}_{r,d} \to \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{T}_{r,d}} f(x) e^{ik\cdot x}, \qquad k \in \mathbb{T}^*_{r,d}. \tag{1.13}$$

Then, the main condition used in [9, 10] to investigate the critical behavior on general high-dimensional tori is the following:

Assumption 1.5 (The random walk condition [10]). *There exists $\beta > 0$ such that*

$$\max_{x \in \mathbb{T}_{r,d}} D(x) \le \beta \qquad \text{and} \qquad \frac{1}{V} \sum_{k \in \mathbb{T}^*_{r,d}\colon k \neq 0} \frac{\hat{D}(k)^2}{\big(1 - \hat{D}(k)\big)^3} \le \beta. \tag{1.14}$$

As explained in more detail in [10, Proposition 1.2 and Section 2], Assumption 1.5 is satisfied in the following cases:

(i) the d-cube $\mathbb{T}_{2,d}$ as $d \to \infty$,

(ii) the complete graph (Hamming torus with $d = 1$ and $r \to \infty$),

(iii) nearest-neighbor percolation on $\mathbb{T}_{r,d}$ with $d \ge 7$ and $r^d \to \infty$ in any fashion, including d fixed and $r \to \infty$, r fixed and $d \to \infty$, or $r, d \to \infty$ simultaneously,

(iv) periodic approximations to range-L percolation on $\mathbb{Z}^d$ for fixed $d \ge 7$ and fixed large L.

We now extend the results of Theorems 1.2–1.4 to the above general setting. In the statements below, we require β in Assumption 1.5 to be sufficiently small, which, in case (iii) above means that d is sufficiently large.

Theorem 1.6 (General high-dimensional tori at criticality). *Suppose Assumption 1.5 hold for the torus $\mathbb{T}_{r,d}$ for some $\beta > 0$ sufficiently small, and also that λ in (1.11) is small enough. Then the results in Theorems 1.2–1.4 hold as well for $\mathbb{T}_{r,d}$ if $p_c(\mathbb{Z}^d)$ is replaced by $p_c(\mathbb{T}_{r,d})$.*

1.4 Discussion

In this section, we discuss our results and state some open questions. Here, and throughout the paper, we make use of the following notation: we write $f(x) = O(g(x))$ for functions $f, g \ge 0$ and x converging to some limit, if there exists a constant $C > 0$ such that $f(x) \le Cg(x)$ in the limit, and $f(x) = o(g(x))$ if $g(x) \neq O(f(x))$. Furthermore, we write $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

Random graph asymptotics at criticality. In this paper, we study properties of the largest components for various high-dimensional tori, within the critical window. We focus on the behavior of the largest connected components, by studying their sizes, their diameter and the mixing time for random walks on them. Indeed, Theorems 1.1–1.4 show that the largest clusters of percolation on the high-dimensional torus behave as they do on the Erdős-Rényi random graph; this can be seen as the take-home message of this paper. For results on the Erdős-Rényi random graph we refer to the monographs [7, 28] and the references therein.

Nachmias and Peres [34] give a beautiful short proof for the bound (1.4) in the context of Erdős-Rényi random graphs. Aldous [4] proved that, for Erdős-Rényi random graphs, the vector

$$V^{-2/3}\big(|\mathcal{C}_{(1)}|, |\mathcal{C}_{(2)}|, \dots, |\mathcal{C}_{(m)}|\big)$$

converges in distribution, as $V \to \infty$, to a random vector

$$(|\gamma_1|, \dots, |\gamma_m|),$$

where $|\gamma_j|$ are the excursion lengths (in decreasing order) of reflected Brownian motion. Nachmias and Peres [33, Thm. 5] prove the same limit (apart from a multiplication with an explicit constant) for random d-regular graphs (for which the critical value equals $(d-1)^{-1}$). In light of our Theorems 1.1–1.4, we conjecture that the same limit, multiplied by an appropriate constant as in [33, Thm. 5], arises for the ordered largest critical components for percolation on various large high-dimensional tori, i.e., in the generalized setting of Theorem 1.6.

The role of boundary conditions. We next remark on the impact of boundary conditions on the size of $\mathcal{C}_{\max}$. The combined results of Aizenman [1] and Hara et al. [19, 20] show that a box of width r under *bulk* boundary conditions in high dimension ($d \geq 19$ in the nearest-neighbor case, and $d > 6$ in the spread-out case) satisfies $|\mathcal{C}_{\max}| \approx r^4$, which is much smaller then $V^{2/3}$. This immediately implies an upper bound on $|\mathcal{C}_{\max}|$ under *free* boundary conditions. Aizenman [1] conjectures that, under periodic boundary conditions, $|\mathcal{C}_{\max}| \approx V^{2/3}$. This conjecture was proven in [24] with a logarithmic correction in the lower bound. The present paper (improving the lower bound) is the ultimate confirmation of the conjecture in [1].

The critical probability for percolation on the torus. One particularly interesting feature of Theorem 1.2 is its implications for the critical value in (1.11). Indeed, the definition of the critical value in (1.11) is somewhat indirect, and it is not obvious that $p_c(\mathbb{T}_{r,d})$ really *is* the correct definition. In Theorem 1.2, however, we prove that any weak limit of $|\mathcal{C}_{\max}|V^{-2/3}$ is non-degenerate, which is the *hallmark of critical behavior*. Thus, Theorem 1.2 can be seen as yet another justification for the choice of $p_c(\mathbb{T}_{r,d})$ in (1.11). For $d \leq 6$, the work in [12] suggests that in the critical regime, $|\mathcal{C}_{\max}|$ is of order $V^{\delta/(\delta+1)}$, where $\delta > 0$ is the critical exponent related to the cluster tails, and $\delta = 2$ in the high-dimensional setting (see e.g. (2.5) below). We shall remark further on possible alternative choices for the critical value below.

The supercritical phase and the role of geometry. The present paper, together with [9, 10, 24], give a relatively complete picture of the behavior of random subgraphs of finite graphs in the *subcritical* and *critical* regimes, where, as argued above, the scaling is intimately connected to that on the Erdős-Rényi random graph. The picture in the *supercritical regime* is far less complete. Indeed, lower bounds on the largest connected component size in the general setting of high-dimensional tori are only proved in rather specific settings, such as Hamming graphs [25, 26, 32] or the n-cube [11]. In the supercritical phase, we expect that *geometry* plays a crucial role. For example, van der Hofstad and Redig [27] show that, for *fixed* $p > p_c(\mathbb{Z}^d)$, $|\mathcal{C}_{(2)}|\,(\log V)^{-d/(d-1)}$ is bounded away from 0 and ∞. Consequently, if we keep p *fixed* as $V \to \infty$, then $p_c(\mathbb{Z}^d)$ is indeed the unique maximizer of $|\mathcal{C}_{(2)}|$. For the Erdős-Rényi random graph, on the other hand, in the supercritical phase where $\varepsilon_n = np - 1 \gg n^{-1/3}$, the second largest component has size $\Theta(\varepsilon_n^{-2}\log(n\varepsilon_n^3))$, which, for *fixed* $\varepsilon > 0$ is $\Theta(\log n)$. As a result, geometry enters in the scaling of the size of the second largest component, even in the high-dimensional setting. This is related to the *Wulff shape*. Indeed, the proofs in [27] make use of a detailed analysis of $\mathbb{P}_{\mathbb{Z},p}(|\mathcal{C}| = n)$ for large n, which turns out to be of the order $\mathrm{e}^{-c(p)n^{(d-1)/d}(1+o(1))}$ (as proved, in various settings, in [5, 13, 14, 15]). The investigation of the barely supercritical phase of random subgraphs of high-dimensional tori remains a highly interesting open problem!

Alternatives for the critical probability for percolation on the torus. We now elaborate on alternative definitions of the critical value, and consequences of our results for this problem. Nachmias and Peres [35, Section 7.2] argue that a decent choice of critical value should be such that *at* criticality $|\mathcal{C}_{\max}|$ is not concentrated, and that $|\mathcal{C}_{(2)}|$ has maximal expected value. Indeed, our results show that $|\mathcal{C}_{\max}|$ *is* non-degenerate for both $p_c(\mathbb{Z}^d)$ and $p_c(\mathbb{T}_{r,d})$. However, if we allow p to vary with V, then our results are not strong enough to rule out the possibility of a barely supercritical regime where $|\mathcal{C}_{(2)}|$ is still of the order $V^{2/3}$ (or even larger). More precisely, the upper bound on the critical window is not known to be sharp, and, as argued above, we expect geometry to play a fundamental role in this problem.

Another alternative might be to require the diameter of the random subgraph to be *maximal*, formalizing the statement that "critical graphs have the most complex structure". It is reasonable to expect that the diameter of $\mathcal{C}_{\max}$ is $o(V^{1/3})$ if $p > p_c(\mathbb{Z}^d)$, showing that p_c maximizes diameters. However, even for the Erdős-Rényi random graph, it is not clear that this leads to the correct critical window. Indeed, by [35], when $\varepsilon_n = np-1 = \Theta(n^{-1/3})$, the diameter of the largest connected component is $\Theta(n^{1/3})$, while, by [8, 16] the diameter of the Erdős-Rényi random graph is $\Theta(\log n)$ when $p = \lambda/n$ with $\lambda \neq 1$. When $\varepsilon_n = np - 1 = o(1)$ is such that $n\varepsilon_n^3 \to \infty$, Riordan and Wormald [36] recently proved that the diameter is $\Theta(\log(n\varepsilon_n^3))$ under some (extremely mild) lower bound on $n\varepsilon_n^3$. When $\varepsilon_n = np-1 = o(1)$ is such that $n\varepsilon_n^3 \to -\infty$, Łuczak [30] proves that the diameter is $\Theta(\log(n|\varepsilon_n|^3)/|\varepsilon_n|)$. In all these cases, the precise constant was also determined. Summarizing, the above shows that the diameter of the Erdős-Rényi random graph is maximal inside the scaling window. It would be of interest to extend this result to more general critical random graphs, such as percolation on random d-regular graphs, as in [33], or the general setting of percolation on high-dimensional tori as studied here.

2 Proof of Theorem 1.1

The following relation between the two critical values $p_c(\mathbb{Z}^d)$ (which is 'inherited' from the infinite lattice) and $p_c(\mathbb{T}_{r,d})$ (as defined in (1.11)) is crucial for our proof.

Theorem 2.1 (The $\mathbb{Z}^d$ critical value is inside the $\mathbb{T}_{r,d}$ critical window)**.** *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then there exists $C_{|-|} > 0$ such that $p_c(\mathbb{Z}^d)$ and $p_c(\mathbb{T}_{r,d})$ satisfy*

$$\Big|p_c(\mathbb{Z}^d) - p_c(\mathbb{T}_{r,d})\Big| \leq C_{|-|}\, V^{-1/3}. \tag{2.1}$$

In other words, $p_c(\mathbb{Z}^d)$ lies in a critical window of order $V^{-1/3}$ around $p_c(\mathbb{T}_{r,d})$. By the work of Borgs, Chayes, van der Hofstad, Slade and Spencer [9, 10], Theorem 2.1 has immediate consequences for the size of the largest cluster, and various other quantities:

Corollary 2.2 (Borgs et al. [9, 10])**.** *Under the conditions of Theorem 2.1, there exist constants $b, C > 0$, such that for all $\omega \geq C$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega}. \tag{2.2}$$

Furthermore,

$$c\, V^{2/3} \leq \mathbb{E}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}_{\max}|\big) \leq C\, V^{2/3} \tag{2.3}$$

and

$$c_\chi V^{1/3} \leq \mathbb{E}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}|\big) \leq C_\chi V^{1/3} \tag{2.4}$$

for some $c, C, c_\chi, C_\chi > 0$. Finally, there are positive constants $b_{|\mathcal{C}|}$, $c_{|\mathcal{C}|}$, $C_{|\mathcal{C}|}$ such that for $k \leq b_{|\mathcal{C}|}V^{2/3}$,

$$\frac{c_{|\mathcal{C}|}}{\sqrt{k}} \leq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}| \geq k\big) \leq \frac{C_{|\mathcal{C}|}}{\sqrt{k}}. \tag{2.5}$$

All of these statements hold uniformly as $r \to \infty$.

The reader may verify that Corollary 2.2 indeed follows from Theorem 2.1 by using [9, Thm. 1.3] in conjunction with [10, Prop. 1.2 and Thm. 1.3]. Note that (2.2) in particular proves (1.4) in Theorem 1.1. We are now turning towards the proof of Theorem 2.1. To this end, we need the following lemma:

Lemma 2.3. *For percolation on $\mathbb{Z}^d$ with $p = p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$, there exists a positive constant $\tilde{C}$ (depending on d and K, but not on V), such that*

$$\sum_{\substack{u,v\in\mathbb{Z}^d, u\neq v\\ u-v\in r\mathbb{Z}^d}} \tau_p(u)\,\tau_p(v) \leq \tilde{C}\,V^{-1/3}. \tag{2.6}$$

The lemma makes use of a number of results on high-dimensional percolation on $\mathbb{Z}^d$, to be summarized in the following theorem.

Theorem 2.4 ($\mathbb{Z}^d$-percolation in high dimension [18, 19, 20, 21].)**.** *Under the conditions in Theorem 1.1, there exist constants $c_\tau, C_\tau, c_\xi, C_\xi, c_{\xi_2}, C_{\xi_2} > 0$ such that*

$$\frac{c_\tau}{(|x|+1)^{d-2}} \leq \tau_{\mathbb{Z},p_c(\mathbb{Z}^d)}(x) \leq \frac{C_\tau}{(|x|+1)^{d-2}}. \tag{2.7}$$

Furthermore, for any $p < p_c(\mathbb{Z}^d)$,

$$\tau_{\mathbb{Z},p}(x) \leq \mathrm{e}^{-\frac{\|x\|_\infty}{\xi(p)}}, \tag{2.8}$$

where the correlation length $\xi(p)$ *is defined by*

$$\xi(p)^{-1} = -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{\mathbb{Z},p}\big((0,\ldots,0)\leftrightarrow(n,0,\ldots,0)\big), \tag{2.9}$$

and satisfies

$$c_\xi\left(p_c(\mathbb{Z}^d)-p\right)^{-1/2} \leq \xi(p) \leq C_\xi\left(p_c(\mathbb{Z}^d)-p\right)^{-1/2} \qquad \text{as } p\nearrow p_c(\mathbb{Z}^d). \tag{2.10}$$

For the mean-square displacement*,*

$$\xi_2(p) := \left(\frac{\sum_{v\in\mathbb{Z}^d}|v|^2\tau_{\mathbb{Z},p}(v)}{\sum_{v\in\mathbb{Z}^d}\tau_{\mathbb{Z},p}(v)}\right)^{1/2}, \tag{2.11}$$

we have

$$c_{\xi_2}\left(p_c(\mathbb{Z}^d)-p\right)^{-1/2} \leq \xi_2(p) \leq C_{\xi_2}\left(p_c(\mathbb{Z}^d)-p\right)^{-1/2} \qquad \text{as } p\nearrow p_c(\mathbb{Z}^d). \tag{2.12}$$

Finally, there exists a positive constant $\tilde{C}_\chi$, such that the expected cluster size $\chi_{\mathbb{Z}}(p)$ *obeys*

$$\frac{1}{\Omega\,(p_c(\mathbb{Z}^d)-p)} \leq \chi_{\mathbb{Z}}(p) \leq \frac{\tilde{C}_\chi}{\Omega\,(p_c(\mathbb{Z}^d)-p)} \qquad \text{as } p\nearrow p_c(\mathbb{Z}^d). \tag{2.13}$$

Some of these bounds express that certain critical exponents exist and take on their mean-field value. For example, (2.7) means that the $\eta = 0$, and similarly (2.13) can be rephrased as $\gamma = 1$. The power-law bound (2.7) is due to Hara [19] for the nearest-neighbor case, and to Hara, van der Hofstad and Slade [20] for the spread-out case. For the exponential bound (2.8), see e.g. Grimmett [17, Prop. 6.47]. Hara [18] proves the bound (2.10), and Hara and Slade [21] prove (2.12) and (2.13) (the latter in conjunction with Aizenman and Newman [3]). The proof of all of the above results uses the lace expansion.

Proof of Lemma 2.3. We split the sum on the left-hand side of (2.6) in parts, and treat each part separately with different methods:

$$\sum_{\substack{u,v\in\mathbb{Z}^d:\\ u\neq v\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v) \le 2\sum_{v}\sum_{\substack{u:\,u\neq v\\ |u|\le|v|\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v) = 2\Big((A)+(B)+(C)+(D)\Big), \tag{2.14}$$

where

$$\begin{aligned} (A) &= \sum_{v}\sum_{\substack{2r\le|u|\le|v|\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v), \qquad (B) = \sum_{|v|>MV^{1/6}\log V}\ \sum_{\substack{u:\,|u|\le 2r\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v)\\ (C) &= \sum_{2r<|v|\le MV^{1/6}\log V}\ \sum_{\substack{u:\,|u|\le 2r\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v), \qquad (D) = \sum_{|v|\le 2r}\ \sum_{\substack{u:\,|u|\le 2r\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v) \end{aligned} \tag{2.15}$$

and M is a (large) constant to be fixed later in the proof. We proceed by showing that each of the four summands is bounded by a constant times $V^{-1/3}$, in that showing (2.6).

Consider (A) first. To this end, we prove for fixed $v\in\mathbb{Z}^d$,

$$\sum_{\substack{2r\le|u|\le|v|\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u) \le C_\tau\,\frac{|v|^2}{V}. \tag{2.16}$$

Indeed,

$$\sum_{\substack{2r\le|u|\le|v|\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u) \le \sum_{\substack{2\le|u|\le\frac{|v|}{r}+1\\ u\in\mathbb{Z}^d}} \tau_{p_c}\big(ru+(v \bmod r)\big). \tag{2.17}$$

By (2.7), this is bounded above by

$$C_\tau \sum_{2\le|u|\le\frac{|v|}{r}+1} \Big(r\big(|u|-1\big)+1\Big)^{-(d-2)} \le \frac{C_\tau}{r^{d-2}} \sum_{1\le|u|\le\frac{|v|}{r}} |u|^{-(d-2)}. \tag{2.18}$$

The discrete sum is dominated by the integral

$$C_\tau\, r^{-(d-2)} \int_{0\le|u|\le\frac{|v|}{r}} |u|^{-(d-2)}\,\mathrm{d}u \le C_\tau\, C_\circ\, r^{-d}\,\frac{|v|^2}{2} \le C_\tau\, C_\circ\,\frac{|v|^2}{V}, \tag{2.19}$$

as desired (with $C_\circ$ denoting the surface of the $(d-1)$-dimensional hypersphere). Consequently, using (2.16),

$$(A) \le \frac{C_\tau\,C_\circ}{V}\sum_{v}|v|^2\tau_{\mathbb{Z},p}(v) \le \frac{C_\tau\,C_\circ}{V}\,\xi_2(p)^2\,\chi_{\mathbb{Z}}(p) \le \frac{C_\tau\,C_\circ\,C_{\xi_2}^2\,\tilde{C}_\chi}{V}\,\big(p_c(\mathbb{Z}^d)-p\big)^{-2} \tag{2.20}$$

by the bounds in Theorem 2.4. Inserting $p=p_c(\mathbb{Z}^d)-K\Omega^{-1}V^{-1/3}$ yields the desired upper bound $(A)\le C\,V^{-1/3}$.

For the bound on (B) we start by calculating

$$\sum_{u:\,|u|\le 2r} \tau_{\mathbb{Z},p}(u) \le \sum_{u:\,|u|\le 2r} \tau_{p_c(\mathbb{Z}^d)}(u) \le \sum_{u:\,|u|\le 2r} \frac{C_\tau}{(|u|+1)^{d-2}} \le O(r^2). \tag{2.21}$$

For the sum over v we use the exponential bound of Theorem 2.4: From (2.9)–(2.10) and our choice of p it follows that $\tau_{\mathbb{Z},p}(v) \le \exp\big\{-C\,|v|\,V^{-1/6}\big\}$ for some constant $C>0$. Consequently,

$$\sum_{\substack{|v|>MV^{1/6}\log V\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(v) \le \sum_{|v|>\frac{M}{r}V^{1/6}\log V} \tau_{\mathbb{Z},p}\big(rv+(u \bmod r)\big) \le \sum_{|v|>\frac{M}{r}V^{1/6}\log V} \exp\big\{-r\big(|v|-1\big)CV^{-1/6}\big\}. \tag{2.22}$$

This sum is dominated by the integral

$$\int_{|v|>\frac{M}{r}V^{1/6}\log V} \exp\big\{-r\,|v|\,CV^{-1/6}\big\}\ \exp\big\{r\,C\,V^{-1/6}\big\}\,\mathrm{d}v, \tag{2.23}$$

which can be shown by partial integration as being less or equal to

$$\mathrm{const}(C,M,d)\,\frac{V^{d/6}}{V}\,(\log V)^d \exp\Big\{-\frac{M}{C}\log V\Big\}\ \exp\big\{r\,C\,V^{-1/6}\big\}. \tag{2.24}$$

This expression equals

$$\mathrm{const}(C,M,d)\,V^{d/6-1-M/C+C(1/d-1/6)}\,(\log V)^d\,. \tag{2.25}$$

We now fix M large enough such that the exponent of V is less than $-(1/3+2/d)$. This finally yields

$$(B) \le \sum_{u\colon |u|\le 2r}\ \sum_{\substack{|v|>MV^{1/6}\log V\\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v) \le \mathrm{const}(C,M,d)\,r^2\,o\big(V^{-(1/3+2/d)}\big) = o\big(V^{-1/3}\big). \tag{2.26}$$

In order to bound (C) we proceed similarly by bounding

$$(C) \le C_\tau^2 \sum_{u\colon |u|<2r} (|u|+1)^{-(d-2)} \sum_{\substack{2r\le|v|\le MV^{1/6}\log V\\ u-v\in r\mathbb{Z}^d}} (|v|+1)^{-(d-2)}\,. \tag{2.27}$$

A domination by integrals as in (2.17)–(2.19) allows for the upper bound

$$C\,r^2\,\frac{M^2\,V^{1/3}\,(\log V)^2}{V}, \tag{2.28}$$

and this is $o\big(V^{-1/3}\big)$ if $d>6$ for any $M>0$.

The final summand (D) is bounded as in (2.27) by

$$C_\tau^2 \sum_{u\colon |u|<2r} (|u|+1)^{-(d-2)} \sum_{\substack{v\colon |v|\le 2r\\ u-v\in r\mathbb{Z}^d}} (|v|+1)^{-(d-2)}\,. \tag{2.29}$$

The second sum can be bounded uniformly in u by

$$\sum_{\substack{v\colon |v|\le 2r\\ u-v\in r\mathbb{Z}^d}} (|v|+1)^{-(d-2)} \le (2r)^{-(d-2)}\ \#\{v\colon |v|\le 2r,\ u-v\in r\mathbb{Z}^d\} \le (2r)^{-(d-2)}\,5^d, \tag{2.30}$$

while the first sum is bounded by $C\,r^2$. Together, this yields the upper bound $C\,r^{-(d-4)}$, and this is $o\big(V^{-1/3}\big)$ for $d>6$.

Finally, we have proved that $(A)\le C\,V^{-1/3}$, and that (B), (C), (D) are of order $o\big(V^{-1/3}\big)$. This completes the proof of Lemma 2.3. □

Proof of Theorem 2.1. Assume that the conditions of Theorem 1.1 are satisfied. Then by [24, Corol. 4.1] there exists a constant $\Lambda > 0$ such that, when $r \to \infty$,

$$p_c(\mathbb{Z}^d) - p_c(\mathbb{T}_{r,d}) \leq \frac{\Lambda}{\Omega} V^{-1/3}. \tag{2.31}$$

It therefore suffices to prove a matching lower bound.

We take $p = p_c(\mathbb{Z}^d) - K\Omega^{-1}V^{-1/3}$. The following bound is proven in [24]:

$$\chi_{\mathbb{T}}(p) \quad \geq \quad \chi_{\mathbb{Z}}(p) \left(1 - \left(\frac{1}{2} + p\,\Omega^2\,\chi_{\mathbb{Z}}(p)\right) \sum_{\substack{u,v\in\mathbb{Z}^d, u\neq v \\ u-v\in r\mathbb{Z}^d}} \tau_{\mathbb{Z},p}(u)\,\tau_{\mathbb{Z},p}(v)\right). \tag{2.32}$$

Indeed, this bound is obtained by substituting [24, (5.9)] and [24, (5.13)] into [24, (5.5)]. Furthermore, by our choice of p and (2.13), $K^{-1}V^{1/3} \leq \chi_{\mathbb{Z}}(p) \leq \tilde{C}_\chi K^{-1}V^{1/3}$. Together with (2.6),

$$\chi_{\mathbb{T}}(p) \quad \geq \quad K^{-1}V^{1/3}\left(1 - \left(1/2 + p\,\Omega^2\,K^{-1}\,\tilde{C}_\chi\,V^{1/3}\right)\tilde{C}V^{-1/3}\right) \quad \geq \quad \tilde{c}_K V^{1/3}, \tag{2.33}$$

where $\tilde{c}_K$ is a small (though positive) constant. Under the conditions of Theorem 1.1, also the following bound holds by Borgs et al. [9]: For $q \geq 0$,

$$\chi_{\mathbb{T}}\big(p_c(\mathbb{T}_{r,d}) - \Omega^{-1}q\big) \leq \frac{2}{q}; \tag{2.34}$$

cf. the upper bound in [9, (1.15)]. The upper bound (2.31) allows K be so large that $p < p_c(\mathbb{T}_{r,d})$. Consequently, the conjunction of (2.33) and (2.34) obtains

$$\frac{2}{\Omega(p_c(\mathbb{T}_{r,d}) - p_c(\mathbb{Z}^d) + KV^{-1/3})} \geq \chi_{\mathbb{T}}(p) \geq \widetilde{c}_K V^{1/3}. \tag{2.35}$$

This implies

$$p_c(\mathbb{Z}^d) \geq p_c(\mathbb{T}_{r,d}) + \left(K - \frac{2}{\widetilde{c}_K\Omega}\right) V^{-1/3}, \tag{2.36}$$

as desired. □

The proof of Theorem 2.1 concludes the proof of (1.4) in Theorem 1.1, and it remains to prove (1.5).

Proof of (1.5). The proof uses the exponential bound proven by Aizenman and Newman [3] that, for any $k \geq \chi_{\mathbb{T}}(p)^2$,

$$\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big) \leq \left(\frac{\mathrm{e}}{k}\right)^{1/2} \exp\left\{-\frac{k}{2\,\chi_{\mathbb{T}}(p)^2}\right\}. \tag{2.37}$$

In order to apply this bound on the torus, we bound

$$\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}_{\max}| \geq k\big) \quad \leq \quad \frac{1}{k}\sum_{v\in\mathbb{V}} \mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}_{\max}| \geq k,\, v \in \mathcal{C}_{\max}\big) \quad \leq \quad \frac{V}{k}\,\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big). \tag{2.38}$$

Together with (2.37), we obtain for $\omega > \chi_{\mathbb{T}}(p)^2 V^{-2/3}$,

$$\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}_{\max}| \geq \omega V^{2/3}\big) \leq \frac{\mathrm{e}^{1/2}}{\omega^{3/2}}\ \exp\left\{-\frac{\omega V^{2/3}}{2\,\chi_{\mathbb{T}}(p)^2}\right\}. \tag{2.39}$$

We now choose $p = p_c(\mathbb{Z}^d)$ and use that $\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \leq C_\chi V^{1/3}$ to see that indeed, for $\omega > C_\chi^2$,

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}_{\max}| \geq \omega V^{2/3}\big) \quad \leq \quad \frac{e^{1/2}}{\omega^{3/2}} \exp\Bigg\{-\frac{\omega}{2\tilde{C}_\chi^2}\Bigg\} \tag{2.40}$$

by (2.13). □

Proof of Theorem 1.1 in the generalized context of Theorem 1.6. The bound (1.4) is proven for $p_c(\mathbb{T}_{r,d})$ in [9, Thm. 1.3(ii)]. For the proof of (1.5) we proceed as above, but use $p = p_c(\mathbb{T}_{r,d})$ and $\chi_{\mathbb{T}}(p) = \lambda V^{1/3}$, cf. (1.11). □

3 Proof of Theorem 1.2

Proof of (1.6). The upper bounds on $|\mathcal{C}_{(i)}|$ in Theorem 1.2 follow immediately from the upper bound on $|\mathcal{C}_{\max}|$ in Theorem 1.1. Thus, we only need to establish the lower bound.

Given $k > 0$, let

$$Z_{\geq k} = \sum_{x\in\mathbb{V}} \mathbb{1}_{\{|\mathcal{C}(x)|\geq k\}} \tag{3.1}$$

denote the number of vertices that lie in clusters of size k or larger. Then

$$\mathbb{E}_p(Z_{\geq k}) = V\, \mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big). \tag{3.2}$$

By construction, $|\mathcal{C}_{\max}| \geq k$ if and only if $Z_{\geq k} \geq k$. We shall make essential use of properties of the sequence of random variables $\{Z_{\geq k}\}$ proved in [9]. Indeed, [9, Lemma 7.1] states that, for all p and all k,

$$\mathrm{Var}_p(Z_{\geq k}) \leq V\chi_{\mathbb{T}}(p). \tag{3.3}$$

When we take $p = p_c(\mathbb{Z}^d)$, then, by (2.4) in Corollary 2.2 above, we have that $\chi_{\mathbb{T}}(p_c(\mathbb{Z}^d)) \leq O(V^{1/3})$. As a result we have that there exists a constant C_Z such that

$$\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{\geq k}) \leq C_Z V^{4/3} \tag{3.4}$$

uniformly in k. Now, further, by (2.5) in Corollary 2.2, there exists $C_{|\mathcal{C}|} > 0$ such that

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}| \geq k\big) \geq \frac{2C_{|\mathcal{C}|}}{\sqrt{k}}. \tag{3.5}$$

Take $k = V^{2/3}/\omega$, for some $\omega \geq 1$ sufficiently large. Together with the identity in (3.2),

$$\mathbb{E}_{p_c(\mathbb{Z}^d)}(Z_{\geq k}) \geq 2C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}. \tag{3.6}$$

Thus, by the Chebychev inequality,

$$\begin{aligned}\mathbb{P}_{p_c(\mathbb{Z}^d)}\big(Z_{\geq k} \leq C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}\big) &\leq \mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\big|Z_{\geq k} - \mathbb{E}_{p_c(\mathbb{Z}^d)}(Z_{\geq k})\big| \geq C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}\Big)\\ &\leq C_{|\mathcal{C}|}^{-2}\,\omega^{-1}\,V^{-4/3}\,\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{\geq k}) \;\leq\; \frac{C_Z}{C_{|\mathcal{C}|}^2\omega}.\end{aligned} \tag{3.7}$$

We take $\omega > 0$ large. Then, the event $Z_{\geq k} > C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}$ holds with high probability. On this event, there are two possibilities. Either $|\mathcal{C}_{\max}| \geq C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}/i$, or $|\mathcal{C}_{\max}| < C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}/i$, in which case there are at least $C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}/|\mathcal{C}_{\max}| \geq i$ distinct clusters of size at least $k = \omega^{-1}V^{2/3}$. We conclude that

$$\begin{aligned}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}_{(i)}| \leq \omega^{-1}V^{2/3}\big) &\leq \mathbb{P}_{p_c(\mathbb{Z}^d)}\big(Z_{\geq k} \leq C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}\big) + \mathbb{P}_{p_c(\mathbb{Z}^d)}\big(|\mathcal{C}_{\max}| \geq C_{|\mathcal{C}|}\omega^{1/2}V^{2/3}/i\big)\\ &\leq \frac{C_Z}{C_{|\mathcal{C}|}^2\omega} + \frac{i\tilde{b}C_{|\mathcal{C}|}}{\omega},\end{aligned} \tag{3.8}$$

where $\tilde{b}$ is chosen appropriately from the exponential bound in (1.5). This identifies b_i as $b_i = i\tilde{b}C_{|\mathcal{C}|} + C_Z/C_{|\mathcal{C}|}^2$, and proves (1.6). □

We complete this section with the proof that any weak limit of $|\mathcal{C}_{\max}|V^{-2/3}$ is non-degenerate. Theorem 1.1 proves that the sequence $|\mathcal{C}_{\max}|V^{-2/3}$ is *tight*, and, therefore, any subsequence of $|\mathcal{C}_{\max}|V^{-2/3}$ has a further subsequence that converges in distribution.

Proposition 3.1 ($|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated)**.** *Under the conditions of Theorem 1.1, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated.*

In order to prove Proposition 3.1, we start by establishing a *lower bound* on the variance of $Z_{\geq k}$. That is the content of the following lemma:

Lemma 3.2 (A lower bound on the variance of $Z_{\geq k}$)**.** *For each $k \geq 1$,*

$$\mathrm{Var}_p(Z_{\geq k}) \geq V\,\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big)\big[k - V\,\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big)\big]. \tag{3.9}$$

Proof. We have that

$$\mathrm{Var}_p(Z_{\geq k}) = \sum_{u,v}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u)| \geq k, |\mathcal{C}(v)| \geq k\big) - \big[V\,\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big)\big]^2. \tag{3.10}$$

Now, we trivially bound

$$\sum_{u,v}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u)| \geq k, |\mathcal{C}(v)| \geq k\big) \geq \sum_{u,v}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u)| \geq k, u \leftrightarrow v\big) = V\,\mathbb{E}[|\mathcal{C}|\mathbb{1}_{\{|\mathcal{C}|\geq k\}}] \geq V\,k\,\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}| \geq k\big). \tag{3.11}$$

Rearranging terms proves Lemma 3.2. □

Lemma 3.3 (An upper bound on the third moment of $Z_{\geq k}$)**.** *For each $k \geq 1$,*

$$\mathbb{E}_p[Z_{\geq k}^3] \leq V\chi_{\mathbb{T}}(p)^3 + 3\,\mathbb{E}_p[Z_{\geq k}]\,V\,\chi_{\mathbb{T}}(p) + \mathbb{E}_p[Z_{\geq k}]^3. \tag{3.12}$$

Proof. We compute

$$\begin{aligned}\mathbb{E}_p[Z_{\geq k}^3] &= \sum_{u_1,u_2,u_3}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u_1)| \geq k, |\mathcal{C}(u_2)| \geq k, |\mathcal{C}(u_3)| \geq k\big)\\ &= \sum_{u_1,u_2,u_3}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u_1)| \geq k, u_1 \leftrightarrow u_2, u_3\big)\\ &\quad + 3\sum_{u_1,u_2,u_3}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u_1)| \geq k, u_1 \leftrightarrow u_2, |\mathcal{C}(u_3)| \geq k, u_1 \not\leftrightarrow u_3\big)\\ &\quad + \sum_{u_1,u_2,u_3}\mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u_1)| \geq k, |\mathcal{C}(u_2)| \geq k, |\mathcal{C}(u_3)| \geq k, u_i \not\leftrightarrow u_j \forall i \neq j\big)\\ &= (I) + 3\,(II) + (III).\end{aligned} \tag{3.13}$$

We shall bound these terms one by one, starting with (I),

$$(I) \leq \sum_{u_1,u_2,u_3} \mathbb{P}_{\mathbb{T},p}\big(|\mathcal{C}(u_1)| \geq k, u_1 \leftrightarrow u_2, u_3\big) = V\,\mathbb{E}_p[|\mathcal{C}|^2 \mathbb{1}_{\{|\mathcal{C}|\geq k\}}] \leq V\,\mathbb{E}_p[|\mathcal{C}|^2] \leq V \chi_{\mathbb{T}}(p)^3, \tag{3.14}$$

by the tree-graph inequality (see [3]). We proceed with (II), for which we use the BK-inequality, to bound

$$\begin{aligned}(II) &\leq \sum_{u_1,u_2,u_3} \mathbb{P}_{\mathbb{T},p}(\{|\mathcal{C}(u_1)| \geq k, u_2 \in \mathcal{C}(u_1)\} \circ \{|\mathcal{C}(u_3)| \geq k\}\big)\\ &\leq \sum_{u_1,u_2,u_3} \mathbb{P}_{\mathbb{T},p}(|\mathcal{C}(u_1)| \geq k|, u_2 \in \mathcal{C}(u_1))\mathbb{P}_{\mathbb{T},p}(\mathcal{C}(u_2)| \geq k)\\ &= V\,\mathbb{E}_p[|\mathcal{C}|\mathbb{1}_{\{|\mathcal{C}|\geq k\}}]\,\mathbb{E}_p[Z_{\geq k}] \leq \mathbb{E}_p[Z_{\geq k}]\,V\,\chi_{\mathbb{T}}(p). \end{aligned}\tag{3.15}$$

We complete the proof by bounding (III), for which we use again the BK-inequality, to obtain

$$\begin{aligned}(III) &\leq \sum_{u_1,u_2,u_3} \mathbb{P}_{\mathbb{T},p}(\{|\mathcal{C}(u_1)| \geq k\} \circ \{|\mathcal{C}(u_2)| \geq k\} \circ \{|\mathcal{C}(u_3)| \geq k\}\big)\\ &\leq \sum_{u_1,u_2,u_3} \mathbb{P}_{\mathbb{T},p}(|\mathcal{C}(u_1)| \geq k|)\,\mathbb{P}_{\mathbb{T},p}(\mathcal{C}(u_2)| \geq k)\,\mathbb{P}_{\mathbb{T},p}(|\mathcal{C}(u_3)| \geq k) = \mathbb{E}_p[Z_{\geq k}]^3. \end{aligned}\tag{3.16}$$

This completes the proof. □

Now we are ready to complete the proof of Proposition 3.1:

Proof of Proposition 3.1. By Theorem 1.1, we know that the sequence $|\mathcal{C}_{\max}|V^{-2/3}$ is tight, and so is $V^{2/3}/|\mathcal{C}_{\max}|$. Thus, there exists a subsequence of $|\mathcal{C}_{\max}|V^{-2/3}$ that converges in distribution, and the weak limit, which we shall denote by X^*, is strictly positive and finite with probability 1. Thus, we are left to prove that X^* is non-degenerate. For this, we shall show that there exists a $\omega > 0$ such that $\mathbb{P}(X^* > \omega) \in (0,1)$.

To prove this, we choose an ω that is not a discontinuity point of the distribution function of X^* and note that

$$\mathbb{P}(X^* > \omega) = \lim_{n\to\infty} \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}|V_n^{-2/3} > \omega), \tag{3.17}$$

where the subsequence along which $|\mathcal{C}_{\max}|V^{-2/3}$ converges is denoted by $\{V_n\}_{n=1}^{\infty}$. Now, using (3.1), we have that

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}_{\max}|V_n^{-2/3} > \omega) = \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V^{2/3}} > \omega V^{2/3}\big). \tag{3.18}$$

The probability $\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V^{2/3}} > \omega V^{2/3}\big)$ is monotone decreasing in ω. By the Markov inequality and (2.5), for $\omega \geq 1$ large enough and uniformly in V,

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V^{2/3}} > \omega V^{2/3}\big) \leq \omega^{-1}V^{-2/3}V\,\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}| \geq \omega V^{2/3}\big) \leq \frac{C_{|\mathcal{C}|}}{\omega^{3/2}} < 1. \tag{3.19}$$

In particular, the sequence $Z_{>\omega V^{2/3}}V^{-2/3}$ is tight, so we can extract a further subsequence $\{V_{n_l}\}_{l=1}^{\infty}$ so that also $Z_{>\omega V^{2/3}}V^{-2/3}$ converges in distribution, say to Z^*_ω. Then, (3.19) implies that

$$\begin{aligned}\mathbb{P}(Z^*_\omega = 0) = 1 - \mathbb{P}(Z^*_\omega > 0) &= 1 - \lim_{l\to\infty} \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V_{n_l}^{2/3}} > 0\big)\\ &= 1 - \lim_{l\to\infty} \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V_{n_l}^{2/3}} > \omega V_{n_l}^{2/3}\big) > 0. \end{aligned}\tag{3.20}$$

Further, by Lemma 3.2,

$$\begin{aligned}\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V^{2/3}}V^{-2/3}) &\geq V^{1/3}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}|>\omega V^{2/3}\big)\big[\omega V^{2/3}-V\,\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}|>\omega V^{2/3})\big]\\ &\geq V^{1/3}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}|>\omega V^{2/3}\big)\big[\omega - C_{|\mathcal{C}|}\omega^{-1/2}\big],\end{aligned} \tag{3.21}$$

which remains uniformly positive for $\omega \geq 1$ sufficiently large, by (2.5). Since there is also an upper bound on $\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V^{2/3}}V^{-2/3})$ (this follows from (3.4)), it is possible to take a further subsequence $\{V_{n_{l_k}}\}_{k=1}^{\infty}$ for which $\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V^{2/3}}V^{-2/3})$ converges to $\sigma^2(\omega)>0$. Since, by Lemma 3.3, the third moment of $Z_{>\omega V^{2/3}}V^{-2/3}$ is bounded, the random variable $(Z_{>\omega V^{2/3}}V^{-2/3})^2$ in uniformly integrable, and, thus, along the subsequence for which $Z_{>\omega V^{2/3}}V^{-2/3}$ weakly converges and $\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V^{2/3}}V^{-2/3})$ converges in distribution to Z^*_ω, we have

$$\mathrm{Var}(Z^*_\omega) = \lim_{k\to\infty}\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V_{n_{l_k}}^{2/3}}V_{n_{l_k}}^{-2/3}) = \sigma^2(\omega)>0. \tag{3.22}$$

Since $\mathrm{Var}(Z^*_\omega)>0$, we must have that $\mathbb{P}(Z^*_\omega=0)<1$. Thus, by (3.20) and the above, we obtain that $\mathbb{P}(Z^*_\omega=0)\in(0,1)$, so that

$$\begin{aligned}\mathbb{P}(X^*>\omega) &= \lim_{n\to\infty}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}_{\max}|V_n^{-2/3}>\omega\big) = \lim_{k\to\infty}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(Z_{>\omega V_{n_{l_k}}^{2/3}}V_{n_{l_k}}^{-2/3}>0\big)\\ &= \mathbb{P}(Z^*_\omega>0)\in(0,1).\end{aligned} \tag{3.23}$$

This proves Proposition 3.1. □

Proof of Theorem 1.2 in the general context of Theorem 1.6. The proof above only makes use of the asymptotics in (2.4) and (2.5), which are proved in [9, 10] under the assumptions of Theorem 1.6. □

4 Proof of Theorems 1.3 and 1.4

For a (bond) percolation configuration on the graph $\mathbb{G}$ we define the random sets

$$\mathcal{C}_{\leq r}(x) := \{u\colon d_{\mathcal{C}}(x,u)\leq r\}, \tag{4.1}$$

$$\mathcal{C}_{=r}(x) := \{u\colon d_{\mathcal{C}}(x,u)= r\}, \tag{4.2}$$

where $d_{\mathcal{C}}$ denotes the graph metric (or *intrinsic* metric) on the percolation cluster $\mathcal{C}$. We basically use the following criterion for the characterization of the diameter by Nachmias and Peres [35, Thm. 6.1].

Theorem 4.1 (Nachmias–Peres [35])**.** *Consider bond percolation on the graph $\mathbb{G}$ with vertex set $\mathbb{V}$, $V=|\mathbb{V}|<\infty$, with percolation parameter $p\in(0,1)$. If for all subgraphs $\mathbb{G}'\subset\mathbb{G}$ with vertex set $\mathbb{V}'$,*

(a) $\mathbb{E}_{\mathbb{G}',p}\,|\mathcal{E}(\mathcal{C}_{\leq k}(v))|\leq d_1 k, \qquad v\in\mathbb{V}'$;

(b) $\mathbb{P}_{\mathbb{G}',p}\,(\mathcal{C}_{=k}(v)\neq\varnothing)\leq d_2/k, \qquad v\in\mathbb{V}'$,

($\mathcal{E}(\mathcal{C})$ denotes the number of edges in the cluster $\mathcal{C}$), then

$$\mathbb{P}_{\mathbb{G},p}\big(\exists v\in\mathbb{V}\colon |\mathcal{C}(v)|\geq M, \mathrm{diam}(\mathcal{C}(v))\geq k\big) \leq \frac{d_1V}{kM}, \tag{4.3}$$

$$\mathbb{P}_{\mathbb{G},p}\big(\exists v\in\mathbb{V}\colon |\mathcal{C}(v)|\geq M, \mathrm{diam}(\mathcal{C}(v))\leq k\big) \leq \frac{d_2kV}{M^2}. \tag{4.4}$$

By Theorem 1.2, we know that $|\mathcal{C}_{(i)}| \geq \omega^{-1/3} V^{2/3}$ with high probability, for some $\omega > 1$ sufficiently large. Thus, we obtain

$$
\begin{aligned}
&\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{1/3} \leq \mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}\Big) \\
&\quad \geq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{1/3} \leq \mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega V^{1/3}, |\mathcal{C}_{(i)}| \geq \omega^{-1/3}V^{2/3}\Big) \\
&\quad = \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{(i)}| \geq \omega^{-1/3}V^{2/3}\Big) - \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\mathrm{diam}(\mathcal{C}_{(i)}) \geq \omega V^{1/3}, |\mathcal{C}_{(i)}| \geq \omega^{-1/3}V^{2/3}\Big) \\
&\qquad - \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\mathrm{diam}(\mathcal{C}_{(i)}) \leq \omega^{-1}V^{1/3}, |\mathcal{C}_{(i)}| \geq \omega^{-1/3}V^{2/3}\Big) \\
&\quad \geq 1 - \frac{b_i}{\omega^{1/3}} - \frac{d_1}{\omega^{2/3}} - \frac{d_2}{\omega^{1/3}}.
\end{aligned} \tag{4.5}
$$

This reduces the proof of Theorem 1.3 subject to Theorem 4.1 to the verification of the bounds in Theorem 4.1(a) and (b).

For completeness, we shall give a simple proof of the diameter bounds in Theorem 4.1. Indeed, for (4.3), we define the random variable

$$X = |\{v \in \mathbb{V} : \mathrm{diam}(\mathcal{C}(v)) \geq k\}|. \tag{4.6}$$

We then have

$$\mathbb{E}_{\mathbb{G},p}[X] = V\mathbb{P}_{\mathbb{G},p}\left(\mathcal{C}_{=k}(v) \neq \varnothing\right) \leq d_1 V/r \tag{4.7}$$

by Theorem 4.1(a). Moreover, when there exists a $v \in \mathbb{V}$ such that $|\mathcal{C}(v)| \geq M$ and $\mathrm{diam}(\mathcal{C}(v)) \geq k$, then $X \geq M$. Whence,

$$\mathbb{P}_{\mathbb{G},p}\left(\exists v \in \mathbb{V} : |\mathcal{C}(v)| \geq M, \mathrm{diam}(\mathcal{C}(v)) \geq k\right) \;\leq\; \mathbb{P}_{\mathbb{G},p}(X \geq M) \;\leq\; \mathbb{E}_{\mathbb{G},p}[X]/M \;\leq\; \frac{d_1 V}{kM}. \tag{4.8}$$

For (4.4), instead, we define the random variable

$$Y = \sum_{v \in \mathbb{V}} |\mathcal{C}_{\leq k}(v)|. \tag{4.9}$$

Then, Theorem 4.1(b),

$$\mathbb{E}_{\mathbb{G},p}[Y] = V\,\mathbb{E}_{\mathbb{G},p}|\mathcal{C}_{\leq k}(v)| \leq d_2 V k. \tag{4.10}$$

Moreover, when there exists a $v \in \mathbb{V}$ such that $|\mathcal{C}(v)| \geq M$ and $\mathrm{diam}(\mathcal{C}(v)) \leq k$, then $Y \geq M^2$. Indeed, for *each* $v' \in \mathcal{C}_{\leq k}(v)$, we have that $|\mathcal{C}_{\leq k}(v')| = |\mathcal{C}(v)| \geq M$, and there are at least M vertices v' such that $v' \in \mathcal{C}_{\leq k}(v)$. Therefore,

$$\mathbb{P}_{\mathbb{G},p}\left(\exists v \in \mathbb{V} : |\mathcal{C}(v)| \geq M, \mathrm{diam}(\mathcal{C}(v)) \leq k\right) \;\leq\; \mathbb{P}_{\mathbb{G}',p}(Y \geq M^2) \;\leq\; \frac{\mathbb{E}_{\mathbb{G},p}[Y]}{M^2} \;\leq\; \frac{d_2 V k}{M^2}. \tag{4.11}$$

□

Our result for the *mixing time* in Theorem 1.4 can be refined as follows:

Proposition 4.2. *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. Then, for every $m = 1, 2, \ldots,$ there exist constants $c_1, \ldots, c_m > 0$, such that for all $\omega \geq 1$, and all $i = 1, \ldots, m$,*

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(T_{\mathrm{mix}}(\mathcal{C}_{(i)}) > \omega V\Big) \;\leq\; \frac{c_i}{\omega^{1/6}}, \tag{4.12}$$

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1} V > T_{\mathrm{mix}}(\mathcal{C}_{(i)})\Big) \;\leq\; \frac{c_i}{\omega^{1/34}} \qquad \textit{as } r \to \infty. \tag{4.13}$$

In order to proof the proposition we use a criterion similar to the one for the diameter result.

Theorem 4.3 (Nachmias–Peres [35]).
(i) For any finite graph $\mathcal{G}$ with edge set $\mathcal{E}$,

$$T_{\text{mix}}(\mathcal{G}) \leq 8\,|\mathcal{E}|\,\text{diam}(\mathcal{G}). \tag{4.14}$$

(ii) If percolation on a finite graph $\mathbb{G}$ with parameter $p \in [0,1]$ satisfies conditions (a) and (b) of Theorem 4.1, then there exist constants C_1, $C_2 > 0$ such that for any $\beta > 0$, $D > 0$,

$$\mathbb{P}_{\mathbb{G},p}\Big(\exists v \in \mathbb{V}\colon |\mathcal{C}(v)| > \beta V^{2/3}, T_{\text{mix}}(\mathcal{C}(v)) < \frac{\beta^{21}}{1000\,D^{13}} V\Big) \leq D^{-1}\left(C_1 + C_2\beta^3 D^{-2}\right). \tag{4.15}$$

For the bound (4.14) we refer to [35, Corol. 4.2], and the bound (4.15) is obtained by combining [35, (5.4)] with the display thereafter.

We proceed by proving Proposition 4.2 subject to conditions (a) and (b) in Theorem 4.1. We first consider (4.12). Fix $\omega \geq 1$. We apply (4.14) for $\mathcal{C}_{(i)}$, and bound the number of edges in $\mathcal{C}_{(i)}$ by $\Omega\,|\mathcal{C}_{(i)}|$ (recall that Ω is the degree in the underlying torus). If both $T_{\text{mix}}(\mathcal{C}_{(i)}) > \omega V$ and $|\mathcal{C}_{(i)}| \leq \omega^{1/2} V^{2/3}$ occur, then by (4.14),

$$\omega V < T_{\text{mix}}(\mathcal{C}_{(i)}) \leq 8\Omega\,|\mathcal{C}_{(i)}|\,\text{diam}(\mathcal{C}_{(i)}) \leq 8\Omega\,\omega^{1/2}\,V^{2/3}\,\text{diam}(\mathcal{C}_{(i)}). \tag{4.16}$$

Consequently,

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(T_{\text{mix}}(\mathcal{C}_{(i)}) > \omega V^{1/3}, |\mathcal{C}_{(i)}| \leq \omega^{1/2} V^{2/3}\Big) \leq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\text{diam}(\mathcal{C}_{(i)}) > \frac{\omega^{1/2}}{8\Omega} V^{1/3}\Big) \leq \frac{2c_i \Omega^{1/3}}{\omega^{1/6}} \tag{4.17}$$

by Theorem 1.3. Consequently, by Theorem 1.2,

$$\begin{aligned}
&\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(T_{\text{mix}}(\mathcal{C}_{(i)}) > \omega V\Big)\\
&\quad\leq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{(i)}| > \omega^{1/2} V^{2/3}\Big) + \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(T_{\text{mix}}(\mathcal{C}_{(i)}) > \omega V^{1/3}, |\mathcal{C}_{(i)}| \leq \omega^{1/2} V^{2/3}\Big)\\
&\quad\leq \frac{c_1}{\omega^{1/2}} + \frac{c_2}{\omega^{1/6}}
\end{aligned} \tag{4.18}$$

for positive constants c_1 and c_2.

For the upper bound (4.13), we proceed in a similar fashion. For any $\omega \geq 1$ and any $\beta > 0$ (to be determined later as a function of ω), we use

$$\begin{aligned}
&\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(T_{\text{mix}}(\mathcal{C}_{(i)}) \geq \omega^{-1} V\Big)\\
&\quad\geq \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{(i)}| \geq \beta V^{2/3}\Big) - \mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}_{(i)}| \geq \beta V^{2/3}, T_{\text{mix}}(\mathcal{C}_{(i)}) < \omega V^{1/3}\Big).
\end{aligned} \tag{4.19}$$

An application of (1.6) and (4.15) with $D = \beta^{21/13}\,\omega^{1/13}\,1000^{-1/13}$ obtains the lower bound

$$1 - b_i\beta - \frac{1000^{1/13}}{\beta^{21/13}\,\omega^{1/13}}\left(C_1 + C_2\frac{1000^{2/13}}{\omega^{2/13}\,\beta^{3/13}}\right). \tag{4.20}$$

Choosing

$$\beta = \omega^{-x} \qquad \text{with } x = \frac{1}{34} = \frac{13}{21}\left(\frac{1}{13} - \frac{1}{34}\right)$$

proves (4.13).

We complete the proof of Theorems 1.3 and 1.4 by verifying that the conditions in Theorem 4.1(a) and (b) indeed hold for critical percolation on the high-dimensional torus:

Verification of Theorem 4.1(a). In [24, Proposition 2.1], a coupling between the cluster of v in the torus and the cluster of v in $\mathbb{Z}^d$ was presented, which proves that $\mathcal{C}(v)$ can be obtained by identifying points which agree modulo r in a subset of the cluster of v in $\mathbb{Z}^d$. A careful inspection of this construction shows that this coupling is such that it *preserves* graph distances. Since $\mathcal{C}_{\leq k}(v)$ is monotone in the number of edges used, the result in Theorem 4.1(a) for the torus follows from the bound $\mathbb{E}_p |\mathcal{E}(\mathcal{C}_{\leq k}(v))| \leq d_1 k$ for critical percolation on $\mathbb{Z}^d$. This bound was proved in [29, Theorem 1.2(i)]. □

Verification of Theorem 4.1(b). For percolation on $\mathbb{Z}^d$, this bound was proved in [29, Theorem 1.2(ii)]. However, the event that $\mathcal{C}_{=k}(v) \neq \varnothing$ is not monotone, and, therefore, this does not prove our claim. However, a close inspection of the proof of [29, Theorem 1.2(ii)] shows that it only relies on the bound that

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}(v)| \geq k) \leq C_1/k^{1/2} \tag{4.21}$$

(see in particular, [29, Section 3.2]). To see (4.21) we use the argument in (2.37)–(2.40). Alternatively, one obtains (4.21) from the corresponding $\mathbb{Z}^d$-bound (proven by Barsky–Aizenman [6] and Hara–Slade [21]), together with the fact that $\mathbb{Z}^d$-clusters stochastically dominate $\mathbb{T}_{r,d}$-clusters by [24, Prop. 2.1]. This completes the verification of Theorem 4.1(b). □

Proof of Theorems 1.3 and 1.4 in the general context of Theorem 1.6. The proof above makes use of Theorems 4.1 and 4.3 of [35], both apply also in the general setting of Theorem 1.6 for $p_c(\mathbb{T}_{r,d})$. It remains to check that conditions (a) and (b) in Theorem 4.1 hold under the conditions of Theorem 1.6.

We first verify condition (b). As explained before, the proof of [29, Thm. 1.2(ii)] generalizes such that the bound

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(|\mathcal{C}(v)| \geq k) \leq C_1/k^{1/2} \tag{4.22}$$

is sufficient for condition (b). As before, (4.22) is proven in the generalized setting of Theorem 1.6 by adapting the argument in (2.37)–(2.40), now using (1.11).

For the verification of condition (a) we use the arguments given by Kozma and Nachmias [29, Section 3.1] for the infinite lattice. Their proof works in fact for any transitive graph provided that certain requirements on the triangle diagram hold. We shall now explain what these requirements are, and why they are actually satisfied in the setting of Theorem 1.6. Borgs et al. use the lace expansion to prove the following triangle condition on finite tori [10, Thm. 1.3]: There is a constant $\beta_0 > 0$ such that if the conditions of Theorem 1.6 are met for $\beta < \beta_0$ and $\lambda^3 < \beta_0$ (λ appears as a factor in (1.11)), then

$$\sum_{z,z' \in \mathbb{T}_{r,d}} \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(u \leftrightarrow z)\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(z \leftrightarrow z')\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(z' \leftrightarrow v) < \delta_{u,v} + 13\beta + 10\lambda. \tag{4.23}$$

For the proof of [29, Lemma 3.2] it is required that

$$\sum_{z,z' \in \mathbb{T}_{r,d}} \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(u \leftrightarrow z)\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(z \leftrightarrow z')\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(z' \leftrightarrow v) < 1 \tag{4.24}$$

if $|u - v|$ is sufficiently large. Indeed, by (4.23), this holds whenever $u \neq v$ (and β and λ are small enough).

Similarly, the proof of [29, Lemma 3.1] makes use of the fact that

$$\sum_{\substack{u' : |u'| \geq K \\ v' : |v'| \geq K}} \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(0 \leftrightarrow u')\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(u' \leftrightarrow v')\, \mathbb{P}_{\mathbb{T},p_c(\mathbb{T}_{r,d})}(v' \leftrightarrow 0) \tag{4.25}$$

converges to 0 as K increases. Again this follows from (4.23).

Consequently, Lemmas 3.1 an 3.2 in [29] hold for the torus with $p = p_c(\mathbb{T}_{r,d})$, and so does condition [29, Thm. 1.2(i)], which is equivalent to condition Theorem 4.1(a). $\square$

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