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INSTITUT
MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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J. Peterson and T. Seppäläinen

REPORT No. 16, 2008/2009, spring
ISSN 1103-467X
ISRN IML-R- -16-08/09- -SE+spring

Current Fluctuations of a System of One-dimensional Random Walks in Random Environment

Jonathon Peterson *
University of Wisconsin-Madison

Timo Seppäläinen †
University of Wisconsin-Madison

April 29, 2009

Abstract

We study the current of particles that move independently in a common static random environment on the one-dimensional integer lattice. A two-level fluctuation picture appears. On the central limit scale the quenched mean of the current process converges to a Brownian motion. On a smaller scale the current process centered at its quenched mean converges to a mixture of Gaussian process. These Gaussian processes are similar to those arising from classical random walks, but the environment makes itself felt through an additional Brownian random shift in the spatial argument of the limiting current process.

1 Introduction

We investigate the effect of a random environment on the fluctuations of particle current in a system of many particles. We take the standard model of random walk in random environment (RWRE) on the one-dimensional integer lattice, and let a large number of particles evolve independently of each other but in a common, fixed environment ω. On the level of the averaged (annealed) distribution particles interact with each other through the environment.

We set the parameters of the model so that an individual particle has a positive asymptotic speed v_P and satisfies a central limit theorem around this limiting velocity under the averaged distribution. There is also a quenched central limit theorem that requires an environment-dependent correction $Z_n(\omega)$ to the asymptotic value $n\mathrm{v}_P$. We scale space and time by the same factor n. We consider initial particle configurations whose distribution may depend on the environment, but in a manner that respects spatial shifts. Under a fixed environment the initial occupation variables are required to be independent.

We find a two-tier fluctuation picture. On the scale $n^{1/2}$ the quenched mean of the current process behaves like a Brownian motion. In fact, up to $o(n^{1/2})$ deviations, this quenched mean coincides with the quenched CLT correction $Z_n(\omega)$ multiplied by the mean density of particles. Around its quenched mean, the current process fluctuates on the scale $n^{1/4}$. These fluctuations are described by the same self-similar Gaussian processes that arise for independent particles performing classical random walks. But the environment-determined correction $Z_n(\omega)$ appears again, this time as an extra shift in the spatial argument of the limit process of the current.

*J. Peterson was partially supported by National Science Foundation grant DMS-0802942.
†T. Seppäläinen was partially supported by National Science Foundation grant DMS-0701091 and by the Wisconsin Alumni Research Foundation.
Much of this work was done while both authors were visiting the Institut Mittag-Leffler (Djursholm, Sweden) for the progam "Discrete Probability."
Mathematics Subject Classification: 60K37, 60K35
Keywords: Random walk in random environment, current fluctuations, central limit theorem
Short title: Current Fluctuations of RWRE

The broader context for this paper is the ongoing work to elucidate the patterns of universal current fluctuations in one-dimensional driven particle systems. A key object is the flux function $H(\mu)$ that gives the average rate of mass flow past a fixed point in space when the system is in a stationary state with mean density μ. Known rigorous results have confirmed the following delineation. If H is strictly convex or concave then current fluctuations have magnitude $n^{1/3}$ and limit distributions are related to Tracy-Widom distributions from random matrix theory. If H is linear then the magnitude of current fluctuations is $n^{1/4}$ and limit distributions are Gaussian.

The RWRE model has a linear flux. Our results show that in a sense it confirms the prediction stated above, but with additional features coming from the random environment. Limit processes possess covariances that are similar to those that arise for independent classical random walks. However, when the environment is averaged out, limit distributions can fail to be Gaussian.

Literature. A standard reference on the basic RWRE model is [16]. Further references to RWRE work follow below when we review basic results. Earlier related results for current fluctuations of independent particles appeared in papers [3], [8] and [14]. A central model for the study of fluctuations in the case of a concave flux is the exclusion process. Key papers include [1], [4] and [6].

Organization of the paper. We define the model and state the results for the current process and its quenched mean in Section 2. Section 3 reviews known central limit results for the walk itself that we need for the proof. Sections 4 and 5 prove the fluctuation theorems for the current. An appendix proves a uniform integrability result for the walk that is used in the proofs.

2 Description of the model and main results

We begin with the standard RWRE model on $\mathbb{Z}$ with the extra feature that we admit infinitely many particles. Let $\Omega := [0,1]^{\mathbb{Z}}$ be the space of environments. For any environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ and any $x \in \mathbb{Z}$, let $\{X_{\bullet}^{m,i}\}_{m,i}$ be a family of Markov chains with distribution P_ω given by the following properties:

1. $\{X_{\bullet}^{m,i}\}_{m\in\mathbb{Z},i\in\mathbb{N}}$ are independent under the measure P_ω.
2. $P_\omega(X_0^{m,i} = m) = 1$, for all $m \in \mathbb{Z}$ and $i \in \mathbb{N}$.
3. The transition probabilities are given by

$$P_\omega(X_{n+1}^{m,i} = x+1 | X_n^{m,i} = x) = 1 - P_\omega(X_{n+1}^{m,i} = x-1 | X_n^{m,i} = x) = \omega_x.$$

A system of random walks in a random environment may then be constructed by first choosing an environment ω according to a probability distribution P on Ω and then constructing the system of random walks $\{X_{\bullet}^{m,i}\}$ as described above. The distribution P_ω of the random walks given the environment ω is called the *quenched law*. The *averaged law* $\mathbb{P}$ (also called the annealed law) is obtained by averaging the quenched law over all environments. That is, $\mathbb{P}(\cdot) := \int_\Omega P_\omega(\cdot) P(d\omega)$.

Often we will be considering events that only concern the behavior of a single random walk started at location m, and so we will use the notation X_n^m in place of $X_n^{m,1}$. Moreover, if the random walk starts at the origin we will further abbreviate the notation by X_n in place of X_n^0. Expectations with respect to the measures P, P_ω and $\mathbb{P}$ will be denoted by E_P, E_ω, and $\mathbb{E}$, respectively, and variances with respect to the measure P_ω will be denoted by Var_ω. Generic probabilities and expectations not defined in the RWRE model are denoted by $\mathbf{P}$ and $\mathbf{E}$.

For the remainder of the paper we will make the following assumptions on the distribution P of the environments.

Assumption 1. *The distribution on environments is i.i.d. and uniformly elliptic. That is, the variables $\{\omega_x\}_{x\in\mathbb{Z}}$ are independent and identically distributed under the measure P, and there exists a $\kappa > 0$ such that $P(\omega_x \in [\kappa, 1-\kappa]) = 1$.*

Assumption 2. $E_P(\rho_0^2) < 1$, *where* $\rho_x := \frac{1-\omega_x}{\omega_x}$.

The above assumptions on the distribution P on environments imply that the RWRE are transient to $+\infty$ with strictly positive speed v_P [15]. That is,

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - E_P \rho_0}{1 + E_P \rho_0} =: \mathrm{v}_P > 0, \qquad \mathbb{P} - a.s. \tag{2.1}$$

Moreover, Assumptions 1 and 2 imply that a quenched central limit theorem holds with a random (depending on the environment) centering. That is, there exists an explicit function of the environment $Z_n(\omega)$ and a constant $\sigma_1 > 0$ such that for $P - a.e.$ environment ω,

$$\lim_{n\to\infty} P_\omega \left(\frac{X_n - n\mathrm{v}_P + Z_n(\omega)}{\sigma_1 \sqrt{n}} \le x \right) = \Phi(x), \qquad \forall x \in \mathbb{R},$$

where Φ is the standard normal distribution function. The environment-dependent centering in the above quenched central limit theorem cannot be replaced by a deterministic centering since it is known that there exists a constant $\sigma_2 > 0$ such that the process $t \mapsto \frac{Z_{nt}(\omega)}{\sigma_2 \sqrt{n}}$ converges weakly to a standard Brownian motion. Definitions of σ_1, σ_2 and $Z_n(\omega)$ are provided in Section 3 where we give a more detailed review of the known limit distribution results for RWRE under Assumptions 1 and 2.

In this paper, we will be concerned with a system of RWRE in a common environment with a finite (random) number of walks started at each site $x \in \mathbb{Z}$. Let $\eta_0(x)$ be the number of walks started from $x \in \mathbb{Z}$. We will allow the law of the initial configurations to depend on the environment (in a measurable way). Let θ be the shift operator on environments defined by $(\theta^x \omega)_y = \omega_{x+y}$. We will assume that our initial configurations are stationary in the following sense.

Assumption 3. *The distribution of* η_0 *is such that* $\omega \mapsto P_\omega(\eta_0(0) = k)$ *is a measurable function of* ω *for any* $k \in \mathbb{N}$*, and the law of* η_0 *respects the shifts of the environment:* $P_\omega(\eta_0(x) = k) = P_{\theta^x \omega}(\eta_0(0) = k)$*. Also, given the environment* ω*, the* $\{\eta_0(x)\}$ *are independent and independent of the paths of the random walks.*

We will also need the following moment assumptions.

Assumption 4. *For some* $\varepsilon > 0$,

$$E_P[\, E_\omega(\eta_0(x))^{2+\varepsilon} + \mathrm{Var}_\omega(\eta_0(x))^{2+\varepsilon}] < \infty. \tag{2.2}$$

To simplify notation some we will let $\bar{\mu}(\omega) := E_\omega[\eta_0(0)]$. Note that Assumption 3 implies that $E_\omega[\eta_0(m)] = \bar{\mu}(\theta^m \omega)$. Let $\mu := E_P[\bar{\mu}(\omega)] = \mathbb{E}\eta_0(0)$ be the average density of the initial configuration of particles, and let $\sigma_0^2 = E_P[\mathrm{Var}_\omega(\eta_0(x))]$.

The law of large numbers (2.1) implies that each random walk moves with asymptotic speed v_P. The main object of study in this paper is the following two-parameter process. For $t \ge 0$ and $r \in \mathbb{R}$, let

$$Y_n(t,r) = \sum_{m>0} \sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} \le nt\mathrm{v}_P + r\sqrt{n}\,\} - \sum_{m\le 0} \sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} > nt\mathrm{v}_P + r\sqrt{n}\,\}. \tag{2.3}$$

$Y_n(t,r)$ is similar to what was called the space-time current process in [8] and studied in a constant environment (that is, particles performing independent classical random walks). We altered the definition because the limit process of this version has a more natural description. The process studied earlier in [8] equals

$$\begin{aligned} &Y_n(t,r) - Y_n(0,r) \\ &= \sum_{m>r\sqrt{n}} \sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} \le nt\mathrm{v}_P + r\sqrt{n}\,\} - \sum_{m\le r\sqrt{n}} \sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} > nt\mathrm{v}_P + r\sqrt{n}\,\}. \end{aligned} \tag{2.4}$$

This process $Y_n(\cdot, r) - Y_n(0, r)$ is the net right-to-left particle current seen by an observer who starts at $r\sqrt{n}$ and moves with deterministic speed v_P. Adapting the proof of [8] to our definition of $Y_n(t, r)$ gives this theorem:

Theorem 2.1 (Kumar [8])**.** *Assume that the environment is non-random. That is, there exists a* $p \in (0, 1)$ *such that* $P(\omega_x = p, \forall x \in \mathbb{Z}) = 1$*. Let* $\mathbb{E}(\eta_0) = \mu$ *and* $\mathbb{V}\mathrm{ar}(\eta_0) = \sigma_0^2$*, and assume that* $\mathbb{E}(\eta_0^{12}) < \infty$*. Then, the process* $n^{-1/4}(Y_n(\cdot, \cdot) - \mathbb{E}Y_n(\cdot, \cdot))$ *converges in distribution on the D-space of two-parameter cadlag processes. The limit is the mean zero Gaussian process* $V^0(\cdot, \cdot)$ *with covariance*

$$\mathbf{E}\big[V^0(s, q)V^0(t, r)\big] = \Gamma((s, q), (t, r)), \tag{2.5}$$

where the covariance function Γ *is defined below in* (2.9)*.*

The theorem above uses the higher moment assumption $\mathbb{E}(\eta_0^{12}) < \infty$ for process-level tightness. We have not proved such tightness, hence we get by with the moments assumed in (2.2). We turn to discuss the results in the random environment.

The random environment adds a new layer of fluctuations to the current. These larger fluctuations are of order $\sqrt{n}$ and depend only on the environment. This is summarized by our first main result. The process $Z_{nt}(\omega)$ in the statement below is the correction required in the quenched central limit theorem of the walk, defined in (3.2) in Section 3.

Theorem 2.2. *For any* $\varepsilon > 0$*,* $0 < R, T < \infty$*,*

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T],\, r\in[-R,R]} \big|E_\omega Y_n(t, r) - \mu r\sqrt{n} - \mu Z_{nt}(\omega)\big| \geq \varepsilon\sqrt{n}\right) = 0. \tag{2.6}$$

Moreover, since $\{n^{-1/2}Z_{nt}(\omega) : t \in \mathbb{R}_+\}$ *converges weakly to* $\{\sigma_2 W(t) : t \in \mathbb{R}_+\}$*, where* $W(\cdot)$ *is a standard Brownian motion, then the two-parameter process* $\{n^{-1/2}E_\omega Y_n(t, r) : t \in \mathbb{R}_+,\ r \in \mathbb{R}\}$ *converges weakly to* $\{\mu\sigma_2 W(t) + \mu r : t \in \mathbb{R}_+,\ r \in \mathbb{R}\}$*.*

To see the next order of fluctuations we center the current at its quenched mean. Define

$$\begin{aligned} V_n(t, r) &= Y_n(t, r) - E_\omega Y_n(t, r) \\ &= \sum_{m>0}\Bigg(\sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} \leq nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega(\eta_0(m))P_\omega\{X_{nt}^m \leq nt\mathrm{v}_P + r\sqrt{n}\}\Bigg) \\ &\qquad - \sum_{m\leq 0}\Bigg(\sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} > nt\mathrm{v}_P + r\sqrt{n}\} - E_\omega(\eta_0(m))P_\omega\{X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}\}\Bigg). \end{aligned} \tag{2.7}$$

The fluctuations of $V_n(t, r)$ are of order $n^{1/4}$ and the same as the current fluctuations in a deterministic environment, up to a random shift coming from the environment. We need to introduce some notation. For any $\alpha > 0$, let $\phi_{\alpha^2}(\cdot)$ and $\Phi_{\alpha^2}(\cdot)$ be the density and distribution function, respectively, for a Gaussian distribution with mean zero and variance α^2. Also, let

$$\Psi_{\alpha^2}(x) := \alpha^2\phi_{\alpha^2}(x) - x\Phi_{\alpha^2}(-x), \quad \text{and} \quad \Psi_0(x) := \lim_{\alpha\to 0}\Psi_{\alpha^2}(x) = x^-. \tag{2.8}$$

Then, for any $(s, q), (t, r) \in \mathbb{R}_+ \times \mathbb{R}$ define the covariance function

$$\begin{aligned} \Gamma((s, q), (t, r)) := \mu &\Big(\Psi_{\sigma_1^2(s+t)}(q - r) - \Psi_{\sigma_1^2|s-t|}(q - r)\Big) \\ &+ \sigma_0^2\Big(\Psi_{\sigma_1^2 s}(-q) + \Psi_{\sigma_1^2 t}(r) - \Psi_{\sigma_1^2(s+t)}(q - r)\Big) \end{aligned} \tag{2.9}$$

where σ_1 is the scaling factor in the quenched central limit theorem (see (3.3) in Section 3 for a formula). Given the above definitions, let $(V, Z) = (V(t, r), Z(t) : t \in \mathbb{R}_+, r \in \mathbb{R})$ be the process whose joint distribution is defined as follows:

(i) Marginally, $Z(\cdot) = \sigma_2 W(\cdot)$ for a standard Brownian motion $W(\cdot)$, and σ_2 is the scaling factor in the central limit theorem of the correction $Z_{nt}(\omega)$ (see (3.4) in Section 3 for a formula).

(ii) Conditionally on the path $Z(\cdot) \in C(\mathbb{R}_+, \mathbb{R})$, V is the mean zero Gaussian process indexed by $\mathbb{R}_+ \times \mathbb{R}$ with covariance

$$\mathbf{E}[V(s,q)V(t,r) \,|\, Z(\cdot)] = \Gamma\big((s, q+Z(s)), (t, r+Z(t))\big) \quad \text{for } (s,q),(t,r) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.10}$$

The next theorem gives joint convergence of the centered current process and the environment-dependent shift.

Theorem 2.3. *Under the averaged probability $\mathbb{P}$, as $n \to \infty$, the finite-dimensional distributions of the joint process $\big\{(n^{-1/4}V_n(t,r), n^{-1/2}Z_{nt}(\omega)) : t \in \mathbb{R}_+, r \in \mathbb{R}\big\}$ converge to those of the process (V, Z).*

Our proof shows additionally that

$$\lim_{n\to\infty} E_P \Big| \, E_\omega \exp\Big\{ i n^{-1/4} \sum_{k=1}^{N} \alpha_k V_n(t_k, r_k) \Big\} \; - \; \mathbf{E} \exp\Big\{ i \sum_{i=1}^{N} \alpha_k V(t_k, r_k) \Big\} \Big| = 0$$

for any choice of time-space points $(t_1, r_1), \ldots, (t_N, r_N) \in \mathbb{R}_+ \times \mathbb{R}$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. (See (5.20) below.) This falls short of a quenched limit for $n^{-1/4}V_n$ (a limit for a fixed ω) but it does imply that if a quenched limit exists, the limit process is the one that we describe. We suspect, however, that no quenched limit exists since the techniques of this paper can be used to show that the quenched covariances of the process $n^{-1/4}V_n(\cdot,\cdot)$ do not converge $P-a.s.$

The mean zero Gaussian process $\{u(t,r) : t \in \mathbb{R}_+, r \in \mathbb{R}\}$ with covariance $\mathbf{E}[u(s,q)u(t,r)] = \Gamma\big((s,q),(t,r)\big)$ from (2.9) can be represented as the sum of two integrals:

$$u(t,r) = \sqrt{\mu} \iint_{[0,t]\times\mathbb{R}} \phi_{\sigma_1^2(t-s)}(r-x)\, dW(s,x) + \sigma_0 \int_{\mathbb{R}} \phi_{\sigma_1^2 t}(r-x) B(x)\, dx \tag{2.11}$$

where W is a two-parameter Brownian motion on $\mathbb{R}_+ \times \mathbb{R}$ (Brownian sheet) and B an independent two-sided one-parameter Browian motion on $\mathbb{R}$. The process $u(t,r)$ is also a weak solution of the stochastic heat equation with initial data given by Brownian motion:

$$u_t = \tfrac{\sigma_1^2}{2} u_{rr} + \sqrt{\mu}\, \dot{W}, \quad u(0,r) = \sigma_0 B(r), \quad (t,r) \in \mathbb{R}_+ \times \mathbb{R}. \tag{2.12}$$

This type of process we obtain if we define $u(t,r) = V(t, r - Z(t))$ by regarding the random path $-Z(\cdot)$ as the new spatial origin.

We next remark on the distribution of the limiting process $V(t,r)$ in a couple of special cases. First we consider the case when $\sigma_0 = 0$ (this includes the case of deterministic initial configurations). If $\sigma_0 = 0$, then (2.10) and (2.9) imply that for any fixed $t \geq 0$, the one-parameter process $V(t,\cdot)$ has conditional covariance

$$E[V(t,q)V(t,r) \,|\, Z(\cdot)] = \Gamma\big((t, q+Z(t)), (t, r+Z(t))\big) = \mu \left(\Psi_{2\sigma_1^2 t}(q-r) - \Psi_0(q-r) \right)$$

In particular, the covariances of $V(t,\cdot)$ do not depend on the process $Z(\cdot)$ and are the same as in the classical random walk case.

Corollary 2.4. *If $\sigma_0 = 0$, then for any fixed $t \geq 0$ the (averaged) finite dimensional distributions of the one parameter process $\{n^{-1/4}V_n(t,r) \;:\; r \in \mathbb{R}\}$ converge to those of the one parameter mean zero Gaussian process $V^0(t,\cdot)$ with covariances given by (2.5) with $s = t$.*

A second special case worth considering is when $\mu = \sigma_0^2$. In the case of classical random walks, $\mu = \sigma_0^2$ implies that

$$\mathbf{E}[V^0(s,0)V^0(t,0)] = \frac{\mu\sigma_1}{\sqrt{2\pi}} (\sqrt{s} + \sqrt{t} - \sqrt{|s-t|}),$$

so that $V^0(\cdot,0)$ is a fractional Brownian motion with Hurst parameter 1/4. For RWRE, $\mu=\sigma_0^2$ implies that

$$\mathbf{E}[V(s,0)V(t,0)\,|\,Z(\cdot)] = \mu\left(\Psi_{\sigma_1^2 s}(-Z(s)) + \Psi_{\sigma_1^2 t}(Z(t)) - \Psi_{\sigma_1^2|s-t|}(Z(s)-Z(t))\right). \tag{2.13}$$

Since the right hand side of (2.13) is a non-constant random variable, the marginal distribution of $V(t,0)$ is non-Gaussian. Taking expectations of (2.13) with respect to $Z(\cdot)$ gives that

$$\mathbf{E}[V(s,0)V(t,0)] = \frac{\mu\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}}(\sqrt{s}+\sqrt{t}-\sqrt{|s-t|}). \tag{2.14}$$

Thus, we have the following.

Corollary 2.5. *If $\mu=\sigma_0^2$ then the process $V(\cdot,0)$ has covariances like that of a fractional Brownian motion, but is not a Gaussian process.*

Remark 2.6. The condition that $\mu=\sigma_0^2$ is important because it includes the case when the configuration of particles is stationary under the dynamics of the random walks. For classical random walks, the stationary distribution on configurations of particles is when the $\eta_0(x)$ are i.i.d. Poisson(μ) random variables. Consider now the case where, given ω, the $\eta_0(x)$ are independent and

$$\eta_0(x) \sim \text{Poisson}(\mu f(\theta^x\omega)), \quad \text{where} \quad f(\omega) = \frac{\mathrm{v}_P}{\omega_0}\left(1+\sum_{i=1}^{\infty}\prod_{j=1}^{i}\rho_j\right). \tag{2.15}$$

It was shown in [12] that given ω, the above distribution on the configuration of particles is stationary under the dynamics of the random walks. Note that in this case, $E_\omega\eta_0(0) = \text{Var}_\omega\,\eta_0(0) = \mu f(\omega)$. Moreover, Assumptions 1 and 2 imply that $E_P\rho_0^{2+\varepsilon} < 1$ for some $\varepsilon>0$, and thus it can be shown that $E_P f(\omega)^{2+\varepsilon}<\infty$. Therefore, Assumptions 3 and 4 are fulfilled in this special case.

It is intuitively evident but not a corollary of our theorem that if the environment-dependent shift is introduced in the current process itself, the random shift Z disappears from the limit process V. For the sake of completeness, we state this result too. For $(t,r)\in\mathbb{R}_+\times\mathbb{R}$ define

$$\begin{aligned} Y_n^{(q)}(t,r) = &\sum_{m>0}\sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} \le nt\mathrm{v}_P - Z_{nt}(\omega) + r\sqrt{n}\,\} \\ &- \sum_{m\le 0}\sum_{k=1}^{\eta_0(m)} \mathbf{1}\{X_{nt}^{m,k} > nt\mathrm{v}_P - Z_{nt}(\omega) + r\sqrt{n}\,\} \end{aligned} \tag{2.16}$$

and its centered version

$$V_n^{(q)}(t,r) = Y_n^{(q)}(t,r) - E_\omega Y_n^{(q)}(t,r).$$

The process $V_n^{(q)}$ has the same limit as classical random walks. As above, let $V^0 = \{V^0(t,r): (t,r)\in\mathbb{R}_+\times\mathbb{R}\}$ be the mean zero Gaussian process with covariance (2.5).

Theorem 2.7. *Under the averaged probability $\mathbb{P}$, as $n\to\infty$, the finite-dimensional distributions of the joint process $\big\{(n^{-1/4}V_n^{(q)}(t,r), n^{-1/2}Z_{nt}(\omega)) : t\in\mathbb{R}_+, r\in\mathbb{R}\big\}$ converge to those of the process (V^0,Z) where V^0 and Z are independent.*

It can be shown, using the techniques of this paper, that $n^{-1/2}E_\omega Y_n^{(q)}(t,r)$ converges to zero in probability for any fixed t and r. We suspect that the fluctuations of $E_\omega Y_n^{(q)}(t,r)$ are at most of order $n^{-1/4}$, but at this point we have no result.

3 Review of CLT for RWRE

In this section, we review some of the limiting distribution results for one-dimensional RWRE implied by Assumptions 1 and 2. Before stating a theorem which summarizes what is known, we introduce some notation. Let $T_x := \inf\{n \geq 0 : X_n = x\}$ be the hitting time of the site $x \in \mathbb{Z}$ of a RWRE started at the origin, and for $x \in \mathbb{Z}$ let

$$h(x, \omega) := \begin{cases} \mathrm{v}_P \sum_{i=0}^{x-1} (E_{\theta^i \omega} T_1 - \mathbb{E} T_1) & x \geq 1 \\ 0 & x = 0 \\ -\mathrm{v}_P \sum_{i=x}^{-1} (E_{\theta^i \omega} T_1 - \mathbb{E} T_1) & x \leq -1. \end{cases} \tag{3.1}$$

Define also

$$Z_{nt}(\omega) := h(\lfloor nt\mathrm{v}_P \rfloor, \omega). \tag{3.2}$$

Theorem 3.1 ([5, 7, 10, 16])**.** *Let Assumptions 1 and 2 hold. Then, the following hold:*

1. *The RWRE satisfies a quenched functional central limit theorem with a random (depending on the environment) centering. For $n \in \mathbb{N}$ and $t \geq 0$, let*

$$B^n(t) := \frac{X_{nt} - nt\mathrm{v}_P + Z_{nt}(\omega)}{\sigma_1 \sqrt{n}}, \qquad \text{where } \sigma_1^2 := \mathrm{v}_P^3 E_P(\mathrm{Var}_\omega T_1). \tag{3.3}$$

Then, for $P-a.e.$ environment ω, under the quenched measure P_ω, $B^n(\cdot)$ converges weakly to standard Brownian motion as $n \to \infty$.

2. *Let*

$$\zeta^n(t) := \frac{Z_{nt}(\omega)}{\sigma_2 \sqrt{n}}, \qquad \text{where } \sigma_2^2 := \mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1). \tag{3.4}$$

Then, under the measure P on environments, $\zeta^n(\cdot)$ converges weakly to standard Brownian motion as $n \to \infty$.

3. *The RWRE satisfies an averaged functional central limit theorem. Let*

$$\mathbb{B}^n(t) := \frac{X_{nt} - nt\mathrm{v}_P + Z_{nt}(\omega)}{\sigma \sqrt{n}}, \qquad \text{where } \sigma^2 = \sigma_1^2 + \sigma_2^2.$$

Then, under the averaged measure $\mathbb{P}$, $\mathbb{B}^n(\cdot)$ converges weakly to standard Brownian motion.

Remark 3.2. The conclusions of Theorem 3.1 still may hold if the law on environments is not uniformly elliptic or i.i.d. but satisfies certain mixing properties [5, 7, 9, 10, 16]. However, if the environment is i.i.d., the requirement that $E_P \rho_0^2 < 1$ in Assumption 2 cannot be relaxed in order for Theorem 3.1 to hold [7, 11, 13].

Let $B_\cdot$ denote a standard Brownian motion with distribution $\mathbf{P}$. The quenched functional central limit theorem implies that, $P - a.s.$, for any $s, t \geq 0$ and $x, y \in \mathbb{R}$,

$$\lim_{n\to\infty} P_\omega \left(\frac{X_{ns} - ns\mathrm{v}_P + Z_{ns}(\omega)}{\sigma_1\sqrt{n}} \leq x, \ \frac{X_{nt} - nt\mathrm{v}_P + Z_{nt}(\omega)}{\sigma_1\sqrt{n}} \leq y \right) = \mathbf{P}(B_s \leq x, \ B_t \leq y), \tag{3.5}$$

where $B_\cdot$ is a standard Brownian motion. Moreover, for fixed $s, t > 0$, the convergence in (3.5) is uniform in x and y. In [16], only an averaged central limit theorem is proved. However, since $\mathbb{B}^n(t) = \frac{\sigma_1}{\sigma} B^n(t) + \frac{\sigma_2}{\sigma} \zeta^n(t)$, the averaged functional central limit theorem can be derived from the previous two parts of Theorem 3.1. Indeed, it follows immediately that the finite dimensional distributions of $\mathbb{B}^n(t)$ converge to those of a Brownian motion (as in [16] this uses that convergences of terms like (3.5) hold uniformly in x and y). Thus, it only remains to show that $\mathbb{B}^n(\cdot)$ is tight, but this is not too difficult.

The random centering $nt\mathrm{v}_P - Z_{nt}(\omega)$ in the quenched CLT is more convenient than centering by the quenched mean $E_\omega X_{\lfloor nt \rfloor}$. Both centerings are essentially the same in the sense that they do not differ on the scale of $\sqrt{n}$:

$$\lim_{n\to\infty} P\left(\sup_{k \leq n} |E_\omega X_k - k\mathrm{v}_P + Z_k(\omega)| \geq \varepsilon\sqrt{n} \right) = 0, \qquad \forall \varepsilon > 0. \tag{3.6}$$

Moreover, $Z_{nt}(\omega)$ is convenient since $Z_{nt}(\omega) = h(\lfloor nt\mathrm{v}_P \rfloor, \omega)$ and $h(x,\omega)$ is defined in terms of partial sums of the random variables $E_{\theta^i\omega}T_1$ for which there is an explicit formula in terms of the environment ω (see [16] or [10]). We note the following Lemma due to Goldsheid [5] which we will use in several places in the remainder of the paper.

Lemma 3.3. *Let Assumptions 1 and 2 hold. Then there exists an $\eta > 0$ and a constant $C < \infty$ such that*

$$E_P\left[\sup_{1\le k\le n} |h(k,\omega)|^{2+2\eta}\right] \le Cn^{1+\eta}, \qquad \forall n \in \mathbb{N}. \tag{3.7}$$

We conclude this section by stating a new result on the uniform integrability (under the averaged measure) of $n^{-1/2}(X_n - n\mathrm{v}_P)$.

Proposition 3.4. *Let σ_1^2 and σ_2^2 be defined as in Theorem 3.1. Then,*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left(X_n - n\mathrm{v}_P\right)^2 = \sigma_1^2 + \sigma_2^2. \tag{3.8}$$

Moreover, there exists a constant $C < \infty$ such that

$$\mathbb{E}\left[\sup_{k\le n}(X_k - k\mathrm{v}_P)^2\right] \le Cn. \tag{3.9}$$

The proof of Proposition 3.4 is given in Appendix A. It should be noted that while the statement (3.6) does not appear anywhere in the literature (at least that we know of), it is included in the proof of Proposition 3.4.

4 Fluctuations of the quenched mean of the current

In this section we prove Theorem 2.2 for the quenched mean of $Y_n(t,r)$. Introduce the notation

$$\begin{aligned} W_n(t,r) &:= E_\omega Y_n(t,r) - \mu r\sqrt{n} \qquad (4.1)\\ &= \sum_{m>0} E_\omega[\eta_0(m)]P_\omega(X_{nt}^m \le nt\mathrm{v}_P + r\sqrt{n}) - \sum_{m\le 0} E_\omega[\eta_0(m)]P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}) - \mu r\sqrt{n}. \end{aligned}$$

The task is to show that $\frac{1}{\sqrt{n}}W_n(t,r)$ can be approximated by $\frac{1}{\sqrt{n}}Z_{nt}(\omega)$ uniformly in both $r \in [-R,R]$ and $t \in [0,T]$ with probability tending to one. The main work goes towards approximation uniformly in $t \in [0,T]$ for a fixed r. Uniformity in $r \in [-R,R]$ then comes easily at the end of this section, completing the proof of Theorem 2.2.

Before the main work we prove two lemmas that remove a few technical difficulties. One technical difficulty is presented by small times t. For any fixed $\delta > 0$ and $t \ge \delta$ we will use the quenched central limit theorem to approximate the probabilities in the definition of $W_n(t,r)$. However, we cannot do this approximation for arbitrarily small t all at once. The following lemma will be used later to handle the small values of t.

Lemma 4.1. *There exists a constant $C < \infty$ such that for any $r \in \mathbb{R}$ and $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} E_P\left[\sup_{t\in[0,\delta]} |W_n(t,r)|\right] \le C\sqrt{\delta}.$$

Proof. The triangle inequality implies that

$$\frac{1}{\sqrt{n}}E_P\left[\sup_{t\in[0,\delta]}|W_n(t,r)|\right] \le \frac{1}{\sqrt{n}}E_P\left[|W_n(0,r)|\right] + \frac{1}{\sqrt{n}}E_P\left[\sup_{t\in[0,\delta]}|W_n(t,r) - W_n(0,r)|\right]. \tag{4.2}$$

For $r > 0$,

$$\frac{1}{\sqrt{n}}W_n(0,r) = \frac{1}{\sqrt{n}}\sum_{0<m\le r\sqrt{n}} E_\omega(\eta_0(m)) - \mu r = \frac{1}{\sqrt{n}}\sum_{0<m\le r\sqrt{n}} \bar{\mu}(\theta^m\omega) - \mu r.$$

A similar equality holds for $r \leq 0$. Therefore, the ergodic theorem implies that the first term on the right hand side of (4.2) vanishes as $n \to \infty$, and so it remains only to show that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} E_P \left[\sup_{t\in[0,\delta]} |W_n(t,r) - W_n(0,r)| \right] \leq C\sqrt{\delta}. \tag{4.3}$$

Recalling (2.4) and the fact that $W_n(t,r) = E_\omega Y_n(t,r) - \mu r\sqrt{n}$, we obtain that

$$\begin{aligned} &W_n(t,r) - W_n(0,r) \\ &= \sum_{m>r\sqrt{n}} E_\omega[\eta_0(m)] P_\omega(X_{nt}^m \leq nt\mathrm{v}_P + r\sqrt{n}) - \sum_{m\leq r\sqrt{n}} E_\omega[\eta_0(m)] P_\omega(X_{nt}^m > nt\mathrm{v}_P + r\sqrt{n}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\sup_{t\in[0,\delta]} |W_n(t,r) - W_n(0,r)| \\ &\leq \sum_{m>r\sqrt{n}} E_\omega[\eta_0(m)] \sup_{t\in[0,\delta]} P_{\theta^m\omega}(X_{nt} - nt\mathrm{v}_P \leq r\sqrt{n} - m) \\ &\quad + \sum_{m\leq r\sqrt{n}} E_\omega[\eta_0(m)] \sup_{t\in[0,\delta]} P_{\theta^m\omega}(X_{nt} - nt\mathrm{v}_P > r\sqrt{n} - m) \\ &\leq \sum_{m>r\sqrt{n}} E_\omega[\eta_0(m)] P_{\theta^m\omega}\left(\inf_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P) \leq r\sqrt{n} - m\right) \\ &\quad + \sum_{m\leq r\sqrt{n}} E_\omega[\eta_0(m)] P_{\theta^m\omega}\left(\sup_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P) > r\sqrt{n} - m\right) \end{aligned}$$

Then, the shift invariance of P implies that

$$\begin{aligned} &E_P\left\{\sup_{t\in[0,\delta]} |W_n(t,r) - W_n(0,r)|\right\} \\ &\leq E_P\left\{E_\omega[\eta_0(0)]\left[\sum_{m>r\sqrt{n}} P_\omega\left(\inf_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P) \leq r\sqrt{n} - m\right)\right.\right. \\ &\qquad\qquad \left.\left. + \sum_{m\leq r\sqrt{n}} P_\omega\left(\sup_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P) > r\sqrt{n} - m\right)\right]\right\} \\ &\leq E_P\left\{E_\omega[\eta_0(0)]\left[E_\omega\left(\sup_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P)^-\right) + E_\omega\left(\sup_{t\in[0,\delta]} (X_{nt} - nt\mathrm{v}_P)^+\right) + 1\right]\right\} \\ &\leq 2\mathbb{E}_P\left\{E_\omega[\eta_0(0)] E_\omega\left(\sup_{t\in[0,\delta]} |X_{nt} - nt\mathrm{v}_P|\right)\right\} + \mu \end{aligned}$$

The Cauchy-Schwartz inequality along with Assumption 4 and Proposition 3.4 imply that the right hand side is bounded above by $C\sqrt{n\delta} + \mu$. Dividing by $\sqrt{n}$ and taking $n \to \infty$ we obtain (4.3). ☐

A second technical difficulty in the analysis of $W_n(t,r)$ is restricting the sums in the definition of $W_n(t,r)$ to $[-a(n)\sqrt{n}, a(n)\sqrt{n}]$, where $a(n)$ is some sequence tending to ∞ slowly (to be specified later, but at least slower than any polynomial in n). Let $W_n(t,r) = W_{n,1}(t,r) +$

$W_{n,2}(t,r)$, where

$$W_{n,1}(t,r) = \sum_{m=1}^{\lfloor a(n)\sqrt{n}\rfloor} E_\omega[\eta_0(m)] P_\omega(X^m_{nt} \le nt\mathrm{v}_P + r\sqrt{n}) \\ - \sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{0} E_\omega[\eta_0(m)] P_\omega(X^m_{nt} > nt\mathrm{v}_P + r\sqrt{n}) - \mu r\sqrt{n}.$$

The next lemma implies that the main contributions to $W_n(t,r)$ come from $W_{n,1}(t,r)$.

Lemma 4.2. *For any $\varepsilon > 0$, $T < \infty$, and $r \in \mathbb{R}$,*

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \frac{1}{\sqrt{n}} |W_{n,2}(t,r)| \ge \varepsilon\right) = 0.$$

Proof. It is enough to show that $E_P|\sup_{t\in[0,T]} W_{n,2}(t,r)| = o(\sqrt{n})$. Similarly to the proof of Lemma 4.1, we obtain that

$$\sup_{t\in[0,T]} |W_{n,2}(t,r)| \le \sum_{m>\lfloor a(n)\sqrt{n}\rfloor} E_\omega[\eta_0(m)] P_{\theta^m\omega}\left(\inf_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n}\right) \le -m\right) \\ + \sum_{m\le -\lfloor a(n)\sqrt{n}\rfloor} E_\omega[\eta_0(m)] P_{\theta^m\omega}\left(\sup_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n}\right) > -m\right),$$

and the shift invariance of P implies that

$$E_P\left\{\sup_{t\in[0,T]} |W_{n,2}(t,r)|\right\} \\ \le E_P\Bigg\{E_\omega[\eta_0(0)]\Bigg[\sum_{m>\lfloor a(n)\sqrt{n}\rfloor} P_\omega\left(\inf_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n}\right) \le -m\right) \\ + \sum_{m\le -\lfloor a(n)\sqrt{n}\rfloor} P_\omega\left(\sup_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n}\right) > -m\right)\Bigg]\Bigg\} \\ \le E_P\Bigg\{E_\omega[\eta_0(0)]\Bigg[E_\omega\left(\sup_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n} + \lfloor a(n)\sqrt{n}\rfloor\right)^-\right) \\ + E_\omega\left(\sup_{t\in[0,T]} \left(X_{nt} - nt\mathrm{v}_P - r\sqrt{n} - \lfloor a(n)\sqrt{n}\rfloor\right)^+\right)\Bigg]\Bigg\} \\ \le E_P\Bigg\{E_\omega[\eta_0(0)]\Bigg[E_\omega\Bigg(\sup_{t\in[0,T]} |X_{nt} - nt\mathrm{v}_P - r\sqrt{n}| \\ \times \mathbf{1}\{\sup_{t\in[0,T]} \left|X_{nt} - nt\mathrm{v}_P - r\sqrt{n}\right| \ge a(n)\sqrt{n}\}\Bigg)\Bigg]\Bigg\}.$$

Let $p = 2+\varepsilon$ for some $\varepsilon > 0$ satisfying Assumption 4, and let $1/p + 1/q = 1$. Note that $p > 2$

implies that $q \in (1,2)$. Then, Hölder's inequality implies that

$$E_P\left\{\sup_{t\in[0,T]}|W_{n,2}(t,r)|\right\}$$

$$\le CE_P\left\{E_\omega\left(\sup_{t\in[0,T]}|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}|\mathbf{1}\{\sup_{t\in[0,T]}\left|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}\right|\ge a(n)\sqrt{n}\}\right)^q\right\}^{1/q}$$

applying the Cauchy-Schwartz inequality to the inner expectation

$$\le CE_P\left\{E_\omega\left(\sup_{t\in[0,T]}|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}|^2\right)^{q/2}\right.$$

$$\left.\times P_\omega\left(\sup_{t\in[0,T]}\left|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}\right|\ge a(n)\sqrt{n}\right)^{q/2}\right\}^{1/q}$$

by Hölder's inequality again and because probabilities are bounded above by 1

$$\le C\mathbb{E}\left(\sup_{t\in[0,T]}|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}|^2\right)^{1/2}\mathbb{P}\left(\sup_{t\in[0,T]}\left|X_{nt}-nt\mathrm{v}_P-r\sqrt{n}\right|\ge a(n)\sqrt{n}\right)^{(2-q)/2q}. \tag{4.4}$$

Proposition 3.4 implies that (for a fixed $T<\infty$ and $r\in\mathbb{R}$) the first term on (4.4) is $\mathcal{O}(\sqrt{n})$, and the averaged functional central limit theorem (Part 3 of Theorem 3.1) implies that the last term in (4.4) vanishes as $n\to\infty$. This completes the proof of the Lemma. □

The majority of this section is devoted the the proof of the following Proposition which is a slightly weaker version of Theorem 2.2.

Proposition 4.3. *For any $\varepsilon>0$, $T<\infty$, and $r\in\mathbb{R}$,*

$$\lim_{n\to\infty}P\left(\sup_{t\in[0,T]}\frac{1}{\sqrt{n}}|W_n(t,r)-\mu Z_{nt}(\omega)|\ge\varepsilon\right)=0.$$

Therefore, $\frac{1}{\sqrt{n}}W_n(\cdot,r)$ converges in distribution to $\mu\sigma_2 W(\cdot)$, where $W(\cdot)$ is a standard Brownian motion.

Proof. For any $\delta>0$,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\frac{1}{\sqrt{n}}|W_n(t,r)-\mu Z_{nt}(\omega)|\ge\varepsilon\right)$$

$$\le\mathbb{P}\left(\sup_{t\in[0,\delta]}|W_n(t,r)|\ge\frac{\varepsilon}{2}\sqrt{n}\right)+\mathbb{P}\left(\sup_{t\in[0,\delta]}\mu|Z_{nt}(\omega)|\ge\frac{\varepsilon}{2}\sqrt{n}\right)$$

$$+\mathbb{P}\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}|W_n(t,r)-\mu Z_{nt}(\omega)|\ge\varepsilon\right)$$

$$\le\frac{2}{\varepsilon\sqrt{n}}\mathbb{E}\left[\sup_{t\in[0,\delta]}|W_n(t,r)|\right]+\mathbb{P}\left(\sup_{t\in[0,\delta]}\mu|Z_{nt}(\omega)|\ge\frac{\varepsilon}{2}\sqrt{n}\right) \tag{4.5}$$

$$+\mathbb{P}\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}|W_{n,2}(t,r)|\ge\frac{\varepsilon}{2}\right) \tag{4.6}$$

$$+\mathbb{P}\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}|W_{n,1}(t,r)-\mu Z_{nt}(\omega)|\ge\frac{\varepsilon}{2}\right) \tag{4.7}$$

Letting $n \to \infty$, Lemma 4.1 and the fact that $Z_{nt}(\omega)/\sqrt{n}$ converges to Brownian motion imply that the two terms in (4.5) can be made arbitrarily small by taking $\delta \to 0$. Also, Lemma 4.2 implies that the term in (4.6) vanishes as $n \to \infty$. Thus, it is enough to show that for any $\delta > 0$, (4.7) vanishes as $n \to \infty$. For this, we need the following lemmas whose proofs we defer for now.

Lemma 4.4. *Let*

$$\widetilde{W}_{n,1}(t,r) := \sum_{m=1}^{\lfloor a(n)\sqrt{n}\rfloor} E_\omega(\eta_0(m))\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\theta^m\omega)-m}{\sqrt{n}}+r\right) - \sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{0} E_\omega(\eta_0(m))\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\theta^m\omega)-m}{\sqrt{n}}-r\right) - \mu r\sqrt{n}.$$

Then, for any $\varepsilon > 0$, $r \in \mathbb{R}$, and $0 < \delta < T < \infty$,

$$\lim_{n\to\infty} P\left(\sup_{t\in[\delta,T]} \frac{1}{\sqrt{n}}\left|W_{n,1}(t,r) - \widetilde{W}_{n,1}(t,r)\right| \geq \varepsilon\right) = 0.$$

Lemma 4.5. *Let*

$$\widehat{W}_{n,1}(t,r) := \sum_{m=1}^{\lfloor a(n)\sqrt{n}\rfloor} E_\omega(\eta_0(m))\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}+r\right) - \sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{0} E_\omega(\eta_0(m))\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)-m}{\sqrt{n}}-r\right) - \mu r\sqrt{n}.$$

Then, for any $\varepsilon > 0$, $r \in \mathbb{R}$, and $0 < \delta < T < \infty$,

$$\lim_{n\to\infty} P\left(\sup_{t\in[\delta,T]} \frac{1}{\sqrt{n}}\left|\widetilde{W}_{n,1}(t,r) - \widehat{W}_{n,1}(t,r)\right| \geq \varepsilon\right) = 0.$$

Lemma 4.6. *Let*

$$\overline{W}_{n,1}(t,r) := \sum_{m=1}^{\lfloor a(n)\sqrt{n}\rfloor} \mu\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}+r\right) - \sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{0} \mu\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)-m}{\sqrt{n}}-r\right) - \mu r\sqrt{n}.$$

Then, for any $\varepsilon > 0$, $r \in \mathbb{R}$, and $0 < \delta < T < \infty$,

$$\lim_{n\to\infty} P\left(\sup_{t\in[\delta,T]} \frac{1}{\sqrt{n}}\left|\widehat{W}_{n,1}(t) - \overline{W}_{n,1}(t)\right| \geq \varepsilon\right) = 0.$$

Assuming for now Lemmas 4.4, 4.5, and 4.6, to finish the proof of Proposition 4.3, it remains to compare $\overline{W}_{n,1}(t,r)$ with $\mu Z_{nt}(\omega)$. Since $\Phi_{\sigma_1^2 t}(\cdot)$ is strictly increasing and bounded above by 1, we have using a Riemann sum approximation that for any $t \in [0,T]$,

$$\left|\frac{\overline{W}_{n,1}(t,r)}{\sqrt{n}} + \mu r - \mu\int_0^{a(n)} \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}}+r-x\right) - \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)}{\sqrt{n}}-r-x\right)dx\right| \leq \frac{2\mu}{\sqrt{n}}. \tag{4.8}$$

It is an easy exercise in calculus to show that for any $z \in \mathbb{R}$ and $A > 0$,

$$\int_0^A \Phi_{\alpha^2}(z-x) - \Phi_{\alpha^2}(-z-x)\,dx = z + \Psi_{\alpha^2}(A+z) - \Psi_{\alpha^2}(A-z),$$

where $\Psi_{\alpha^2}(x)$ is defined in (2.8). Therefore,

$$\int_0^{a(n)} \Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}} + r - x\right) - \Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)}{\sqrt{n}} - r - x\right) dx$$
$$= \frac{Z_{nt}(\omega)}{\sqrt{n}} + r + \Psi_{\sigma_1^2 t}\left(a(n) + \frac{Z_{nt}(\omega)}{\sqrt{n}} + r\right) - \Psi_{\sigma_1^2 t}\left(a(n) - \frac{Z_{nt}(\omega)}{\sqrt{n}} - r\right).$$

Recalling (4.8), this implies that for $\varepsilon > 0$ and n sufficiently large,

$$P\left(\sup_{t\in[0,T]}\left|\frac{\overline{W}_{n,1}(t,r)}{\sqrt{n}} - \frac{Z_{nt}(\omega)}{\sqrt{n}}\right| \geq \varepsilon\right)$$
$$\leq P\left(\sup_{t\in[0,T]}\left|\Psi_{\sigma_1^2 t}\left(a(n) + \frac{Z_{nt}(\omega)}{\sqrt{n}} + r\right) - \Psi_{\sigma_1^2 t}\left(a(n) - \frac{Z_{nt}(\omega)}{\sqrt{n}} - r\right)\right| \geq \frac{\varepsilon}{2\mu}\right) \tag{4.9}$$

A simple calculation shows that $\Psi'_{\alpha^2}(x) = -\Phi_{\alpha^2}(-x) < 0$, and so $\Psi_{\alpha^2}(x)$ is decreasing in x. Another direct calculation shows that $\frac{d}{d\alpha}\Psi_{\alpha^2}(x) = \alpha\phi_{\alpha^2}(x) > 0$. Thus, $\Psi_{\alpha^2}(x)$ is increasing in α. Thus, if $|Z_{nt}(\omega)| \leq a(n)\sqrt{n}/2$ and $t \leq T$,

$$\left|\Psi_{\sigma_1^2 t}\left(a(n) + \frac{Z_{nt}(\omega)}{\sqrt{n}} + r\right) - \Psi_{\sigma_1^2 t}\left(a(n) - \frac{Z_{nt}(\omega)}{\sqrt{n}} - r\right)\right| \leq 2\Psi_{\sigma_1^2 t}\left(a(n)/2 - |r|\right)$$
$$\leq 2\Psi_{\sigma_1^2 T}\left(a(n)/2 - |r|\right).$$

Since $\lim_{x\to\infty}\Psi_{\sigma_1^2 T}(x) = 0$, then $\Psi_{\sigma_1^2 T}(a(n)/2 - |r|) < \frac{\varepsilon}{2}$ for all n large enough. Thus, recalling (4.9), we obtain that for any $\varepsilon > 0$ and n sufficiently large,

$$P\left(\sup_{t\in[0,T]}\left|\frac{\overline{W}_{n,1}(t,r)}{\sqrt{n}} - \frac{Z_{nt}(\omega)}{\sqrt{n}}\right| \geq \varepsilon\right) \leq P\left(\sup_{t\in[0,T]}\left|\frac{Z_{nt}(\omega)}{\sqrt{n}}\right| \geq \frac{a(n)}{2}\right).$$

Since $t \mapsto \frac{Z_{nt}(\omega)}{\sqrt{n}}$ converges in distribution to a Brownian motion, this last probability tends to zero as $n \to \infty$. This completes the proof of Proposition 4.3. ☐

We now return to the proofs of Lemmas 4.4 – 4.6.

Proof of Lemma 4.4.
Let

$$D(n,\omega) := \sup_{x\in\mathbb{R}}\left|P_\omega\left(\frac{X_n - n\mathrm{v}_P + Z_n(\omega)}{\sqrt{n}} \leq x\right) - \Phi_{\sigma_1^2}(x)\right|, \quad \text{and} \quad \bar{D}(n,\omega) := \sup_{k\geq n} D(k,\omega).$$

Theorem 3.1 implies that $\lim_{n\to\infty}\bar{D}(n,\omega) = 0$, $P-a.s.$, and so by the bounded convergence theorem, $\lim_{n\to\infty}E_P[\bar{D}(n,\omega)^p] = 0$ for any $p > 0$. Thus, it is possible to choose the sequence $a(n)$ tending to infinity slowly enough so that

$$\lim_{n\to\infty} a(n)\left(E_P\left[\bar{D}(\delta n,\omega)^2\right]\right)^{1/2} = 0, \qquad \forall\delta > 0.$$

(For example, let $a(n) = (E_P\left[\bar{D}(\sqrt{n},\omega)^2\right])^{-1/4}$.) The definition of $D(n,\omega)$ implies that for any $t > 0$,

$$\left|W_{n,1}(t,r) - \widetilde{W}_{n,1}(t,r)\right| \leq \sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor} E_{\theta^m\omega}(\eta_0(0))D(nt,\theta^m\omega).$$

Therefore,

$$
\begin{aligned}
& P\left(\sup_{t\in[\delta,T]}\left|W_{n,1}(t,r)-\widetilde{W}_{n,1}(t,r)\right|\geq \varepsilon\sqrt{n}\right) \\
& \leq P\left(\sup_{t\in[\delta,T]}\sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor} E_{\theta^m\omega}(\eta_0(0))D(nt,\theta^m\omega)\geq\varepsilon\sqrt{n}\right) \\
& \leq P\left(\sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor} E_{\theta^m\omega}(\eta_0(0))\bar{D}(\delta n,\theta^m\omega)\geq\varepsilon\sqrt{n}\right) \\
& \leq \frac{2a(n)}{\varepsilon}E_P[E_\omega(\eta_0(0))\bar{D}(\delta n,\omega)] \\
& \leq \frac{2a(n)}{\varepsilon}\left(E_P[(E_\omega\eta_0(0))^2]\right)^{1/2}\left(E_P[\bar{D}(\delta n,\omega)^2]\right)^{1/2}
\end{aligned}
$$

where the next to last inequality follows from Chebyshev's inequality and the shift invariance of P. Our choice of the sequence $a(n)$ ensures that this last term vanishes as $n\to\infty$. □

Proof of Lemma 4.5.
Note that the mean value theorem implies

$$
\left|\Phi_{\sigma_1^2 t}(x)-\Phi_{\sigma_1^2 t}(y)\right|\leq\left(\sup_{z\in\mathbb{R}}\Phi'_{\sigma_1^2 t}(z)\right)|x-y|=\frac{1}{\sigma_1\sqrt{2\pi t}}|x-y|,\qquad\forall x,y\in\mathbb{R}.
$$

Therefore,

$$
\begin{aligned}
\sup_{t\in[\delta,T]}|\widetilde{W}_{n,1}(t,r)-\widehat{W}_{n,1}(t,r)| &\leq \sup_{t\in[\delta,T]}\sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor}\bar{\mu}(\theta^m\omega)\frac{1}{\sigma_1\sqrt{2\pi t}}\left|\frac{Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)}{\sqrt{n}}\right| \\
&\leq \frac{2a(n)}{\sigma_1\sqrt{2\pi\delta}}\sup_{t\in[\delta,T]}\max_{|m|\leq a(n)\sqrt{n}}|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)| \\
&\quad\times\left(\frac{1}{2a(n)\sqrt{n}}\sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor}\bar{\mu}(\theta^m\omega)\right).
\end{aligned}
$$

The ergodic theorem implies that the averaged sum on the last line converges to μ, $P-a.s.$ Thus, to finish the proof of the lemma it is enough to show that

$$
\lim_{n\to\infty}P\left(\sup_{t\in[\delta,T]}\max_{|m|\leq a(n)\sqrt{n}}|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)|\geq\frac{\varepsilon\sqrt{n}}{a(n)}\right)=0,\qquad\forall\varepsilon>0.
$$

Since $Z_{nt}(\theta^m\omega)=h(m+\lfloor nt\mathrm{v}_P\rfloor,\omega)-h(m,\omega)$,

$$
|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)|\leq|h(m,\omega)|+|h(m+\lfloor nt\mathrm{v}_P\rfloor,\omega)-h(\lfloor nt\mathrm{v}_P\rfloor,\omega)|.
$$

Thus,

$$
\begin{aligned}
&\sup_{t\in[\delta,T]}\max_{|m|\leq a(n)\sqrt{n}}|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)| \\
&\quad\leq 2\max_{x\in[0,nT]}\max_{1\leq m\leq a(n)\sqrt{n}}|h(x+m,\omega)-h(x,\omega)| \\
&\quad\leq 6\max_{0\leq i\leq\sqrt{n}T/a(n)}\max_{1\leq m\leq a(n)\sqrt{n}}|h(i\lfloor a(n)\sqrt{n}\rfloor+m,\omega)-h(i\lfloor a(n)\sqrt{n}\rfloor,\omega)|
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&P\left(\sup_{t\in[\delta,T]}\max_{|m|\le a(n)\sqrt{n}}|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)|\ge\frac{\varepsilon\sqrt{n}}{a(n)}\right)\\
&\quad\le P\left(\max_{0\le i\le\sqrt{n}T/a(n)}\max_{1\le m\le a(n)\sqrt{n}}|h(i\lfloor a(n)\sqrt{n}\rfloor+m,\omega)-h(i\lfloor a(n)\sqrt{n}\rfloor,\omega)|\ge\frac{\varepsilon\sqrt{n}}{6a(n)}\right)\\
&\quad\le\frac{\sqrt{n}T}{a(n)}P\left(\max_{1\le m\le a(n)\sqrt{n}}|h(m,\omega)|\ge\frac{\varepsilon\sqrt{n}}{6a(n)}\right),
\end{aligned}
$$

where the last inequality is from a union bound and the shift invariance of P. Recalling Lemma 3.3, there exist constants $C,\eta>0$ such that for any fixed $\varepsilon>0$ and $0<\delta<T<\infty$,

$$
\begin{aligned}
P\left(\sup_{t\in[\delta,T]}\max_{|m|\le a(n)\sqrt{n}}|Z_{nt}(\theta^m\omega)-Z_{nt}(\omega)|\ge\frac{\varepsilon\sqrt{n}}{a(n)}\right)&\le\frac{\sqrt{n}T}{a(n)}\left(\frac{6a(n)}{\varepsilon\sqrt{n}}\right)^{2+2\eta}C\left(a(n)\sqrt{n}\right)^{1+\eta}\\
&=\mathcal{O}\left(a(n)^{2+3\eta}n^{-\eta/2}\right).
\end{aligned}
$$

Since $a(n)$ grows slower than polynomially in n, this last term vanishes as $n\to\infty$. □

Proof of Lemma 4.6. For any integer R let

$$
\begin{aligned}
\widehat{W}^R_{n,1}(t,r):=&\sum_{m=1}^{\lfloor R\sqrt{n}\rfloor}E_\omega(\eta_0(m))\Phi_{\sigma_1^2t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}+r\right)\\
&-\sum_{m=-\lfloor R\sqrt{n}\rfloor+1}^{0}E_\omega(\eta_0(m))\Phi_{\sigma_1^2t}\left(-\frac{Z_{nt}(\omega)-m}{\sqrt{n}}-r\right)-\mu r\sqrt{n}.
\end{aligned}
$$

and

$$
\overline{W}^R_{n,1}(t,r):=\sum_{m=1}^{\lfloor R\sqrt{n}\rfloor}\mu\Phi_{\sigma_1^2t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}+r\right)-\sum_{m=-\lfloor R\sqrt{n}\rfloor+1}^{0}\mu\Phi_{\sigma_1^2t}\left(-\frac{Z_{nt}(\omega)-m}{\sqrt{n}}-r\right)-\mu r\sqrt{n}.
$$

Then, it is enough to show that

$$
\lim_{n\to\infty}\frac{1}{\sqrt{n}}E_P\left[\sup_{t\in[\delta,T]}\left|\widehat{W}^R_{n,1}(t,r)-\overline{W}^R_{n,1}(t,r)\right|\right]=0,\qquad\forall R<\infty,\tag{4.10}
$$

and that

$$
\lim_{R\to\infty}\limsup_{n\to\infty}P\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}\left|\widehat{W}_{n,1}(t,r)-\widehat{W}^R_{n,1}(t,r)\right|\ge\varepsilon\right)=0,\qquad\forall\varepsilon>0,\tag{4.11}
$$

and

$$
\lim_{R\to\infty}\limsup_{n\to\infty}P\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}\left|\overline{W}_{n,1}(t,r)-\overline{W}^R_{n,1}(t,r)\right|\ge\varepsilon\right)=0,\qquad\forall\varepsilon>0.\tag{4.12}
$$

To bound (4.10), we fix another parameter L and then divide the interval $(-\lfloor R\sqrt{n}\rfloor,\lfloor R\sqrt{n}\rfloor]$ into $2RL$ intervals, each of length approximately $\sqrt{n}/L$. For ease of notation, let $B_{n,L}(\ell):=\{m\in\mathbb{Z}:\frac{(\ell-1)\sqrt{n}}{L}<m\le\frac{\ell\sqrt{n}}{L}\}$. Now, for any $m\in B_{n,L}(\ell)$ and $t\in[\delta,T]$,

$$
\Phi_{\sigma_1^2t}\left(\frac{Z_{nt}(\omega)-m}{\sqrt{n}}+r\right)-\Phi_{\sigma_1^2t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}}-\frac{\ell}{L}+r\right)\le\frac{1}{\sqrt{2\pi t}}\left|\frac{m}{\sqrt{n}}-\frac{\ell}{L}\right|\le\frac{C}{L},
$$

where the constant C depends only on $\delta > 0$. Thus,

$$\frac{1}{\sqrt{n}}\widehat{W}^R_{n,1}(t,r) = \sum_{\ell=1}^{RL}\left(\frac{1}{\sqrt{n}}\sum_{m\in B_{n,L}(\ell)}\bar{\mu}(\theta^m\omega)\right)\Phi_{\sigma_1^2 t}\left(\frac{Z_{nt}(\omega)}{\sqrt{n}} - \frac{\ell}{L} + r\right)$$
$$- \sum_{\ell=-RL+1}^{0}\left(\frac{1}{\sqrt{n}}\sum_{m\in B_{n,L}(\ell)}\bar{\mu}(\theta^m\omega)\right)\Phi_{\sigma_1^2 t}\left(-\frac{Z_{nt}(\omega)}{\sqrt{n}} + \frac{\ell}{L} - r\right)$$
$$+ \frac{1}{\sqrt{n}}\sum_{m=1}^{\lfloor R\sqrt{n}\rfloor}\bar{\mu}(\theta^m\omega)\mathcal{O}(L^{-1}) - \frac{1}{\sqrt{n}}\sum_{m=-\lfloor R\sqrt{n}\rfloor+1}^{0}\bar{\mu}(\theta^m\omega)\mathcal{O}(L^{-1}).$$

A similar equality also holds for $\overline{W}^R_{n,1}(t,r)$ with $\bar{\mu}(\theta^m\omega)$ replaced by μ. Therefore, using the fact that Φ_{α^2} is bounded by 1, we obtain that

$$\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}|\widehat{W}^R_{n,1}(t,r) - \overline{W}^R_{n,1}(t,r)|$$
$$\leq \sum_{\ell=-RL+1}^{RL}\left|\frac{1}{\sqrt{n}}\sum_{m\in B_{n,L}(\ell)}(\bar{\mu}(\theta^m\omega)-\mu)\right| + \mathcal{O}(L^{-1})\left(\frac{1}{\sqrt{n}}\sum_{m=-\lfloor R\sqrt{n}\rfloor+1}^{\lfloor R\sqrt{n}\rfloor}\bar{\mu}(\theta^m\omega) + 2R\mu\right).$$

Note that we were able to include the supremum over t in the above inequality since the constant in the $\mathcal{O}(L^{-1})$ term is valid for any $t \geq \delta$. Taking expectations of the above with respect to the measure P and letting $n \to \infty$, the ergodic theorem implies that the first term vanishes and the second term has lim sup less than $4R\mu\mathcal{O}(L^{-1})$. Thus, taking $L \to \infty$ proves (4.10).

To bound (4.11), let

$$G_{n,R} := \left\{\omega : \sup_{t\in[\delta,T]}\left|\frac{Z_{nt}(\omega)}{\sqrt{n}} + r\right| \leq \frac{R}{2}\right\}.$$

Since $t \mapsto Z_{nt}(\omega)/\sqrt{n}$ converges to Brownian motion, $\lim_{R\to\infty}\lim_{n\to\infty}P(G_{n,R}) = 1$ for any fixed $r \in \mathbb{R}$. Thus,

$$\lim_{R\to\infty}\limsup_{n\to\infty}P\left(\sup_{t\in[\delta,T]}\frac{1}{\sqrt{n}}\left|\widehat{W}_{n,1}(t,r) - \widehat{W}^R_{n,1}(t,r)\right| \geq \varepsilon\right)$$
$$\leq \lim_{R\to\infty}\limsup_{n\to\infty}\frac{1}{\varepsilon\sqrt{n}}E_P\left[\sup_{t\in[\delta,T]}\left|\widehat{W}_{n,1}(t,r) - \widehat{W}^R_{n,1}(t,r)\right|\mathbf{1}_{G_{n,R}}\right] \tag{4.13}$$

If $\omega \in G_{n,R}$, $|m| \geq R\sqrt{n}$, and $t \leq T$, then $\Phi_{\sigma_1^2 t}\left(\left|\frac{Z_{nt}(\omega)}{\sqrt{n}} + r\right| - \frac{|m|}{\sqrt{n}}\right) \leq \Phi_{\sigma_1^2 T}\left(\frac{R}{2} - \frac{|m|}{\sqrt{n}}\right)$. Therefore,

$$\frac{1}{\sqrt{n}}E_P\left[\sup_{t\in[\delta,T]}\left|\widehat{W}_{n,1}(t) - \widehat{W}^R_{n,1}(t)\right|\mathbf{1}_{G_{n,R}}\right]$$
$$\leq \frac{1}{\sqrt{n}}\sum_{m=\lfloor R\sqrt{n}\rfloor+1}^{\lfloor a(n)\sqrt{n}\rfloor}\mu\Phi_{\sigma_1^2 T}\left(\frac{R}{2} - \frac{m}{\sqrt{n}}\right) + \frac{1}{\sqrt{n}}\sum_{m=-\lfloor a(n)\sqrt{n}\rfloor+1}^{-\lfloor R\sqrt{n}\rfloor}\mu\Phi_{\sigma_1^2 T}\left(\frac{R}{2} + \frac{m}{\sqrt{n}}\right)$$
$$\leq \mu\int_R^\infty \Phi_{\sigma_1^2 T}(R/2 - x)\,dx + \mu\int_{-\infty}^{-R}\Phi_{\sigma_1^2 T}(R/2 + x)\,dx + \frac{\mu}{\sqrt{n}},$$

where the last inequality is from a Riemann sum approximation. Since the integrals in the last line can be made arbitrarily small by taking $R \to \infty$, recalling (4.13) finishes the proof of (4.11). The proof of (4.12) is similar. □

We conclude this section with the proof of Theorem 2.2.

Proof of Theorem 2.2.
To prove Theorem 2.2 from Proposition 4.3, we need to justify the ability to include a supremum over $r \in [-R, R]$ inside the probability in the statement of Proposition 4.3. A simple union bound implies that we may include a supremum over a finite set of r values inside the probability in the statement of Proposition 4.3. That is, for $N < \infty$ and $r_1, r_2, \dots r_N \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\max_{k\le N}\sup_{t\in[0,T]}\frac{1}{\sqrt{n}}\left|E_\omega Y_n(t,r_k)-\mu r_k\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\varepsilon\right)=0. \tag{4.14}$$

Now, the definition of $Y_n(t,r)$ implies that $Y_n(t,r)$ is non-decreasing in r. Therefore, for any fixed t, $E_\omega Y_n(t,r) - \mu Z_{nt}(\omega)$ is non-decreasing in r. Choose $-R = r_1 < r_2 < \ldots < r_{N-1} < r_N = R$ such that $r_{k+1} - r_k \le \frac{\varepsilon}{2\mu}$ for $k = 1, \ldots, N-1$. Then, if $r \in [r_k, r_{k+1}]$,

$$\begin{aligned}&\left\{\left|E_\omega Y_n(t,r)-\mu r\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\varepsilon\sqrt{n}\right\}\\&\subset\left\{\left|E_\omega Y_n(t,r_k)-\mu r_k\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\frac{\varepsilon}{2}\sqrt{n}\right\}\cup\left\{\left|E_\omega Y_n(t,r_{k+1})-\mu r_{k+1}\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\frac{\varepsilon}{2}\sqrt{n}\right\}.\end{aligned}$$

Taking unions over $r \in [-R, R]$ and $t \in [0, T]$ implies that

$$\begin{aligned}&\left\{\sup_{r\in[-R,R]}\sup_{t\in[0,R]}\left|E_\omega Y_n(t,r)-\mu r\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\varepsilon\sqrt{n}\right\}\\&\subset\left\{\max_{k\le N}\sup_{t\in[0,R]}\left|E_\omega Y_n(t,r_k)-\mu r_k\sqrt{n}-\mu Z_{nt}(\omega)\right|\ge\frac{\varepsilon}{2}\sqrt{n}\right\}\end{aligned}$$

Recalling (4.14) finishes the proof of Theorem 2.2. □

5 Fluctuations of the centered current

Theorems 2.3 and 2.7 are proved in a similar way. We spell out some details for Theorem 2.3 and restrict to a few remarks on Theorem 2.7. The following representation of the covariance function $\Gamma((s,q),(t,r))$ will be convenient (proof by calculus). Recall that $B_\cdot$ denotes standard Brownian motion.

$$\begin{aligned}\Gamma\big((s,q),(t,r)\big) &= \mu\int_{-\infty}^{\infty}\Big(\mathbf{P}[B_{\sigma_1^2 s}\le q-x]\mathbf{P}[B_{\sigma_1^2 t}>r-x]-\mathbf{P}[B_{\sigma_1^2 s}\le q-x, B_{\sigma_1^2 t}>r-x]\Big)\,dx\\&\quad+\ \sigma_0^2\Big\{\int_0^{\infty}\mathbf{P}[B_{\sigma_1^2 s}\le q-x]\mathbf{P}[B_{\sigma_1^2 t}\le r-x]\,dx\\&\qquad\qquad+\int_{-\infty}^{0}\mathbf{P}[B_{\sigma_1^2 s}>q-x]\mathbf{P}[B_{\sigma_1^2 t}>r-x]\,dx\ \Big\}.\end{aligned} \tag{5.1}$$

Pick time-space points $(t_1, r_1), \ldots, (t_N, r_N) \in \mathbb{R}_+ \times \mathbb{R}$ and $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \mathbb{R}$. Form the linear combinations

$$(\bar{V}_n,\bar{Z}_n)=\Big(n^{-1/4}\sum_{i=1}^{N}\alpha_i V_n(t_i,r_i)\,,\ n^{-1/2}\sum_{i=1}^{N}\beta_i Z_{nt_i}\Big)$$

and

$$(\bar{V},\bar{Z})=\Big(\sum_{i=1}^{N}\alpha_i V(t_i,r_i)\,,\sum_{i=1}^{N}\beta_i Z(t_i)\Big).$$

Theorem 2.3 is proved by showing $(\bar{V}_n, \bar{Z}_n) \xrightarrow{\mathcal{D}} (\bar{V}, \bar{Z})$ for an arbitrary choice of $\{t_i, r_i, \alpha_i, \beta_i\}$.

We can work with $\bar{V}_n$ alone for a while because much of its analysis is done under a fixed ω, and then $Z_{n,\bullet}$ is not random.

$$\bar{V}_n = n^{-1/4} \sum_{i=1}^N \alpha_i V_n(t_i, r_i) = n^{-1/4} \sum_{x \in \mathbb{Z}} \sum_{i=1}^N \alpha_i \bigl[\mathbf{1}_{\{x>0\}} \phi_{x,i} \; - \; \mathbf{1}_{\{x \le 0\}} \psi_{x,i} \bigr] \tag{5.2}$$

where

$$\phi_{x,i} = \sum_{k=1}^{\eta_0(x)} \mathbf{1}\{ X^{x,k}_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \} - E_\omega(\eta_0(x)) P_\omega \{ X^x_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \}$$

and

$$\psi_{x,i} = \sum_{k=1}^{\eta_0(x)} \mathbf{1}\{ X^{x,k}_{nt_i} > nt_i \mathrm{v}_P + r_i \sqrt{n} \} - E_\omega(\eta_0(x)) P_\omega \{ X^x_{nt_i} > nt_i \mathrm{v}_P + r_i \sqrt{n} \}.$$

Equation (5.2) expresses $\bar{V}_n = n^{-1/4} \sum_{x \in \mathbb{Z}} u(x)$ as a sum of random variables

$$u(x) = \sum_{i=1}^N \alpha_i \bigl[\mathbf{1}_{\{x>0\}} \phi_{x,i} \; - \; \mathbf{1}_{\{x \le 0\}} \psi_{x,i} \bigr]$$

that are independent and mean zero under the quenched measure P_ω. They satisfy

$$|u(x)| \le \sum_{i=1}^N |\alpha_i| \bigl(\eta_0(x) + E_\omega(\eta_0(x)) \bigr). \tag{5.3}$$

Again we will pick $a(n) \nearrow \infty$ and define

$$\bar{V}^*_n = n^{-1/4} \sum_{|x| \le a(n)\sqrt{n}} u(x).$$

We first show that the rest of the sum can be ignored.

Lemma 5.1. $\lim_{n \to \infty} \mathbb{E}\bigl[(\bar{V}_n - \bar{V}^*_n)^2 \bigr] = 0.$

Proof. By the independence of the $\{u(x)\}$ under P_ω,

$$\begin{aligned} \mathbb{E}\bigl[(\bar{V}_n - \bar{V}^*_n)^2 \bigr] &= n^{-1/2} E_P \sum_{|x| > a(n)\sqrt{n}} E_\omega[u(x)^2] \\ &\le C n^{-1/2} \sum_{i=1}^N \sum_{|x| > a(n)\sqrt{n}} E_P \Biggl[\mathbf{1}_{\{x>0\}} \operatorname{Var}_\omega \Biggl(\sum_{k=1}^{\eta_0(x)} \mathbf{1}\{ X^{x,k}_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \} \Biggr) \\ &\qquad + \; \mathbf{1}_{\{x \le 0\}} \operatorname{Var}_\omega \Biggl(\sum_{k=1}^{\eta_0(x)} \mathbf{1}\{ X^{x,k}_{nt_i} > nt_i \mathrm{v}_P + r_i \sqrt{n} \} \Biggr) \Biggr]. \end{aligned} \tag{5.4}$$

Consider the first type of variance above:

$$\begin{aligned} &\operatorname{Var}_\omega \Biggl(\sum_{k=1}^{\eta_0(x)} \mathbf{1}\{ X^{x,k}_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \} \Biggr) \\ &\quad = E_\omega(\eta_0(x)) \operatorname{Var}_\omega \bigl(\mathbf{1}\{ X^x_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \} \bigr) + \operatorname{Var}_\omega(\eta_0(x)) P_\omega \{ X^x_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \}^2 \\ &\quad \le \bigl[E_\omega(\eta_0(x)) + \operatorname{Var}_\omega(\eta_0(x)) \bigr] P_\omega \{ X^x_{nt_i} \le nt_i \mathrm{v}_P + r_i \sqrt{n} \}. \end{aligned}$$

The upshot is that to show the vanishing of (5.4) we need to control terms of the type

$$n^{-1/2} \sum_{x > a(n)\sqrt{n}} E_P\big[\big(E_\omega(\eta(x)) + \mathrm{Var}_\omega(\eta_0(x))\big) P_\omega\{X_n^x \le n\mathrm{v}_P + r\sqrt{n}\,\}\big] \tag{5.5}$$

as $a(n) \to \infty$, together with its counterpart for $x < -a(n)\sqrt{n}$. For convenience we replaced time points nt_i with n and r represents $\max r_i$. We treat the part in (5.5) with the variance and omit the rest. Let $a_1(n) = a(n) - r$.

$$\begin{aligned}
&n^{-1/2} \sum_{x > a(n)\sqrt{n}} E_P\big[\mathrm{Var}_\omega(\eta(x))\, P_\omega\{X_n^x \le n\mathrm{v}_P + r\sqrt{n}\,\}\big] \\
&\quad = n^{-1/2} \sum_{x > a(n)\sqrt{n}} E_P\big[\mathrm{Var}_\omega(\eta(x))\, P_{\theta^x\omega}\{X_n \le n\mathrm{v}_P + r\sqrt{n} - x\,\}\big] \\
&\quad \le n^{-1/2} \sum_{y > a_1(n)\sqrt{n}} E_P\big[\mathrm{Var}_\omega(\eta(0))\, P_\omega\{X_n - n\mathrm{v}_P \le -y\,\}\big] \\
&\quad = E_P\bigg[\mathrm{Var}_\omega(\eta(0))\, E_\omega\bigg\{\bigg(\frac{X_n - n\mathrm{v}_P}{\sqrt{n}} + a_1(n)\bigg)^{-}\,\bigg\}\bigg] \\
&\quad \le \Big\{E_P[(\mathrm{Var}_\omega(\eta(0)))^p]\Big\}^{1/p} \bigg\{E_P\bigg[\bigg(E_\omega\bigg\{\bigg(\frac{X_n - n\mathrm{v}_P}{\sqrt{n}} + a_1(n)\bigg)^{-}\,\bigg\}\bigg)^{q}\,\bigg]\bigg\}^{1/q}.
\end{aligned}$$

for some $p > 2$ and hence $q = p/(p-1) < 2$. By assumption (2.2) the first factor above is a constant if we take $2 < p < 2 + \varepsilon$. Then by the $L^2(\mathbb{P})$ boundedness of $n^{-1/2}(X_n - n\mathrm{v}_P)$ (Prop. 3.4) the second factor vanishes as $a(n) \to \infty$. □

Assume now by a truncation that for $\bar{V}_n^*$ the initial occupations satisfy

$$\eta_0(x) \le n^{1/4-\delta} \tag{5.6}$$

for a small $\delta > 0$. Let momentarily $\widetilde{V}_n^*$ denote the variable with truncated occupations $\tilde{\eta}_0(x) = \lfloor \eta_0(x) \wedge n^{1/4-\delta} \rfloor$.

Lemma 5.2. *If $a(n) \nearrow \infty$ slowly enough, $\mathbb{E}\big[\,|\bar{V}_n^* - \widetilde{V}_n^*|^2\,\big] \to 0$.*

Proof. With $A_i^{x,k}$ denoting the random walk events that appear in $\phi_{x,i}$ and $B_i^{x,k}$ the ones in $\psi_{x,i}$,

$$\begin{aligned}
\bar{V}_n^* - \widetilde{V}_n^* = \sum_{i=1}^N \alpha_i\, n^{-1/4} \Bigg[&\sum_{0 < x \le a(n)\sqrt{n}} \Bigg(\sum_{k=\tilde{\eta}_0(x)+1}^{\eta_0(x)} \mathbf{1}_{\{A_i^{x,k}\}} - E_\omega(\eta_0(x) - \tilde{\eta}_0(x)) P_\omega(A_i^x) \Bigg) \\
&- \sum_{a(n)\sqrt{n} \le x \le 0} \Bigg(\sum_{k=\tilde{\eta}_0(x)+1}^{\eta_0(x)} \mathbf{1}_{\{B_i^{x,k}\}} - E_\omega(\eta_0(x) - \tilde{\eta}_0(x)) P_\omega(B_i^x) \Bigg) \Bigg].
\end{aligned}$$

Square and use independence across sites as in the beginning of the proof of Lemma 5.1 to get

$$\begin{aligned}
E_\omega|\bar{V}_n^* - \widetilde{V}_n^*|^2 &\le Cn^{-1/2} \sum_{|x| \le a(n)\sqrt{n}} \big[\mathrm{Var}_\omega(\eta_0(x) - \tilde{\eta}_0(x)) + E_\omega(\eta_0(x) - \tilde{\eta}_0(x))\big] \\
&\le Cn^{-1/2} \sum_{|x| \le a(n)\sqrt{n}} E_\omega\big(\eta_0(x)^2 \mathbf{1}\{\eta_0(x) \ge n^{1/4-\delta}\}\big).
\end{aligned}$$

By shift-invariance

$$\mathbb{E}|\bar{V}_n^* - \widetilde{V}_n^*|^2 \le Ca(n)\mathbb{E}[\eta_0(0)^2 \mathbf{1}\{\eta_0(0) \ge n^{1/4-\delta}\}].$$

Assumption (2.2) implies that $\mathbb{E}(\eta_0(0)^2) < \infty$ and hence the last expectation tends to 0 as $n \to \infty$. The lemma follows. □

Consequently Theorem 2.3 is not affected by this truncation. For the remainder of this proof we work with the truncated occupation variables that satisfy (5.6) without indicating it explicitly in the notation.

Recall that for complex numbers such that $|z_i|, |w_i| \le 1$,

$$\Big|\prod_{i=1}^m z_i \; - \; \prod_{i=1}^m w_i\Big| \le \sum_{i=1}^m |z_i - w_i|. \tag{5.7}$$

Let

$$\sigma^2_{n,\omega}(x) = n^{-1/2} E_\omega[u(x)^2].$$

By (5.3) and the truncation (5.6)

$$\sigma^2_{n,\omega}(x) \le C n^{-1/2} E_\omega[\eta_0(x)^2] \le C n^{-2\delta} \tag{5.8}$$

which is < 1 for large enough n. Then

$$\begin{aligned} &\Big| E_\omega[e^{i\bar{V}^*_n}] - \prod_{|x|\le a(n)\sqrt{n}} \big(1 - \tfrac12 \sigma^2_{n,\omega}(x)\big)\Big| \qquad (5.9)\\ &\le \sum_{|x|\le a(n)\sqrt{n}} \big| E_\omega(e^{in^{-1/4}u(x)}) - (1 - \tfrac12 \sigma^2_{n,\omega}(x))\big| \end{aligned}$$

by an expansion of the exponential, as in the proof of the Lindeberg-Feller theorem in [2, Sect. 2.4.b, p. 115]

$$\le \frac{C\varepsilon(n)}{\sqrt{n}} \sum_{|x|\le a(n)\sqrt{n}} E_\omega[u(x)^2] + \frac{C}{\sqrt{n}} \sum_{|x|\le a(n)\sqrt{n}} E_\omega\big[u(x)^2 \mathbf{1}\{|u(x)| \ge n^{1/4}\varepsilon(n)\}\big]. \tag{5.10}$$

for some $0 < \varepsilon(n) \searrow 0$ that we can choose. If $\varepsilon(n) n^\delta \to \infty$ then the truncation (5.6) makes the second sum on line (5.10) vanish. Take E_P expectation over the inequalities from (5.9) to (5.10). Since $E_\omega[u(x)^2] \le C E_\omega[\eta_0(x)^2]$, moment assumption (2.2) gives

$$\frac{C\varepsilon(n)}{\sqrt{n}} \sum_{|x|\le a(n)\sqrt{n}} \mathbb{E}[u(x)^2] \le C a(n)\varepsilon(n). \tag{5.11}$$

Thus if $a(n) \nearrow \infty$ slowly enough so that $\varepsilon(n) = a(n)^{-2} >> n^{-\delta}$, (5.10) vanishes as $n \to \infty$.

We have reached this intermediate conclusion:

$$\lim_{n\to\infty} E_P \Big| E_\omega[e^{i\bar{V}^*_n}] - \prod_{|x|\le a(n)\sqrt{n}} \big(1 - \tfrac12 \sigma^2_{n,\omega}(x)\big)\Big| = 0. \tag{5.12}$$

The main technical work is encoded in the following proposition. Recall the definition of Γ from (5.1).

Proposition 5.3. *There exist bounded continuous functions g_n on $\mathbb{R}^N$ with these properties.*

(a) $\sup_n \|g_n\|_\infty < \infty$ *and $g_n \to g$ uniformly on compact subsets of $\mathbb{R}^N$ where g is also bounded, continuous and satisfies*

$$g(z_1, \dots, z_N) = \sum_{1\le i,j\le N} \alpha_i \alpha_j \Gamma\big((t_i, r_i + z_i), (t_j, r_j + z_j)\big) \qquad \text{for } z = (z_1, \dots, z_N) \in \mathbb{R}^N. \tag{5.13}$$

(b) *The following limit holds in P-probability as $n \to \infty$:*

$$\Big| \sum_{|x|\le a(n)\sqrt{n}} \sigma^2_{n,\omega}(x) \; - \; g_n\big(n^{-1/2} Z_{nt_1}, \dots, n^{-1/2} Z_{nt_N}\big)\Big| \longrightarrow 0. \tag{5.14}$$

Proof of Theorem 2.3 assuming Proposition 5.3. By virtue of Lemma 5.1, it remains to show

$$\big|\,\mathbb{E}[e^{i\bar V_n^* + i\bar Z_n}] \,-\, \mathbf{E}[e^{i\bar V + i\bar Z}]\,\big| \to 0. \tag{5.15}$$

(We need not put coefficients in front of $\bar V_n^*$ and $\bar Z_n$ because these coefficients can be subsumed in the α_i, β_i coefficients.) Define the random N-vectors

$$\mathbf{z}_n^{1,N} = (n^{-1/2}Z_{nt_1}, \dots, n^{-1/2}Z_{nt_N}) \quad \text{and} \quad \mathbf{z}^{1,N} = (Z(t_1), \dots, Z(t_N)).$$

Then the conditional distribution of V given Z, described in conjunction with (2.10) above, together with (5.13) gives

$$\mathbf{E}[e^{i\bar V + i\bar Z}] = \mathbf{E}[e^{-\frac12 g(\mathbf{z}^{1,N}) + i\bar Z}].$$

Now bound the absolute value in (5.15) by

$$\begin{aligned}
&\big|\, E_P\big[E_\omega(e^{i\bar V_n^*})\, e^{i\bar Z_n(\omega)}\big] \,-\, \mathbf{E}[e^{-\frac12 g(\mathbf{z}^{1,N}) + i\bar Z}]\,\big| \\
&\le E_P\big|E_\omega(e^{i\bar V_n^*}) - e^{-\frac12 g_n(\mathbf{z}_n^{1,N})}\,\big| \,+\, \big|E_P[e^{-\frac12 g_n(\mathbf{z}_n^{1,N}) + i\bar Z_n(\omega)}] \,-\, \mathbf{E}[e^{-\frac12 g(\mathbf{z}^{1,N}) + i\bar Z}]\,\big|.
\end{aligned} \tag{5.16}$$

The last absolute values expression above vanishes as $n \to \infty$ by the invariance principle $n^{-1/2}Z_{n\cdot} \xrightarrow{\mathcal{D}} Z(\cdot)$ (Theorem 3.1, part 2) and by a simple property of weak convergence stated in Lemma 5.4 after this proof. The second last term is bounded as follows.

$$E_P\big|E_\omega(e^{i\bar V_n^*}) - e^{-\frac12 g_n(\mathbf{z}_n^{1,N})}\,\big| \;\le\; E_P\Big|E_\omega(e^{i\bar V_n^*}) - \prod_{|x|\le a(n)\sqrt n} \big(1 - \tfrac12 \sigma^2_{n,\omega}(x)\big)\Big| \tag{5.17}$$

$$+\, E_P\Big|\prod_{|x|\le a(n)\sqrt n} \big(1 - \tfrac12 \sigma^2_{n,\omega}(x)\big) \,-\, \exp\Big\{-\tfrac12 \sum_{|x|\le a(n)\sqrt n} \sigma^2_{n,\omega}(x)\Big\}\,\Big| \tag{5.18}$$

$$+\, E_P\Big|\,\exp\Big\{-\tfrac12 \sum_{|x|\le a(n)\sqrt n} \sigma^2_{n,\omega}(x)\Big\} - \exp\big(-\tfrac12 g_n(\mathbf{z}_n^{1,N})\big)\,\Big|. \tag{5.19}$$

Let $n \to \infty$. Line (5.17) after the inequality vanishes by (5.12). Line (5.18) vanishes by the inequalities

$$\exp\Big(-\tfrac12(1+n^{-2\delta}) \sum_{|x|\le a(n)\sqrt n} \sigma^2_{n,\omega}(x)\Big) \le \prod_{|x|\le a(n)\sqrt n} \big(1 - \tfrac12\sigma^2_{n,\omega}(x)\big) \le \exp\Big(-\tfrac12 \sum_{|x|\le a(n)\sqrt n} \sigma^2_{n,\omega}(x)\Big),$$

where we used (5.8) and $-y - y^2 \le \log(1-y) \le -y$ for small $y > 0$. Finally, line (5.19) vanishes by (5.14).

We have shown that line (5.16) vanishes as $n \to \infty$ and thereby verified (5.15). This completes the proof of Theorem 2.3, assuming Proposition 5.3. □

Lines (5.17)–(5.19), $\mathbf{z}_n^{1,N} \xrightarrow[n\to\infty]{\mathcal{D}} \mathbf{z}^{1,N}$ and $g_n \to g$ uniformly on compacts show that

$$E_P\big|E_\omega(e^{i\bar V_n^*}) - e^{-\frac12 g(\mathbf{z}^{1,N})}\,\big| \to 0. \tag{5.20}$$

This verifies the remark stated after Theorem 2.3.

The next lemma was used in the proof above. We omit its short and simple proof.

Lemma 5.4. *Suppose $\zeta_n \xrightarrow{\mathcal{D}} \zeta$ for random variables with values in some Polish space S. Let f_n, f be bounded, continuous functions on S such that $\sup_n \|f_n\|_\infty < \infty$ and $f_n \to f$ uniformly on compact sets. Then $f_n(\zeta_n) \xrightarrow{\mathcal{D}} f(\zeta)$.*

We turn to the proof of the main technical proposition, Proposition 5.3.

Proof of Proposition 5.3. Consider n large enough so that $a(n) > \max_i |r_i|$.

$$\begin{aligned}
\sum_{|x|\le a(n)\sqrt{n}} \sigma^2_{n,\omega}(x) &= n^{-1/2} \sum_{|x|\le a(n)\sqrt{n}} E_\omega[u(x)^2] = n^{-1/2} \sum_{|x|\le a(n)\sqrt{n}} \mathrm{Cov}_\omega[u(x), u(x)] \\
&= \sum_{1\le i,j\le N} \alpha_i\alpha_j\, n^{-1/2} \sum_{|x|\le a(n)\sqrt{n}} \Big[\mathbf{1}_{\{x>0\}} \,\mathrm{Cov}_\omega(\phi_{x,i},\, \phi_{x,j}) \;+\; \mathbf{1}_{\{x\le 0\}} \,\mathrm{Cov}_\omega(\psi_{x,i},\, \psi_{x,j}) \Big].
\end{aligned} \tag{5.21}$$

Whenever we work with a fixed (i, j) we let $((s, q), (t, r))$ represent $((t_i, r_i), (t_j, r_j))$ to avoid excessive subscripts. To each term above apply the formula for the covariance of two random sums, with $\{Z_i\}$ i.i.d. and independent of K:

$$\mathrm{Cov}\Big(\sum_{i=1}^{K} f(Z_i)\,, \sum_{j=1}^{K} g(Z_j) \Big) = EK\; \mathrm{Cov}(f(Z), g(Z)) + \mathrm{Var}(K)\, Ef(Z)\, Eg(Z).$$

The first covariance on the last line of (5.21) develops as

$$\begin{aligned}
\mathrm{Cov}_\omega(\phi_{x,i},\, \phi_{x,j}) &= E_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n},\, X^x_{nt} \le nt\mathrm{v}_P + r\sqrt{n}\,\} \\
&\quad - E_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n}\,\} P_\omega\{X^x_{nt} \le nt\mathrm{v}_P + r\sqrt{n}\,\} \\
&\quad + \mathrm{Var}_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n}\,\} P_\omega\{X^x_{nt} \le nt\mathrm{v}_P + r\sqrt{n}\,\} \\
&= -\, E_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n},\, X^x_{nt} > nt\mathrm{v}_P + r\sqrt{n}\,\} \\
&\quad +\; E_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n}\,\} P_\omega\{X^x_{nt} > nt\mathrm{v}_P + r\sqrt{n}\,\} \\
&\quad + \mathrm{Var}_\omega(\eta_0(x)) P_\omega\{X^x_{ns} \le ns\mathrm{v}_P + q\sqrt{n}\,\} P_\omega\{X^x_{nt} \le nt\mathrm{v}_P + r\sqrt{n}\,\}.
\end{aligned} \tag{5.22}$$

Develop the second covariance in a similar vein, and then collect the terms:

$$\begin{aligned}
&\sum_{|x|\le a(n)\sqrt{n}} \sigma^2_{n,\omega}(x) \\
&= \sum_{1\le i,j\le N} \alpha_i\alpha_j \Bigg[n^{-1/2} \sum_{|x|\le a(n)\sqrt{n}} E_\omega(\eta_0(x)) \\
&\qquad \times \Big(P_\omega\{X^x_{nt_i} \le nt_i\mathrm{v}_P + r_i\sqrt{n}\,\} P_\omega\{X^x_{nt_j} > nt_j\mathrm{v}_P + r_j\sqrt{n}\,\} && (5.23) \\
&\qquad\qquad -\; P_\omega\{X^x_{nt_i} \le nt_i\mathrm{v}_P + r_i\sqrt{n},\, X^x_{nt_j} > nt_j\mathrm{v}_P + r_j\sqrt{n}\,\} \Big) && (5.24) \\
&+ n^{-1/2} \sum_{|x|\le a(n)\sqrt{n}} \mathrm{Var}_\omega(\eta_0(x)) \Big(\mathbf{1}_{\{x>0\}} P_\omega\{X^x_{nt_i} \le nt_i\mathrm{v}_P + r_i\sqrt{n}\,\} P_\omega\{X^x_{nt_j} \le nt_j\mathrm{v}_P + r_j\sqrt{n}\,\} && (5.25) \\
&\qquad\qquad + \mathbf{1}_{\{x\le 0\}} P_\omega\{X^x_{nt_i} > nt_i\mathrm{v}_P + r_i\sqrt{n}\,\} P_\omega\{X^x_{nt_j} > nt_j\mathrm{v}_P + r_j\sqrt{n}\,\} \Big) \Bigg]. && (5.26)
\end{aligned}$$

The function $g_n(z_1, \ldots, z_N)$ required for Proposition 5.3 is defined as the linear combination of integrals of Brownian probabilities that match up with the terms of the sum above. For

$(z_1, \dots, z_N) \in \mathbb{R}^N$,

$$
\begin{aligned}
g_n(z_1, \dots, z_N) = \sum_{1 \le i,j \le N} \alpha_i \alpha_j \Bigg[\mu \int_{-a(n)}^{a(n)} \Big(&\mathbf{P}[B_{\sigma_1^2 t_i} \le z_i + r_i - x] \mathbf{P}[B_{\sigma_1^2 t_j} > z_j + r_j - x] \\
&- \mathbf{P}[B_{\sigma_1^2 t_i} \le z_i + r_i - x, B_{\sigma_1^2 t_j} > z_j + r_j - x] \Big) \, dx \\
+ \sigma_0^2 \Big\{ \int_0^{a(n)} &\mathbf{P}[B_{\sigma_1^2 t_i} \le z_i + r_i - x] \mathbf{P}[B_{\sigma_1^2 t_j} \le z_j + r_j - x] \, dx \\
+ \int_{-a(n)}^0 &\mathbf{P}[B_{\sigma_1^2 t_i} > z_i + r_i - x] \mathbf{P}[B_{\sigma_1^2 t_j} > z_j + r_j - x] \, dx \Big\} \Bigg].
\end{aligned} \tag{5.27}
$$

Let $g(z_1, \dots, z_N)$ be the function defined by the above sum of integrals with $a(n)$ replaced by ∞. Then (5.13) holds by direct comparison with definition (5.1). Part (a) of Proposition 5.3 is now clear.

To prove limit (5.14) in part (b) of Proposition 5.3, namely that

$$
\Big| \sum_{|x| \le a(n)\sqrt{n}} \sigma^2_{n,\omega}(x) \; - \; g_n\big(n^{-1/2} Z_{nt_1}, \dots, n^{-1/2} Z_{nt_N}\big) \Big| \xrightarrow{P} 0,
$$

we approximate the sums on lines (5.23)–(5.26) with the corresponding integrals from (5.27). The steps are the same for each sum. We illustrate this reasoning with the sum of the terms on line (5.24), given by

$$
U_n(\omega) = \sum_{|m| \le a(n)\sqrt{n}} E_\omega(\eta_0(m)) P_\omega\{X^m_{ns} \le ns\mathrm{v}_P + q\sqrt{n}, \, X^m_{nt} > nt\mathrm{v}_P + r\sqrt{n}\,\} \tag{5.28}
$$

and the corresponding part of (5.27), defined by

$$
U^*_n(\omega) = \mu \int_{-a(n)}^{a(n)} \mathbf{P}\Big\{ B_{\sigma_1^2 s} \le \frac{Z_{ns}(\omega)}{\sqrt{n}} - x + q\,,\, B_{\sigma_1^2 t} > \frac{Z_{nt}(\omega)}{\sqrt{n}} - x + r \Big\} \, dx. \tag{5.29}
$$

The goal is to show

$$
\lim_{n\to\infty} |n^{-1/2} U_n(\omega) - U^*_n(\omega)| = 0 \quad \text{in } P\text{-probability.}
$$

The steps are the same as those employed in the proofs of Lemmas 4.4–4.6. First approximate U_n with

$$
\widetilde{U}_n(\omega) = \sum_{|m| \le a(n)\sqrt{n}} E_\omega(\eta_0(m)) \mathbf{P}\Big\{ B_{\sigma_1^2 s} \le \frac{Z_{ns}(\theta^m\omega)}{\sqrt{n}} - \frac{m}{\sqrt{n}} + q\,,\, B_{\sigma_1^2 t} > \frac{Z_{nt}(\theta^m\omega)}{\sqrt{n}} - \frac{m}{\sqrt{n}} + r \Big\}. \tag{5.30}
$$

This approximation is similar to the proof of Lemma 4.4 and uses the fact that for a fixed $s, t > 0$, the limits of the form (3.5) are uniform in $x, y \in \mathbb{R}$. Then remove the shift from $Z_n(\omega)$ by defining

$$
\widehat{U}_n(\omega) = \sum_{|m| \le a(n)\sqrt{n}} E_\omega(\eta_0(m)) \mathbf{P}\Big\{ B_{\sigma_1^2 s} \le \frac{Z_{ns}(\omega)}{\sqrt{n}} - \frac{m}{\sqrt{n}} + q\,,\, B_{\sigma_1^2 t} > \frac{Z_{nt}(\omega)}{\sqrt{n}} - \frac{m}{\sqrt{n}} + r \Big\} \tag{5.31}
$$

and showing that $\lim_{n\to\infty} n^{-1/2} |\widetilde{U}_n - \widehat{U}_n| = 0$, in $P-$probability. For the last step, to show $\lim_{n\to\infty} |n^{-1/2}\widehat{U}_n(\omega) - U^*_n(\omega)| = 0$ in $P-$probability, truncate the sum (5.31) and the integral (5.29), use a Riemann approximation of the sum, introduce an intermediate scale for further partitioning and appeal to the ergodic theorem, as was done in Lemma 4.6. We omit these details since the corresponding steps were spelled out in full in Section 4.

We have verified the part of the desired limit (5.14) that comes from pairing up the sum on line (5.24) with the second line of (5.27). The remaining parts are handled similarly. This completes the proof of Proposition 5.3. □

Theorem 2.3 has now been proved. Proof of Theorem 2.7 goes essentially the same way. The crucial difference comes at the point (5.31) where $\widehat{U}_n$ is introduced. Instead of $n^{-1/2}Z_{ns}(\omega)$ and $n^{-1/2}Z_{nt}(\omega)$ inside the Brownian probability $\mathbf{P}$, one has $n^{-1/2}\big(Z_{ns}(\omega) - Z_{ns}(\theta^m\omega)\big)$ and $n^{-1/2}\big(Z_{nt}(\omega) - Z_{nt}(\theta^m\omega)\big)$. These vanish on the scale considered here, with $|m| \le a(n)\sqrt{n}$, by the arguments used in the proof of Lemma 4.5.

Consequently, in the subsequent approximation by U_n^* at (5.29), the terms $n^{-1/2}Z_{ns}(\omega)$ and $n^{-1/2}Z_{nt}(\omega)$ have disappeared. Then in limit (5.15) in Proposition 5.3 we can take $g_n(0,\dots,0)$.

A Uniform integrability of $\sup_{k\le n}(X_k - k\mathrm{v}_P)/\sqrt{n}$

In this Appendix, we give the proof of Proposition 3.4. The main tool used in the proof is a martingale representation that was given in the proof of the averaged central limit theorem in [16]. Recall the definition of $h(x,\omega)$ in (3.1), and let $\mathcal{F}_n := \sigma(X_i : i \le n)$. Then, $M_n := X_n - n\mathrm{v}_P + h(X_n,\omega)$ is an $\mathcal{F}_n$-martigale under the measure P_ω. The correction term $h(X_n,\omega)$ may further be decomposed as $h(X_n,\omega) = Z_n(\omega) + R_n$, where $Z_n(\omega) = h(\lfloor n\mathrm{v}_P \rfloor,\omega)$ and $R_n := h(X_n,\omega) - Z_n(\omega)$. The main contributions to $X_n - n\mathrm{v}_P$ come from M_n and $Z_n(\omega)$, while the term R_n contributes on a scale of order less than $\sqrt{n}$. M_n accounts for the fluctuations due to the randomness of the walk in a fixed environment, and $Z_n(\omega)$ accounts for the fluctuations due to randomness of the environment.

Using the above notation, we then have

$$\mathbb{E}\left(X_n - n\mathrm{v}_P\right)^2 = \mathbb{E}M_n^2 + E_P Z_n(\omega)^2 + \mathbb{E}R_n^2 - 2\mathbb{E}[M_nR_n] + 2\mathbb{E}[Z_n(\omega)R_n]. \tag{A.1}$$

Note that the term $\mathbb{E}[M_nZ_n(\omega)]$ is missing on the right hand side above. This is because $Z_n(\omega)$ depends only on the environment and M_n is a martingale under P_ω and thus $\mathbb{E}[M_nZ_n(\omega)] = E_P\left[Z_n(\omega)E_\omega(M_n)\right] = 0$. Since Hölder's inequality implies that

$$\mathbb{E}[M_nR_n] + \mathbb{E}[Z_n(\omega)R_n] \le \Big((\mathbb{E}M_n^2)^{1/2} + (E_P Z_n(\omega)^2)^{1/2}\Big)(\mathbb{E}R_n^2)^{1/2},$$

to complete the proof of (3.8) it is enough to show

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}M_n^2 = \sigma_1^2, \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{n}E_P Z_n(\omega)^2 = \sigma_2^2, \tag{A.2}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}R_n^2 = 0. \tag{A.3}$$

Since $Z_n(\omega) = h(\lfloor n\mathrm{v}_P\rfloor,\omega)$, to prove the second statement in (A.2) it is enough to show that

$$\lim_{n\to\infty}\frac{1}{n}E_P[h(n,\omega)^2] = \mathrm{v}_P\,\mathrm{Var}(E_\omega T_1) = \frac{1}{\mathrm{v}_P}\sigma_2^2$$

However, since $h(n,\omega)$ is the sum of mean zero terms,

$$\begin{aligned} E_P[h(n,\omega)^2] = \mathrm{Var}(h(n,\omega)) &= \mathrm{v}_P^2\sum_{i=0}^{n-1}\mathrm{Var}(E_\omega T_1) + 2\mathrm{v}_P^2\sum_{0\le i<j\le n-1}\mathrm{Cov}\left(E_{\theta^i\omega}T_1, E_{\theta^j\omega}T_1\right) \\ &= n\mathrm{v}_P^2\,\mathrm{Var}(E_\omega T_1) + 2\mathrm{v}_P^2\sum_{k=1}^{n-1}(n-k)\,\mathrm{Cov}\left(E_\omega T_1, E_{\theta^k\omega}T_1\right), \end{aligned}$$

where the last equality is due to the shift invariance of environments. Since $E_{\theta^k\omega}T_1 = 1 + \rho_k + \rho_k E_{\theta^{k-1}\omega}T_1$ (see the derivation of a formula for $E_\omega T_1$ in [15] or [16]), the fact that P is an i.i.d. law on environments implies that

$$\mathrm{Cov}\left(E_\omega T_1, E_{\theta^k\omega}T_1\right) = (E_P\rho_0)\,\mathrm{Cov}\left(E_\omega T_1, E_{\theta^{k-1}\omega}T_1\right).$$

Iterating this computation, we get that $\mathrm{Cov}\left(E_\omega T_1, E_{\theta^k\omega} T_1\right) = (E_P\rho_0)^k \,\mathrm{Var}(E_\omega T_1)$. Therefore,

$$\begin{aligned}
E[h(n,\omega)^2] &= n\mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1) + 2\mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1) \sum_{k=1}^{n-1} (n-k)(E_P\rho_0)^k \\
&= n\mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1) \left(1 + 2\sum_{k=1}^{n-1} (E_P\rho_0)^k \right) - 2\mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1) \sum_{k=1}^{n-1} k(E_P\rho_0)^k.
\end{aligned}$$

Since $E_P\rho_0 < 1$, this implies that

$$\lim_{n\to\infty} \frac{1}{n} E_P[h(n,\omega)^2] = \mathrm{v}_P^2 \,\mathrm{Var}(E_\omega T_1) \left(1 + 2\frac{E_P\rho_0}{1-E_P\rho_0} \right) = \mathrm{v}_P \,\mathrm{Var}(E_\omega T_1), \tag{A.4}$$

where the last equality is from the explicit formula for v_P given in (2.1). Thus, we have proved the second statement in (A.2).

We now turn to the proof of the first statement in (A.2). Let

$$V_n := \sum_{k=1}^n E_\omega\left[(M_{k+1} - M_k)^2 | \mathcal{F}_k \right].$$

Note that $E_\omega V_n = E_\omega M_n^2$ since M_n is a martingale under P_ω. Thus, the first statement in (A.2) is equivalent to $\lim_{n\to\infty} \mathbb{E} V_n/n = \sigma_1^2$. A direct computation (see the proof of the averaged central limit theorem on page 211 of [16]) yields that $E_\omega\left[(M_{k+1}-M_k)^2|\mathcal{F}_k\right] = g(\theta^{X_k}\omega)$, where

$$g(\omega) = \mathrm{v}_P^2 \left(\omega_0 (E_\omega T_1 - 1)^2 + (1-\omega_0)(E_{\theta^{-1}\omega} T_1 + 1)^2 \right).$$

Recall the definition of $f(\omega)$ in (2.15), and let Q be a measure on environments defined by $\frac{dQ}{dP}(\omega) = f(\omega)$, where $f(\omega)$ is defined in (2.15). Under the averaged measure $\mathbb{Q}(\cdot) = E_Q[P_\omega(\cdot)]$, the sequence $\{\theta^{X_k}\omega\}_{k\in\mathbb{N}}$ is stationary and ergodic. Therefore, $\frac{V_n}{n} = \frac{1}{n}\sum_{k=1}^n g(\theta^{X_k}\omega)$ converges in $L^1(\mathbb{Q})$ to

$$E_Q[g(\omega)] = E_P\left[\frac{dQ}{dP}(\omega) g(\omega) \right] = \mathrm{v}_P^3 E_P[\mathrm{Var}_\omega T_1] = \sigma_1^2,$$

where the second to last equality follows from the formulas for $\frac{dQ}{dP}(\omega)$ and $g(\omega)$ given above, the explicit formula for $\mathrm{Var}_\omega T_1$ shown in [10], and the shift invariance of the law P. Since $\frac{dQ}{dP}(\omega) = f(\omega) \geq \mathrm{v}_P$, we obtain that

$$\mathbb{E}\left| V_n/n - \sigma_1^2 \right| = E_Q\left[\frac{dP}{dQ}(\omega) E_\omega \left| V_n/n - \sigma_1^2 \right| \right] \leq \frac{1}{\mathrm{v}_P} E_{\mathbb{Q}} \left| V_n/n - \sigma_1^2 \right| \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, since V_n/n converges in $L^1(\mathbb{Q})$ to σ_1^2, V_n/n also converges to σ_1^2 in $L^1(\mathbb{P})$.

Finally, we turn to the proof of (A.3). Fix a $\beta \in (1/2, 1)$. Since $R_n = h(X_n,\omega) - h(\lfloor n\mathrm{v}_P \rfloor, \omega)$,

$$E_\omega R_n^2 \leq \sup_{x:|x-\lfloor n\mathrm{v}_P\rfloor| \leq n^\beta} |h(x,\omega) - h(\lfloor n\mathrm{v}_P\rfloor,\omega)|^2 + \sup_{|x|\leq n} 4|h(x,\omega)|^2 P_\omega(|X_n - \lfloor n\mathrm{v}_P \rfloor| > n^\beta).$$

Then, the shift invariance of the measure P and Hölder's inequality imply that for any $\delta > 0$,

$$\begin{aligned}
\mathbb{E}R_n^2 &\leq 2E_P\left[\sup_{|x|\leq n^\beta} h(x,\omega)^2 \right] + 4E_P\left[\sup_{|x|\leq n} h(x,\omega)^2 P_\omega(|X_n - \lfloor n\mathrm{v}_P\rfloor| > n^\beta) \right] \\
&\leq E_P\left[\sup_{|x|\leq n^\beta} h(x,\omega)^2 \right] + 4\left(E_P\left[\sup_{|x|\leq n} h(x,\omega)^{2+2\delta} \right] \right)^{1/(1+\delta)} \mathbb{P}(|X_n - \lfloor n\mathrm{v}_P\rfloor| > n^\beta)^{\delta/(1+\delta)} \\
&\leq Cn^\beta + Cn\mathbb{P}(|X_n - \lfloor n\mathrm{v}_P\rfloor| > n^\beta)^{\delta/(1+\delta)},
\end{aligned}$$

where the last inequality follows from Lemma 3.3. The first term on the right above is $o(n)$ since $\beta < 1$, and the second term on the right is $o(n)$ because $\beta > 1/2$ and the averaged central limit theorem implies that $\mathbb{P}(|X_n - \lfloor n\mathrm{v}_P \rfloor| > n^\beta)$ tends to zero. This completes the proof of (A.3) and thus also the first part of Proposition 3.4.

To prove the second part of Proposition 3.4, we again use the representation $X_n - n\mathrm{v}_P = M_n - Z_n(\omega) - R_n$. Then,

$$\mathbb{E}\left[\sup_{k\leq n}(X_k - k\mathrm{v}_P)^2\right] \leq 3\mathbb{E}\left[\sup_{k\leq n} M_k^2\right] + 3\mathbb{E}\left[\sup_{k\leq n} Z_k(\omega)^2\right] + 3\mathbb{E}\left[\sup_{k\leq n} R_k^2\right].$$

Since M_n is a martingale, Doob's inequality and the first statement in (A.2) imply that

$$\mathbb{E}\left[\sup_{k\leq n} M_k^2\right] \leq 4\mathbb{E}\left[M_n^2\right] = \mathcal{O}(n).$$

The same argument given above which showed that $\mathbb{E}R_n^2 = o(n)$ can be repeated to show that for any $\beta \in (1/2, 1)$, there exists a constant $C < \infty$ such that

$$\mathbb{E}\left[\sup_{k\leq n} R_k^2\right] \leq Cn^\beta + Cn\mathbb{P}\left(\sup_{k\leq n}|X_k - k\mathrm{v}_P| \geq n^\beta\right) = o(n),$$

where in the last equality we used the averaged functional central limit theorem. To finish the proof of (3.9) we need to show that $E_P\left[\sup_{k\leq n} Z_k(\omega)^2\right] = \mathcal{O}(n)$. Since $Z_n(\omega) = h(\lfloor n\mathrm{v}_P\rfloor, \omega)$, this is equivalent to showing that $E_P\left[\sup_{k\leq n} h(k,\omega)^2\right] = \mathcal{O}(n)$. However, Hölder's inequality and (3.7) imply that there exists an $\eta > 0$ and $C < \infty$ such that

$$E_P\left[\sup_{k\leq n} h(k,\omega)^2\right] \leq \left(E_P\left[\sup_{k\leq n}|h(k,\omega)|^{2+2\eta}\right]\right)^{1/(1+\eta)} \leq Cn.$$

This completes the proof of Proposition 3.4.

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Jonathon Peterson
University of Wisconsin - Madison
Mathematics Department
480 Lincoln Drive
Madison, WI 53706-1388

Timo Seppäläinen
University of Wisconsin - Madison
Mathematics Department
480 Lincoln Drive
Madison, WI 53706-1388