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CORRELATIONS FOR PATHS IN RANDOM ORIENTATIONS OF $G(n, p)$

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ABSTRACT. We study the random graph $G(n, p)$ with a random orientation. For three fixed vertices s, a, b in $G(n, p)$ we study the correlation of the events $\{a \to s\}$ and $\{s \to b\}$. We prove that for a fixed $p < 1/2$ the correlation is negative for large enough n and for $p > 1/2$ the correlation is positive for large enough n. We present exact recursions to compute $P(a \to s)$ and $P(a \to s, s \to b)$. We conjecture that for a fixed $n \geq 27$ the correlation changes sign three times for three critical values of p.

1. Introduction

Given a graph $G = (V, E)$ we orient each edge with equal probability for the two possible directions and independent of all other edges. This model has been studied previously in for instance [1, 4, 7]. Let s, a, b be fixed vertices in the graph. Let $\{a \to s\}$ denote the event that is there exists a directed path from a to s. The object of this paper is to study the correlation of the two events $\{a \to s\}$ and $\{s \to b\}$ in a random graph. One might intuitively guess that they are negatively correlated in any graph, that is $P(a \to s, s \to b) - P(a \to s)P(s \to b) < 0$. This is however not true for all graphs. In fact, the smallest counter example is the graph on four vertices with all edges except $\{a, b\}$. In [1] we proved that for the complete graph K_n they are negatively correlated for $n = 3$, independent for $n = 4$ and positively correlated for $n \geq 5$. The results in [1] suggested that the correlation seemed more likely to be positive in dense graphs, which led us to further investigate the correlation in $G(n, p)$ as presented in the present paper. In the random graph $G(n, p)$ there are n vertices and every edge exists with probability p independent of other edges. Our main result in Section 2 implies that for a fixed p the events are positively correlated when $p > 1/2$ for large enough n, but negatively correlated when $p < 1/2$ for large enough n. To be more precise, in Theorem 2.6 we prove that

$$\frac{P(a \nrightarrow s, s \nrightarrow b) - P(a \nrightarrow s)P(s \nrightarrow b)}{P(a \nrightarrow s, s \nrightarrow b)} \to \frac{2p-1}{3} \text{ as } n \to \infty.$$

Note that the covariance is bilinear so that $P(a \to s, s \to b) - P(a \to s)P(s \to b) = P(a \nrightarrow s, s \nrightarrow b) - P(a \nrightarrow s)P(s \nrightarrow b)$. Thus we may instead study the complementary events $\{a \nrightarrow s\}$, that there does not exist a directed path from a to s, and $\{s \nrightarrow b\}$,

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which turns out to be more practical. The change of sign for the correlation at $p = 1/2$ was unexpected for us. We challenge the reader to give a good intuitive explanation for this.

In Section 3 we give exact recursions for the probabilities $P(a \nrightarrow s), P(s \nrightarrow b)$ and $P(a \nrightarrow s, s \nrightarrow b)$. This is done by studying the size of the set of the vertices that can be reached along a directed path from a given vertex s and the size of the set of vertices from which there is a directed path to s. We give recursions for the joint distribution of these sizes, which could be of independent interest.

The situation seems however to be much more complicated than what is suggested by Theorem 2.6. In Section 4 we report our results from exact computations using the recursions. It turns out that for $n \geq 27$ the difference $P(a \to s, s \to b) - P(a \to s) \cdot P(s \to b)$ seems to change sign three times. First twice for small p (roughly $constant/n$) and then it goes from negative to positive again just before $p = 1/2$. We find this behavior mysterious but conjecture it to be true in general and plan to study this further in a coming paper.

The background of the questions studied in this paper is as follows. Let G be any graph and a, b, s, t distinct vertices of G. Further, assign a random orientation to the graph as described above, that is each edge is given its direction independent of the other edges. In [7] it was proved that then the events $\{s \to a\}$ and $\{s \to b\}$ are positively correlated. This was shown to be true also if we first conditioned on $\{s \nrightarrow t\}$, i.e. $P(s \to a, s \to b | s \nrightarrow t) \geq P(s \to a | s \nrightarrow t) \cdot P(s \to b | s \nrightarrow t)$. Note that it is not intuitively clear why this is so. It is for instance no longer true of we instead condition on $\{s \to t\}$. In another direction, it was proved that $P(s \to b, a \to t | s \nrightarrow t) \leq P(s \to b | s \nrightarrow t) \cdot P(a \to t | s \nrightarrow t)$. The proofs in [7] relied heavily on the results in [2] and [3] where similar statements were proved for edge percolation on a given graph. These questions has to a large extent been inspired by an interesting conjecture due to Kasteleyn named the Bunkbed conjecture by Häggström [5], see also [6] and Remark 5 in [2].

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2. Main Theorem

Let $G(n, p)$ be the random graph where each edge has probability p of being present independent of other edges. We further orient each present edge either way independently with probability $\frac{1}{2}$. It turns out to be easier to study the negated events $A := \{a \nrightarrow s\}$ and $B := \{s \nrightarrow b\}$. We say that the events A and B are negatively correlated if

$$P(A \cap B) - P(A)P(B) < 0,$$

and positively correlated if the inequality goes the other way. If A and B are negatively correlated, then A and $\neg B$ are positively correlated. Thus A and B are negatively correlated if and only if $\{a \to s\}$ and $\{s \to b\}$ are negatively correlated. Trivially, $P(A) = P(B)$.

The case $p = 1$ corresponds to random orientations of the complete graph K_n. In [1] it was proved that A and B are negatively correlated in K_3, independent in K_4 and positively correlated in K_n for $n \geq 5$. It was in fact proved that the relative covariance $(P(A \cap B) - P(A)P(B))/P(A \cap B)$ converged to $\frac{1}{3}$ as $n \to \infty$.

For the rest of this paper, we fix the notation $x := p/2$ and $y := 1 - x = 1 - p/2$. It will be important to understand that x is the probability that a certain edge exists and is directed a certain way, whereas y is the probability that the edge is not directed a certain way.

We first estimate the size of $P(A)$.

Lemma 2.1. *For fixed p and all sufficiently large n,*

$$2y^{n-1}\left(1-\frac{1}{2}\Big(1-\frac{p}{2}\Big)^{n-2}\right) \le P(A) \le 2y^{n-1}\left(1+\Big(1-\frac{p^2}{5}\Big)^{n-1}\right).$$

Proof. A necessary condition for A is that the edge between a and s, if it exists, is not directed from a to s. Let C, with $P(C) = y$, denote this event.

Let O_a and O_s denote the sets of points in $[n] \setminus \{a, s\}$ that can be reached from a and s respectively in one step. Similarly, let I_a and I_s denote the sets of points in $[n] \setminus \{a, s\}$ that can reach a and s respectively in one step.

For the lower bound, note that $C \cap (O_a = \emptyset) \Rightarrow A$ and $C \cap (I_s = \emptyset) \Rightarrow A$, so that

$$\begin{aligned} P(A) &\ge P((O_a = \emptyset) \cup (I_s = \emptyset)) \cdot y = \left(y^{n-2} + y^{n-2} - y^{2n-4}\right) \cdot y \\ &= 2y^{n-1}\left(1 - \frac{1}{2}y^{n-2}\right). \end{aligned}$$

For the upper bound, note that $A \Rightarrow C \cap (O_a \cap I_s = \emptyset)$, and that no existing edge may be directed from O_a to I_s. With $k = |O_a|$ and $m = |I_s|$, there are km such edges and we get

$$\begin{aligned} P(A) &\le y \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} x^k y^{n-2-k} \sum_{m=0}^{n-2-k} \binom{n-2-k}{m} x^m y^{n-2-m} \cdot y^{km} \\ &= y^{n-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \sum_{m=0}^{n-2-k} \binom{n-2-k}{m} x^{k+m} y^{n-2-k-m} y^{k\cdot m} \\ &= y^{n-1} \sum_{m=0}^{n-2} \binom{n-2}{m} x^m y^{n-2-m} + y^{n-1} \sum_{k=1}^{n-2} \binom{n-2}{k} x^k y^{n-2-k} \\ &\quad + y^{n-1} \sum_{k=1}^{n-3} \binom{n-2}{k} x^k \sum_{m=1}^{n-2-k} \binom{n-2-k}{m} x^m y^{n-2-k-m} y^{k\cdot m} \\ &= y^{n-1} \cdot 1 + y^{n-1} \cdot (1 - y^{n-2}) \\ &\quad + y^{n-1} \sum_{k=1}^{n-3} \binom{n-2}{k} x^k \left(\left(y + xy^k\right)^{n-2-k} - y^{n-2-k}\right) \\ &\le 2y^{n-1} + y^{n-1} \cdot h_p(n-2) \le y^{n-1}\left(2 + \Big(1 - \frac{p^2}{5}\Big)^{n-2}\right) \\ &\le 2y^{n-1}\left(1 + \Big(1 - \frac{p^2}{5}\Big)^{n-1}\right), \end{aligned}$$

where h_p is defined in Lemma 2.2 below. ∎

Lemma 2.2. *For fixed p and all sufficiently large n,*

$$h_p(n) := \sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k} \left(\left(1 + xy^{k-1}\right)^{n-k} - 1 \right) \leq \left(1 - \frac{p^2}{5}\right)^n.$$

Proof. Let $\delta > 0$ be arbitrary. Then

$$\begin{aligned} \left(1 + xy^{k-1}\right)^{n-k} - 1 &= \exp\left\{(n-k)\log\left(1 + xy^{k-1}\right)\right\} - 1 \\ &\leq \exp\left\{nxy^{k-1}\right\} - 1 \leq ny^k \cdot e, \end{aligned}$$

if $ny^k < 1$, i.e. $k > \alpha_y := -\log n / \log y$. Here we have used the inequalities: $x \leq y$, $\log(1+x) \leq x$ and $x < 1 \implies e^x - 1 < xe$. We now split the sum as follows.

$$\begin{aligned} \sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k} \left(\left(1 + xy^{k-1}\right)^{n-k} - 1 \right) &\leq \sum_{k=1}^{\alpha_y} \binom{n}{k} x^k y^{n-k} \left(\left(1 + xy^{k-1}\right)^{n-k} - 1 \right) \\ &\quad + \sum_{k=\alpha_y}^{n-1} \binom{n}{k} x^k y^{n-k} \left(\left(1 + xy^{k-1}\right)^{n-k} - 1 \right) \\ &\leq \alpha_y \cdot n^{\alpha_y} y^n (1+x)^n + n \cdot e \sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k} y^k \\ &\leq (\alpha_y \cdot n^{\alpha_y} + n \cdot e)\left(1 - \frac{p^2}{4}\right)^n \\ &\leq \left(1 - \frac{p^2}{4} + \delta\right)^n \leq \left(1 - \frac{p^2}{5}\right)^n, \end{aligned}$$

if n is sufficiently large. □

Theorem 2.3. *For fixed p*

$$\lim_{n\to\infty} \frac{P(A)}{2y^{n-1}} = 1.$$

Proof. Follows immediately from Lemma 2.1. □

The second step in evaluating the covariance is to understand the behavior of $P(A \cap B)$.

Lemma 2.4. *Assume that $0 \leq p \leq 1$ is fixed. For all $n \geq 3$,*

$$y^{2n-3}\left(3 - 2y^{n-3}\right) \leq P(A \cap B).$$

As $n \to \infty$,

$$P(A \cap B) \leq y^{2n-3}\left(3 + o(1)\right).$$

Proof. In this proof, let C be the event that there is not an edge directed from a to s, not one from a to b and not one from s to b. Thus $P(C) = y^3$.

Further, let O_a, O_s and O_b denote the sets of points in $[n] \setminus \{a, s, b\}$ that can be reached from a, s and b respectively in one step. Similarly, let I_a, I_s and I_b denote the sets of points in $[n] \setminus \{a, s, b\}$ that can reach a, s and b respectively in one step. Note that

$C \cap (O_a = O_s = \emptyset) \Rightarrow A \cap B$, $C \cap (O_a = I_b = \emptyset) \Rightarrow A \cap B$ and $C \cap (I_s = I_b = \emptyset) \Rightarrow A \cap B$, so that

$$\begin{aligned} P(A \cap B) &\geq y^3 \cdot P((O_a = O_s = \emptyset) \cup (O_a = I_b = \emptyset) \cup (I_s = I_b = \emptyset)) \\ &= y^3 \left(3y^{2n-6} - 2y^{3n-9} - y^{2n-6}(1-p)^{n-3} + y^{2n-6}(1-p)^{n-3}\right) \\ &= y^{2n-3}\left(3 - 2y^{n-3}\right), \end{aligned}$$

by inclusion-exclusion.

For the upper bound it is more complicated than in the proof of Theorem 2.3. To see this, observe for instance that if the edges $\{a,s\}, \{b,s\}$ are not present than we may have an edge directed from a to b. This forces us to distinguish between two cases.

Case 1: We have $O_a \cap I_b = \emptyset$ and there is no edge directed from $a \cup O_a$ to $b \cup I_b$. This event will be called D.

Case 2: At least one of these conditions is violated, this is called $\neg D$.

Let $C_1 := P(A \cap B \cap D)$ and $C_2 := P(A \cap B \cap \neg D)$ We want to estimate $P(A \cap B) = C_1 + C_2$ by studying the two cases. Let $|O_a| = k$, $|I_b| = m$, $|O_s \setminus O_a| = i$, $|I_s \setminus I_b| = j$ and $n' := n - 3$.

For Case 1, first note that $A \cap B \cap D \implies C$, which gives a factor y^3 in the formula below. We sum over all possible values of k and m. By definition, edges will be directed from a to O_a, but no other edges will be directed away from a. Similarly, edges will exist and be directed from I_b to b. Event D implies that no edge may be directed from O_a to I_b which gives y^{km}. We then sum over all possible values for i and j and consider the possible edges at s. These edges will be directed to $O_s \setminus O_a$, from $I_s \setminus I_b$, not from O_a, not to I_b and there will be no edges between s and $[n] \setminus (\{a,b,s\} \cup O_a \cup O_s \cup I_b \cup I_s)$. Event B also implies that there must not be any edge directed from $O_s \setminus O_a$ to I_b and event A implies that there is no edge directed from O_a to $I_s \setminus I_b$. We get the following:

$$\begin{aligned} C_1 \leq y^3 &\sum_{k=0}^{n'} \binom{n'}{k} x^k y^{n'-k} \sum_{m=0}^{n'-k} \binom{n'-k}{m} x^m y^{n'-m} y^{km} \sum_{i=0}^{n'-k-m} \binom{n'-k-m}{i} \\ &\cdot \sum_{j=0}^{n'-k-m-i} \binom{n'-k-m-i}{j} x^i x^j y^k y^m (1-2x)^{n'-k-m-i-j} y^{im} y^{jk} \\ &= y^{2n-3} \sum_{k=0}^{n'} \binom{n'}{k} x^k \sum_{m=0}^{n'-k} \binom{n'-k}{m} x^m y^{km} \\ &\cdot \sum_{i=0}^{n'-k-m} \binom{n'-k-m}{i} x^i y^{im} (1-2x+xy^k)^{n'-k-m-i} \\ &= y^{2n-3} \sum_{k=0}^{n'} \binom{n'}{k} x^k \sum_{m=0}^{n'-k} \binom{n'-k}{m} x^m y^{km} (1-2x+xy^k+xy^m)^{n'-k-m}. \end{aligned}$$

We now continue by splitting into four subcases: (1) $k = m = 0$, (2) $k = 0, m \geq 1$, (3) $k \geq 1, m = 0$ and (4) $k, m \geq 1$. Let $C_{11}, C_{12}, C_{13}, C_{14}$ be the values of the formula corresponding to these cases. It is easy to see that $C_{11} = y^{2n-3}$ and

$$\begin{aligned}C_{13} = C_{12} &= y^{2n-3} \sum_{m=1}^{n'} \binom{n'}{m} x^m (1 - x + xy^m)^{n'-m} \\ &= y^{2n-3} \left(\sum_{m=1}^{n'} \binom{n'}{m} x^m y^{n'-m} \Big((1 + xy^{m-1})^{n'-m} - 1 \Big) + 1 - y^{n'} \right) \\ &\le y^{2n-3} \left(1 + \Big(\frac{1-p^2}{5} \Big)^{n'} - y^{n'} \right),\end{aligned}$$

for sufficiently large n' by Lemma 2.2.

In subcase (4) we first consider the terms for which $y^m + y^k < 1$. We get

$$\begin{aligned}& y^{2n-3} \sum_{k,m\ge 1, k+m\le n', y^k+y^m<1} \binom{n'}{k} x^k \binom{n'-k}{m} x^m y^{km} (1 - 2x + xy^k + xy^m)^{n'-k-m} \\ &\le y^{2n-3} \sum_{k,m\ge 1, k+m\le n'} \binom{n'}{k} x^k \binom{n'-k}{m} (xy^k)^m y^{n'-k-m} \\ &\le y^{2n-3} \sum_{k=1}^{n'} \binom{n'}{k} x^k \Big((y + xy^k)^{n'-k} - y^{n'-k} \Big) \\ &= y^{2n-3} \sum_{k=1}^{n'} \binom{n'}{k} x^k y^{n'-k} \Big((1 + xy^{k-1})^{n'-k} - 1 \Big) \\ &\le y^{2n-3} \left(\frac{1-p^2}{5} \right)^{n'},\end{aligned}$$

by Lemma 2.2.

To deal with the terms where $y^k + y^m \ge 1$, let L be such that $y^k \le x$ when $k \ge L$. If both $k \ge L$ and $m \ge L$, then $y^k + y^m \le x + x = 2x < 1$, so we need only consider the cases when at least one of k and m are less than L. Further, by the symmetry between k and m, it is sufficient to study two cases, call the corresponding values C_{141}, C_{142}. Let $Q_r(n)$ denote a polynomial in n of degree r.

$$\begin{aligned}\frac{C_{141}}{y^{2n-3}} &= \sum_{k=1}^{L} \binom{n'}{k} x^k \sum_{m=1}^{L} \binom{n'-k}{m} x^m y^{km} (1 - 2x + xy^k + xy^m)^{n'-k-m} \\ &\le Q_{2L}(n')(1 - 2x + 2xy)^{n'-2L} = Q_{2L}(n')(1 - 2x^2)^{n'} = Q_{2L}(n') \left(1 - \frac{p^2}{2} \right)^{n'},\end{aligned}$$

$$\begin{aligned}
\frac{C_{142}}{y^{2n-3}} &= \sum_{k=1}^{L}\binom{n'}{k}x^k\sum_{m=L}^{n'-k}\binom{n'-k}{m}x^m y^{km}(1-2x+xy^k+xy^m)^{n'-k-m} \\
&\le \sum_{k=1}^{L}\binom{n'}{k}\sum_{m=0}^{n'-k}\binom{n'-k}{m}x^m y^m\left(1-2x+xy+x^2\right)^{n'-k-m} \\
&= \sum_{k=1}^{L}\binom{n'}{k}\left(xy+1-2x+xy+x^2\right)^{n'-k} = Q_L(n')\left(1-x^2\right)^{n'-L} \\
&= Q_L(n')\left(1-\frac{p^2}{4}\right)^{n'}.
\end{aligned}$$

Collecting the estimates, we get, for sufficiently large n,

$$C_1 \le y^{2n-3}\left(3+2\left(\frac{1-p^2}{5}\right)^{n'}+Q_{2L}(n')\left(1-\frac{p^2}{2}\right)^{n'}+Q_L(n')\left(1-\frac{p^2}{4}\right)^{n'}\right).$$

Thus $C_1 = y^{2n-3}(3+o(1))$ as $n\to\infty$.

For Case 2, let $|O_a\cup O_b| = k, |I_a\cup I_b| = m, |(O_a\cup O_b)\cap(I_a\cup I_b)| = i, |O_s\setminus(O_a\cup O_b)| = r, |I_s\setminus(I_a\cup I_b)| = t, |O_s\cap(O_a\cup O_b)| = u, |I_s\cap(I_a\cup I_b)| = v$. Let also $z := 1-p = 1-2x$, denote the probability that an edge is not present. Note that since we have assumed there to be a (short) path from a to b, then we must avoid not only paths from a to s and s to b, but also from s to a and from b to s. Thus we know that $O_s\cap(I_a\cup I_b)=\emptyset$ and $I_s\cap(O_a\cup O_b)=\emptyset$. For a vertex v to be in $(O_a\cup O_b)\setminus(I_a\cup I_b)$, it must either have both edges from a and b directed to v or one of them directed to v and the other edge not present, which gives probability $x^2+2xz = 2x-3x^2$. For v to be in $(O_a\cup O_b)\cap(I_a\cup I_b)$ both edges to a and b must be present and directed opposite ways (two possibilities), which gives probability $2x^2$. We get the following upper bound

$$\begin{aligned}
C_2 \le z^2 &\sum_{k=0}^{n'}\binom{n'}{k}\sum_{i=0}^{k}\binom{k}{i}\sum_{m=i}^{n'-k+i}\binom{n'-k}{m-i}(2x-3x^2)^{k-i}(2x-3x^2)^{m-i}(2x^2)^i z^{2(n'-k-m+i)} \\
&\cdot\sum_{u=0}^{k-i}\binom{k-i}{u}\sum_{r=0}^{n'-m-k+i}\binom{n'-m-k+i}{r}x^{r+u}\sum_{v=0}^{m-i}\binom{m-i}{v} \qquad (1)\\
&\cdot\sum_{t=0}^{n'-m-k+i-r}\binom{n'-m-k+i-r}{t}x^{t+v}z^{n'-r-u-t-v}y^{(r+u)m+(t+v)k-uv},
\end{aligned}$$

where the last factor comes from the fact that there must not be any edges directed from O_s to $I_a\cup I_b$ or from $O_a\cup O_b$ to I_s.

This septuple sum is somewhat laborious to estimate. We have collected the necessary calculations in Lemma 5.1 in the Appendix. From that lemma we conclude that $C_2 \le y^{2n-3}o(1)$ and the upper bound in the lemma follows. □

Theorem 2.5.

$$\lim_{n\to\infty}\frac{P(A\cap B)}{y^{2n-3}} = 3.$$

Proof. Follows immediately from Lemma 2.4. □

We may now deduce our main theorem.

Theorem 2.6. *For a fixed p we have the following limit of the relative covariance*

$$\lim_{n\to\infty} \frac{P(A\cap B) - P(A)P(B)}{P(A\cap B)} = \frac{2p-1}{3}.$$

Proof. Follows from Theorem 2.3 and Theorem 2.5 as $P(A) = P(B)$ and $y = 1 - \frac{p}{2}$. □

Remark 2.7. *Theorem 2.6 shows that the events $A = \{a \nrightarrow s\}$ and $B = \{s \nrightarrow b\}$ are negatively correlated when $p < \frac{1}{2}$ is fixed and n sufficiently large, but positively correlated when $p > \frac{1}{2}$ and n sufficiently large. From this follows the same statement for the events $\{a \to s\}$ and $\{s \to b\}$.*

Note that the functions $P(A)$ and $P(A \cap B)$ by Lemmas 3.1 and Lemma 3.3 below are polynomials in p and hence continuous.

3. Exact recursions

In this section we will give exact recursions to compute

$$f_p(n) := P_{G(n,p)}(s \nrightarrow b) \text{ and } g_p(n) := P_{G(n,p)}(a \nrightarrow s, s \nrightarrow b).$$

For a vertex $v \in V(G)$, let $\overrightarrow{C}_v \subset V(G)$ be the (random) set of all vertices u for which there is a directed path from v to u. We will call this the **out-cluster** from v. Let also the **in-cluster**, $\overleftarrow{C}_v \subset V(G)$ be the (random) set of all vertices u for which there is a directed path from u to v. Note that we will use the convention that $v \in \overleftarrow{C}_v \cap \overrightarrow{C}_v$. As before we let $x := p/2$ denote the probability that an edge exists and has a certain direction. Also $y := 1 - x$ is the probability that an edge does not exist with a certain direction, and $q := 1 - p$ is the probability that there is no edge at all.

For $n \geq 2$, $s \in X \subset [n]$ and $|X| = k$ define:

$$d_p(n,k) := P_{G(n,p)}(\overrightarrow{C}_s = X),$$

where in particular $d_p(1,1) = 1$.

Lemma 3.1. *We have the following recursions*

1. $d_p(n,k) = d_p(k,k)y^{k(n-k)}$, *for* $n > k \geq 1$,
2. $d_p(k,k) = 1 - \sum_{i=1}^{k-1} \binom{k-1}{i-1} d_p(i,i) y^{i(k-i)}$, *and*
3. $f_p(n) = \sum_{k=1}^{n-1} \binom{n-2}{k-1} d_p(k,k) y^{k(n-k)}$.

Proof. If $n > k$ there is a vertex $w \notin X$. The only restriction on w is that it must not have any edge directed from X so $P_{G(n,p)}(\overrightarrow{C}_s = X) = P_{G(n-1,p)}(\overrightarrow{C}_s = X) \cdot y^k$ and (1) follows by induction. Clearly $\sum_{X:s\in X\subseteq[n]} P_{G(n,p)}(\overrightarrow{C}_s = X) = 1$, which gives formula (2) after using equation (1). To get the third equation we sum over all sets X that contain s but not b, i.e. $f_p(n) = \sum_X P_{G(n,p)}(\overrightarrow{C}_s = X) = \sum_{k=1}^{n-1} \binom{n-2}{k-1} d_p(n,k)$, which using (1) gives (3). □

Note that, by symmetry, also $P_{G(n,p)}(\overleftarrow{C}_s = X) = d_p(n,k)$.

Remark 3.2. *Note that by Lemma 2.1 in* [7]*, which is a special case of a theorem by McDiarmid,* [8]*, our recursion for $d_p(k,n)$ also gives a formula for the probability that a certain set of vertices X, with $|X| = k$ is the connected component (or open cluster) containing s in $G(n,p/2)$.*

We now want to do something similar for the more complicated case of $g_p(n)$. For $n \geq 2$, $s \in X \subset [n]$, $s \in Y \subset [n]$ with $|X| = k, |Y| = m$ and $|[n] \setminus (Y \cup X)| = r$ define:

$$M_p(n,k,m,r) := P_{G(n,p)}(\overrightarrow{C}_s = X, \overleftarrow{C}_s = Y),$$

where in particular $M_p(1,1,1,0) = 1$.

Lemma 3.3. *We have the following recursions for M_p, where $k + m > n - r \geq k, m$ and $k, m \geq 1$*

(1) $M_p(n,k,m,r) = M_p(n-r,k,m,0)q^{r(r+k+m-n)}y^{r(2n-2r-k-m)}$, *for* $r > 0$,

(2) $M_p(n,k,m,0) = \sum_{j=1}^{n-k} \binom{n-k-1}{j-1} M_p(n-j,k,m-j,0)d_p(j,j)y^{j(n-m)} \cdot \big(y^{m+k-n} - y^{n-k-j}q^{m+k-n}\big)q^{(j-1)(m+k-n)}y^{(j-1)(n-k-j)}$, *for* $n > k, n \geq m$,

(3) $M_p(n,k,m,r) = M_p(n,m,k,r)$,

(4) $M_p(n,n,n,0) = 1 - \sum_{r=1}^{n-1} \binom{n-1}{r-1} \sum_{k=r}^{n} \binom{n-r}{k-r} \sum_{m=r}^{n-k+r} \binom{n-k}{m-r} M(n,k,m,n-m-k+r)$ *and*

(5) $g_p(n) = \sum_{r=1}^{n-2} \binom{n-3}{r-1} \cdot \Big(\sum_{k=r}^{n-2} \binom{n-2-r}{k-r} \sum_{m=r}^{n-k+r-1} \binom{n-k-1}{m-r} M_p(n,k,m,n-m-k+r) + \sum_{k=r+1}^{n-1} \binom{n-2-r}{k-r-1} \sum_{m=r}^{n-k+r} \binom{n-k}{m-r} M_p(n,k,m,n-m-k+r)\Big)$.

Proof. Assume, as given for the first equation, that $r > 0$. All vertices in $[n] \setminus (Y \cup X)$ must not have any edge directed to Y or from X. This means that there must be no edge at all to $Y \cap X$, which gives probability $q^{|[n]\setminus Y\cup X|\cdot|Y\cap X|} = q^{r(r+k+m-n)}$. There must not be any edge directed to $(Y \setminus X)$ and there must not be any edge directed from $(X \setminus Y)$. This gives a factor of $y^{|[n]\setminus Y\cup X|\cdot|(Y\setminus X)\cup(X\setminus Y)|} = y^{r(2n-2r-k-m)}$.

For equation (2), note that $n > k$ implies that there exist a vertex $w \in Y \setminus X$. Let G be any directed graph on n vertices with $\overrightarrow{C}_s = X$ and $\overleftarrow{C}_s = Y$. If we remove vertex w and all its edges from G the resulting graph will still have $\overrightarrow{C}_s = X$ since $w \notin X$, whereas $\overleftarrow{C}_s = Y'$, for some Y' such that $Y \cap X \subseteq Y' \subseteq Y \setminus \{w\}$. Let $j = |Y \setminus Y'|$ and sum over all possible Y'. The probability is $M_p(n-j,k,m-j,0)$ that the subgraph on $[n] \setminus (Y \setminus Y')$ is as needed. The subgraph on $Y \setminus Y'$ must have $\overleftarrow{C}_w = Y \setminus Y'$ which gives probability $d_p(j,j)$. There must not be any edge directed from $X \setminus Y$ to $Y \setminus Y'$, since the vertices of the latter do not belong to X. The other direction is legal and this gives the factor $y^{(n-m)j}$. There must by the definition of Y' not be any edge at all between $X \cap Y$ and $Y \setminus (Y' \cup \{w\})$, which gives the factor $q^{(k+m-n)(j-1)}$. There must also not be any edge directed from $Y \setminus (Y' \cup \{w\})$ to $Y' \setminus (Y \cap X)$, which gives the factor $y^{(n-k-j)(j-1)}$. The only possible edges left to consider have one endpoint in w. The edges between $Y' \setminus (Y \cap X)$ and w could have any direction and there must not be any edge directed from $Y \cap X$ to w (since $w \notin X$), but there must be at least one edge directed from w to Y' (since $w \in Y$). This gives probability $(y^{m+k-n} \cdot 1^{n-k-j} - q^{m+k-n} \cdot y^{n-k-j})$. Putting all this together gives formula (2).

The third equation is obtained from the symmetry of reversing all directions.

The fourth equation follows from the fact that

$$\sum_{X,Y:s\in Y,X\subseteq[n]} P_{G(n,p)}(\overrightarrow{C}_s = X, \overleftarrow{C}_s = Y) = 1.$$

Here $r = |Y \cap X|$ and recall that $s \in Y \cap X$ is a necessary condition.

The last equation is obtained by summing over all possible pairs Y, X such that $a \notin Y, b \notin X$. Again $r = |Y \cap X|$ and the formula is split into the cases when $a \notin X$ and $a \in X$, respectively. □

4. Computations and Conjectures

We have used MAPLE to compute the functions $f_p(n)$ and $g_p(n)$ for $n \leq 30$. Figure 1 displays the relative covariance $(g_p(n) - f_p(n)^2)/g_p(n)$. All curves start with being mildly negative. They then turn positive and for $n < 27$ they stay positive. For $n \geq 27$ however, they go below the p-axis again for some time.

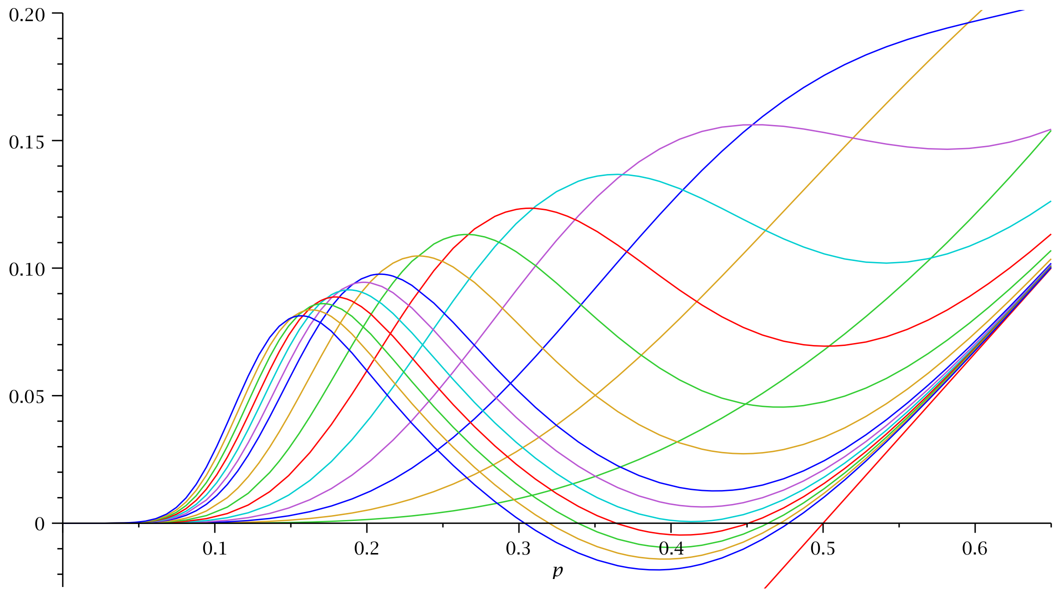

Figure 1. The relative correlation $(P(A \cap B) - P(A)P(B))/P(AB)$ for $n = 8, 10, 12, 14, 16, 18, 20, 22, 24, 25, 26, 27, 28, 29, 30,$ and the asymptote $(2p-1)/3$. All curves are negative for very small p. For $n \geq 27$ we get three zeros.

For larger values of n it becomes infeasible for our computers to obtain the exact functions. We have instead for various fixed values of p used the recursions to obtain the value of the quotient $(g_p(n) - f_p(n)^2)/g_p(n)$ for $n \leq 300$. Based on these calculations we conjecture the following.

Conjecture 4.1. *For $n \geq 27$, the relative covariance changes sign at three critical probabilities $p_1(n) < p_2(n) < p_3(n)$.*

Conjecture 4.2. *Asymptotically, $p_1(n) = c/n$ and $p_2(n) = C/n$ for some constants c and C.*

The computations indicate that very rough estimates of c and C are 0.36 and 7.5 respectively.

Conjecture 4.3. *For all $n \geq 27$, $p_3(n) < 1/2$ and $p_3(n)$ converges to $1/2$ exponentially fast.*

Conjecture 4.4. *For $n \geq 8$, the relative covariance $\frac{P(a \not\to s, s \not\to b) - P(a \not\to s)P(s \not\to b)}{P(a \not\to s, s \not\to b)} > \frac{2p-1}{3}$.*

Note that Conjecture 4.4 implies the first half ($p_3 < \frac{1}{2}$) of Conjecture 4.3 and the following conjecture.

Conjecture 4.5. *For $p = \frac{1}{2}$, and $n \geq 6$ the relative covariance $\frac{P(a \not\to s, s \not\to b) - P(a \not\to s)P(s \not\to b)}{P(a \not\to s, s \not\to b)}$ is positive.*

There is strong numerical support for the following conjecture.

Conjecture 4.6. *For $p > \frac{1}{2}$, the relative covariance approaches the asymptote $\frac{2p-1}{3}$ exponentially fast.*

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5. Appendix

In this appendix we prove the upper bounds needed for Lemma 2.4

Lemma 5.1. *For fixed p, as $n \to \infty$ we have*

$$C_2 := P(A \cap B \cap \neg D) \leq y^{2n-3} \cdot o(1).$$

Proof. The last sum of inequality (1) equals

$$x^v \cdot z^{m+k-i-u-v} \cdot y^{(s+u)m+vk-uv} \sum_{t=0}^{n'-m-k+i-s} \binom{n'-m-k+i-s}{t} x^t y^{kt} z^{n'-m-k+i-s-t}$$
$$= x^v \cdot z^{m+k-i-u-v} \cdot y^{(s+u)m+vk-uv} \cdot (z+xy^k)^{n'-m-k+i-s},$$

which inserted into the bound (1) for C_2 gives

$$C_2 \leq z^2 \sum_{k=0}^{n'} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2x^2)^i \sum_{m=i}^{n'-k+i} \binom{n'-k}{m-i} (2x-3x^2)^{m-i} z^{2(n'-k-(m-i))}$$
$$\cdot \sum_{u=0}^{k-i} \binom{k-i}{u} x^u y^{mu} z^{k-u} \sum_{s=0}^{n'-k-(m-i)} \binom{n'-k-(m-i)}{s} x^s y^{ms} (z+xy^k)^{n'-k-(m-i)-s}$$
$$\cdot \sum_{v=0}^{m-i} \binom{m-i}{v} (xy^{k-u})^v z^{m-i-v} \qquad (\text{Let } r = m-i.)$$
$$= z^2 \sum_{k=0}^{n'} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2x^2)^i z^i \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x(y^k+y^{i+r}))^{n'-k-r} \cdot \sum_{u=0}^{k-i} \binom{k-i}{u} (xy^{i+r})^u z^{k-i-u} \cdot (z+xy^{k-u})^r$$
$$\leq z^2 \sum_{k=0}^{n'} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i},$$

as $y < 1$ and $y^{k-u} \leq y^i$ when $u \leq k-i$.

We want to show that $C_2 \leq y^{2n-3} \cdot o(1)$, as $n \to \infty$. Let, for fixed $p > 0$ (and thus fixed $x = p/2 > 0$ and $y = 1-x < 1$) L be such that $y^k \leq x$, when $k \geq L$. We will split the triple sum in the following way

$$\sum_k \sum_i \sum_r = \left(\sum_{k<L} + \sum_{k=L}^{n'-L} + \sum_{k>n'-L} \right) \left(\sum_{i<L} + \sum_{i \geq L} \right) \left(\sum_{r<L} + \sum_{r \geq L} \right).$$

Note that not all twelve combinations are possible, e.g. $i \geq L$ can not occur when $k < L$ as $i \leq k$. Numbering the subcases tuv according to k case ($t = 1,2,3$), i case ($u = 1,2$) and r case ($v = 1,2$), we note that the combinations $t = 1, u = 2$ and $t = 3, v = 2$ are empty, so we are left with the following eight subcases

$$C_2 \leq C_{2111} + C_{2112} + C_{2211} + C_{2212} + C_{2221} + C_{2222} + C_{2311} + C_{2321},$$

which we will now estimate. Note that $z+x=y$ and that $z+2x=1$. Let $Q_m(n)$ denote a polynomial in n of degree m and K denote a constant.

$$\begin{aligned}
C_{2111} &\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=0}^{L} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} \\
&\quad \cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i} \\
&\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=0}^{L} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} \\
&\quad \cdot 1 \cdot y^r \cdot y^{k-i} \\
&\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i} y^{k-i} (2zx^2)^i \sum_{r=0}^{L} \binom{n'-k}{r} z^{2(n'-2L)} \\
&\le Q_L(n') \cdot z^{2n'} \sum_{k=0}^{L} \binom{n'}{k} (y(2x-3x^2)+2zx^2)^k \\
&\le Q_L(n') \cdot z^{2n'} \sum_{k=0}^{L} \binom{n'}{k} (2x-3x^2-x^3)^k \le Q_{2L}(n') \cdot (z^2)^{n'} \\
&= y^{2n-3} \cdot Q_{2L}(n') \cdot \left(\frac{z^2}{y^2}\right)^{n'} = o(y^{2n-3}),
\end{aligned}$$

as $2x-3x^2-x^3<1$ and $z/y = 1-p/(2-p) < 1$.

$$\begin{aligned}
C_{2112} &\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=L}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} \\
&\quad \cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i} \\
&\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \\
&\quad \cdot \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} (z+x(y^L+1))^{n'-k-r} \cdot y^r \cdot (z+xy^L)^{k-i} \\
&\le z^2 \sum_{k=0}^{L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(z+x^2)^{k-i}(2zx^2)^i \\
&\quad \cdot (y(2x-3x^2)+z^2(y+x^2))^{n'-k} \\
&\le z^2 \sum_{k=0}^{L} \binom{n'}{k} (y(2x-3x^2)+z^2(y+x^2))^{n'-k}((2x-3x^2)(z+x^2)+2zx^2)^k
\end{aligned}$$

$$= z^2 \sum_{k=0}^{L} \binom{n'}{k} (1-3x+4x^2-5x^3+4x^4)^{n'-k}(2x-5x^2+4x^3-3x^4)^k$$
$$\leq Q_L(n')(1-3x+4x^2-5x^3+4x^4)^{n'-L} = y^{2n-3}Q_L(n')\left(1-\frac{x-3x^2+5x^3-4x^4}{y^2}\right)^{n'}$$
$$= o(y^{2n-3}),$$

as $x-3x^2+5x^3-4x^4>0$ and $2x-5x^2+4x^3-3x^4<1$.

$$C_{2211} \leq z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=0}^{L} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i}$$
$$\leq z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{k-L} \sum_{r=0}^{L} \binom{n'-k}{r} z^{2(n'-k-L)}(z+x(1+x))^{n'-k-L}$$
$$\leq Q_{2L}(n') \sum_{k=0}^{n'} \binom{n'}{k} (2x-3x^2)^k (z^2(y+x^2))^{n'-k}$$
$$= Q_{2L}(n')(2x-3x^2+z^2(y+x^2))^{n'} = Q_{2L}(n')(1-3x+6x^2-8x^3+4x^4)^{n'}$$
$$= y^{2n-3}Q_{2L}(n')\left(1-\frac{x-5x^2+8x^3}{y^2}\right) = o(y^{2n-3}),$$

as $x-5x^2+8x^3>0$.

To estimate C_{2212}, we need to partition the r sum into two parts and use that $y^{2L} \leq x^2$.

$$C_{2212} \leq z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=L}^{2L} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i}$$
$$+ z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{k-i}(2zx^2)^i \sum_{r=2L}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i}$$
$$\leq Q_L(n')Q_{2L}(n') \sum_{k=0}^{n'} \binom{n'}{k} (2x-3x^2)^{k-L} z^{2(n'-k-2L)}(z+2x^2)^{n'-k-2L}(z+x^2)^{k-L}$$
$$+ Q_L(n') \sum_{k=0}^{n'} \binom{n'}{k} (2x-3x^2)^{k-L} \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)}$$
$$\cdot (z+x^3+x^2)^{n'-k-r} \cdot y^r \cdot (z+x^3)^{k-L}$$

$$
\begin{aligned}
&\le Q_{3L}(n')((2x-3x^2)(z+x^2)+z^2(z+2x^2))^{n'}\\
&\quad + Q_L(n')\sum_{k=0}^{n'}\binom{n'}{k}(2x-3x^2)^k(z+x^3)^k\big(y(2x-3x^2)+z^2(z+x^2+x^3)\big)^{n'-k}\\
&= Q_{3L}(n')(1-4x+7x^2-8x^3+5x^4)^{n'}\\
&\quad + Q_L(n')\big((2x-3x^2)(z+x^3)+y(2x-3x^2)+z^2(z+x^2+x^3)\big)^{n'}\\
&= Q_{3L}(n')(1-4x+7x^2-8x^3+5x^4)^{n'}+Q_L(n')(1-2x+x^2-2x^3+2x^4+x^5)^{n'}\\
&= y^{2n-3}\left(Q_{3L}(n')\Big(1-\frac{2x-6x^2+8x^3-5x^4}{y^2}\Big)^{n'}+Q_L(n')\Big(1-\frac{2x^3-2x^4-x^5}{y^2}\Big)^{n'}\right)\\
&= o(y^{2n-3}),
\end{aligned}
$$

as $2x-6x^2+8x^3-5x^4>0$ and $2x^3-2x^4-x^5>0$.

$$
\begin{aligned}
C_{2221} &\le z^2\sum_{k=L}^{n'-L}\binom{n'}{k}\sum_{i=L}^{k}\binom{k}{i}(2x-3x^2)^{k-i}(2zx^2)^i\sum_{r=0}^{L}\binom{n'-k}{r}(2x-3x^2)^r z^{2(n'-k-r)}\\
&\quad\cdot(z+x(y^{i+r}+y^k))^{n'-k-r}\cdot(z+xy^i)^r\cdot(z+xy^{i+r})^{k-i}\\
&\le z^2\sum_{k=L}^{n'-L}\binom{n'}{k}\sum_{i=L}^{k}\binom{k}{i}(2x-3x^2)^{k-i}(2zx^2)^i\sum_{r=0}^{L}\binom{n'-k}{r}z^{2(n'-k-L)}\\
&\quad\cdot(z+2x^2))^{n'-k-L}\cdot(z+x^2)^{k-i}\\
&\le Q_L(n')\sum_{k=0}^{n'}\binom{n'}{k}z^{2(n'-k)}(z+2x^2))^{n'-k}\sum_{i=0}^{k}\binom{k}{i}(2x-3x^2)^{k-i}(2zx^2)^i(z+x^2)^{k-i}\\
&= Q_L(n')\sum_{k=0}^{n'}\binom{n'}{k}z^{2(n'-k)}(z+2x^2))^{n'-k}\big((2x-3x^2)(z+x^2)+2zx^2\big)^k\\
&= Q_L(n')\big(z^2(z+2x^2)+(2x-3x^2)(z+x^2)+2zx^2\big)^{n'}\\
&= Q_L(n')(1-4x+9x^2-12x^3+5x^4)^{n'}\\
&= y^{2n-3}Q_L(n')\left(1-\frac{2x-8x^2+12x^3-5x^4}{y^2}\right)^{n'}=o(y^{2n-3}),
\end{aligned}
$$

as $2x-8x^2+12x^3-5x^4>0$.

$$
\begin{aligned}
C_{2222} &\le z^2\sum_{k=L}^{n'-L}\binom{n'}{k}\sum_{i=L}^{k}\binom{k}{i}(2x-3x^2)^{k-i}(2zx^2)^i\sum_{r=L}^{n'-k}\binom{n'-k}{r}(2x-3x^2)^r z^{2(n'-k-r)}\\
&\quad\cdot(z+x(y^{i+r}+y^k))^{n'-k-r}\cdot(z+xy^i)^r\cdot(z+xy^{i+r})^{k-i}
\end{aligned}
$$

$$\begin{aligned}
&\leq z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=L}^{k} \binom{k}{i} (2x-3x^2)^{k-i} (2zx^2)^i (z+x^3)^{k-i} \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r \\
&\quad \cdot z^{2(n'-k-r)} (z+x(x^2+x))^{n'-k-r} (z+x^2)^r \\
&\leq z^2 \sum_{k=L}^{n'-L} \binom{n'}{k} \sum_{i=0}^{k} \binom{k}{i} (2x-3x^2)^{k-i} (2zx^2)^i (z+x^3)^{k-i} \\
&\quad \cdot \big((z+x^2)(2x-3x^2) + z^2(z+x^2+x^3)\big)^{n'-k} \\
&\leq z^2 \sum_{k=0}^{n'} \binom{n'}{k} \big((z+x^2)(2x-3x^2) + z^2(z+x^2+x^3)\big)^{n'-k} \\
&\quad \cdot \big((2x-3x^2)(z+x^3) + 2zx^2\big)^k \\
&= K \cdot \big((z+x^2)(2x-3x^2) + z^2(z+x^2+x^3) + (2x-3x^2)(z+x^3) + 2zx^2\big)^{n'} \\
&= K \cdot (1-2x+x^2-x^3-x^4+x^5)^{n'} = K \cdot y^{2n-3} \left(1 - \frac{x^3+x^4-x^5}{y^2}\right)^{n'} \\
&= o(y^{2n-3}),
\end{aligned}$$

as $x^3+x^4-x^5>0$. (The constant $K = z^2/y^3$.)

$$\begin{aligned}
C_{2311} &\leq z^2 \sum_{k=n'-L}^{n'} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{k-i} (2zx^2)^i \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} \\
&\quad \cdot (z + x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i} \\
&\leq z^2 \sum_{k=n'-L}^{n'} \binom{n'}{k} \sum_{i=0}^{L} \binom{k}{i} (2x-3x^2)^{n'-2L} \sum_{r=0}^{n'-k} \binom{n'-k}{r} y^{n'-2L} \\
&= Q_{3L}(n')(y(2x-3x^2))^{n'} = y^{2n-3} Q_{3L}(n') \left(\frac{2x-3x^2}{y}\right)^{n'} \\
&= y^{2n-3} Q_{3L}(n') \left(1 - \frac{1-3x+3x^2}{y}\right)^{n'} = o(y^{2n-3}),
\end{aligned}$$

as $1-3x+3x^2>0$.

$$\begin{aligned}
C_{2321} &\leq z^2 \sum_{k=n'-L}^{n'} \binom{n'}{k} \sum_{i=L}^{k} \binom{k}{i} (2x-3x^2)^{k-i} (2zx^2)^i \sum_{r=0}^{n'-k} \binom{n'-k}{r} (2x-3x^2)^r z^{2(n'-k-r)} \\
&\quad \cdot (z + x(y^{i+r}+y^k))^{n'-k-r} \cdot (z+xy^i)^r \cdot (z+xy^{i+r})^{k-i}
\end{aligned}$$

$$
\begin{aligned}
&\leq z^2 \sum_{k=n'-L}^{n'} \binom{n'}{k} \sum_{i=L}^{k} \binom{k}{i} (2x-3x^2)^{k-i}(z+x^2)^{k-i}(2zx^2)^i \sum_{r=0}^{n'-k} \binom{n'-k}{r} \\
&\leq Q_L(n') \sum_{k=n'-L}^{n'} \binom{n'}{k} \big((2x-3x^2)(z+x^2)+2zx^2\big)^k \\
&\leq Q_{2L}(n')\big((2x-3x^2)(z+x^2)+2zx^2\big)^{n'} = Q_{2L}(n')(2x-5x^2+4x^3-3x^4)^{n'} \\
&= y^{2n-3} Q_{2L}(n') \left(1 - \frac{1-4x+6x^2-4x^3+3x^4}{y^2}\right)^{n'} = o(y^{2n-3}),
\end{aligned}
$$

as $1-4x+6x^2-4x^3+3x^4 > 0$. □

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