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# Diffusion approximation for the components in critical inhomogeneous random graphs of rank 1.

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## Abstract

Consider the random graph on  $n$  vertices  $1, \dots, n$ . Each vertex  $i$  is assigned a type  $X_i$  with  $X_1, \dots, X_n$  being independent identically distributed as a nonnegative discrete random variable  $X$ . We assume that  $\mathbf{E}X^3 < \infty$ . Given types of all vertices, an edge exists between vertices  $i$  and  $j$  independent of anything else and with probability  $\min\{1, \frac{X_i X_j}{n} (1 + \frac{a}{n^{1/3}})\}$ . We study the critical phase, which is known to take place when  $\mathbf{E}X^2 = 1$ . We prove that normalized by  $n^{-2/3}$  the asymptotic joint distributions of component sizes of the graph equals the joint distribution of the excursions of a reflecting Brownian motion  $B^a(s)$  with diffusion coefficient  $\sqrt{\mathbf{E}X\mathbf{E}X^3}$  and drift  $a - \frac{\mathbf{E}X^3}{\mathbf{E}X}s$ . This shows that finiteness of  $\mathbf{E}X^3$  is the necessary condition for the diffusion limit. In particular, we conclude that the size of the largest connected component is of order  $n^{2/3}$ .

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## 1 Introduction.

### 1.1 The Model.

We study here a rank 1 case of a general inhomogeneous random graph model introduced by Bollobás, Janson and Riordan [2]. We shall define a random graph  $G^\mathcal{V}(n)$  with a vertex space

$$\mathcal{V} = (S, \mu, (x_1^{(n)}, \dots, x_n^{(n)})_{n \geq 1}),$$

where  $S = \{0, 1, \dots\}$  and  $\mu$  is a probability on  $S$ . No relationship is assumed between  $x_i^{(n)}$  and  $x_i^{(n')}$ , but to simplify notations we shall write further  $(x_1, \dots, x_n) = (x_1^{(n)}, \dots, x_n^{(n)})$ . For each  $n$  let  $(x_1, \dots, x_n)$  be a deterministic or random sequence of points in  $S$ , such that for any  $A \subseteq S$

$$\frac{\#\{i : x_i \in A\}}{n} \xrightarrow{P} \mu(A). \tag{1.1}$$

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Given the sequence  $x_1, \dots, x_n$ , we let  $G^\mathcal{V}(n)$  be the random graph on  $\{1, \dots, n\}$ , such that any two vertices  $i$  and  $j$  are connected by an edge independently of the others and with a probability

$$p_n(x_i, x_j) = \min \left\{ \frac{x_i x_j}{n} \left( 1 + \frac{a}{n^{1/3}} \right), 1 \right\}, \quad (1.2)$$

where  $a$  is any fixed real constant.

Let  $X$  denote a random variable with values in  $S$  and probability function  $\mu$ . It is proved in [2] that the phase transition occurs when  $\mathbf{E}X^2 = 1$ . Our aim here is to derive the asymptotic behaviour of the sizes of the connected components.

We shall assume that

$$\mathbf{E}X^2 = 1, \quad (1.3)$$

and

$$\mathbf{E}X^3 < \infty. \quad (1.4)$$

Let  $x_1, \dots, x_n$  be *i.i.d.* as random variable  $X$ . Then by assumption (1.4) there is an increasing unbounded function  $\omega(x)$  such that bound

$$\max_{1 \leq i \leq n} x_i \leq \frac{n^{1/3}}{\omega(n)}, \quad (1.5)$$

holds with probability tending to one as  $n \rightarrow \infty$ .

**Remark 1.1.** *It will be clear that one can extend all the results for the case of non - i.i.d. random variables, assuming, however, (1.5) and some uniformity in the convergence (1.1).*

Let  $C_1(n), C_2(n), \dots$  denote the ordered sizes of the connected components in  $G^\mathcal{V}(n)$  with  $C_1(n)$  being the largest one. We shall find the weak limit of

$$n^{-2/3} (C_1(n), C_2(n), \dots).$$

Let  $(W(s), s \geq 0)$  be the standard Brownian motion. Define

$$W^a(s) = \sqrt{\mathbf{E}X\mathbf{E}X^3} W(s) + as - \frac{\mathbf{E}X^3}{2\mathbf{E}X} s^2. \quad (1.6)$$

Let  $\gamma_1, \gamma_2, \dots$  denote the ordered lengths of the excursions of the process

$$B(s) = W^a(s) - \min_{0 \leq s' \leq s} W^a(s'), \quad s \geq 0. \quad (1.7)$$

To formulate the convergence result define  $l^2$  to be the set of infinite sequences  $\mathbf{x} = (x_1, x_2, \dots)$  with  $x_1 \geq x_2 \geq \dots \geq 0$  and  $\sum_i x_i^2 < \infty$ , and give  $l^2$  metric  $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_i - y_i)^2}$ .

**Theorem 1.1.** *The convergence in distribution*

$$n^{-2/3} (C_1(n), C_2(n), \dots) \xrightarrow{d} (\gamma_1, \gamma_2, \dots) \quad (1.8)$$

*holds with respect to the  $l^2$  topology.*

The statement of this result is very much inspired by the earlier work of Aldous [1], who was the first to prove rigorously this theorem not only for the case of homogeneous random graph  $G_{n,p}$  (in our setting this corresponds the case when  $x_i \equiv 1$ ), but for a particular nonuniform case as well. Notice here, that Theorem 1.1 and the result of Proposition 4 in [1] for a nonuniform random graph only partially overlap: there are cases which are covered by one theorem, but not by another. It should be also mentioned that in the nonuniform case Aldous derives in [1] asymptotic for the *sums of types of vertices* in the components, while we state the result directly for the components sizes. However, this is a minor difference, since most likely both objects behave in a similar fashion: the critical values for the phase transition coincide, and the phase transitions are qualitatively very similar (at least for the case of  $X$  with exponential tail it is explicitly derived in [7]).

The main difference between Theorem 1.1 and result of Aldous for a nonuniform random graph (Proposition 4 in [1]) is in the relations between the assumptions on the graph model and the coefficients of the process  $W^a$ . In the proof of Theorem 1.1 we combine approach of Aldous in [1] with idea of Martin-Löf in [6] on a construction of diffusion approximation for a critical epidemics. This allows us to derive in a straightforward way the coefficients of the diffusion process  $W^a$ , which gives a number of advantages. In particular, it is transparent in our proof (see Remark 2.1) that the phase transition occurs indeed when  $\mathbf{E}X^2 = 1$ , and that the famous scaling  $n^{2/3}$  is the proper one here.

Theorem 1.1 places all the rank 1 graphs with a finite third moment into the same universality class as a homogeneous  $G_{n,p}$  model, as long as the scaling  $n^{2/3}$  concerns. This fact was also observed by van der Hofstad [5] who (under some additional assumptions about the distribution of  $X$ ) obtained good bounds for the probability of order (in  $n$ ) of the largest component, and classified possible critical scalings when only  $\mathbf{E}X^2 < \infty$ .

Our proof also shows that finiteness of  $\mathbf{E}X^3$  is the necessary condition for the diffusion limit, and it indicates the possibility to find another scaling and a corresponding weak limit when  $\mathbf{E}X^3 < \infty$  is not the case.

## 1.2 The breadth-first walk for inhomogeneous random graph.

Following ideas from [1] we construct here a process associated with revealing connected components in inhomogeneous random graph. The basic procedure is the same as in homogeneous random graphs:

Given a graph  $G^\nu(n)$  and a set

$$V_n = \{x_1, \dots, x_n\}$$

choose a random (size-biased, as we explain later) vertex in  $\{1 \dots, n\}$ , and mark it by  $v_1$ . Reveal all the vertices connected to the marked vertex  $v_1$  in the graph  $G^\nu(n)$ . If the set of non-marked revealed vertices is not empty pick a random uniform vertex among this set, mark it  $v_2$ , and find all the vertices connected to it but which have not been used previously. We continue this process until we end up with a tree of marked vertices. Then choose again randomly size-biased vertex among the unrevealed ones and start the same process again until we use all the vertices of the graph.

We shall now introduce a Markov chain

$$(\bar{U}(i), \bar{I}(i), x(i)), \quad 1 \leq i \leq n, \quad (1.9)$$

associated with this algorithm, where at any step  $i \geq 1$

$\bar{U}(i) = (U^x(i), x \in S)$  and  $U^x(i)$  denotes the number of unrevealed vertices of type  $x$ ;

$\bar{I}(i) = (I^x(i), x \in S)$  and  $I^x(i)$  denotes the number of revealed but non-marked vertices of type  $x$ ;

$x(i) = x_{v_i}$  is the type of the vertex marked at step  $i$ .

Notice that all these variables depend on  $n$ . Revealed but non-marked vertices at step  $i$  we shall simply call "active vertices at step  $i$ ".

For the initial state we set

$$\begin{aligned} I^x(1) &= 0, \\ U^x(1) &= \#\{1 \leq i \leq n : x_i = x\}, \quad x \in S, \end{aligned} \quad (1.10)$$

and let  $x(1)$  be a random variable with *size-biased* distribution

$$\mathbf{P}\{x(1) = x \mid \bar{U}(1)\} = \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)}. \quad (1.11)$$

Denote further for any  $i \geq 1$

$$\begin{aligned} I(i) &= \sum_{x \in S} I^x(i), \\ U(i) &= \sum_{x \in S} U^x(i). \end{aligned}$$

Then for any  $i \geq 1$  conditionally on  $(x(i), \bar{I}(i), \bar{U}(i))$  the number of new " $x$  - neighbours" of  $v_i$  in the graph  $G^\nu(n)$  is distributed as

$$\mathcal{N}_n^x(i) \in \text{Bin}\left(U^x(i) - \mathbf{1}_{I(i)=0}\mathbf{1}_{x(i)=x}, p_n(x(i), x)\right). \quad (1.12)$$

Therefore conditionally on  $(x(i), \bar{I}(i), \bar{U}(i))$  we set

$$I^x(i+1) = I^x(i) + \mathcal{N}_n^x(i) - \mathbf{1}_{I(i)>0}\mathbf{1}_{x(i)=x}, \quad (1.13)$$

$$U^x(i+1) = U^x(i) - \mathcal{N}_n^x(i) - \mathbf{1}_{I(i)=0}\mathbf{1}_{x(i)=x}$$

for all  $x \in S$ . Then given  $(\bar{I}(i+1), \bar{U}(i+1))$ , choose  $v_{i+1}$  *uniformly* among the revealed unmarked vertices, unless this set is empty; in the latter case choose  $v_{i+1}$  *size-biased* among the unrevealed vertices as long as  $U(j) > 0$ , otherwise, stop the algorithm. In other words, we let  $x(j) = x_{v_j}$  have the following distribution

$$\mathbf{P}\{x(j) = x \mid \bar{I}(j), \bar{U}(j)\} = \begin{cases} \frac{I^x(j)}{I(j)}, & \text{if } I(j) > 0, \\ \frac{xU^x(j)}{\sum_{x \in S} xU^x(j)}, & \text{otherwise,} \end{cases} \quad (1.14)$$

for all  $x \in S$ .

We shall use Markov chain (1.9) to define the sizes of the components. We start with the component containing vertex  $v_1$ . By our algorithm vertex  $v_i$ ,  $i \geq 2$ , belongs to the same component as  $v_1$  if and only if  $I(i) = \sum_{x \in S} I^x(i) > 0$ . Hence, the size of the component containing  $v_1$  is exactly

$$\min\{j \geq 2 : I(j) = 0\} - 1. \quad (1.15)$$

Given the states of Markov chain (1.9) let us define a process which gives a useful representation for  $\tau_1$  as well as for the sizes of other components in the graph. Set

$$z^x(1) = 0, \quad (1.16)$$

$$z^x(i+1) = z^x(i) - \mathbf{1}_{x(i)=x} + \mathcal{N}_n^x(i), \quad i \geq 1,$$

and consider

$$z(i) := \sum_{x \in S} z^x(i),$$

which by (1.16) satisfies

$$z(1) = 0, \quad (1.17)$$

$$z(i+1) = z(i) - 1 + \sum_{x \in S} \mathcal{N}_n^x(i).$$

Notice that as long as  $1 < i \leq \min\{j > 1 : I(j) = 0\}$  we simply have

$$z(i) = I(i) - 1. \quad (1.18)$$

Indeed, by (1.13) and (1.10)

$$\begin{aligned} I(1) &= 0, \\ I(i+1) &= I(i) + \sum_{x \in S} \mathcal{N}_n^x(i) - \mathbf{1}_{I(i) > 0}, \quad 1 \leq i \leq n, \end{aligned} \quad (1.19)$$

which together with (1.17) gives (1.18). Then by (1.18) and (1.15) the size of the first revealed component (containing  $v_1$ ) is

$$\min\{i > 1 : z(i) = -1\} - 1. \quad (1.20)$$

Furthermore, setting for all  $k \geq 1$

$$\tau_k := \min\{i \geq 1 : z(i) = -k\}, \quad (1.21)$$

it is easy to check by induction that the size of the  $k$ -th revealed component,  $k \geq 2$ , is given by

$$\tau_k - \tau_{k-1}. \quad (1.22)$$

By the construction of the breadth-first process Theorem 1.1 should follow (at least intuitively) from the next stated theorem, which is the main result here.

**Theorem 1.2.** *Define for  $s \geq 0$*

$$Z_n(s) = n^{-1/3} z([n^{2/3}s] + 1). \quad (1.23)$$

*Then  $Z_n(s) \xrightarrow{d} W^a(s)$  as  $n \rightarrow \infty$ .*

Observe that process  $\{z(i), i \geq 1\}$  is not Markov. However, it converges (after scaling) to a Markov process. The reason is that we consider the rescaled process up to time  $n^{2/3}$ , and within this time we explore, roughly speaking only small amount of vertices.

It will be proved in Lemma 2.1 that in our algorithm of revealing components the ordering  $x(i), i \geq 1$ , is size-biased. By Theorem 1.2 the limiting process is  $W^a(s)$ , which differs from  $W(s) + as - \frac{1}{2}s^2$  considered in [1] only by the constant coefficients. Therefore Theorem 1.1 follows from Theorem 1.2 by the general theory of the multiplicative coalescent developed by Aldous (Lemma 14 and Proposition 15 in [1]).

## 2 Proof of Theorem 1.2.

### 2.1 Weak convergence.

For any function  $f : \mathbf{Z} \rightarrow R$  we shall denote

$$\Delta f(i) = f(i+1) - f(i), \quad i \in \{1, 2, \dots\}.$$

Define a martingale sequence  $\mathcal{M}_n(k)$ ,  $k \geq 1$ , by setting  $\mathcal{M}_n(1) = z(1)$  and

$$\Delta \mathcal{M}_n(k) = \Delta z(k) - \mathbf{E} \{ \Delta z(k) \mid \mathcal{F}_k^z \}, \quad (2.24)$$

where

$$\mathcal{F}_k^z = \sigma \{ z(j) : j \leq k \}.$$

Observe, that since each  $z(j)$  is defined as (Borel) function of states of the Markov chain  $((\bar{I}(i), \bar{U}(i), x(i)), i \leq j)$ , we have

$$\mathcal{F}_k^{\mathcal{M}} := \sigma \{ \mathcal{M}_n(j) : j \leq k \} \subseteq \mathcal{F}_k^z \subseteq \mathcal{F}_k := \sigma \{ (\bar{U}(k), \bar{I}(k), x(k)) : j \leq k \}. \quad (2.25)$$

Then we can write

$$z(k) = \mathcal{M}_n(k) + \sum_{j=1}^{k-1} \mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j^z \} =: \mathcal{M}_n(k) + \mathcal{D}_n(k). \quad (2.26)$$

Rescale to define for all  $s > 0$

$$\tilde{\mathcal{D}}_n(s) = n^{-1/3} \mathcal{D}_n(1 + [n^{2/3}s]),$$

$$\tilde{\mathcal{M}}_n(s) = n^{-1/3} \mathcal{M}_n(1 + [n^{2/3}s]).$$

Our aim is to show that (Lemma 2.3)

$$\tilde{\mathcal{M}}_n(s) \xrightarrow{d} \sqrt{\mathbf{E}X\mathbf{E}X^3} W(s), \quad (2.27)$$

while (Lemma 2.2)

$$\tilde{\mathcal{D}}_n(s) \xrightarrow{P} as - \frac{1}{2} \frac{\mathbf{E}X^3}{\mathbf{E}X} s^2 \quad (2.28)$$

uniformly on a finite interval as  $n \rightarrow \infty$ . Then Theorem 1.2 follows by (2.27) and (2.28).

First we shall study the summands in "the drift term"  $\mathcal{D}_n(k)$  making use of the Markov chain introduced above.



From now on we assume that  $n$  is so large that

$$-1/2 < \varepsilon_n := \frac{a}{n^{1/3}} < 1/2.$$

Also, since we are dealing here with convergence in probability or weak convergence, we may restrict ourselves on the event of asymptotically high probability, where condition (1.5), i.e.,

$$\max_{1 \leq i \leq n} x_i \leq \frac{n^{\frac{1}{3}}}{\omega(n)} \quad (2.29)$$

holds. (By the assumption the probability of this event converges to one as  $n \rightarrow \infty$ .)

**Proposition 2.1.** *For all  $k \geq 1$*

$$\begin{aligned} \mathbf{E} \{ \Delta z(k) \mid \mathcal{F}_k^z \} &= -1 + \mathbf{E} \left\{ x(k) \frac{1+\varepsilon_n}{n} \sum_{x \in S} x U^x(1) \left( 1 - \frac{z^x(k)}{U^x(1)} \right) \mid \mathcal{F}_k^z \right\} \\ &\quad - \frac{1+\varepsilon_n}{n} \mathbf{E} \left\{ x(k) \sum_{i=1}^k x(i) (1 + \mathbf{1}_{I(i)=0}) - x(k)^2 \mid \mathcal{F}_k^z \right\}. \end{aligned} \quad (2.30)$$

**Proof.** By the definitions (1.17) and (1.13) we have

$$\Delta z^x(i) = -\mathbf{1}_{x(i)=x} + \mathcal{N}_n^x(i) = -\Delta^x U(i) - \mathbf{1}_{x(i)=x} - \mathbf{1}_{I(i)=0} \mathbf{1}_{x(i)=x}. \quad (2.31)$$

Denote

$$\mathbf{E}_k \{ \cdot \} = \mathbf{E} \{ \cdot \mid \mathcal{F}_k \},$$

center  $\Delta z^x(k)$  and put

$$\begin{aligned} M^x(1) &= z^x(1), \\ \Delta M^x(k) &= \Delta z^x(k) - \mathbf{E}_k \Delta z^x(k) \end{aligned} \quad (2.32)$$

for all  $x \in S$  and  $k \geq 1$ , where by (1.12) and (2.31)

$$\mathbf{E}_k \Delta z^x(k) = -\mathbf{1}_{x(k)=x} + (U^x(k) - \mathbf{1}_{I(k)=0} \mathbf{1}_{x(k)=x}) p_n(x(k), x). \quad (2.33)$$

Then we derive recursively

$$\begin{aligned} z^x(k+1) &= z^x(k) + \Delta M^x(k) + \mathbf{E}_k \Delta z^x(k) \\ &= z^x(k) + \Delta M^x(k) - \mathbf{1}_{x(k)=x} + (U^x(k) - \mathbf{1}_{I(k)=0} \mathbf{1}_{x(k)=x}) p_n(x(k), x). \end{aligned} \quad (2.34)$$

Taking into account that by (2.31)

$$U^x(k+1) = U^x(1) - z^x(k+1) - \sum_{i=1}^k \mathbf{1}_{x(i)=x} (1 + \mathbf{1}_{I(i)=0}), \quad (2.35)$$

we obtain from (2.34)

$$z^x(k+1) = z^x(k) + \Delta M^x(k) - \mathbf{1}_{x(k)=x} \quad (2.36)$$

$$+ p_n(x(k), x) \left( U^x(1) - z^x(k) - \sum_{i=1}^{k-1} \mathbf{1}_{x(i)=x} (1 + \mathbf{1}_{I(i)=0}) - \mathbf{1}_{I(k)=0} \mathbf{1}_{x(k)=x} \right).$$

This allows us to derive

$$\begin{aligned} \mathbf{E} \{ \Delta z(k) \mid \mathcal{F}_k^z \} &= \mathbf{E} \left\{ \sum_{x \in S} \Delta z^x(k) \mid \mathcal{F}_k^z \right\} \\ &= \mathbf{E} \{ (\Delta z(j) - \mathbf{E}_k \Delta z(k)) \mid \mathcal{F}_k^z \} - 1 \\ &\quad + \mathbf{E} \left\{ \sum_{x \in S} p_n(x(k), x) (U^x(1) - z^x(k)) \mid \mathcal{F}_k^z \right\} \\ &\quad - \sum_{i=1}^{k-1} \mathbf{E} \left\{ p_n(x(k), x(i)) (1 + \mathbf{1}_{I(i)=0}) \mid \mathcal{F}_k^z \right\} \\ &\quad - \mathbf{E} \left\{ p_n(x(k), x(k)) \mathbf{1}_{I(k)=0} \mid \mathcal{F}_k^z \right\}. \end{aligned} \quad (2.37)$$

Note that

$$\mathbf{E} \{ (\Delta z(j) - \mathbf{E}_k \Delta z(k)) \mid \mathcal{F}_k^z \} = 0$$

due to (2.25). Therefore (2.37) yields

$$\begin{aligned} \mathbf{E} \{ \Delta z(k) \mid \mathcal{F}_k^z \} &= -1 + \mathbf{E} \left\{ x(k) \frac{1+\varepsilon_n}{n} \sum_{x \in S} x U^x(1) \left( 1 - \frac{z^x(k)}{U^x(1)} \right) \mid \mathcal{F}_k^z \right\} \\ &\quad - \frac{1+\varepsilon_n}{n} \mathbf{E} \left\{ x(k) \sum_{i=1}^k x(i) (1 + \mathbf{1}_{I(i)=0}) - x(k)^2 \mid \mathcal{F}_k^z \right\}, \end{aligned} \quad (2.38)$$

which proves the Proposition.  $\square$

In our analysis we will use the following result.

**Proposition 2.2.** *Uniformly in  $j \leq sn^{2/3}$  (for any fixed  $s > 0$ )*

$$\frac{1}{n} \sum_{x \in S} U^x(j) \rightarrow 1, \quad (2.39)$$

$$\frac{1}{n} \sum_{x \in S} x U^x(j) \rightarrow \mathbf{E}X, \quad (2.40)$$

and

$$\frac{1}{n} \sum_{x \in S} x^2 U^x(j) \rightarrow \mathbf{E}X^2. \quad (2.41)$$

in  $L^1$  as  $n \rightarrow \infty$ .

**Proof.** We shall establish (2.40); the rest follows exactly by the same argument. By (1.13)

$$U^x(j) = U^x(1) - \sum_{i=1}^{j-1} (\mathcal{N}_n^x(i) + \mathbf{1}_{I(i)=0} \mathbf{1}_{x(i)=x})$$

for all  $x \in S$ . Hence

$$\begin{aligned} \frac{1}{n} \sum_{x \in S} x U^x(j) &= \frac{1}{n} \sum_{x \in S} x U^x(1) - \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x (\mathcal{N}_n^x(i) + \mathbf{1}_{I(i)=0} \mathbf{1}_{x(i)=x}) \\ &= \frac{1}{n} \sum_{x \in S} x U^x(1) - \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x \mathcal{N}_n^x(i) - \frac{1}{n} \sum_{i=1}^{j-1} x(i) \mathbf{1}_{I(i)=0}, \end{aligned} \quad (2.42)$$

where the last term for all  $j = O(n^{2/3})$

$$\frac{1}{n} \sum_{i=1}^{j-1} x(i) \mathbf{1}_{I(i)=0} \leq \frac{1}{n} \sum_{i=1}^{j-1} x(i) \leq \frac{O(n^{2/3})}{n} \max_{1 \leq i \leq n} x_i \rightarrow 0 \quad (2.43)$$

in  $L^1$  as  $n \rightarrow \infty$ , since by the assumption (1.5) we have

$$\mathbf{E} \max_{1 \leq i \leq n} x_i = o(n^{1/3}). \quad (2.44)$$

We will show that the first summand on the right of (2.42) gives the main contribution. Consider

$$\begin{aligned} \frac{1}{n} \sum_{x \in S} x U^x(1) &= \frac{1}{n} \sum_{x \in S} x \#\{i : x_i = x\} = \frac{1}{n} \sum_{x \in S} x \sum_{i=1}^n \mathbf{1}_{\{x_i=x\}} \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{x \in S} x \mathbf{1}_{\{x_i=x\}}. \end{aligned} \quad (2.45)$$

Under assumption that  $x_i$  are *i.i.d.* with finite third moment, the ergodic theorem gives us convergence in  $L^1$  and *a.s.*

$$\frac{1}{n} \sum_{i=1}^n \sum_{x \in S} x \mathbf{1}_{\{x_i=x\}} \rightarrow \mathbf{E} \sum_{x \in S} x \mathbf{1}_{\{x_1=x\}} = \mathbf{E}X,$$

and thus by (2.45) we have convergence in  $L^1$  and *a.s.*

$$\frac{1}{n} \sum_{x \in S} x U^x(1) \rightarrow \mathbf{E} \sum_{x \in S} x \mathbf{1}_{\{x_1=x\}} = \mathbf{E}X. \quad (2.46)$$

Finally, we bound with help of (1.12)

$$\mathbf{E} \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x \mathcal{N}_n^x(i) \leq \mathbf{E} \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x p_n(x(i), x) U^x(i) \leq \mathbf{E} \frac{1 + \varepsilon_n}{n} \sum_{i=1}^{j-1} x(i) \sum_{x \in S} x^2 \frac{U^x(1)}{n}.$$

Using representation (2.45) we derive from here

$$\mathbf{E} \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x \mathcal{N}_n^x(i) \leq \frac{O(n^{2/3})}{n} \sum_{x \in S} x^2 \frac{1}{n} \sum_{m=1}^n \mathbf{E} \left( \max_{1 \leq i \leq j} x(i) \right) \mathbf{1}_{\{x_m = x\}}. \quad (2.47)$$

Separately we compute taking into account (2.44)

$$\mathbf{E} \left( \max_{1 \leq i \leq j} x(i) \right) \mathbf{1}_{\{x_m = x\}} \leq \mathbf{E} \mathbf{1}_{\{x_m = x\}} (x_m + \mathbf{E} \max_{1 \leq i \leq n: i \neq m} x_i) = \mathbf{P}\{x_m = x\} (x + o(n^{1/3})).$$

Substituting this into (2.47) we immediately derive

$$\mathbf{E} \frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x \mathcal{N}_n^x(i) \leq \frac{O(n^{2/3})}{n} \sum_{x \in S} x^2 \mathbf{P}\{X = x\} (x + o(n^{1/3})) \rightarrow 0.$$

This confirms that

$$\frac{1}{n} \sum_{i=1}^{j-1} \sum_{x \in S} x \mathcal{N}_n^x(i) \rightarrow 0 \quad (2.48)$$

in  $L^1$  uniformly in  $j \leq sn^{2/3}$ . Convergence of the terms (2.46), (2.48) and (2.43), together with formula (2.42) yield statement (2.40).

Statements (2.41) and (2.39) can be proved exactly in the same way.  $\square$

**Lemma 2.1.** *Let  $x_n(i) = x(i)$ ,  $1 \leq i \leq k$ , be the sequence of random variables defined in (1.9), and let  $k \leq sn^{2/3}$  for some constant  $s > 0$ . Then*

$$\begin{aligned} \mathbf{P}\{x_n(i) = y\} &= (1 + o(1)) \mathbf{E} \frac{x U^x(i)}{\sum_{x \in S} x U^x(i)} \\ &= (1 + o(1)) \mathbf{E} \frac{x U^x(1)}{\sum_{x \in S} x U^x(1)}, \end{aligned} \quad (2.49)$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$  uniformly in  $1 \leq i \leq k$ , and

$$\mathbf{P}\{x_n(i) = y \mid \bar{U}(i)\} = (1 + o_{L_1}(1)) \frac{x U^x(i)}{\sum_{x \in S} x U^x(i)}, \quad (2.50)$$

where  $o_{L_1}(1) \rightarrow 0$  in  $L_1$  as  $n \rightarrow \infty$  uniformly in  $1 \leq i \leq k$ .

We postpone the proof of this lemma till the end of this section.

Lemma 2.1 together with Proposition 2.2 yield immediately the following corollary.

**Corollary 2.1.** 1. *Uniformly in*  $1 \leq i \leq sn^{2/3}$

$$x_n(i) \xrightarrow{d} \tilde{X}, \quad \text{as } n \rightarrow \infty \quad (2.51)$$

where  $\tilde{X}$  is a random variable with a distribution

$$\mathbf{P}\{\tilde{X} = y\} = \frac{y \mathbf{P}\{X = y\}}{\mathbf{E}X}. \quad (2.52)$$

2. *For all*  $k \leq sn^{2/3}$

$$\frac{1}{k} \sum_{i=1}^k \mathbf{E}x_n(i) \rightarrow \frac{\mathbf{E}X^2}{\mathbf{E}X}, \quad (2.53)$$

and

$$\frac{1}{k} \sum_{i=1}^k \mathbf{E}x_n^2(i) \rightarrow \frac{\mathbf{E}X^3}{\mathbf{E}X} \quad (2.54)$$

as  $n \rightarrow \infty$ .

Now we can prove statement (2.28).

**Lemma 2.2.** *Uniformly in*  $0 \leq s \leq s_0$

$$\tilde{\mathcal{D}}_n(s) \xrightarrow{P} as - \frac{1}{2} \frac{\mathbf{E}X^3}{\mathbf{E}X} s^2 \quad (2.55)$$

as  $n \rightarrow \infty$ .

**Proof.** By (2.26) and Proposition 2.1

$$\begin{aligned} \mathcal{D}_n(k+1) &= \sum_{j=1}^k \mathbf{E} \left\{ \Delta z(j) \mid \mathcal{F}_j^z \right\} \\ &= -k + \sum_{j=1}^k \mathbf{E} \left\{ x(j) \frac{1+\varepsilon_n}{n} \sum_{x \in S} x U^x(1) \left( 1 - \frac{z^x(j)}{U^x(1)} \right) \mid \mathcal{F}_j^z \right\} \end{aligned} \quad (2.56)$$

$$- \frac{1+\varepsilon_n}{n} \sum_{j=2}^k \mathbf{E} \left\{ x(j) \sum_{i=1}^{j-1} x(i) \mid \mathcal{F}_j^z \right\} - \delta_n(k),$$

where

$$\delta_n(k) = \frac{1+\varepsilon_n}{n} \sum_{j=1}^k \mathbf{E} \left\{ x(j) \sum_{i=1}^j x(i) \mathbf{1}_{I(i)=0} \mid \mathcal{F}_j^z \right\}. \quad (2.57)$$

We shall use the following fact.

**Proposition 2.3.** *Uniformly in  $j \leq s_0 n^{2/3}$*

$$\mathbf{E} \frac{|z^x(j)|}{U^x(1)} = o\left(\frac{x}{n^{1/3}}\right).$$

**Proof.** Observe, that

$$|z^x(j)| \leq \sum_{i=1}^{j-1} (\Delta^x z(i))^+ + \sum_{i=1}^{j-1} (\Delta z^x(i))^- , \quad (2.58)$$

where  $a^+ = \max\{0, a\}$  and  $a^- = -\min\{0, a\}$ . By the definition (1.16)

$$\sum_{i=1}^{j-1} (\Delta z^x(i))^+ \leq \sum_{i=1}^{j-1} \mathcal{N}_n^x(i) \quad (2.59)$$

and

$$\sum_{i=1}^{j-1} (\Delta z^x(i))^- \leq \sum_{0 \leq i \leq j-1: I(i)=0} \mathbf{1}_{x(i+1)=x}. \quad (2.60)$$

Recall also that by the definition (1.14) in our algorithm at step  $i$  when  $I(i) = 0$  we choose  $x(i+1)$  size-biased among the unrevealed vertices. Then (2.58) together with (2.59) and (2.60) give us

$$\mathbf{E} \frac{|z^x(j)|}{U^x(1)} \leq \sum_{i=1}^{j-1} \mathbf{E} \frac{\mathcal{N}_n^x(i)}{U^x(1)} + \mathbf{E} \frac{1}{U^x(1)} \sum_{0 \leq i \leq j-1: I(i)=0} \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)}.$$

Since  $U^x(i)$  is non-increasing in  $i$ , the last bound yields

$$\begin{aligned} \mathbf{E} \frac{|z^x(j)|}{U^x(1)} &\leq (1 + \varepsilon_n) \left( \sum_{i=1}^{j-1} \mathbf{E} \frac{x(i)x}{n} + \mathbf{E} \frac{x}{\sum_{x \in S} xU^x(j)} \#\{0 \leq i \leq j-1 : I(i) = 0\} \right) \\ &\leq 2 \frac{xj}{n} \left( \frac{1}{j} \sum_{i=1}^{j-1} \mathbf{E} x(i) + \mathbf{E} \frac{n}{\sum_{x \in S} xU^x(j)} \right). \end{aligned}$$

Using Proposition 2.2 and Corollary 2.1 we derive from here

$$\mathbf{E} \frac{|z^x(j)|}{U^x(1)} \leq c \frac{xj}{n}$$

for some positive constant  $c$ . The statement of Proposition 2.3 follows now by the assumptions on  $j$ .  $\square$

With a help of Proposition 2.3 and bound (1.5) we derive from (2.56)

$$\begin{aligned} \mathcal{D}_n(k+1) &= -k + (1 + o_P(1)) \sum_{j=1}^k \mathbf{E} \left\{ x(j) \frac{1+\varepsilon_n}{n} \sum_{x \in S} x U^x(1) \mid \mathcal{F}_j^z \right\} \\ &\quad - \frac{1+\varepsilon_n}{n} \sum_{j=2}^k \mathbf{E} \left\{ x(j) \sum_{i=1}^{j-1} x(i) \mid \mathcal{F}_j^z \right\} - \delta_n(k). \end{aligned}$$

This together with Proposition 2.2 and Lemma 2.1 gives us

$$\begin{aligned} \mathcal{D}_n(k+1) &= -k + (1 + \varepsilon_n + o_P(1)) (\mathbf{E}X) \sum_{j=1}^k \mathbf{E} \left\{ x(j) \mid \mathcal{F}_j^z \right\} \\ &\quad - \frac{1 + \varepsilon_n}{n} \sum_{j=2}^k \mathbf{E} \left\{ x(j) \sum_{i=1}^{j-1} x(i) \mid \mathcal{F}_j^z \right\} - \delta_n(k). \end{aligned}$$

Again using Lemma 2.1 we derive from here

$$\begin{aligned} \mathcal{D}_n(k+1) &= -k + \mathbf{E}X \left( k \frac{\mathbf{E}X^2}{\mathbf{E}X} + \frac{k^2}{2n} \left( \left( \frac{\mathbf{E}X^2}{\mathbf{E}X} \right)^3 - \frac{\mathbf{E}X^2 \mathbf{E}X^3}{(\mathbf{E}X)^2} \right) \right) (1 + \varepsilon_n + o_P(1)) \\ &\quad - \frac{k^2}{2n} \left( \frac{\mathbf{E}X^2}{\mathbf{E}X} \right)^2 (1 + o_P(1)) - \delta_n(k). \end{aligned} \tag{2.61}$$

**Remark 2.1.** Formula (2.61) shows that whenever  $\mathbf{E}X^2 \neq 1$  the principal term in the drift  $\mathcal{D}_n(k)$  is linear in  $k$ , which makes out further analysis inapplicable. Thus we get here once again a confirmation that the case  $\mathbf{E}X^2 = 1$  is critical.

Using now assumptions  $\varepsilon_n = a/n^{1/3}$  and  $\mathbf{E}X^2 = 1$  we get from (2.61)

$$\begin{aligned} \mathcal{D}_n(k+1) &= a \frac{k}{n^{1/3}} (1 + o_p(1)) - \frac{k^2}{2n} \frac{\mathbf{E}X^3}{\mathbf{E}X} (1 + o_p(1)) - \delta_n(k) \\ &= n^{1/3} \left( a \frac{k}{n^{2/3}} - \frac{k^2}{2n^{4/3}} \frac{\mathbf{E}X^3}{\mathbf{E}X} \right) (1 + o_p(1)) - \delta_n(k) \end{aligned} \tag{2.62}$$

Finally, we shall find an upper bound for  $|\delta_n(k)|$  defined in (2.57). First we derive

$$\delta_n(k) \leq \frac{2}{n} \sum_{j=1}^k \mathbf{E} \left\{ x(j) \left( \max_{1 \leq i \leq n} x_i \right) \left( \sum_{i=1}^j \mathbf{1}_{I(i)=0} \right) \mid \mathcal{F}_j^z \right\}. \tag{2.63}$$

Recall that by our construction of process  $z(i)$

$$\sum_{i=1}^j \mathbf{1}_{I(i)=0} \leq - \min_{0 \leq i \leq j} z(i).$$

Therefore we derive from (2.63)

$$\delta_n(k) \leq \frac{\max_{1 \leq i \leq n} x_i}{n} \sum_{j=1}^k \left( \max_{0 \leq i \leq j} |z(i)| \right) \mathbf{E} \{x(j) \mid \mathcal{F}_j^z\}.$$

This together with Lemma 2.1 yields for  $k = O(n^{2/3})$

$$\delta_n(k) \leq (c + o_P(1)) \frac{\max_{1 \leq i \leq n} x_i}{n^{1/3}} \max_{1 \leq i \leq k} |z(i)| \quad (2.64)$$

for some positive constant  $c$ .

**Proposition 2.4.**

$$n^{-1/3} \max_{k \leq n^{2/3} s_0} |z(k)| \text{ is stochastically bounded as } n \rightarrow \infty. \quad (2.65)$$

**Proof.** We shall use the idea from Aldous [1]. Fix a constant  $K$  and define

$$\begin{aligned} T_n^* &= \min\{s : |z(s)| > Kn^{1/3}\}, \\ T_n &= \min(T_n^*, \lceil n^{2/3} s_0 \rceil). \end{aligned} \quad (2.66)$$

Then using representation (2.26)

$$z(k) = \mathcal{M}_n(k) + \mathcal{D}_n(k),$$

we get

$$\begin{aligned} \mathbf{P} \left\{ \sup_{t \leq n^{2/3} s_0} |z(t)| > Kn^{1/3} \right\} &= \mathbf{P} \{ |z(T_n)| > Kn^{1/3} \} \\ &\leq \mathbf{P} \{ |\mathcal{M}_n(T_n)| + |\mathcal{D}_n(T_n)| > Kn^{1/3} \} \\ &\leq \frac{\mathbf{E} |\mathcal{M}_n(T_n)| + \mathbf{E} |\mathcal{D}_n(T_n)|}{Kn^{1/3}} \leq \frac{(\mathbf{E} \mathcal{M}_n^2(T_n))^{1/2} + \mathbf{E} |\mathcal{D}_n(T_n)|}{Kn^{1/3}}. \end{aligned} \quad (2.67)$$



Next we shall find bounds for  $\mathbf{E}\mathcal{M}_n^2(T_n)$  and  $\mathbf{E}|\mathcal{D}_n(T_n)|$  separately.

Consider first  $\mathbf{E}\mathcal{M}_n^2(T_n)$ . Set

$$A_n(k) = \sum_{j=1}^k \mathbf{E} \{ (\Delta\mathcal{M}_n(j))^2 \mid \mathcal{F}_j^{\mathcal{M}} \}. \quad (2.68)$$

Then

$$\mathcal{M}_n^2(k) - A_n(k), \quad k \geq 1,$$

is a martingale and by the optional sampling theorem we have

$$\mathbf{E}\mathcal{M}_n^2(T_n) = \mathbf{E}A_n(T_n) \leq \sum_{j=1}^{\lfloor n^{2/3}s_0 \rfloor} \mathbf{E} (\Delta\mathcal{M}_n(j))^2. \quad (2.69)$$

By the definition

$$\Delta\mathcal{M}_n(j) = \Delta z(j) - \mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j^z \}, \quad (2.70)$$

and thus

$$\mathbf{E} (\Delta\mathcal{M}_n(j))^2 = \mathbf{E} \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j^z \}. \quad (2.71)$$

Using  $\mathcal{F}_j$  associated with the Markov chain (1.9) we derive

$$\begin{aligned} & \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j^z \} \\ &= \mathbf{E} \{ \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j \} \mid \mathcal{F}_j^z \} + \mathbf{Var} \{ \mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j \} \mid \mathcal{F}_j^z \}. \end{aligned} \quad (2.72)$$

Notice  $\mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j^z \}$  in the definition (2.70) is  $\mathcal{F}_j^z$ -measurable by (2.25). Hence, we obtain from (2.72)

$$\begin{aligned} & \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j^z \} \\ &= \mathbf{E} \{ \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j \} \mid \mathcal{F}_j^z \} + \mathbf{Var} \{ \mathbf{E} \{ (\Delta z(j) \mid \mathcal{F}_j) - \mathbf{E} \{ \Delta z(j) \} \mid \mathcal{F}_j^z \} \mid \mathcal{F}_j^z \} \\ &= \mathbf{E} \{ \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j \} \mid \mathcal{F}_j^z \} + \mathbf{Var} \{ \mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j \} \mid \mathcal{F}_j^z \}. \end{aligned} \quad (2.73)$$

Conditionally on  $\mathcal{F}_j$  we have by (2.31)

$$\Delta z(j) = \sum_{x \in S} \Delta z^x(j) = -1 + \sum_{x \in S} \mathcal{N}_n^x(j), \quad (2.74)$$

where  $\mathcal{N}_n^x(j)$  are independent Binomial random variables for different  $x$ . Hence, by (1.12)

$$\mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j \} = (1 + o(1)) \frac{1}{n} \sum_{x \in S} U^x(j) x(j) x = x(j) \mathbf{E}X (1 + o_{L_1}(1)), \quad (2.75)$$

where to derive the last equality we used Proposition 2.2. Similarly we get

$$\mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j \} = (1 + o(1)) \frac{1}{n} \sum_{x \in S} U^x(j) x(j) x = x(j) \mathbf{E} X (1 + o_{L_1}(1)). \quad (2.76)$$

Substituting (2.75) and (2.76) into (2.73) gives us

$$\begin{aligned} & \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j^z \} \\ &= \mathbf{E} \{ x(j) \mid \mathcal{F}_j^z \} \mathbf{E} X (1 + o_{L_1}(1)) + \mathbf{Var} \{ x(j) \mid \mathcal{F}_j^z \} (\mathbf{E} X)^2 (1 + o_{L_1}(1)). \end{aligned} \quad (2.77)$$

Now with a help of (2.77) we can derive from (2.71) for all large  $n$

$$\begin{aligned} \mathbf{E} (\Delta \mathcal{M}_n(j))^2 &\leq 2 (\mathbf{E} x(j) \mathbf{E} X + \mathbf{E} \mathbf{Var} \{ x(j) \mid \mathcal{F}_j^z \} (\mathbf{E} X)^2) \\ &\leq 2 (\mathbf{E} x(j) \mathbf{E} X + \mathbf{E} x^2(j) (\mathbf{E} X)^2), \end{aligned}$$

which by Lemma 2.1 is uniformly bounded, so that

$$\mathbf{E} (\Delta \mathcal{M}_n(j))^2 \leq c \quad (2.78)$$

for some  $c > 0$ , which together with (2.69) implies

$$\mathbf{E} \mathcal{M}_n^2(T_n) \leq cn^{2/3} s_0. \quad (2.79)$$

Let us bound now term  $\mathbf{E} |\mathcal{D}_n(T_n)|$  in (2.67). Using formula (2.62), we derive

$$|\mathcal{D}_n(T_n)| \leq 2|a| \frac{n^{2/3} s_0}{n^{1/3}} + 2 \frac{(n^{2/3} s_0)^2}{2n} \left( \frac{1}{\mathbf{E} X} \right)^2 + |\delta_n(T_n - 1)|,$$

where by (2.64) and definition of  $T_n$

$$\delta_n(T_n - 1) \leq \frac{1}{\omega_1(n)} \max_{1 \leq i \leq T_n - 1} |z(i)| \leq \frac{Kn^{1/3}}{\omega_1(n)}.$$

Hence,

$$\mathbf{E} |\mathcal{D}_n(T_n)| \leq c_1 n^{1/3} \quad (2.80)$$

for some positive  $c_1$ . Substituting bounds (2.80) and (2.79) into (2.67), we get

$$\mathbf{P} \left\{ \sup_{t \leq n^{2/3} s_0} |z(t)| > Kn^{1/3} \right\} \leq \frac{\sqrt{cs_0} + c_1}{K},$$

and therefore establish (2.65).  $\square$

Proposition 2.4 and bound (2.64) yield

$$\frac{\delta_n(s_0 n^{2/3})}{n^{1/3}} \rightarrow_P 0.$$

This together with (2.62) implies

$$\tilde{\mathcal{D}}_n(s) = n^{-1/3} \mathcal{D}_n([n^{2/3}s] + 1) \xrightarrow{P} as - \frac{1}{2} \frac{\mathbf{E}X^3}{\mathbf{E}X} s^2,$$

which completes the proof of Lemma 2.2.  $\square$

Next we shall study the martingale part  $\mathcal{M}_n(k)$  in the representation (2.26) and prove (2.27).

**Lemma 2.3.**

$$\tilde{\mathcal{M}}_n(s) \xrightarrow{d} \sqrt{\mathbf{E}X\mathbf{E}X^3} W(s) \quad (2.81)$$

as  $n \rightarrow \infty$ .

**Proof.** To establish convergence (2.81) we shall apply the functional CLT for martingales (see [4], Theorem 1.4 (b) and Remark 1.5). Rescale  $A_n$  defined in (2.68):

$$\tilde{A}_n(s) = n^{-2/3} A_n([n^{2/3}s]) = n^{-2/3} \sum_{j=1}^{[n^{2/3}s]} \mathbf{E} \{ (\Delta \mathcal{M}_n(j))^2 \mid \mathcal{F}_j^{\mathcal{M}} \},$$

where

$$\Delta \mathcal{M}_n(j) = (\Delta z(j) - \mathbf{E} \{ \Delta z(j) \mid \mathcal{F}_j^z \}), \quad j > 1,$$

with

$$\mathcal{M}_n(1) = z(1) = 0.$$

We have to verify the following conditions: for each  $s > 0$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \sup_{t \leq s} |\tilde{A}_n(t) - \tilde{A}_n(t-)| \right] = 0, \quad (2.82)$$

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[ \sup_{t \leq s} |\tilde{\mathcal{M}}_n(t) - \tilde{\mathcal{M}}_n(t-)|^2 \right] = 0, \quad (2.83)$$

and

$$\tilde{A}_n(s) \xrightarrow{P} \mathbf{E}X\mathbf{E}X^3 s. \quad (2.84)$$

We start with (2.84). Note that by (2.25)

$$\begin{aligned} \tilde{A}_n(s) &= n^{-2/3} \sum_{j=1}^{\lfloor n^{2/3}s \rfloor} \mathbf{E} \{ (\Delta\mathcal{M}_n(j))^2 \mid \mathcal{F}_j^{\mathcal{M}} \} \\ &= n^{-2/3} \sum_{j=1}^{\lfloor n^{2/3}s \rfloor} \mathbf{E} \{ \mathbf{Var} \{ \Delta z(j) \mid \mathcal{F}_j^z \} \mid \mathcal{F}_j^{\mathcal{M}} \}. \end{aligned} \quad (2.85)$$

Substituting here formula (2.77), we obtain

$$\begin{aligned} \tilde{A}_n(s) &= n^{-2/3} (1 + o_P(1)) \sum_{j=1}^{\lfloor n^{2/3}s \rfloor} (\mathbf{E} \{ x(j) \mid \mathcal{F}_j^{\mathcal{M}} \} \mathbf{E}X \\ &\quad + \mathbf{E} \{ \mathbf{Var} \{ x(j) \mid \mathcal{F}_j^z \} \mid \mathcal{F}_j^{\mathcal{M}} \} \mathbf{E}X^2) \\ &= n^{-2/3} \sum_{j=1}^{\lfloor n^{2/3}s \rfloor} \mathbf{E} \{ x(j) \mid \mathcal{F}_j^{\mathcal{M}} \} \mathbf{E}X (1 + o_P(1)) \\ &\quad + n^{-2/3} \sum_{j=1}^{\lfloor n^{2/3}s \rfloor} \left( \mathbf{E} \{ x(j)^2 \mid \mathcal{F}_j^{\mathcal{M}} \} - \mathbf{E} \left\{ \left( \mathbf{E} \{ x(j) \mid \mathcal{F}_j^z \} \right)^2 \mid \mathcal{F}_j^{\mathcal{M}} \right\} \right) (\mathbf{E}X)^2 (1 + o_P(1)). \end{aligned} \quad (2.86)$$

Using Lemma 2.1, one can derive from here

$$\tilde{A}_n(s) = s(1 + o_P(1)) (\mathbf{E}X^2 + (\mathbf{E}X^3)\mathbf{E}X - (\mathbf{E}X^2)^2).$$

Under assumption (1.3) that  $\mathbf{E}X^2 = 1$  this yields

$$\tilde{A}_n(s) \rightarrow_P s\mathbf{E}X^3\mathbf{E}X,$$

which is the statement (2.84).

Next we prove (2.83). Consider

$$\mathbf{E} \left[ \sup_{t \leq s} |\tilde{\mathcal{M}}_n(t) - \tilde{\mathcal{M}}_n(t-)|^2 \right] = \frac{1}{n^{2/3}} \mathbf{E} \sup_{j \leq sn^{2/3}} (\Delta\mathcal{M}_n(j))^2, \quad (2.87)$$

where by the definition

$$\Delta\mathcal{M}_n(j) = \Delta z(j) - \mathbf{E} \left\{ \Delta z(j) \mid \mathcal{F}_j^{\mathcal{M}} \right\} = \sum_{x \in S} \mathcal{N}_n^x(j) - \mathbf{E} \left\{ \sum_{x \in S} \mathcal{N}_n^x(j) \mid \mathcal{F}_j^{\mathcal{M}} \right\}.$$

Then statement (2.83) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2/3}} \mathbf{E} \left( \sup_{j \leq sn^{2/3}} \left| \sum_{x \in S} \mathcal{N}_n^x(j) - \mathbf{E} \left\{ \sum_{x \in S} \mathcal{N}_n^x(j) \mid \mathcal{F}_j^{\mathcal{M}} \right\} \right| \right)^2 = 0. \quad (2.88)$$

Observe that  $\sum_{x \in S} \mathcal{N}_n^x(j)$  conditionally on  $x(j)$  for  $j \leq sn^{2/3}$  can be well approximated by Poisson  $Po(x(j)\mathbf{E}X)$  random variable, where by Corollary 2.1  $x(j) \rightarrow_d \tilde{X}$ . Therefore we shall use the following result, which is straightforward to obtain.

**Proposition 2.5.** *Let  $\xi$  be a random variable such that conditionally on  $\tilde{X} = x$  the distribution of  $\xi$  is  $Po(x\mathbf{E}X)$ . Let  $\xi_i$ ,  $1 \leq i \leq k$ , be independent copies of  $\xi$ , and define*

$$Y_k = \max_{1 \leq i \leq k} |\xi_i - \mathbf{E}\xi_i|.$$

Then under condition (1.5) (i.e., when  $\mathbf{E}\tilde{X}^2 < \infty$ ) we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \mathbf{E}Y_k^2 = 0. \quad (2.89)$$

□

As a corollary to this Proposition we get convergence (2.88), and therefore we establish (2.83).

Finally, let us check condition (2.82). By (2.86)

$$\begin{aligned} \mathbf{E} \left[ \sup_{t \leq s} |A_n(t) - A_n(t-)| \right] &= n^{-2/3} \mathbf{E} \max_{1 \leq j \leq sn^{2/3}} \left( \mathbf{E} \{x(j) \mid \mathcal{F}_j^{\mathcal{M}}\} \mathbf{E}X \right. \\ &\quad \left. + \mathbf{E} \{x(j)^2 \mid \mathcal{F}_j^{\mathcal{M}}\} - \mathbf{E} \left\{ \left( \mathbf{E} \{x(j) \mid \mathcal{F}_j^z\} \right)^2 \mid \mathcal{F}_j^{\mathcal{M}} \right\} \right) (\mathbf{E}X)^2 (1 + o_P(1)). \end{aligned} \quad (2.90)$$

Using statement (2.50) from Lemma 2.1 and Corollary 2.1 we see that the expectation of the maximum on the right is bounded uniformly in  $n$ . Therefore condition (2.82) follows. This finishes the proof of Lemma 2.3. □

## 2.2 Proof of Lemma 2.1.

Let us consider  $\mathbf{P}\{x(i) = x\}$ . When  $i = 1$  by the definition (1.11) we have

$$\mathbf{P}\{x(1) = x\} = \mathbf{E} \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)}, \quad (2.91)$$

and for all  $i > 1$  by the definition (1.14) we have

$$\mathbf{P}\{x(i) = x\} = \mathbf{E} \left( \frac{I^x(i)}{I(i)} \mathbf{1}\{I(i) > 0\} + \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)} \mathbf{1}\{I(i) = 0\} \right). \quad (2.92)$$

*Claim.* For all  $1 < i \leq sn^{2/3}$

$$\mathbf{E} \frac{I^x(i)}{I(i)} \mathbf{1}\{I(i) > 0\} = (1 + o(1)) \mathbf{E} \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)} \mathbf{1}\{I(i) > 0\} \quad (2.93)$$

where  $o(1) \rightarrow 0$  uniformly in  $i \leq sn^{2/3}$  as  $n \rightarrow \infty$ .

Before we prove this, let us observe that (2.93) together with (2.92) and (2.91) would give us

$$\mathbf{P}\{x(i) = x\} = (1 + o(1)) \mathbf{E} \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)}, \quad (2.94)$$

where  $o(1) \rightarrow 0$  uniformly in  $i \leq sn^{2/3}$  as  $n \rightarrow \infty$ , which is the statement of Lemma 2.1.

*Proof of the Claim.* Recall that by the definition (1.13) for all  $i > 1$  we have

$$\mathbf{E} \frac{I^x(i)}{I(i)} \mathbf{1}\{I(i) > 0\} \quad (2.95)$$

$$= \mathbf{E} \frac{I^x(i-1) + \mathcal{N}_n^x(i-1) - \mathbf{1}_{I(i-1) > 0} \mathbf{1}_{x(i-1)=x}}{I(i-1) + \sum_{x \in S} \mathcal{N}_n^x(i-1) - \mathbf{1}_{I(i-1) > 0}} \mathbf{1}\{I(i) > 0\}.$$

Then we observe that conditionally on  $x(i-1)$  and  $\bar{U}(i-1)$  the distribution of  $\mathcal{N}_n^x(i-1)$  is binomial

$$\mathcal{N}_n^x(i-1) \in \text{Bin} \left( U^x(i-1) - \mathbf{1}_{I(1)=0} \mathbf{1}_{x(i-1)=x}, p_n(x(i-1), x) \right).$$

Since this can be well approximated by a Poisson distribution, we shall make use of the following fact. Let  $Z^x(t) \in Po(\lambda_x)$ ,  $x \in S$ ,  $t \geq 0$ , be independent Poisson processes with parameters  $\lambda_x > 0$  such that  $\sum_{x \in S} \lambda_x < \infty$ . Define  $Z(t) := \sum_{x \in S} Z^x(t)$  to be a

superposition of these processes. Then conditionally on  $Z(t)$  the distribution of  $Z^x(t)$  equals the distribution of thinned Poisson process  $Z(t)$ , with a probability of thinning

$$\frac{\lambda_x}{\sum_{x \in S} \lambda_x}.$$

In particular, this implies

$$\mathbf{E}\{Z^x(1) \mid Z(1)\} = \frac{\lambda_x}{\sum_{x \in S} \lambda_x} Z(1), \quad (2.96)$$

which yields

$$\begin{aligned} \mathbf{E} \frac{Z^x(1)}{\sum_{x \in S} Z^x(1)} \mathbf{1} \left\{ \sum_{x \in S} Z^x(1) > 0 \right\} &= \mathbf{E} \mathbf{E} \left\{ \frac{Z^x(1)}{Z(1)} \mathbf{1} \{Z(1) > 0\} \mid Z(1) \right\} \\ &= \frac{\lambda_x}{\sum_{x \in S} \lambda_x} \mathbf{P} \{Z(1) > 0\}, \end{aligned} \quad (2.97)$$

and also for any  $a, b \geq 0$

$$\begin{aligned} \mathbf{E} \frac{a + Z^x(1)}{b + Z(1)} \mathbf{1} \{b + Z(1) > 0\} \\ = \mathbf{E} \frac{a + (\lambda_x / \sum_{x \in S} \lambda_x) Z(1)}{a + Z(1)} \mathbf{1} \{b + Z(1) > 0\}. \end{aligned} \quad (2.98)$$

Return to our binomial variables: with a help of Poisson approximation we will show that a statement similar to (2.98) holds (up to a small error term) as well for the binomial variables  $\mathcal{N}_n^x(i)$ .

**Proposition 2.6.** *For any  $a, b \geq 0$*

$$\begin{aligned} \mathbf{E} \frac{a + \mathcal{N}_n^x(i)}{b + \sum_{x \in S} \mathcal{N}_n^x(i)} \mathbf{1} \left\{ b + \sum_{x \in S} \mathcal{N}_n^x(i) > 0 \right\} \\ = (1 + o(1)) \mathbf{E} \frac{a + \lambda_i(x) \sum_{x \in S} \mathcal{N}_n^x(i)}{b + \sum_{x \in S} \mathcal{N}_n^x(i)} \mathbf{1} \left\{ b + \sum_{x \in S} \mathcal{N}_n^x(i) > 0 \right\}, \end{aligned} \quad (2.99)$$

where

$$\lambda_i(x) := \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)},$$

and  $o(1) \rightarrow 0$  uniformly in  $i \leq sn^{2/3}$ .

In particular, if  $a = b = 0$

$$\begin{aligned} & \mathbf{E} \frac{\mathcal{N}_n^x(i)}{\sum_{x \in S} \mathcal{N}_n^x(i)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(i) > 0 \right\} \\ &= (1 + o(1)) \mathbf{E} \frac{xU^x(i)}{\sum_{x \in S} xU^x(i)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(i) > 0 \right\}, \end{aligned} \quad (2.100)$$

where  $o(1) \rightarrow 0$  uniformly in  $i \leq sn^{2/3}$ .

Before we proceed with the proof of this proposition, we shall derive with its help statement (2.93) of the Claim. In fact, statement (2.93) is a particular case (when  $a = b = 0$ ) of the following corollary.

**Corollary 2.2.** *For any  $a, b \geq 0$  we have*

$$\mathbf{E} \frac{a + I^x(i)}{b + I(i)} \mathbf{1} \{b + I(i) > 0\} = (1 + o(1)) \mathbf{E} \frac{a + \lambda_i(x)I(i)}{b + I(i)} \mathbf{1} \{b + I(i) > 0\}, \quad (2.101)$$

where  $o(1) \rightarrow 0$  uniformly in  $i \leq sn^{2/3}$ .

**Proof.** We shall use induction argument. Recall that by the definitions (1.10) and (1.13)

$$I^x(2) = \mathcal{N}_n^x(1) \quad \text{and} \quad I(2) = \sum_{x \in S} \mathcal{N}_n^x(1).$$

Therefore, (2.101) holds for  $i = 2$  simply by (2.99).

Assume now that (2.101) holds for all  $2 \leq i < j < sn^{2/3}$ . We shall deduce that then it holds for  $i = j$  as well.

By definition (1.13) we have

$$\begin{aligned} & \mathbf{E} \frac{a + I^x(j)}{b + I(j)} \mathbf{1} \{b + I(j) > 0\} \\ &= \mathbf{E} \frac{a + I^x(j-1) + \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0} \mathbf{1}_{x(j-1)=x}}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \mathbf{1} \{b + I(j) > 0\} \end{aligned} \quad (2.102)$$

which by (2.99)

$$= (1 + o(1)) \mathbf{E} \frac{a + I^x(j-1) + \lambda_{j-1}(x) \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0} \mathbf{1}_{x(j-1)=x}}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \mathbf{1} \{b + I(j) > 0\}.$$



Now using (1.14) we derive first

$$\begin{aligned} & \mathbf{E} \mathbf{E} \left\{ \frac{\mathbf{1}_{I(j-1)>0} \mathbf{1}_{x(j-1)=x}}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \mathbf{1}\{b + I(j) > 0\} \mid I(j-1), I^x(j-1) \right\} \\ &= \mathbf{E} \frac{\mathbf{1}\{b + I(j) > 0\}}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \mathbf{1}_{I(j-1)>0} \frac{I^x(j-1)}{I(j-1)}. \end{aligned} \quad (2.103)$$

Substituting (2.103) into (2.102), and then using the assumption of the induction, i.e., formula (2.101) for  $i = j - 1$ , we derive

$$\begin{aligned} & \mathbf{E} \frac{a + I^x(j)}{b + I(j)} \mathbf{1}\{b + I(j) > 0\} \\ &= (1 + o(1)) \mathbf{E} \frac{a + I^x(j-1) (1 - \mathbf{1}_{I(j-1)>0}/I(j-1)) + \lambda_{j-1}(x) \sum_{x \in S} \mathcal{N}_n^x(j-1)}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \\ & \quad \times \mathbf{1}\{b + I(j) > 0\} \\ &= (1 + o(1))^2 \mathbf{E} \frac{a + \lambda_{j-1}(x) (I(j-1) - \mathbf{1}_{I(j-1)>0}) + \lambda_{j-1}(x) \sum_{x \in S} \mathcal{N}_n^x(j-1)}{b + I(j-1) + \sum_{x \in S} \mathcal{N}_n^x(j-1) - \mathbf{1}_{I(j-1)>0}} \\ & \quad \times \mathbf{1}\{b + I(j) > 0\} \\ &= (1 + o(1)) \mathbf{E} \frac{a + \lambda_{j-1}(x) I(j)}{b + I(j)} \mathbf{1}\{b + I(j) > 0\}. \end{aligned} \quad (2.104)$$

Recall that

$$\lambda_{j-1}(x) = \frac{xU^x(j-1)}{\sum_{x \in S} xU^x(j-1)} = (1 + o(1)) \frac{xU^x(j)}{\sum_{x \in S} xU^x(j)} = (1 + o(1)) \lambda_j(x),$$

which holds uniformly in  $j \leq sn^{2/3}$  by Proposition 2.2. Hence, we readily get from (2.104) that

$$\mathbf{E} \frac{a + I^x(j)}{b + I(j)} \mathbf{1}\{b + I(j) > 0\} = (1 + o(1)) \mathbf{E} \frac{a + \lambda_j(x) I(j)}{b + I(j)} \mathbf{1}\{b + I(j) > 0\},$$

which confirms (2.101) for  $i = j$ . This completes the proof of the Corollary, and therefore the statement (2.93) follows.  $\square$

**Proof of Proposition 2.6.** We shall start with statement (2.100). Note that it is enough to establish (2.100) for  $i = 1$ , since the difference with the general case is in using  $\frac{xU^x(1)}{\sum_{x \in S} xU^x(1)}$  instead of

$$\frac{xU^x(i)}{\sum_{x \in S} xU^x(i)} = (1 + o(1)) \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)}$$

where  $o(1)$  is bounded and  $o(1) \rightarrow 0$  in  $L_1$  uniformly in  $i \leq sn^{2/3}$  by Proposition 2.2.

We shall explore the following relation between the binomial and the Poisson distributions. Let  $Y_{n,p} \in \text{Bin}(n, p)$  and  $Z_\lambda \in \text{Po}(\lambda)$ . Then it is straightforward to derive from the formulas for the corresponding probabilities that for all  $0 < p < 1$  and  $0 \leq k \leq n$

$$\begin{aligned} \mathbf{P}\{Y_{n,p} = k\} &= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{n!}{n^k(n-k)!} \left( (1-p)e^{\frac{p}{1-p}} \right)^n e^{-n\frac{p}{1-p}} \frac{\left( n\frac{p}{1-p} \right)^k}{k!} \\ &= \frac{n!}{n^k(n-k)!} \left( (1-p)e^{\frac{p}{1-p}} \right)^n \mathbf{P}\{Z_{n\frac{p}{1-p}} = k\}. \end{aligned} \quad (2.105)$$

Conditionally on  $\bar{U}(1)$  and  $x(1)$  introduce independent Poisson random variables

$$Z^x \in \text{Po} \left( (U^x(1) - \mathbf{1}_{x(1)=x}) \frac{p_n(x(1), x)}{1 - p_n(x(1), x)} \right), \quad x \in S. \quad (2.106)$$

With a help of (2.105) we derive for fixed values  $\bar{U}(1), x(1)$

$$\begin{aligned} &\mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \\ &= \sum_{0 \leq k_y \leq U^y, y \in S: \sum_{y \in S} k_y > 0} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \left( (1 - p^y) e^{\frac{p^y}{1-p^y}} \right)^{\hat{U}^y} \mathbf{P}\{Z^y = k_y\}, \end{aligned} \quad (2.107)$$

where to simplify notations we set  $p^y = p_n(x(1), y)$  and  $\hat{U}^y = U^y(1) - \mathbf{1}_{x(1)=y}$ . Consider first

$$C(n, \bar{U}, x(1)) := \prod_{y \in S} \left( (1 - p^y) e^{\frac{p^y}{1-p^y}} \right)^{\hat{U}^y} = \exp \left\{ \sum_{y \in S \setminus \{x(1): \hat{U}^y > 0\}} \hat{U}^y \left( \frac{p^y}{1 - p^y} + \log(1 - p^y) \right) \right\}.$$

Notice here for a further reference, that

$$C(n, \bar{U}, x(1)) = \frac{\mathbf{P} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) = 0 \mid \bar{U}(1), x(1) \right\}}{\mathbf{P} \left\{ \sum_{x \in S} Z^x = 0 \mid \bar{U}(1), x(1) \right\}}. \quad (2.108)$$

Recall that whenever  $\hat{U}^y > 0$  we have  $p^y = o(n^{-1/3})$ . Then

$$\begin{aligned}
C(n, \bar{U}, x(1)) &= \exp \left\{ \sum_{y \in S \setminus x(1): \hat{U}^y > 0} \hat{U}^y \left( \frac{p^y}{1-p^y} - p^y + (p^y)^2 + o((p^y)^2) \right) \right\} \\
&= \exp \left\{ \sum_{y \in S \setminus x(1): \hat{U}^y > 0} \hat{U}^y (2(p^y)^2 + o((p^y)^2)) \right\} \\
&= \exp \left\{ \sum_{y \in S \setminus x(1)} \hat{U}^y \left( 2 \frac{(x(1)y)^2}{n^2} + o((p^y)^2) \right) \right\}.
\end{aligned} \tag{2.109}$$

This together with the assumption  $\mathbf{E}X^3 < \infty$  and  $\max x_i = o(n^{-1/3})$  yields

$$C(n, \bar{U}, x(1)) = 1 + o(1) \tag{2.110}$$

uniformly in  $\bar{U}$  and  $x(1)$ . In particular, this together with (2.108) gives us

$$\mathbf{P} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) = 0 \mid \bar{U}(1), x(1) \right\} = (1 + o(1)) \mathbf{P} \left\{ \sum_{x \in S} Z^x = 0 \mid \bar{U}(1), x(1) \right\}. \tag{2.111}$$

Now we rewrite (2.107) with a help of (2.110) as

$$\begin{aligned}
&\mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \\
&= (1 + o(1)) \sum_{0 \leq k_y \leq U^y, y \in S: \sum_{y \in S} k_y > 0} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \mathbf{P} \{ Z^y = k_y \}.
\end{aligned} \tag{2.112}$$

We shall split the sum on the right into two sums, one of which will give the major contribution while the rest will be a small term. Define sets

$$B_1 := \left\{ \bar{k} = (k_y, y \in S) : \sum_{y \in S} k_y > 0, \right. \tag{2.113}$$

$$\left. k_y = 0 \text{ if } \hat{U}^y = 0, \text{ otherwise } 0 \leq k_y \leq \max \left\{ 1, \frac{\hat{U}^y}{n^{1/2} \log n} \right\} \right\}$$

and

$$B_2 := \left\{ \bar{k} = (k_y, y \in S) : \sum_{y \in S_n} k_y > 0, 0 \leq k_y \leq \hat{U}^y, y \in S \right\} \setminus B_1. \quad (2.114)$$

For all  $\bar{k} \in B_1$  we have

$$\begin{aligned} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} &= \prod_{y \in S: 1 < k_y \leq \frac{\hat{U}^y}{n^{1/2} \log n}} \prod_{i=0}^{k_y-1} \left( 1 - \frac{i}{\hat{U}^y} \right) \\ &= 1 + O \left( \sum_{y \in S: 1 < k_y \leq \frac{\hat{U}^y}{n^{1/2} \log n}} \frac{k_y^2}{\hat{U}^y} \right) = 1 + O \left( \sum_{y \in S} \frac{U^y(1)}{n \log n} \right) = 1 + o(1). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \\ &= (1 + o(1)) \left( \sum_{\bar{k} \in B_1} + \sum_{\bar{k} \in B_2} \right) \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \mathbf{P}\{Z^y = k_y\} \\ &= (1 + o(1))^2 \sum_{\bar{k} \in B_1} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\} \\ &+ (1 + o(1)) \sum_{\bar{k} \in B_2} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \mathbf{P}\{Z^y = k_y\} \\ &= (1 + o(1)) \left( \sum_{\bar{k}: \sum_{y \in S} k_y > 0} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\} \right. \\ &\quad \left. - \sum_{\bar{k} \in \{\bar{k}: \sum_{y \in S} k_y > 0\} \setminus B_1} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\} \right) \\ &+ (1 + o(1)) \sum_{\bar{k} \in B_2} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \mathbf{P}\{Z^y = k_y\}. \end{aligned}$$

Hence,

$$\mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \quad (2.115)$$

$$=: (1 + o(1)) \mathbf{E} \left\{ \frac{Z^x}{\sum_{y \in S} Z^y} \mathbf{I} \left\{ \sum_{y \in S} Z^y > 0 \right\} \mid \bar{U}(1), x(1) \right\} + R_n(x, \bar{U}, x(1)),$$

where

$$\begin{aligned} |R_n(x, \bar{U}, x(1))| &\leq (1 + o(1)) \sum_{\bar{k} \in \{\bar{k} : \sum_{y \in S} k_y > 0\} \setminus B_1} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\} \\ &+ (1 + o(1)) \sum_{\bar{k} \in B_2} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \frac{\hat{U}^y!}{(\hat{U}^y)^{k_y} (\hat{U}^y - k_y)!} \mathbf{P}\{Z^y = k_y\}. \end{aligned}$$

Define a set

$$B_3 := \left\{ \bar{k} = (k_y, y \in S) : k_y > \max \left\{ 1, \frac{\hat{U}^y}{n^{1/2} \log n} \right\} \text{ for some } y \in S \right\},$$

which contains both sets  $B_2$  and  $\{\bar{k} : \sum_{y \in S} k_y > 0\} \setminus B_1$ . Then

$$|R_n(x, \bar{U}, x(1))| \leq 3(1 + o(1)) \sum_{\bar{k} \in B_3} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\}. \quad (2.116)$$

It is straightforward to derive

$$\begin{aligned} &\sum_{\bar{k} \in B_3} \frac{k_x}{\sum_{y \in S} k_y} \prod_{y \in S} \mathbf{P}\{Z^y = k_y\} \\ &\leq \sum_{y \in S} \mathbf{P} \left( \left\{ Z^y > \max \left\{ 1, \frac{\hat{U}^y}{n^{1/2} \log n} \right\} \right\} \cap \{Z^x > 0\} \right) \\ &= \mathbf{P}\{Z^x > 0\} \sum_{x \neq y \in S} \sum_{k = \max \left\{ 1, \frac{\hat{U}^y}{n^{1/2} \log n} \right\} + 1}^{\infty} e^{-\hat{U}^y \frac{p^y}{1-p^y}} \frac{\left( \hat{U}^y \frac{p^y}{1-p^y} \right)^k}{k!} \end{aligned} \quad (2.117)$$

$$\begin{aligned}
& + \sum_{k=\max\left\{1, \frac{\hat{U}^x}{n^{1/2} \log n}\right\}+1}^{\infty} e^{-\hat{U}^x \frac{p^x}{1-p^x}} \frac{\left(\hat{U}^x \frac{p^x}{1-p^x}\right)^k}{k!} \\
& \leq (\mathbf{E}Z^x) \sum_{y \in S} \hat{U}^y \left(\frac{p^y}{1-p^y}\right)^2 n^{1/2} \log n \sum_{k=2}^{\infty} e^{-\hat{U}^y \frac{p^y}{1-p^y}} \frac{\left(\hat{U}^y \frac{p^y}{1-p^y}\right)^{k-2}}{(k-1)!} \\
& + \hat{U}^x \left(\frac{p^x}{1-p^x}\right)^2 n^{1/2} \log n \sum_{k=2}^{\infty} e^{-\hat{U}^x \frac{p^x}{1-p^x}} \frac{\left(\hat{U}^x \frac{p^x}{1-p^x}\right)^{k-2}}{(k-1)!} = o(1)\mathbf{E}Z^x.
\end{aligned}$$

Hence, (2.115) together with (2.116) and (2.117) yields

$$\begin{aligned}
& \mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \tag{2.118} \\
& = (1 + o(1)) \mathbf{E} \left\{ \frac{Z^x}{\sum_{x \in S} Z^x} \mathbf{1} \left\{ \sum_{x \in S} Z^x > 0 \right\} \mid \bar{U}(1), x(1) \right\} + o(1)\mathbf{E}Z^x.
\end{aligned}$$

Using (2.97) we derive from here

$$\begin{aligned}
& \mathbf{E} \left\{ \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \mid \bar{U}(1), x(1) \right\} \tag{2.119} \\
& = \frac{x(U^x(1) - \mathbf{1}_{x(1)=x})}{\sum_{x \in S} \frac{x(U^x(1) - \mathbf{1}_{x(1)=x})}{1-p_n(x(1), x)}} \mathbf{P} \left\{ \sum_{x \in S} Z^x > 0 \mid \bar{U}(1), x(1) \right\} + o(1)(U^x(1) - \mathbf{1}_{x(1)=x}) \frac{p_n(x(1), x)}{1-p_n(x(1), x)} \\
& = \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)} \mathbf{P} \left\{ \sum_{x \in S} Z^x > 0 \mid \bar{U}(1), x(1) \right\} + o(1)x(1) \frac{xU^x(1)}{n}.
\end{aligned}$$

Taking expectation on both sides, we get

$$\begin{aligned}
& \mathbf{E} \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \tag{2.120} \\
& = \mathbf{E} \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)} \mathbf{P} \left\{ \sum_{x \in S} Z^x > 0 \mid \bar{U}(1), x(1) \right\} + R_n(x),
\end{aligned}$$

where we denote  $R_n(x)$  the remaining term, which satisfies

$$|R_n(x)| \leq o(1) \mathbf{E}x(1) \frac{xU^x(1)}{n} \leq o(1) \mathbf{E} \frac{xU^x(1)}{n}. \quad (2.121)$$

Recalling also the relation (2.111), we obtain from (2.120)

$$\begin{aligned} & \mathbf{E} \frac{\mathcal{N}_n^x(1)}{\sum_{x \in S} \mathcal{N}_n^x(1)} \mathbf{P} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\} \\ &= (1 + o(1)) \mathbf{E} \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)} \mathbf{E} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \mid \bar{U}(1), x(1) \right\} + o(1) \mathbf{E} \frac{xU^x(1)}{n} \\ &= (1 + o(1)) \mathbf{E} \frac{xU^x(1)}{\sum_{x \in S} xU^x(1)} \mathbf{1} \left\{ \sum_{x \in S} \mathcal{N}_n^x(1) > 0 \right\}. \end{aligned} \quad (2.122)$$

This completes the proof of (2.100).

Statement (2.99) can be proved in the same straightforward (but lengthy) fashion. We shall omit it here for the sake of brevity. This finishes the proof of Proposition 2.6 and therefore Lemma 2.1 is proved.  $\square$

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**Remark.** After this work was completed the author became aware of study of a similar problem by Bhamidi, van der Hofstad and van Leeuwaarden (see [3]).

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