THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

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T. Seierstad

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Finite-dimensional distributions for the size of the core in random graphs

Taral Guldahl Seierstad*
Department of Biostatistics
Institute of Basic Medical Sciences
University of Oslo

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Abstract

In the Erdős–Rényi random graph process it is known that the size of the core at any given time in the supercritical phase converges to a normal distribution as the number of vertices tends to infinity. Proofs of this fact usually study the properties of the random graph at a fixed time. We instead use a dynamic viewpoint, studying how the core evolves as more edges are added to the graph, and show that if we measure of the core at any finite number of times, the size of the core at these times converge jointly to a multivariate normal distribution. This suggests the conjecture that the order of the core, properly normalized, converges to a Gaussian process.

MSC2000: 05C80, 60F05

Key words: Random graph process, core, limit theorem, differential equations

1 Introduction

We consider the random graph process, first studied by Erdős and Rényi, which begins with an empty graph on n vertices, and where edges are added randomly one by one. It was shown by Erdős and Rényi [2] that this graph process undergoes a phase transition when the number of edges $m = m(n)$ is roughly $n/2$. Before the phase transition the graph consists of relatively small components; after the phase transition there is one large component, dubbed the *giant component*, which is much larger than every other component.

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The properties of the giant component have been very well studied, and we now have detailed knowledge about its evolution. For example, Pittel [6] showed that the order of the giant component is asymptotically normally distributed. We will concern ourselves with the *2-core* of the graph after the phase transition. The 2-core, or simply the *core*, of a graph is the maximal subgraph with minimum degree 2. The giant component then consists of a core, plus a *mantle*, consisting of trees rooted at the core. Pittel [6] found the order of the core up to an error term of $o(n)$. It was shown by Pittel and Wormald [7] that the order of the core, properly normalized, also converges to a Gaussian distribution. Janson and Luczak [4] proved a similar theorem for a related random graph model, namely random graphs with a given degree sequence.

In this paper we consider the evolution of the core as the random graph process progresses by adding more and more edges. We prove that if we consider the process at several different points in time, the order of the core — again properly normalized — at these times converges jointly to normal distributions. In view of this result, it is natural to conjecture that the order of the core in fact converges to a Gaussian process. However, proving tightness, which would be sufficient to prove this conjecture, appears to be tricky, so we will be content with proving convergence for the finite-dimensional distributions.

Our methods are different from [7] and [4]: we make a detailed study of how the core acquires new vertices at every step of the graph process, while [7] and [4] make no such attempt, and rather study the structure at one given point in time. Note, however, that we shall require the result in [7] as a basis for our proof, so we do not provide a new method for proving the normality of the core; rather, we show that once we know that the order of the core is normal at one given point in time after the phase transition, it will remain normal at all subsequent times, and in fact that the finite-dimensional distributions converge to a multivariate normal distribution.

In the supercritical phase, the random graph $G(n, m)$ is, like Gaul, divided into three parts: the core, the mantle and the wilderness outside, which consists of a forest and a small number of unicyclic components. We will determine the size of the core, by determining all types of trees in the forest, and all types of rooted trees in the mantle. Since every vertex which is not in the mantle and not in the forest is either in the core or in one of the negligible number of unicyclic components outside the giant component, this will give us the size of the core.

To this end, we will employ the so-called differential equation method. Wormald [11] showed that many random variables defined on a random discrete graph process converge in probability to the solution of certain differential equations, provided that a number of conditions are satisfied. Later, Seierstad [8] showed that if some further conditions are satisfied, the random variables in question actually converge in distribution to a Gaussian distribution. This was used in [9] to show that the order of the giant component is normally distributed, in a general family of random graph processes. In Section 2 we will extend this to show that the random variables actually converge to a Gaussian process. This theorem will then be used to show the following theorem regarding the core in $G(n,m)$.

Theorem 1. *Let $C_{n,m}$ be the random variable denoting the order of the core in $G_{n,m}$. For $c > 1$, let $d = d(c) \in (0,1)$ such that $de^{-d} = ce^{-c}$ and let $\mu(c) = (1-d)(1-d/c)$. Moreover let*

$$\sigma(c) = \frac{d(c-d)(-c^2d + c + 4c^2d^2 - 6cd^3 - 4cd + c^2 - 4cd^2 + 2d^4 - c^3d)}{c^3(1-d)^2}.$$

Fix $c_1, \ldots, c_k$ such that $1 < c_i$ for $i = 1, \ldots, k$, and let $\Sigma = \{\sigma(\min(c_i, c_j))\}_{i,j=1}^k$. If we write $C_n(c) = C_{n,\lfloor cn \rfloor}$ and let $\boldsymbol{C} = [C_n(c_1), \ldots, C_n(c_k)]'$, then $\boldsymbol{C} \xrightarrow{d} \mathcal{N}(0, \Sigma)$.

The value of $\mu(c)$ was found by Pittel [6], and $\sigma(c)$ was found by Pittel and Wormald [7]. The method we use unfortunately does not offer a simple way to calculate $\mu(c)$ and $\sigma(c)$, and we will therefore make no attempt to recalculate these values.

2 The differential equation method

The differential equation method is a technique which can be used to establish convergence of given parameters in a family of random variables by showing that certain, relatively simple, criteria are satisfied. Wormald [11] found criteria which ensure that the given parameters converge in probability to the solution of a system of differential equations. Seierstad [8] used a central limit theorem for near-martingales in McLeish [5] to prove that if further criteria are satisfied, the given parameters converge in distribution to a multivariate normal distribution. Here we will use a stronger theorem from [5] to extend the theorem and show that the parameters actually converge to a Gaussian process.

Theorem 2. *Assume that there is a constant C_0 such that $X_{n,m,k} \leq C_0 n$ for all n, $0 \leq m \leq m_n$ and $1 \leq k \leq q$. Let $f_k, g_{ij} : \mathbb{R}^q \to \mathbb{R}$, $1 \leq i,j,k \leq q$ be functions. Assume that there is a constant vector $\boldsymbol{\mu}_0$ and a constant $q \times q$-matrix Σ_0 such that*

$$\frac{\boldsymbol{X}_0 - n\boldsymbol{\mu}_0}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \Sigma_0).$$

Assume that the following conditions hold.

(i) For some function $\beta = \beta(n)$ with $1 \leq \beta = o(n^{1/12-\varepsilon})$ for some $\varepsilon > 0$, we have $\Delta X_{n,m,k} \leq \beta$ for $1 \leq m \leq m_n$, where $m_n = O(n)$, and $1 \leq k \leq q$.

(ii) For some function $\lambda_1(n) = o(n^{-1/2})$ and all $i \geq 1$,

$$\left|\mathbb{E}[\Delta X_{n,m,i} \mid \mathcal{F}_{m-1}] - f_i(n^{-1}X_{n,m,1}, \ldots, n^{-1}X_{n,m,q})\right| \leq \lambda_1(n).$$

(iii) *For some function* $\lambda_2(n) = o(1)$ *and* $i, j \geq 1$*,*

$$\left|\mathbb{E}[\Delta X_{n,m,i}\Delta X_{n,m,j} \mid \mathcal{F}_{m-1}] - g_{ij}(n^{-1}X_{n,m,1}, \ldots, n^{-1}X_{n,m,q})\right| \leq \lambda_2(n).$$

(iv) *Each function* f_i *is continuous and differentiable, with continuous derivatives, and satisfies a Lipschitz condition on* D*.*

(v) *Each function* g_{ij} *is continuous and satisfies a Lipschitz condition on* D*.*

Then the following is true.

a. *Let* $\boldsymbol{F} : \mathbb{R}^q \to \mathbb{R}^q$ *be the vector-valued function whose* k*th component is* f_k*. There is a unique vector-valued function* $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^q$ *satisfying the differential equation*

$$\frac{d}{dt}\boldsymbol{\alpha}(t) = \boldsymbol{F}(\boldsymbol{\alpha}(t))$$

with $\boldsymbol{\alpha}(0) = \boldsymbol{\mu}_0$*.*

b. *Let* $J(\boldsymbol{z})$ *be the Jacobian matrix of* $\boldsymbol{F}$*, and let* $T(t)$ *be the* $q \times q$*-matrix satisfying the differential equation*

$$\frac{d}{dt}T(t) = -T(t)J(\boldsymbol{\alpha}(t)).$$

Furthermore, let $\boldsymbol{G} : \mathbb{R}^{q\times q} \to \mathbb{R}^q$ *be the function* $\boldsymbol{G}(\boldsymbol{z}) = \{g_{ij}(\boldsymbol{z})\}_{i,j=1}^q$ *and define*

$$\Xi(t) = \Sigma_0 + \int_0^t T(s)(\boldsymbol{G}(\boldsymbol{\alpha}(s)) - \boldsymbol{F}(\boldsymbol{\alpha}(t))\boldsymbol{F}(\boldsymbol{\alpha}(t))')T(s)'ds$$

and $\Sigma(t) = T(t)^{-1}\Xi(t)(T(t)')^{-1}$*. Let* $\boldsymbol{Y}_n(t) = n^{-1/2}(\boldsymbol{X}_{n,\lfloor tn\rfloor} - n\boldsymbol{\mu}(t))$ *and let* $\boldsymbol{W}(t)$ *be a* q*-dimensional Gaussian process with* $\mathbb{E}\boldsymbol{W}(t) = 0$ *and covariances* $\mathbb{E}[\boldsymbol{W}(t)\boldsymbol{W}'(u)] = \Sigma(t)$ *for* $0 \leq t \leq u$*. Then* $\boldsymbol{Y}_n \xrightarrow{d} \boldsymbol{W}$ *on* $D[0,\infty)$ *as* $n \to \infty$*.*

The conditions of this theorem are not stronger than those in the main theorem of [8]. The proof is found in Section 6.

3 Outline of the proof

As we mentioned in the Introduction, we shall partition the graph $G(n, m)$ into the core and the mantle of the giant component, and the forest outside of the giant. Let us for the moment forget about the components containing cycles which lie outside the giant. If v is a vertex in the core, we define *the branch rooted at* v to be the maximal connected subgraph of $G(n, m)$ which contains v, but does not contain any edges from the core. If v does not have any neighbours outside the core, v itself constitutes the entire branch rooted at v, which therefore has order 1. With this definition, the order of the core is equal to the number of branches in the mantle.

We say that every vertex and edge which is contained in the giant, but not in the core, is in the mantle. Thus, the core and the mantle constitute a partition of the giant component. Note, however, even though we talk about the branches in the mantle, the branches are not entirely contained in the mantle, since every branch contains exactly one vertex in the core, namely the root.

In order to prove that the number of branches is asymptotically Gaussian, we will use the differential equation method. It turns out that in order to control the number of branches, we have to control a number of other random variables, namely the number of trees of different types in the forest, and the number of branches of different types in the mantle. Thus, we will introduce a sequence of random variables $X_\bullet$ counting the trees, and a sequence $Y_\bullet$ counting the branches, and we will show that these random variables satisfy the conditions of Theorem 2.

The first question we must settle is what exactly the random variables $X_\bullet$ and $Y_\bullet$ should count. The important thing is that they are defined such that whenever an edge is added to the graph process, the expected change in each of the random variables can be expressed as a function of the values in the previous state.

The first thought may be to let there be variables X_k and Y_k counting the number of trees and branches of order k, respectively. But it quickly turns out that knowing the values of these random variables is not sufficient to express the expected change in the same random variables. For example, when an edge is added between two rooted trees in the mantle, some of the vertices in the rooted trees will be included in the core, while the remaining vertices will remain as part of smaller rooted trees in the mantle. But what kind of rooted trees are formed does not only depend on the size of the original rooted trees, but in fact on their isomorphism classes. Thus, we need to control a much larger number of variables.

For the components outside the giant we need one random variable for each isomorphism class of trees. Let $\mathcal{T}$ be the set of all trees. For every tree $T \in \mathcal{T}$, let X_T be the random variable denoting the number of trees outside the giant which are isomorphic to T.

For the branches in the mantle, we need even more random variables. It is not sufficient to control the number of rooted trees in every isomorphism class. Consider for example branches of order 1 — rooted trees consisting of a single root — and let R be an arbitrary rooted tree in the mantle. Regardless of the isomorphism class of R, if we add an edge between a leaf of R and another vertex in the core, the number of branches of order 1 will change. Thus, if $Y_{1,m}$ is the number of branches of order 1 in $G_{n,m}$, if we naïvely write an expression for the expected value of $\Delta Y_{1,m}$ we will get a sum with an infinite number of summands, one for every isomorphism class of rooted trees.

To get around this obstacle, we will associate to every vertex v in the giant of $G_{n,m}$, a parameter $r_{v,m}$, which we shall call the *recollection parameter*. The intuitive explanation of the recollection parameter is the following: whenever two vertices find themselves in the same branch, they brag to each other about the largest branch they have ever been part of themselves or heard about from other vertices. The technical definition is the following: We will consider a starting value m_0 such that $c_0 = m_0/n$ satisfied $1 < c_0 < c_i$ for all $i = 1, \dots, k$, where the c_i are as in Theorem 1. If v is in the giant of $G_{n,m}$, we let $s_{v,m}$ be the number of vertices in the branch containing v. In the initial graph, G_{n,m_0}, we let the recollection parameter equal the size of the rooted tree containing v; thus $r_{v,m_0} = s_{v,m_0}$. If v is a vertex in the core, we let $R_{v,m}$ be the branch in the mantle in $G_{n,m}$ which is rooted at v. Assume now that $m > m_0$, and let $v_m w_m$ be the mth edge added to the graph. If v_m and w_m both belong to the giant component in $G_{n,m-1}$, we let $r_{x,m} = \max\{r_{v_m,m-1}, r_{w_m,m-1}\}$ for every vertex x in $R_{v_m,m-1}$ and $R_{w_m,m-1}$. If one of the vertices, say v_m, belongs to the giant in $G_{n,m-1}$, and the other one does not, we let $r_{x,m} = \max\{r_{v_m,m-1}, s_{v_m,m}\}$ for all $x \in R_{v_m,m}$. Whenever, for some vertex x, the recollection parameter is not defined by these rules, we let the recollection parameter remain unchanged: that is $r_{x,m} = r_{x,m-1}$. In particular, this is the case if both v_m and w_m belong to the forest in $G_{n,m-1}$.Vertices outside the giant are not assigned a recollection parameter.

Two important properties which are satisfied by the recollection parameter is that it is non-decreasing for any given vertex in the giant, and that for any rooted tree in the mantle of $G_{n,m}$, all the vertices have the same recollection parameter.

Let $\mathcal{R}$ be the set of all isomorphism classes of rooted trees. For every natural number r and rooted tree $R \in \mathcal{R}$, let $Y_{R,r,m}$ be the number of branches in the mantle of $G_{n,m}$ which are isomorphic to R and whose vertices have recollection parameter r. It is now possible to express the order of the core $C_{n,m}$ in $G_{n,m}$ as

$$C_{n,m} = \sum_{R \in \mathcal{R}} \sum_{r=1}^{\infty} Y_{R,r,m}. \tag{1}$$

We will now state a series of lemmas which lead up to the proof of Theorem 1. We will say for short that a set of random processes $V_{1,m}, \dots, V_{q,m}$ for $m \geq 0$ are *asymptotically Gaussian* on an interval J if the normalized versions of the random variables converge to a Gaussian process; that is, there is a q-dimensional vector-valued function $\boldsymbol{\mu}(t)$ such that if $\boldsymbol{V}_m = [V_{1,m}, \dots, V_{q,m}]'$, then $n^{-1/2}(\boldsymbol{V}_{\lfloor tn \rfloor} - n\boldsymbol{\mu}(t))$ converges to a Gaussian process on $D(J)$, which is the interval J endowed with the Skorokhod topology. Similarly, we will say that a set of random variables $V_1, \dots, V_q$ is *asymptotically normal* if the normalized versions converge to a multivariate normal distribution, that is, if $n^{-1/2}(\boldsymbol{V} - n\boldsymbol{\mu}) \xrightarrow{d} \mathcal{N}(0, \Sigma)$, where $\boldsymbol{V} = [V_1, \dots, V_q]'$, $\boldsymbol{\mu}$ is a q-dimensional vector and Σ is a $q \times q$-matrix.

Lemma 3. *Let $\mathcal{T}^\circ$ be a finite subset of $\mathcal{T}$. The set of random variables $\{X_T \mid T \in \mathcal{T}^\circ\}$ is asymptotically Gaussian on $[0, \infty)$.*

The next step is to show the same for the branches in the mantle. In this case we cannot consider the whole time interval $[0, \infty)$, since there is an abrupt change when $c = 1$ which the differential equation method cannot handle. Instead we use the time $c_0 > 1$ as a starting point and rely on Pittel and Wormald [7] to show that the mantle behaves nicely at this point. The following lemma is proved using one of the main results in [7].

Lemma 4. *Let $\mathcal{R}^\circ$ be a finite subset of $\mathcal{R}$. For any $c > 1$, $\{Y_R | R \in \mathcal{R}^\circ\}$ is asymptotically normal.*

We can then use the differential equation method to show that the rooted trees are also asymptotically Gaussian.

Lemma 5. *Let $\mathcal{R}^\circ$ be a finite subset of $\mathcal{R} \times \mathbb{N}$. The set of random variables $\{Y_{R,r,m/n} \mid (R,r) \in \mathcal{R}^\circ\}$ is asymptotically Gaussian on $[c_0, \infty)$ for any $c_0 > 1$.*

Of course, what we are interested in is the sum of all the random variables $Y_{R,r,m}$, so that we can calculate the expression for $C_{n,m}$ in (1). We will add them together in two steps, first by isomorphism class, and then by recollection parameter. Thus, let $Z_{r,m} = \sum_{R \in \mathcal{R}} Y_{R,r,m}$ be the number of branches whose vertices have recollection parameter r. If v is a vertex with recollection parameter r, then by definition it must be in a branch of order at most r. Since there is only a finite number of rooted trees with order at most r, for any fixed r all but a finite number of random variables $Y_{R,r,m}$ are zero, so the sums are well-defined. Then it immediately follows from Lemma 5 that any finite number of these random variables are also asymptotically Gaussian:

Lemma 6. *For any $q \geq 1$, the random vector $[Z_{1,tn}, \dots, Z_{q,tn}]$ is asympotically Gaussian on $[c_0, \infty)$ for any $c_0 > 1$.*

Lemma 7. *There are constants K, γ and ρ, depending only on $c = m/n$ such that*

$$\mathbb{E}[Z_{r,m}] \leq K r^\gamma \rho^r$$

and

$$\operatorname{Var} Z_{r,m} \leq K r^\gamma \rho^r.$$

From this we may then prove Theorem 1.

4 Rooted forest

Since the proofs of Lemmas 3 and 5 are thematically similar, in that they use the differential equation method, we treat them together in the next section, and instead we begin with the proof of Lemma 4, which is based on Pittel and Wormald [7].

We first prove a simple lemma stating that the sum of a random, but normally distributed, number of binomial random variables converges to a normal distribution.

Lemma 8. *Let $X_1, X_2, \dots$ be independent random variables taking values in $\{0,1\}$ such that $\mathbb{P}[X_i = 1] = p$ for all $i \geq 1$. Let $S_n = \sum_{i=1}^{n} X_i$. Let N_n be a random variable such that*

$$n^{-1/2}(N_n - \mu n) \xrightarrow{d} \mathcal{N}(0, \sigma). \tag{2}$$

Then $n^{-1/2}(S_{N_n} - p\mu n) \xrightarrow{d} \mathcal{N}(0, p^2\sigma^2 + p(1-p)\mu)$. Moreover, N_n and S_{N_n} converge jointly.

Proof. Let $M_n = n^{-1/2}(N_n - \mu n)$ and $Z_m = n^{-1/2}(S_m - p\mu n)$. It is sufficient to show that $aZ_{N_n} + bM_n$ is asymptotically normal with the correct variance for all choices of a and b. By assumption $\mathbb{E}[e^{itM_n}] \to e^{-\sigma^2 t^2/2}$ as $n \to \infty$. Moreover,

$$\mathbb{E}[e^{itX_i}] = pe^{it} + (1-p) = 1 + ipt - \frac{pt^2}{2} + O(t^3)$$

when $t \to 0$. For simplicity, we may write N for N_n. Then, using that $\ln(1+t) = t - t^2/2 + o(t^2)$ as $t \to 0$,

$$\begin{aligned}
\mathbb{E}[e^{it(aZ_N + bM_n)}] &= \mathbb{E}[e^{ait(S_N - p\mu n)/\sqrt{n} + bit(N_n - \mu n)/\sqrt{n}}] \\
&= \mathbb{E}[e^{aitS_N/\sqrt{n} - aitp\mu\sqrt{n} + bitN_n/\sqrt{n} - bit\mu\sqrt{n}}] \\
&= \mathbb{E}[e^{it\sum_1^N (aX_i + b)/\sqrt{n}}]e^{-(ap+b)it\mu\sqrt{n}} \\
&= \mathbb{E}\left[\prod_{i=1}^{N} e^{it(aX_i+b)/\sqrt{n}}\right] e^{-(ap+b)it\mu\sqrt{n}} = \mathbb{E}[\mathbb{E}[e^{it(aX_i+b)/\sqrt{n}}]^N]e^{-(ap+b)it\mu\sqrt{n}} \\
&= \mathbb{E}\left[\left(1 + \frac{it(ap+b)}{\sqrt{n}} - \frac{t^2(a^2p(1-p) + (ap+b)^2)}{2n} + o(n^{-1})\right)^N\right] e^{-(ap+b)it\mu\sqrt{n}} \\
&= \mathbb{E}[e^{(M_n\sqrt{n} + \mu n)\left(\frac{it(ap+b)}{\sqrt{n}} - \frac{t^2a^2p(1-p)}{2n} + o(n^{-1})\right)}]e^{-(ap+b)it\mu\sqrt{n}} \\
&= \mathbb{E}[e^{M_n(it(ap+b) + O(n^{-1/2}))}]e^{-\mu a^2 p(1-p)t^2/2} \\
&\to e^{-(\sigma^2(ap+b)^2 + \mu a^2 p(1-p))t^2/2}.
\end{aligned}$$

□

Then we show a similar lemma for multinomial random variables.

Lemma 9. *For a fixed k, let $0 \leq p_i \leq 1$ for $1 \leq i \leq k$ and $\sum_i p_i = 1$. Let N_n be a random variable such that $n^{-1/2}(N_n - \mu n) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$. Let $X_1, \dots, X_{N_n}$ be independent random variables in $\{1, \dots, k\}$ such that $X_i = j$ with probability p_j for all i and j. Then, for some matrix Σ,*

$$n^{-1/2}(\boldsymbol{X} - \mu\boldsymbol{p}) \xrightarrow{d} \mathcal{N}(0, \Sigma).$$

Proof. By Lemma 8, X_i is asymptotically normal for all $1 \leq i \leq k$, so it only remains to show that the convergence is joint. First assume that $k = 2$. By Lemma 8, X_1 and N_n are asymptotically jointly normal. If a and b are real numbers and $c = a - b$, then $aX_1 + bX_2 = (a-b)X_1 + bN_n$ is asymptotically normal, so X_1 and X_2 converge jointly. For $k > 2$ the lemma follows by induction. □

In order to complete the proof of Lemma 4, we shall need two theorems, one regarding the distribution of trees in a random forest proved by Janson [3], and the other being one of the main theorems in Pittel and Wormald [7].

Lemma 10 (Example 3.3 of Janson [3])**.** *Let $F_{n,m}$ be a random forest with $n + m$ vertices, with n roots and m non-roots. Let U_k be the number of trees with k vertices. Then, for any fixed l, the random variables $U_1, \dots, U_l$ are asymptotically jointly normal.*

Lemma 11 (Theorem 6 of Pittel and Wormald [7])**.** *Let $c > 1$ and $m = \lfloor cn \rfloor$. Let $Y_{n,m,1}$ be the order of the core and $Y_{n,m,2}$ be the order of the mantle in $G_{n,m}$. Then the vector $(Y_{n,m,1}, Y_{n,m,2})$ is asymptotically normal.*

Proof of Lemma 4. By Lemma 11 the number of vertices in the core and the number of vertices in the mantle are asymptotically jointly normal. Let us assume that the core contains C vertices, and the mantle M vertices. If we take the giant component and remove all the edges in the core, we are left with a forest, where every tree contains exactly one vertex from the core, and where the total number of vertices is $C + M$. We consider every tree to be rooted at the core vertex, and thus obtain a rooted forest. Every possible rooted forest with C roots and M non-roots is equally likely, and the forest is therefore a random forest with the uniform distribution. By Lemma 10, the random variables $Y_1, \dots, Y_q$ are therefore asymptotically jointly normal.

Consider a rooted tree of order k in the core. Clearly, every rooted tree on k vertices is equally likely. For i with $1 \leq i \leq Y_k$, let $V_{R,i} = 1$ if the ith tree is isomorphic to R and $V_{R,i} = 0$ otherwise. Then $Y_R = \sum_{i=1}^{Y_k} V_{R,i}$. By Lemma 9, the random variables $\{Y_R\}_{R \in \mathcal{R}_k}$ are therefore asymptotically jointly normal. Moreover, when conditioned on Y_k, Y_R is independent of Y'_R whenver R' does not have k vertices, so we conclude that $\{Y_R\}_{R \in \mathcal{R}_{\leq q}}$ are asymptotically jointly normal for any fixed q. □

5 Counting trees

We now turn to Lemmas 3 and 5, which we will prove using the differential equation method. Thus, we need to know how the random variables X_T and $Y_{R,r}$ change as new edges are added in the graph process. Assume that we are in the supercritical phase such that a giant component with a core already exists. When an edge is added to the graph, we classify the edge according to one of the following types of edges.

1. The endpoints belong to distinct trees outside the giant.
2. One endpoint belongs to the giant, while the other belongs to a tree outside the giant.
3. Both endpoints belong to the giant, but are located in distinct rooted trees.
4. None of the above.

The fourth item in the list refers to events with relatively small probability; namely, one of the endpoints may belong to a component outside the largest component, which is not a tree, or both vertices may belong to the same tree outside the giant or the same branch in the giant.

Let $\mathcal{T}^* = \{(T,z); T \in \mathcal{T}, z \in T\}$ be the set of trees with one distinguished vertex. (There is, of course, an obvious isomorphism between $\mathcal{T}^*$ and $\mathcal{R}$, but for our purposes it is convenient to consider these as separate sets.) Similarly we let $\mathcal{R}^* = \{(R,r,z); R \in \mathcal{R}, r \in \mathbb{N}, z \in R\}$ be the set of rooted trees with one distinguished vertex, which may or may not be the root.

Let $\mathcal{E} = (\mathcal{T}^* \cup \mathcal{R}^*) \times (\mathcal{T}^* \cup \mathcal{R}^*)$. We will call the elements of $\mathcal{E}$ *events.* Whenever we add an edge (v,w) that can be classified by one of the types 1–3, it corresponds to an element $(\boldsymbol{a},\boldsymbol{b})$ in $\mathcal{E}$, where $\boldsymbol{a}$ encodes the information about v, and $\boldsymbol{b}$ encodes the information about w. If $\boldsymbol{a} = (T,z)$ with $z \in T \in \mathcal{T}$, it means that v is contained in a component isomorphic to T, and occupies the spot in T^* which is occupied by z in T. Similarly, if $\boldsymbol{a} = (R,r,z)$ with $z \in R \in \mathcal{R}$, it means that v is contained in the mantle of the giant component, in a rooted tree R^* isomorphic to R with recollection parameter r, and occupies the same spot in R^* as z occupies in R.

As we will observe in the proofs of Lemmas 3 and 5, at step m every event $E \in \mathcal{E}$ occurs with a probability expressible up to a small error term by functions of a finite number of the random variables $n^{-1}X_{T,m-1}$ and $n^{-1}Y_{R,r,m-1}$, and thus we will be able to use the differential equation method.

Proof of Lemma 3. We will use the notation and terminology which was introduced above. Let u be any vertex in G, and suppose it has degree d. The probability that $v_m = u$ is then $\frac{1}{2}\frac{n-1-d}{\binom{n}{2}-m+1} = n^{-1} + O(n^{-2})$. The same holds for the probability that $w_m = u$. As we see, this probability depends on d, but the dependence is negligible, so the probability that a particular vertex is chosen at the next step is virtually the same for all vertices. Thus, the probability that $\boldsymbol{a} = (T,z)$ is equal to $\frac{X_{m,T}}{n}(1 + O(1/n))$. Likewise, the probability that $\boldsymbol{b} = (T',z')$ equals $\frac{X_{m,T'}}{n}(1 + O(1/n))$. Moreover, if $\boldsymbol{a} = (T,z)$ and $\boldsymbol{b} = (T',z')$ and v and w do not belong to the same component in $G_{n,m-1}$, then the tree containing v and the tree containing w merge to form a new tree, whose isomorphism class is uniquely determined by $\boldsymbol{a}$ and $\boldsymbol{b}$.

We define a binary operation $\oplus : \mathcal{T}^{*2} \to \mathcal{T}$ such that $(T_1,z_1) \oplus (T_2,z_2)$ is the tree obtained by joining T_1 to T_2 by an edge going from z_1 in T_1 to z_2 in T_2. Moreover, for $T \in \mathcal{T}$, let

$$\mathcal{A}_T = \{((T_1,z_1),(T_2,z_2)) \in \mathcal{T}^{*2} \mid T = (T_1,z_1) \oplus (T_2,z_2)\}.$$

Then

$$\mathbb{E}[\Delta X_{m,T} \mid \mathcal{F}_{m-1}] = -\frac{2|T|X_{m-1,T}}{n} + \sum_{\mathcal{A}_T} \frac{X_{m,T_1}}{n}\frac{X_{m,T_2}}{n} + o(n^{-1/2}).$$

Here $|T|$ denotes the number of vertices in T. The first term in this expression accounts for trees lost because v_m or w_m is contained in a tree isomorphic to T. The terms inside the sum account for trees gained because smaller trees merge to form a tree isomorphic to T. The error term contains discrepancies due to the difference in the degrees of different vertices, and to the event that v_m and w_m belong to the same tree component isomorphic to T.

Next we have to find an expression for $\mathbb{E}[\Delta_{T_1,T_2} \mid \mathcal{F}_{m-1}]$, where $\Delta_{T_1,T_2} = \Delta X_{m,T_1} \Delta X_{m,T_2}$. Let us first assume that T_1 and T_2 are not isomorphic. We can also assume without loss of generality that $|T_1| \geq |T_2|$. In order forΔ_{T_1,T_2} to be nonzero, both the number of trees of type T_1 and T_2 must clearly change. This can only happen in one of the following ways.

1. $T_v \simeq T_1$ and $T_w \simeq T_2$ or vice versa. Then $\Delta_{T_1 T_2} = (-1)^2 = 1$; this happens with probability $2\frac{|T_1|X_{m-1,T_1}}{n}\frac{|T_2|X_{m-1,T_2}}{n} + O(n^{-1})$.
2. $T_v \simeq T_2$ and $x \oplus y = T_1$, or vice versa. Then X_{T_1} increases by 1, while X_{T_2} decreases by 1, so $\Delta_{T_1 T_2} = (+1)(-1) = -1$. Let $\mathcal{B}_{T_1 T_2} = \{(T_2, z), (T', z') \mid T_1 = (T_2, z) \oplus (T', z')\}$. Then the probability of this event is $2\sum_{\mathcal{B}_{T_1 T_2}} \frac{X_{m-1,T_2}}{n}\frac{X_{m-1,T'}}{n} + O(n^{-1})$.
3. $x = (T_2, z_1)$ and $y = (T_2, z_2)$ and $x \oplus y = T_1$. Then $\Delta_{T_1 T_2} = (+1)(-2) = -2$. Let $\mathcal{C} \subset V(T_2)^2$ be the set of pairs (z_1, z_2) of vertices in T_2 such that $(T_2, z_1) \oplus (T_2, z_2) \simeq T_1$. Then the probability of this event is $\sum_{\mathcal{C}} \frac{X_{m-1,T_2}}{n}\frac{X_{m-1,T_2}}{n} + O(n^{-1})$.

Thus

$$\begin{aligned} \mathbb{E}[\Delta X_{m,T_1} \Delta X_{m,T_2} \mid \mathcal{F}_{m-1}] &= 2\frac{|T_1|X_{m-1,T_1}}{n}\frac{|T_2|X_{m-1,T_2}}{n} \\ -2\sum_{\mathcal{B}_{T_1 T_2}} \frac{X_{m-1,T_1}}{n}\frac{X_{m-1,T_2}}{n} &- 2\sum_{\mathcal{C}} \frac{X_{m-1,T_2}}{n}\frac{X_{m-1,T_2}}{n} + o(n^{-1/2}). \end{aligned} \tag{3}$$

The situation is slightly different when $T_1 \simeq T_2 \simeq T$, but by enumerating the possibilities as above, we arrive at the formula

$$\begin{aligned} \mathbb{E}[(\Delta X_{m,T})^2 \mid \mathcal{F}_{m-1}] = 4\left(\frac{|T|X_{m-1,T}}{n}\right)^2 &+ 2\frac{|T|X_{m-1,T}}{n}\left(1 - \frac{|T|X_{m-1,T}}{n}\right) \\ &+ \sum_{\mathcal{A}_T} \frac{X_{m-1,T'}}{n}\frac{X_{m-1,T''}}{n} + o(n^{-1/2}). \end{aligned} \tag{4}$$

The first term accounts for the case that both v and w is in a tree of type T. The second term accounts for the case that one of the vertices is in a T-tree, while the other is not. The sum accounts for the cases that T_v and T_w together form a T-tree.

It is important to note that both in (3) and (4) the number of summands on the right hand side is finite. Thus, we have found functions f_T expressing the value of $\mathbb{E}[\Delta X_{m,T} \mid \mathcal{F}_{m-1}]$ and functions $g_{T_1 T_2}$ expressing the value of $\mathbb{E}[\Delta X_{m,T_1} \Delta X_{m,T_2} \mid$

$\mathcal{F}_{m-1}]$, up to an error term. We let D be the set of points for which all coordinates are between ε and $1+\varepsilon$. Then clearly f_T and $g_{T_1T_2}$ satisfy the requirements of Theorem 2 on D, and we can conclude that the lemma follows from Theorem 2. $\square$

We now turn our attention to the rooted trees in the mantle.

Proof of Lemma 5. Again we use the differential equation method to show that the random variables are asymptotically Gaussian, with Lemma 4 providing a normally distributed starting point. Fix some $q \geq 1$. Let $\mathcal{R}^\circ = \{(R,r) \in \mathcal{R} \times \mathbb{N} \mid |R| \leq q, r \leq q\}$. Clearly it is sufficient to show that the lemma holds for such sets. Moreover, let $\mathcal{T}^\circ = \{T \mid |T| \leq q\}$.

Fix $c_0 > 1$. By Lemma 3 the random variables $X_{\mathcal{T}^\circ}(c_0)$ converge to a multivariate normal distribution as $n \to \infty$. By Lemma 4 the random variables $Y_{\mathcal{R}^\circ}(c_0)$ also converge to a multivariate normal distribution as $n \to \infty$. We can therefore use c_0 as a starting point for the differential equation method.

In order to apply Theorem 2, we must, for any $T \in \mathcal{T}^\circ$ and any $(R,r),(R',r') \in \mathcal{R}^\circ$, find expressions for the expectation of $\Delta Y_{R,r,m}$, $\Delta Y_{R,r,m}\Delta Y_{R',r',m}$ and $\Delta X_{T,m}\Delta Y_{R,r,m}$, which satisfy the conditions of the theorem. (Expressions for $\Delta X_{T,m}$ and $\Delta X_{T,m}\Delta X_{T',m}$ for $T' \in \mathcal{T}^\circ$ were found already in the proof of Lemma 3.)

Let $E_m \in \mathcal{E}$. Let $x_T(E_m)$ be the value of $\Delta X_{T,m}$ if the mth edge to be added is of type E_m, and let $y_{R,r}(E_m)$ be the corresponding value of $\Delta Y_{R,r,m}$.

The first case we consider is that $E_m = ((T,z),(R,r,z'))$; then v_m is in a tree and w_m is in the giant. Thus, $x_T(E_m) = -1$ and $y_{R,r}(E_m) = -1$. On the other hand, let R' be the rooted tree obtained by joining T to R by an edge whose endpoints are z and z' and r' be the number of vertices in R'. Then $y_{R',\max(r,r')}(E_m) = 1$. For all other choices of T and (R,r), $x_T(E_m)$ and $y_{R,r}(E_m)$ are 0.

The second case is that $E_m = ((R,r,z),(R',r',z'))$. Then v_m and w_m both belong to the giant. When we add an edge connecting v_m and w_m, we clearly lose a rooted tree of type (R,r) and one of type (R',r'). Let a_m be the root of the rooted tree containing v_m and b_m be the root of the rooted tree containing w_m, in $G_{n,m-1}$. Then all the vertices on the path between v_m and a_m become part of the core, while any other vertices in $R_{v,m}$ remain in the mantle. In the same way the vertices on the path between w_m and b_m become part of the core, while the remaining vertices in $R_{w,m}$ are in the mantle. Let $z_1,\dots,z_s$ be the vertices on the path from a_m to b_m after the edge v_mw_m is added. Then every z_i, $1 \leq i \leq s$, is part of the core in $G_{n,m}$, and thus is the root of a rooted tree in the mantle, either consisting only of the vertex z_i itself, or containing some of the other vertices from $R_{v_m,m-1}$ or $R_{w_m,m-1}$.

We obtain the formula

$$\mathbb{E}[\Delta Y_{R,r,m} \mid \mathcal{F}_{m-1}] = -2|R|\frac{Y_{R,r,m-1}}{n} + \sum_{E\in\mathcal{E}} \mathbb{P}[E_m = E \mid \mathcal{F}_{m-1}]y_{R,r}(E) + o(n^{-1/2}). \tag{5}$$

The first term takes account of the case that we lose a rooted tree of type (R,r), which happens exactly when one of the vertices v_m or w_m is incident to such a rooted tree. The sum takes into account that we may gain a rooted tree of type (R, r), which may happen either when a tree outside the giant is joined to a rooted tree in the giant, or when two rooted trees in the giant are joined to each other. We have to make sure that the total number of nonzero summands is finite. Since there are only a finite number of rooted trees with fewer vertices than R, it follows that the number of nonzero summands of the first kind is finite. For the second kind, we note that both or the rooted trees that are joined together must have order at most r. Since there are only a finite number of rooted trees of order at most r, it follows that the number of summands is finite.

We note that

$$\mathbb{P}[E_m = ((T, z), (R, r, z')) \mid \mathcal{F}_{m-1}] = \frac{X_{m-1,T}}{n} \frac{Y_{m-1,R,r}}{n} + o(1)$$

and

$$\mathbb{P}[E_m = ((R, r, z), (R', r', z')) \mid \mathcal{F}_{m-1}] = \frac{Y_{m-1,R,r}}{n} \frac{Y_{m-1,R',r'}}{n} + o(1).$$

Thus, (5) is on the form required by Theorem 2. Applying a similar reasoning we can also find expressions on this form for $\mathbb{E}[\Delta Y_{R,r,m} \Delta Y_{R',r',m} \mid \mathcal{F}_{m-1}]$ and $\mathbb{E}[\Delta X_{T,m} \Delta Y_{R,r,m} \mid \mathcal{F}_{m-1}]$. Hence the conditions of Theorem 2 are satisfied and the lemma follows. □

We now aim towards a proof of Lemma 7, and to this end we will need the following two theorems from Spencer and Wormald [10], the first of which was apparently first shown by Cramér in 1920.

Lemma 12 (Theorem 3.2 in Spencer and Wormald [10])**.** *Let K, γ, c be positive reals. Let Z be a nonnegative integer-valued random variable with K, γ, c tail and with $\mathbb{E}Z = \mu < 1$. Let T be the size of the Galton–Watson branching process in which each node independently has Z children. Then there exist positive K^+, c^+, dependent only on μ, K, c such that T has a K^+, c^+ tail.*

Lemma 13 (Theorem 3.3 in Spencer and Wormald [10])**.** *Let $K_1, \gamma_1, \rho_1, K_2, \gamma_2, \rho_2$ be constants. Let X, Y be nonnegative integer random variables with X having a K_1, γ_1, ρ_1 tail and Y having a K_2, γ_2, ρ_2 tail. Consider the two generation branching process in which the root node has X children and then each child independently has Y children. Let Z be the number of grandchildren. Then there exist K, γ, ρ dependent only on $K_1, \gamma_1, \rho_1, K_2, \gamma_2, \rho_2$ such that Z has a K, γ, ρ tail.*

Proof of Lemma 7. Let $S_{r,m}$ be the number of vertices in the giant component of $G_{n,m}$ with recollection parameter at least r. In the course of the graph process, the number of vertices with recollection parameter at least r may increase in two ways. Firstly, a rooted tree of order smaller than r may attach itself to a tree such that the combined size becomes r or greater: all the vertices in the combined tree have recollection parameter r or greater. Secondly, a vertex with recollection parameter at least r may attach itself to any tree: the vertices in the tree then get recollection parameter r or greater. Edges with both endpoints in the giant cannot increase S_r.

We define two random variables as follows. Let $S^{(1)}_{r,m}$ be the number of vertices in $G_{n,m}$ which have acquired recollection parameter at least r in the first manner, and $S^{(2)}_{r,m}$ be the number of vertices in $G_{n,m}$ which have acquired it in the second manner. Then $S_{r,m} = S^{(1)}_{r,m} + S^{(2)}_{r,m}$.

We know that R_k has a $K_1(c), \gamma_1(c), \rho_1(c)$ tail, while T_k has a $K_2(c), \gamma_2(c), \rho_2(c)$ tail. Let $K_i = \max_{c_0 \le c \le m/n} K_i(c)$, $\gamma_i = \max_{c_0 \le c \le m/n} \gamma_i(c)$ and $\rho_i = \max_{c_0 \le c \le m/n} \rho_i(c)$ for $i = 1, 2$. Then clearly R_k has a K_1, γ_1, ρ_1 and T_k has a K_2, γ_2, ρ_2 tail whenever $c_0 \le c \le c_1$. Let us bound the expected size of $\Delta S^{(1)}_{r,m}$. For $\Delta S^{(1)}_{r,m}$ to be nonzero, a rooted tree of order i must be attached to a rooted tree of order j, for some i and j with $i + j \ge r$ and $i < r$, in which case $\Delta S^{(1)}_{r,m} = i + j$. The probability that v is in a rooted tree of order i and w is in an isolated tree of order j is $\frac{R_{m,i}}{n}\frac{T_{m,j}}{n}$. The expected value of $\Delta S^{(1)}_{r,m}$ is then bounded by (where the sum is over all pairs (i, j) satisfying $i + j \ge r$ and $i < r$)

$$\begin{aligned} \sum_{i,j} 2(i+j)\frac{R_{m,i}}{n}\frac{T_{m,j}}{n} &\le 2K_1K_2 \sum_{i,j} (i+j) i^{\gamma_1} j^{\gamma_2} \rho_1^i \rho_2^j \\ &\le 2K_1K_2 \sum_{i,j} (i+j)^{1+\gamma_1+\gamma_2} \rho^{i+j} \\ &\le 2K_1K_2 \sum_{k \ge r} k^{2+\gamma_1+\gamma_2} \rho^k \le K k^{\gamma} \rho^k, \end{aligned}$$

where $K = 2K_1K_2$, $\rho = \max(\rho_1, \rho_2)$ and $\gamma = 2 + \gamma_1 + \gamma_2$. In any finite time interval, the expected number of new vertices with recollection parameter at least r is therefore bounded by $Kr^{\gamma}\rho^r n$.

Let v be an arbitrary vertex in the giant. We define the random variable U_v in the following way. Whenever an edge is added to the process between v and some tree in the forest, then the vertices in the forest are called *children* of v. We define a *descendant* of v to be a child of v, or a child of a descendant of v. Then we let U_v be the total number of descendents of v until the end of the process. The vertices in the giant of G_{n,m_0} do not have any parents, while vertices that join the giant later all have exactly one parent. We want to show that U_v has a K, γ, ρ tail.

If $m = cn$, then the number of isolated trees of order k is roughly $Ck^{-5/2}(2ce^{1-2c})^k$. Thus, if an edge is added from v to the forest at time c, the expected number of children acquired is

$$\sum_k Ck^{-3/2}(2ce^{1-2c})^k.$$

The probability that any given edge attaches v to an isolated tree is bounded by $\frac{2}{n}$. Thus, the expected number of children of v acquired between time u_1 and u_2 is at most

$$\sum_{m=u_1/n}^{u_2/n} \frac{2}{n} \sum_{k\geq 1} Ck^{-3/2}(2ce^{1-2c})^k = 2C \int_{u_1}^{u_2} \sum_{k\geq 1} k^{-3/2}(2ce^{1-2c})^k dc.$$

Whenever $u_1 > 1$, this integral is finite, even if $u_2 = \infty$. Thus, for any c_0 we can find constants $t_1, \dots, t_s$, where s is finite and depends on c_0, such that $\int_{t_i}^{t_{i+1}} f(t)dt < 1$ for $i = 0, \dots, s-1$ and $\int_{t_s}^{\infty} f(t)dt < 1$.

Consider the graph process at time t_i, and let v be a vertex in the giant at this time. Let $V_{v,i}$ be the random variable denoting the number of children acquired by v between time t_i and t_{i+1}. Let $\mu_i = \mathbb{E}[V_{v,i}]$. By the calculations above, $\mu_i < 1$, and furthermore $V_{v,i}$ has a K, γ, ρ tail. Suppose w is a child of v acquired in this time interval. The number of children of w obtained in this interval is then stochastically dominated by the random variable $V_{v,i}$, and the same holds for any other descendant of v. Thus, the total number of descendants acquired by v in this interval is bounded above by the total number of particles generated in a Galton-Watson process where every vertex has a number of children distributed as $V_{v,i}$. Hence, by Lemma 12, the number of descendants of v acquired in this interval has a K', γ', ρ' tail.

From this it follows, using Lemma 13 and induction, that for any vertex v in the giant, at any time $t > 1$, the total number of descendants is a random variable with a K'', γ'', ρ'' tail. Let $A_{r,1}$ be the set of vertices which acquired recollection parameter at least r in the first manner, while $A_{r,2}$ is the set of vertices which acquired it in the second manner. Above we showed that $S_{r,m}^{(1)}$ has a K, γ, ρ tail. Every vertex in $A_{r,2}$ must be a descendant of a vertex in $A_{r,1}$, it follows therefore by Lemma 13 that $S_{r,m}$ has a K''', γ''', ρ''' tail. □

By Lemma 6, for every k there is a function $\zeta_k(c)$ and a Gaussian process $W_k(c)$ on $[c_0, \infty)$ such that $\overline{Z}_k := n^{-1/2}(Z_k(c) - \zeta_k(c)) \to W_k$. We let $\overline{\boldsymbol{Z}}(c) = (Z_1(c), Z_2(c), \dots)$ and $\boldsymbol{W}(c) = (W_1(c), W_2(c), \dots)$.

We want to show that $\overline{\boldsymbol{Z}}(c)$ converges to $\boldsymbol{W}(c)$ in some sense. We first have to define in which space the convergence should occur. Let $\mathcal{L}$ be the Banach space of all sequences $\boldsymbol{z} = (z_1, z_2, \dots) \in \mathbb{R}^\infty$ with the norm

$$\|\boldsymbol{z}\| = \sum_{k=1}^{\infty} |z_k| < \infty.$$

Lemma 14. *Let $c > 1$ be fixed. Then $\boldsymbol{Z}(c) \xrightarrow{d} W(c)$ in $\mathcal{L}$.*

Proof. By Lemma 6, for any fixed k, $(\overline{Z}_1, \dots, \overline{Z}_k) \xrightarrow{d} (W_1, \dots, W_k)$. By Billingsley [1], we must show that for every $\varepsilon > 0$, there is a compact set K_ε such that $\mathbb{P}[\boldsymbol{Z} \notin K_\varepsilon] < \varepsilon$. By Lemma 7, $\mathbb{E}[Z_r] \leq K_1 r^\gamma \rho^r n$ and $\operatorname{Var} Z_r \leq K_1 r^\gamma \rho^r n$. Let $\overline{Z}_r = n^{-1/2}(Z_r - n\zeta_r)$. Let

$$K_\varepsilon = \{\boldsymbol{x} \in \mathcal{L} : |x_r| \leq \varepsilon^{-1} \rho^{r/3}\},$$

which is a compact set. By Chebyshev's inequality,

$$\mathbb{P}[|\overline{Z}_r - \mathbb{E}\overline{Z}_r| > \nu] \le \frac{\operatorname{Var}\overline{Z}_r}{\nu^2} \le \frac{\operatorname{Var} Z_r}{n\nu^2} \le \frac{Kr^\gamma \rho^r}{\nu^2}.$$

Define

$$\overline{Z}^*_{n,k} = \begin{cases} \overline{Z}_{n,k} & \text{if} \quad k \le k_0 = \lfloor n^{1/3} \rfloor, \\ 0 & \text{if} \quad k > k_0, \end{cases}$$

and let $\overline{\boldsymbol{Z}}^*_n = (\overline{Z}^*_{n,1}, \ldots)$. It suffices (Pittel [6]) to verify the tightness condition for $\overline{\boldsymbol{Z}}^*_n$ and to show that

$$\sum_{k>k_0} |\overline{Z}_{n,k}| \xrightarrow{p} 0 \tag{6}$$

as $n \to \infty$. Then

$$\mathbb{P}[\overline{\boldsymbol{Z}}^* \notin K_\varepsilon] \le \sum_{r\ge 1} \mathbb{P}[|\overline{Z}^*_r| > \varepsilon^{-1}\rho^{r/3}] \le \sum_{r\ge 1} \frac{Kr^\gamma \rho^r}{\varepsilon^{-2}\rho^{2r/3} + O\left(\frac{\log^a n}{\sqrt{n}}\right)}$$

$$\sum_{r=1}^{k_0} \frac{Kr^\gamma\rho^r}{\varepsilon^{-2}\rho^{2r/3}} \left(1 + O\left(\frac{\varepsilon^2 \log^a n}{\rho^{2r/3}\sqrt{n}}\right)\right) = O(\varepsilon^2).$$

Since Z_k has a K, γ, ρ tail, (6) follows easily, finishing the proof. □

Proof of Theorem 1. We have $c_1, \ldots, c_k$ fixed. Let $\mathcal{L}^*$ be the Banach space of sequences $\boldsymbol{z}^* = (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots) \in \mathbb{R}^{k^\infty}$ with the norm

$$\|\boldsymbol{z}^*\| = \sum_{i=1}^{\infty} \|\boldsymbol{z}_i\| < \infty,$$

where $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ are k-dimensional vectors.

Again, by Lemma 6 all finite-dimensional distributions converge. Hence, if the vector $[\overline{\boldsymbol{Z}}(c_1), \ldots, \overline{\boldsymbol{Z}}(c_q)]$ converges at all, it must converge to $[\boldsymbol{W}(c_1), \ldots, \boldsymbol{W}(c_q)]$ which has distribution $\mathcal{N}(0, \Sigma)$ for some matrix Σ. Thus, it is sufficient to show tightness. But probability measures on the Cartesian product $\prod_{i=1}^q S_i$ are tight if and only if the marginal distributions are tight on $S_1, \ldots, S_q$. (Exercise 5.9 in Billingsley [1].) Tightness for the marginal distributions was proved in Lemma 14. Theorem 1 follows directly. □

6 Proof of the differential equation method

Let $J = [0, \infty)$, and let $D(J)$ be the space of right continuous real-valued functions on J, endowed with the Skorokhod J_1 topology. For every n we let $k_n(t)$ be an integer-valued, nondecreasing right-continuous function on J, such that $k_n(0) = 0$. We define

$$W_n(t) = \sum_{i=1}^{k_n(t)} X_{n,i}$$

for $t \in J$.

We let B be a standard Brownian motion process on $D(J)$. Recall that $\mathbb{E}B(t) = 0$ and $\mathbb{E}[B(t)B(t')] = t$ whenever $0 \le t \le t' < \infty$.

We shall require the following result, which is Corollary 3.8 in McLeish [5].

Lemma 15. *Let $X_{n,m}$ be an array of random variables, and assume that the following conditions are satisfied.*

(15-i) $\sum_{m=1}^{k_n(t)} \mathbb{E}[(\Delta X_{n,m})^2 I(|\Delta X_{n,m}| > \varepsilon) \mid \mathcal{F}_{n,m-1}] \xrightarrow{p} 0$ *for all* $\varepsilon > 0$.

(15-ii) $\sum_{m=1}^{k_n(t)} \mathbb{E}[(\Delta X_{n,m})^2 \mid \mathcal{F}_{n,m-1}] \xrightarrow{p} t$.

(15-iii) $\sum_{m=1}^{k_n(t)} |\mathbb{E}[\Delta X_{n,m} \mid \mathcal{F}_{n,m-1}]| \xrightarrow{p} 0$ *for all* $t \in J$.

Then $W_n \xrightarrow{d} W$ *on* $D(J)$.

This theorem considers a 1-dimensional random process $\{X_{n,m}\}_{m\ge0}$, whose initial position $X_{n,0}$ is deterministically equal to 0. Next we will generalize it in order to obtain a theorem which can be used in the proof of Theorem 2.

Lemma 16. *Let $\boldsymbol{X}_{n,m}$ be a q-dimensional array of random variables. Assume that there are continuous functions $\sigma_{ij} : J \to \mathbb{R}$ for $1 \le i, j \le q$ such that the following conditions are satisfied.*

(16-i) $\sum_{m=1}^{tn} \mathbb{E}[(\Delta X_{n,i})^2 I(|\Delta X_{n,j}| > \varepsilon) \mid \mathcal{F}_{m,i-1}] \xrightarrow{p} 0$ *for all* $\varepsilon > 0$ *and* $1 \le i, j \le q$.

(16-ii) $\sum_{m=1}^{tn} \mathbb{E}[\Delta X_{n,m,i} \Delta X_{n,m,j} \mid \mathcal{F}_{n,m-1}] \xrightarrow{p} \sigma_{ij}(t)$ *for all* $1 \le i, j \le q$.

(16-iii) $\sum_{m=1}^{tn} |\mathbb{E}[\Delta X_{n,m,i} \mid \mathcal{F}_{n,m-1}] \xrightarrow{p} 0$ *for all* $t \in J$ *and* $1 \le i \le q$.

Then there is a continuous q-dimensional Gaussian process $\boldsymbol{W}$ with $\mathbb{E}\boldsymbol{W}(t) = 0$ for $t \in J$ and covariances $\mathbb{E}[\boldsymbol{W}(t)\boldsymbol{W}(t')] = \Sigma(t)$ for $0 \le t \le t' < \infty$, such that $\boldsymbol{X}_{n,tn} \xrightarrow{d} \boldsymbol{W}$.

Proof. Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_l \in \mathbb{R}^q$ be arbitrary vectors and let $0 = t_0 < t_1 < \ldots < t_l$ be real numbers. Define $\Delta Y_{n,m} = \boldsymbol{a}_{h_m} \cdot \boldsymbol{X}_{n,m}$, where h_m is such that $t_{h_m-1} < m/n \le t_{h_m}$, and let $Y_{n,m} = \sum_{i=1}^{m} \Delta Y_{n,i}$. Then, for every n, $Y_{n,m}$ is a one-dimensional stochastic process. Let us define $k_n(t) = \min\{m : \sum_{i,j} a_{h_m,i} a_{h_m,j} \sigma_{ij}(m/n) \ge t\}$. We will show that with this choice of $k_n(t)$, the stochastic process $Y_{n,m}$ satisfies the conditions of Lemma 15.

We note that $|\Delta Y_{n,m}| > \varepsilon$ implies that $|\Delta X_{n,m,i}| > \varepsilon / a_{h_m,i} q$ for at least one i.

Moreover, $ab \leq \frac{1}{2}(a^2+b^2)$ for all $a, b \in \mathbb{R}$, so

$$
\begin{aligned}
&\sum_{m=1}^{k_n(t)} \mathbb{E}[(\Delta Y_{n,i})^2 I(|Y_{n,i}| > \varepsilon) \mid \mathcal{F}_{m-1}] \\
&= \sum_{m=1}^{k_n(t)} \sum_{i,j} a_{h_m,i} a_{h_m,j} \mathbb{E}[\Delta X_{n,m,i} \Delta X_{n,m,j} I(|Y_{n,m}| > \varepsilon) \mid \mathcal{F}_{m-1}] \\
&\leq \frac{1}{2} \sum_{m=1}^{k_n(t)} \sum_{i,j} a_{h_m,i} a_{h_m,j} \mathbb{E}[((\Delta X_{n,m,i})^2 + (\Delta X_{n,m,j})^2) I(|Y_{n,m}| > \varepsilon) \mid \mathcal{F}_{m-1}] \\
&\leq \frac{1}{2} \sum_{i,j,k} a_{h_m,i} a_{h_m,j} \sum_{m=1}^{k_n(t)} \mathbb{E}[((\Delta X_{n,m,i})^2 + (\Delta X_{n,m,j})^2) I(|X_{n,k}| > \varepsilon / a_{h_m,k} q) \mid \mathcal{F}_{m-1}] \xrightarrow{p} 0,
\end{aligned}
$$

by Condition (i); thus, $Y_{n,m}$ satisfies Condition (i) of Lemma 15. Moreover, we have

$$
\begin{aligned}
\sum_{m=1}^{k_n(t)} \mathbb{E}[(\Delta Y_{n,m})^2 \mid \mathcal{F}_{m-1}] &= \sum_{i,j} a_{h_m,i} a_{h_m,j} \sum_{m=1}^{k_n(t)} \mathbb{E}[\Delta X_{n,m,i} \Delta X_{n,m,j} \mid \mathcal{F}_{m-1}] \\
&\xrightarrow{p} \sum_{i,j} a_{h_m,i} a_{h_m,j} \sigma_{ij}(k_n(t)/n) = t + O(1/n),
\end{aligned}
$$

by the definition of $k_n(t)$, and since $\sigma_{ij}(t)$ are continuous. Thus, Condition (ii) of Lemma 15 is satisfied. Finally,

$$
\sum_{m=1}^{k_n(t)} |\mathbb{E}[\Delta Y_{n,m} \mid \mathcal{F}_{m-1}]| \leq \sum_i |a_{h_m,i}| \sum_{m=1}^{k_n(t)} |\mathbb{E}[\Delta X_{n,m} \mid \mathcal{F}_{m-1}]| \xrightarrow{p} 0,
$$

so Condition (iii) is satisfied. Thus, by Lemma 15, if we define $Y_n^*(t) = Y_{n,k_n(t)}$, then $Y_n^* \xrightarrow{d} B$. We write $Y_n(t) = Y_{n,tn}$. Then $Y_n(t) = Y_n^*(k_n^{-1}(tn)/n)$. Thus, if we let $Z_n(t) = B(k_n^{-1}(tn)/n)$, we see that Y and Z must converge to the same limit. Then $\mathbb{E}Z(t) = 0$ for $t \geq 0$ and if $0 \leq t \leq t' < \infty$, then

$$
\mathbb{E}[Z_n(t) Z_n(t')] = \mathbb{E}[B(k_n^{-1}(tn)/n) B(k_n^{-1}(t'n)/n)],
$$

so $Y(t)$ converges to a Gaussian process W.

Hence, the vectors $\{\boldsymbol{a}_h(X(t_h) - X(t_{h-1}))\}_{1 \leq h \leq l}$ converge jointly to a multivariate normal distribution. We conclude that the finite-dimensional distributions converge.

For tightness, we note that if we consider the one-dimensional process $\{\boldsymbol{X}_{n,m,i}\}$ for any i, $1 \leq i \leq q$, the conditions of Lemma 15 are satisfied for some function

$k_n(t)$. Tightness for these processes was shown in the proof in McLeish [5]. But probability measures on the Cartesian product $\prod_{i=1}^q S_i$ are tight if and only if the marginal distributions are tight on $S_1, \ldots, S_q$. (Exercise 5.9 in Billingsley [1].) This finishes the proof. $\square$

We are now ready to prove Theorem 2. The argument is similar to the proof of the main theorem in [8]. Lemma 16 cannot be applied to the array $\boldsymbol{X}_{n,m}$ from Theorem 2, since Condition (i) clearly cannot be satisfied in general, as we may have $\|\Delta \boldsymbol{X}_{n,m}\| > \varepsilon$. It is natural to instead try to apply the lemma to the array $\boldsymbol{Y}_{n,m} = \boldsymbol{X}_{n,m} - n\boldsymbol{\alpha}(m/n)$; for this array we expect that $\Delta \boldsymbol{Y}_{n,m}$ is close to 0. However, it turns out that $\Delta \boldsymbol{Y}_{n,m} = O_p(n^{-1/2})$ which is not close enough to 0 for the argument to work. Instead, we will consider the array $\boldsymbol{Z}_{n,m} = T(m/n)\boldsymbol{Y}_{n,m}$, where $T(t)$ is the matrix defined in Theorem 2. This linear transformation ensures that $\Delta \boldsymbol{Z}_{n,m} = o(n^{-1/2})$, which is good enough for our purposes.

Lemma 17. *Let $\Xi(t)$ be defined as in Theorem 2, and let $\xi_{ij}(t)$ be the entry in the ith row and jth column of $\Xi(t)$. Then*

$$\mathbb{E}[\Delta \boldsymbol{Z}_{n,m} \mid \mathcal{F}_{m-1}] = o(n^{-1/2}) \tag{7}$$

and for $1 \le i, j \le q$,

$$n^{-1}\mathbb{E}[\Delta Z_{n,m,i}\Delta Z_{n,m,j} \mid \mathcal{F}_{m-1}] \xrightarrow{p} \xi_{ij}(t). \tag{8}$$

Proof. This was proved in [8]. More precisely, (7) was shown in Lemma 2 and (8) in Lemma 3 of [8]. $\square$

Proof of Theorem 2. We will show that $\boldsymbol{M}_{n,m} = n^{-1/2}\boldsymbol{Z}_{n,m}$ satisfies the conditions of Lemma 16. Write $T(t) = \{\tau_{ij}(t)\}_{i,j}^q$. Then $Z_{n,m,i} = \sum_{j=1}^q \tau_{ij}(m/n)Y_{n,m,j}$ and

$$\Delta Z_{n,m,i} = \sum_{j=1}^q \tau_{ij}(m/n)\Delta Y_{n,m,j} + O(1/n)$$

since $\tau_{ij}(t)$ is continuous. By assumption we have $\mathbb{E}[\Delta X_{n,m}] = o(n^{1/12})$ with probability 1, so

$$\mathbb{P}[|\Delta M_{n,m,i}| > \varepsilon] = \mathbb{P}[n^{-1/2}|\Delta Z_{n,m,i}| > \varepsilon] \le \sum_{j=1}^q \mathbb{P}[n^{-1/2}|\tau_{ij}(t)\Delta Y_{n,m,j}| > \varepsilon]$$

$$= \sum_{j=1}^q \mathbb{P}[|\Delta Y_{n,m,j}| > \varepsilon\sqrt{n}/|\tau_{ij}(t)|] \le \sum_{j=1}^q \mathbb{P}[|\Delta X_{n,m,j}| > \varepsilon\sqrt{n}/|\tau_{ij}(t)| + O(1/n)] = 0,$$

where the last equality holds for large enough n. Hence the sum in Condition (i) of Lemma 16 vanishes. Next,

$$\sum_{m=1}^{tn} \mathbb{E}[\Delta M_{n,m,i}\Delta M_{n,m,j} \mid \mathcal{F}_{m-1}] = n^{-1}\sum_{m=1}^{tn} \mathbb{E}[\Delta Z_{n,m,i}\Delta Z_{n,m,j} \mid \mathcal{F}_{m-1}] \xrightarrow{p} \xi_{ij}(m/n),$$

by (8) in Lemma 17 so Condition (ii) is satisfied as well. Finally,

$$\sum_{m=1}^{tn} |\mathbb{E}[\Delta M_{n,m,i} \mid \mathcal{F}_{m-1}]| = n^{-1/2} \sum_{m=1}^{tn} |\mathbb{E}[\Delta Z_{n,m,i} \mid \mathcal{F}_{m-1}]| = n^{-1/2} tn \cdot o(n^{-1/2}) \xrightarrow{p} 0$$

by (7) in Lemma 17. Hence, all the conditions of Lemma 16 are satisfied, and it follows that $\boldsymbol{M} \xrightarrow{d} \boldsymbol{V}$ where $\boldsymbol{V}$ is a q-dimensional Gaussian process such that $\mathbb{E}\boldsymbol{V}(t) = 0$ for $t \geq 0$ and $\mathbb{E}[\boldsymbol{V}(t)\boldsymbol{V}(t')] = \Xi(t)$ for $0 \leq t \leq t'$. Since $\boldsymbol{M}_{n,m} = n^{-1/2}T(m/n)(\boldsymbol{X}_{n,m} - n\boldsymbol{\alpha}(m/n))$, it follows that if $\boldsymbol{W}$ is defined as in the statement of Theorem 2, then $n^{-1/2}(\boldsymbol{X}_{n,m} - n\boldsymbol{\alpha}(m/n))$ converges to $\boldsymbol{W}$ in $D(J)$. □

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