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Perfect matchings and Hamilton cycles in hypergraphs with large degrees

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REPORT No. 35, 2008/2009, spring

ISSN 1103-467X
ISRN IML-R- -35-08/09- -SE+spring

Perfect matchings and Hamilton cycles in hypergraphs with large degrees

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November 13, 2009

Abstract

We establish a new lower bound on the l-wise collective minimum degree which guarantees the existence of a perfect matching in a k-uniform hypergraph, where $1 \leq l < k/2$. For $l = 1$, this improves a long standing bound by Daykin and Häggkvist [4]. Our proof is a modification of the approach of Han, Person, and Schacht from [8].

In addition, we fill a gap left by the results solving a similar question for the existence of Hamilton cycles.

1 Introduction

Recently there has been a lot of interest in Dirac-type properties of uniform hypergraphs. With this name we describe a general class of problems and results relating minimum degrees of k-uniform hypergraphs to the existence of a Hamilton cycle (of some kind) or a perfect (or near perfect) matching, see, e.g., [12], [15], [19], and [14], [17], [20], [2], [21], [8], resp. For some complexity aspects of these problems, see [10], [22], and [11].

Besides the celebrated theorem of Dirac [5] for graphs, the first result of this kind was obtained already by Daykin and Häggkvist in 1981 [4], who proved that in order to have a perfect matching in a k-uniform hypergraph H with n vertices, where n is divisible by k, it is sufficient if the minimum degree in H is greater than $(1 - 1/k)\left(\binom{n-1}{k-1} - 1\right)$, about the $\frac{k-1}{k}$ fraction of the maximum possible vertex degree. They also gave a separate result for the case of k-partite hypergraphs. Recently, it was proved by Han, Person, and Schacht ([8], Theorem 6) that for $k = 3$ the fraction $\frac{2}{3}$ can be replaced by $\frac{5}{9} + \epsilon$, which, moreover, is asymptotically best possible.

Given a k-uniform hypergraph H and an integer l, $0 < l < k$, let $\delta_l(H)$ be the largest integer d such that every l-element set S of vertices of H has degree $deg_H(S) \geq d$, that is, S is contained in at least d edges. In particular, $\delta_1(H) = \delta(H)$ is the ordinary minimum vertex degree.

In [8], the case $1 \leq l < k/2$ is studied. At the other extreme lies the equally interesting case $l = k - 1$, in which the threshold value of $\delta_l(H)$ guaranteeing a perfect matching in H has been determined *precisely* [20]. Later Pikhurko [17] proved that the threshold value of $\delta_l(H)$ for all $l \geq k/2$ is asymptotically $\frac{1}{2}\binom{n-l}{k-l}$. The case of $l < k/2$ seems to be harder. In addition to the above mentioned result for $k = 3$, paper [8] contains the following general theorem, which for $l = 1$ coincides asymptotically with the almost thirty years old bound of Daykin and Häggvist.

Theorem 1 ([8]). *For all integers k and l, where $1 \leq l < k/2$, and all $\epsilon > 0$, there is n_0 such that if H is a k-uniform hypergraph on $n > n_0$ vertices, with n divisible by k and*

$$\delta_l(H) \geq \left(\frac{k-l}{k} + \epsilon\right)\binom{n-l}{k-l},$$

then H contains a perfect matching.

In this paper we improve the above result.

Theorem 2. *For all integers k and l, where $1 \leq l < k/2$, and all $\epsilon > 0$, there is n_0 such that if H is a k-uniform hypergraph on $n > n_0$ vertices, with n divisible by k and*

$$\delta_l(H) \geq \left(\frac{k-l}{k} - \frac{1}{k^{k-l}} + \epsilon\right)\binom{n-l}{k-l},$$

then H contains a perfect matching.

It is conjectured in [8] that the optimal bound on $\delta_l(H)$ guaranteeing a perfect matching in H is asymptotically equal to

$$\max(1/2, 1-(1-1/k)^{k-l})\binom{n-1}{k-l} + o(n^{k-l}). \tag{1}$$

For $l < k/2$ this conjecture is still open except the smallest case $k = 3$ and $l = 1$.

The proof in [8] uses the idea of absorption introduced in [19] and [20]. In this paper we simplify the proof from [8]. Most notably, we do not need Goodman's result on the number of triangles in a dense graph, but instead we use the Erdős counting lemma for partite, uniform hypergraphs (see Lemma 1 below). In addition, in Section 2 we prove a sharp result about edge maximal partite hypergraphs with a given size t of a maximum matching (Theorem 3). When $t = 1$, we obtain a description of the extremal sets in a special case of a result of Frankl [7]. These new tools allow us to extend the method from [8], Theorem 6, to other instances of k and $l < k/2$. The main proof is presented in Sections 3. In Section 4, we further improve our bound in the smallest open case: $k = 4$ and $l = 1$.

In Section 5, we give a small contribution to the solution of a similar question for the existence of a Hamilton cycle. Dirac-type problems for Hamilton cycles are related to those for perfect matchings, both by the results obtained and by the methods of proof. Since they are much harder to tackle, the existing results limit themselves to only one case of δ_l: $l = k-1$. On the other hand, unlike for matchings, there are several notions of a hypercycle. Besides the classic notion of a Berge cycle, the most studied case is that of *(k, r)-cycles*, $0 \leq r \leq k-1$, defined as k-uniform hypergraphs whose vertices can be ordered cyclically in such a way that the edges are segments of that cyclic order and every two consecutive edges share exactly r vertices.

A Hamilton r-cycle is then defined as a (k, r)-cycle in a k-uniform hypergraph H containing all vertices of H. A necessary condition is that $k-r$ divides $|V(H)|$, and for $r = 0$ this is a perfect matching. For k, r, and n, satisfying $k - r|n$, let $h_r(k, n)$ be the smallest integer h such that $\delta_{k-1}(H) \geq h$ implies that an n-vertex k-uniform hypergraph H contains a Hamilton r-cycle.

It was proved in [19] that $h_{k-1}(k, n) \sim \frac{1}{2}n$. Since for $k|n$ we have $\frac{1}{2}n-k \leq h_0(k, n) \leq h_{k-1}(k, n)$ (the lower bound by a simple construction, cf. [14] or [20]), it follows that $h_0(k, n) \sim \frac{1}{2}n$ too (as mentioned above, $h_0(k, n)$ was determined exactly in [20]). Moreover, trivially, if $k - r|k$ then $h_0(k, n) \leq h_r(k, n)$ (take every $\frac{k}{k-r}$th edge of a Hamilton r-cycle), and if $k-r|n$ then $h_r(k, n) \leq h_{k-1}(k, n)$ (take every $(k-r)$th edge of a Hamilton $(k-1)$-cycle). Consequently, if, in addition, $k|n$ then $h_r(k, n) \sim \frac{1}{2}n$ as well.

On the other hand, the results from [15, 9, 13] show that

$$h_r(k, n) \sim \frac{n}{\lceil \frac{k}{k-r} \rceil (k-1)},$$

whenever $k-r \not| k$ (and $k-r|n$, of course). This leaves only a small gap in our knowledge about Dirac thresholds for Hamilton r-cycles in k-uniform hypergraphs. Namely, what is the asymptotic value of $h_r(k, n)$ when $k - r|n$, $k - r|k$ but $k \not| n$ (e.g., $k = 6$, $r = 4$, and $n = 20$)? Note that all counterexamples existing in the literature assume that $k|n$ (cf. [9], the discussion following the proof of Fact 4, and [13], Proposition 2.2). Here we close this gap by providing 'the missing piece in the puzzle'.

Proposition 1. *If $k - r|n$ and $k - r|k$ then $h_r(k, n) \geq \frac{1}{2}n - k$. Consequently, $h_r(k, n) \sim \frac{1}{2}n$, regardless whether $k|n$ or not.*

Throughout the paper k-uniform hypergraphs will be called *k-graphs*.

2 Extremal k-partite k-graphs without matchings of given size

We first determine the maximum number of edges in balanced k-partite k-graphs without a matching of a given size. For $t = 1$ this result follows from a more general theorem of Frankl [7] on intersecting families.

Fact 1. *For all integer $k \geq 1$, $n \geq 1$, and $1 \leq t \leq n-1$, the maximum number of edges in a k-partite k-graph with n vertices in each class and no matching of size $t+1$ is tn^{k-1}.*

Proof. By Theorem 3 of [3] the complete k-partite k-graph $K(n, \ldots, n)$ with n vertices in each part has chromatic index n^{k-1}, that is, it has a factorization. Hence, the edge set of $K(n, \ldots, n)$ can be partitioned into n^{k-1} disjoint perfect matchings M_i, $i = 1, \ldots, n^{k-1}$. If H is a k-partite k-graph with n vertices in each class and more than tn^{k-1} edges, then by the Pigeon-hole Principle for some i we must have $|M_i \cap H| > t$, which yields a matching of size $t+1$ in H, a contradiction.

On the other hand, the k-partite k-graph

$$K_k^t(n) := K(t, n, \ldots, n) \cup (n-t)K_1,$$

containing in one class $n-t$ isolated vertices but otherwise being complete, has exactly tn^{k-1} edges and no perfect matching of size $t+1$. $\square$

In our main proof, we will need a structural result saying that for $n \geq 3$ the above defined hypergraph $K_k^t(n)$ is the only extremal k-partite k-graph. As our next example shows the assumption that $n \geq 3$ is crucial.

Example 1. For $k \geq 3$, k odd, consider a k-partite k-graph H_0 with partition $V(H_0) = V_1 \cup \cdots \cup V_k$, where $V_i = \{u_i, v_i\}$, $i = 1, \ldots, k$, and with the edge set $E(H_0)$ consisting of all k-subsets containing at least $(k+1)/2$ vertices of $\{u_1, \ldots, u_k\}$. Then, the number of edges in H_0 is $\sum_{i=(k+1)/2}^{k} \binom{k}{i} = 2^{k-1}$ and the set of edges is an intersecting family, that is, there is no matching of size 2. Thus, besides $K_k^1(2)$, also H_0 is extremal in this case (note that for $k \geq 3$, $H_0 \ncong K_k^1(2)$). For k even, we include into H_0, in addition, a half of all k-subsets containing precisely $k/2$ vertices of $\{u_1, \ldots, u_k\}$, making sure that no set is included together with its complement, so that H_0 is still intersecting.

Our next result could be reformulated in terms of König's property stating that the size of a maximum matching equals the size of a minimum vertex cover (note that the set of t vertices in $K_k^t(n)$ of maximum degree n^{k-1} forms a unique minimal vertex cover of $K_k^t(n)$). In general, for k-partite k-graphs König's property does not hold, and is replaced by Ryser's conjecture (cf. [1] and [16]).

Theorem 3. *For all integers $k \geq 1$, $n \geq 3$, and $1 \leq t \leq n-1$, the k-graph $K_k^t(n)$ is (up to isomorphism) the only k-partite k-graph with n vertices in each class and tn^{k-1} edges which contains no matching of size $t+1$.*

Proof. We prove the statement by induction on k. For $k=2$, by König's theorem there is a vertex cover in H of size t, but for t vertices to cover all tn edges these vertices have to be in the same partition class. Thus, $H = K_2^t(n)$. Now assume that the statement is true for all $2 \leq k' \leq k-1$ and consider a k-partite k-graph with n vertices in each class, tn^{k-1} edges, and no matching of size $t+1$. Denote the partition classes of H by $V_1, \ldots, V_k$.

For a matching M in the complete $(k-1)$-partite $(k-1)$-graph $K(V_1, \ldots, V_{k-1})$ define an auxiliary bipartite graph G_M with vertex classes M and V_k and such that there is an edge $\{e, v\}$, $e \in M$, $v \in V_k$ if and only if $e \cup \{v\} \in H$.

Let $M_1, \ldots, M_{n^{k-2}}$ be a factorization of $K(V_1, \ldots, V_{k-1})$. For each i put $G_i = G_{M_i}$. As $\sum_i e(G_i) = e(H)$, the average number of edges in G_i's is tn. However, if for some i, we had $e(G_i) > tn$, then, by Fact 1 there would be a matching of size $t+1$ in G_i, and hence, a matching of that size in H, a contradiction. Thus, for all i we have $e(G_i) = tn$ and G_i does not have a matching of size $t+1$. By the induction assumption for $k' = 2$, we have $G_i \cong K_2^t(n)$, that is, there is a vertex cover C_i in G_i of size t such that either $C_i \subset M_i$ or $C_i \subset V_k$.

Since every matching M in $K(V_1, \dots, V_{k-1})$ belongs to a factorization, the above properties of G_i hold also for G_M. That is, for any matching M in $K(V_1, \dots, V_{k-1})$ there is a vertex cover C_M in G_M of size t such that either $C_M \subset M$ (type I) or $C_M \subset V_k$ (type II). Moreover, for any edge e of $K(V_1, \dots, V_{k-1})$, the neighborhood $N_{G_M}(e)$ is the same for all $M \ni e$. Thus, if two matchings M', M'' share an edge then they are of the same type (I or II). Moreover, if they are both of type II then $C_{M'} = C_{M''}$.

We first show that either for all i the matchings M_i are for type I or for all i they are of type II. Indeed, fix $j \neq i$ and let $e \in M_i$ and $e' \in M_j$. Since $n \geq 3$ there exists $e_0 \in K(V_1, \dots, V_{k-1})$ such that $e_0 \cap (e \cup e') = \emptyset$. Let M be a matching in $K(V_1, \dots, V_{k-1})$ containing e and e_0, and let M' be a matching in $K(V_1, \dots, V_{k-1})$ containing e' and e_0. Then, by transitivity, M_i and M_j are of the same type.

If all M_i are of type II then the sets C_i are the same set $C \subset V_k$ which, therefore, is a minimal vertex cover of H.

Finally, consider the case when for all i, $C_i \subset M_i$. Set $H' = \bigcup_{i=1}^{n^{k-2}} C_i$ and notice that H' has tn^{k-2} edges, it is completely connected with V_k in H and thus, the link of H in $V_1 \cup \dots \cup V_{k-1}$ is precisely H'. If H' had a matching of size $t+1$ that matching could be extended to a matching of size $t+1$ in H, again, a contradiction. Thus, there is no matching of size $t+1$ in H' and H' has tn^{k-2} edges. By the induction assumption for $k' = k-1$, we conclude that $H' \cong K_{k-1}^t(n)$ has a vertex cover of size t, which by the construction of H' is a vertex cover of the entire hypergraph H. Hence, $H \cong K_k^t(n)$ $\square$

In view of Theorem 3, it is perhaps interesting to ask how many edges still guarantee that the König property holds. For $t = n-1$ we may ask a weaker question: how many edges guarantee the presence of an isolated vertex, or more generally, a given minimum degree.

For 3-partite 3-graphs without perfect matchings (that is, for $t = n-1$), we undertook a more detailed study of the relation between the minimum degree and the maximum number of edges. We used integer programming. A linear program was created with one binary variable for each edge of the complete 3-partite 3-graph with n vertices in each class. For each perfect matching an inequality was created, stating that at least one edge of the matching must be missing. At the same time, one inequality for each vertex was created, stating that the number of edges at that vertex must be at least δ. Observe that this only gives a lower bound on the actual δ of the hypergraph. Finally the objective was chosen to be maximum number of edges, i.e. the maximum number of variables set to 1.

For $n = 3$ and $n = 4$ the resulting integer program is quite small and can easily be solved by a standard integer programming solver (we used GNU's glpk and verified the results using a commercial solver). The maximum number of edges for each case is shown in Table 1. In particular, and most importantly for us, the smallest number of edges in a $4 \times 4 \times 4$ 3-partite 3-graph without a perfect matching which forces the presence of an isolated vertex is 43.

$\delta\backslash n$	3	4
0	18	48
1	16	42
2	16	42
3	15	42
4	14	40
5	-	37
6		37
7		37
8		32
9		-

Table 1: The maximum number of edges in a 3-partite 3-graph without a perfect matching, having n vertices in each class and given lower bound on the minimum degree δ.

3 The proof of Theorem 2

For two hypergraphs F and Q, let $N(F,Q)$ be the number of copies of Q in F. We will need the following lemma proved, in a slightly different form, by Erdős in [6]. Here we present a version from [18].

Lemma 1. *For every integer $r \geq 2$, every $d > 0$, and every r-partite r-graph Q, there exist $c > 0$ and n_0 such that for every r-graph F on $n \geq n_0$ vertices with $e(F) \geq dn^r$, we have $N(F,Q) \geq cn^{|V(Q)|}$.* □

As a consequence of the absorption lemma proved in [8] (Theorem 10), in order to prove our Theorem 2 it is sufficient to show a seemingly weaker statement. It is analogous to Theorem 16 in [8].

Lemma 2. *For all integers k and l, where $0 < 2l < k$, and all $\gamma > 0$, there is n_0 such that if H is a k-graph on $n > n_0$ vertices with*

$$\delta_l(H) \geq \left(\frac{k-l}{k} - \frac{1}{k^{k-l}} + \gamma\right)\binom{n-l}{k-l},$$

then H contains a matching covering more than $n - \sqrt{n}$ vertices.

We wrote above $n - \sqrt{n}$ but, in fact, we could have any sufficiently large constant instead of $\sqrt{n}$. On the other hand, to deduce Theorem 2, even $\gamma' n$ unmatched vertices for a small constant γ' would be tolerable (as was the case in [8]). Once we prove Lemma 2, it will be quite straightforward to deduce Theorem 2. Just take γ small enough with respect to ϵ and apply Corollary 13 from [8] (as a guideline, see the short proof of Theorem 6 in [8]). Hence, it remains to prove Lemma 2.

Proof of Lemma 2: Let M be a largest matching in H. Assume to the contrary that $n - |V(M)| \geq \sqrt{n}$. Let $X = V(H) \setminus V(M)$. Without loss of generality we may suppose that $x := |X| = \sqrt{n}$ (we omit floors and ceilings for clarity of presentation). Set $m = |M|$.

For every l-element subset $S \subseteq X$ and any submatching M' of M, denote by $L_S(M')$ the $(k-l)$-uniform *link* hypergraph of S, consisting of all $(k-l)$-element sets $T \subseteq V(M')$ such that $S \cup T \in H$ and $|T \cap e| \le 1$ for every edge $e \in M'$. Given S, and taking $M' = M$, the number of edges of H of the form $S \cup T$ and such that $T \notin L_S(M)$, is $o(n^{k-l})$. Hence, by the assumption on $\delta_l(H)$, for every $S \in \binom{X}{l}$,

$$|L_S(M)| = deg_H(S) - o(n^{k-l}) \ge \left(\frac{k-l}{k} - \frac{1}{k^{k-l}} + \gamma - o(1)\right)\binom{n-l}{k-l}. \tag{2}$$

To complete the proof, we will find a set S which violates the above inequality.

For every $S \in \binom{X}{l}$, we break the family $\binom{M}{k-l}$ consisting of the sets $E = \{e_1, \dots, e_{k-l}\}$, where $e_i \in M$, into three parts, according to the properties of the link $L_S(E)$. Namely, we write

$$\binom{M}{k-l} = P(S) \cup A(S) \cup B(S),$$

where

- $P(S) = \left\{E \in \binom{M}{k-l} : L_S(E) \text{ has a matching of size } k-l+1\right\}$
- $A(S) = \left\{E \in \binom{M}{k-l} : |L_S(E)| \le (k-l)k^{k-l-1} - 1\right\}$
- $B(S) = \left\{E \in \binom{M}{k-l} \setminus P(S) : |L_S(E)| = (k-l)k^{k-l-1}\right\}$

The number $(k-l)k^{k-l-1}$ is not magic. By Fact 1 with $n := k$, $k := k-l$ and $t := k-l$, this is the maximum number of edges in a $(k-l)$-partite $(k-l)$-graph with k vertices in each partition class and without a matching of size $k-l+1$. Moreover, by Theorem 3, the only hypergraph which achieves this maximum is one with exactly l isolated vertices, all belonging to the same partition class, that is, $K^{k-l}_{k-l}(k)$. We set $K := K^{k-l}_{k-l}(k)$ for convenience. Let us recall that K is isomorphic to $K_{k-l,k,\dots,k} \cup I$, where $K_{k-l,k,\dots,k}$ is the complete, $(k-l)$-partite $(k-l)$-graph and I is a set of l isolated vertices, disjoint from $V(K_{k-l,k,\dots,k})$. It follows that for every $E \in B(S)$, $L_S(E)$ is a copy of K.

Our ultimate goal is to find a set $S \in \binom{X}{l}$ with

$$\max(|P(S)|, |B(S)|) \le \frac{\gamma}{3}\binom{m}{k-l}. \tag{3}$$

Indeed, then

$$\begin{aligned} |L_S(M)| &\le k^{k-l}(|P(S)| + |B(S)|) + \left((k-l)k^{k-l-1} - 1\right)|A(S)| \\ &\le \left(\frac{2\gamma}{3}k^{k-l} + (k-l)k^{k-l-1} - 1\right)\binom{m}{k-l}, \end{aligned} \tag{4}$$

which, after using the obvious bound $m \le n/k$ yields a contradiction with (2).

We first show that for most $S \in \binom{X}{l}$ we do have $|P(S)| \le \frac{1}{3}\gamma\binom{m}{k-l}$. This is the easier of the two remaining tasks, but at the same time very instructive for the other, more involved case.

Fact 2. *For at most $\gamma\binom{x}{l}$ sets $S \in \binom{X}{l}$ we have $|P(S)| > \frac{1}{3}\gamma\binom{m}{k-l}$.*

Proof. Suppose that at least $\gamma\binom{x}{l}$ sets $S \in \binom{X}{l}$ satisfy $|P(S)| > \frac{1}{3}\gamma\binom{m}{k-l}$. Then, by averaging, there exists $E_0 \in \binom{M}{k-l}$ such that $E_0 \in P(S)$ for at least $\frac{1}{3}\gamma^2\binom{x}{l}$ sets $S \in \binom{X}{l}$. As there are only $O(1)$ different labeled $(k-l)$-graphs on $k(k-l)$ vertices, there exists a particular hypergraph L_0 on the vertex set $\bigcup_{e\in E} e$ and, for some $c = c(\gamma, k) > 0$, at least $c\binom{x}{l}$ sets $S \in \binom{X}{l}$ such that $L_S(E_0) = L_0$. Remembering that $x = \sqrt{n}$, we see that one can choose from among these sets $k-l+1$ disjoints sets $S_1, \dots, S_{k-l+1}$. (We could choose more, but this is what we need.)

Since $E_0 \in P(S_i)$ and $L_{S_i}(E_0) = L_0$ for all $i = 1, \dots, k-l+1$, there is a matching M_0 in L_0 of size $k-l+1$, say $M_0 = \{T_1, \dots, T_{k-l+1}\}$. But then the sets $S_i \cup T_i$, $i = 1, \dots, k-l+1$ form a matching in H of size $k-l+1$ which intersects only $k-l$ edges of M (the edges in E). This is a contradiction with the maximality of M in H. $\square$

Fact 2 alone yields a weaker version of Lemma 2 without the term "$-\frac{1}{k^{k-l}}$", and thus, together with the absorption lemma, it provides an alternative proof of Theorem 1. To prove Theorem 2 we need another, much more involved statement.

Fact 3. *For at most $\gamma\binom{x}{l}$ sets $S \in \binom{X}{l}$ we have $|B(S)| > \frac{1}{3}\gamma\binom{m}{k-l}$.*

Proof. Suppose that at least $\gamma\binom{x}{l}$ sets $S \in \binom{X}{l}$ satisfy $|B(S)| > \frac{1}{3}\gamma\binom{m}{k-l}$. Fix one such S. Let $\mathcal{P}_k$ be a $(k-l)$-graph consisting of $2(k-l)+1$ vertices $e_1, \dots, e_{2(k-l)+1}$ and four edges $E_i = \{e_i, \dots, e_{i+k-l-1}\}$, $i \in \{1, 2, k-l+1, k-l+2\}$. Let $\mathcal{F}$ consist of $k-l$ disjoint copies $\mathcal{P}^1, \mathcal{P}^2, \dots, \mathcal{P}^{k-l}$ of $\mathcal{P}_k$, whose midpoints, $e^1_{k-l+1}, e^2_{k-l+1}, \dots, e^{k-l}_{k-l+1}$ form an edge E_0 (see Fig. 1).

It is time to recall the Erdős counting lemma, Lemma 1, by which there are $\Theta(m^{(k-l)(2(k-l)+3)})$ copies of $\mathcal{F}$ in $B(S)$.

By the same averaging argument as before, we conclude that there exists a copy $\mathcal{F}_0$ of $\mathcal{F}$ and, say, $(k-l)^2+1$ disjoint sets $S_1, \dots, S_{(k-l)^2+1}$ in $\binom{X}{l}$ such that for every edge $E \in \mathcal{F}_0$ and every $q = 1, \dots, (k-l)^2+1$, we have $L_{S_q}(E) = K(E)$, where $K(E)$ is a copy of the critical hypergraph K with the partition classes $e \in E$, one of which contains the set $I(E)$ of l isolated vertices. To get a contradiction with the maximality of M, we have to find a matching M' in $\bigcup_{E\in\mathcal{F}_0} K(E)$ of some size $h \le (k-l)^2+1$ which touches at most $h-1$ edges of M. That matching, combined with the sets $S_1, \dots, S_h$ will yield an enlargement of M.

To show the existence of the required matching, we consider a couple of cases with respect to the location of the sets $I(E)$.

Case 1. If for all $j = 1, \dots, k-l$, $I(E^j_{k-l+1}) \not\subset e^j_{k-l+1}$ then construct M' by taking any edge T of K_{E_0} plus $(k-l)$-matchings $M^j \subset K(E^j_{k-l+1})$, $j = 1, \dots, k-l$, disjoint from T. Matching M' has $(k-l)^2+1$ edges, but it intersects only $(k-l)^2$ edges of M.

Case 2. There exists $j \in \{1, \dots, k-l\}$ such that $I(E^j_{k-l+1}) \subset e^j_{k-l+1}$. Without loss of generality we assume that $j = 1$ and suppress the superscript ^{1} thereafter. We also introduce shorthand notation $I_i = I(E_i)$ and $K_i = K(E_i)$.

Subcase 2a. If $I_2 \subset e_{k-l+1}$ then take as M' a matching M_1 of size $k-l$ in K_1 and a matching M_{k-l+2} of size $k-l$ in K_{k-l+2}, and supplement them by two disjoint edges,

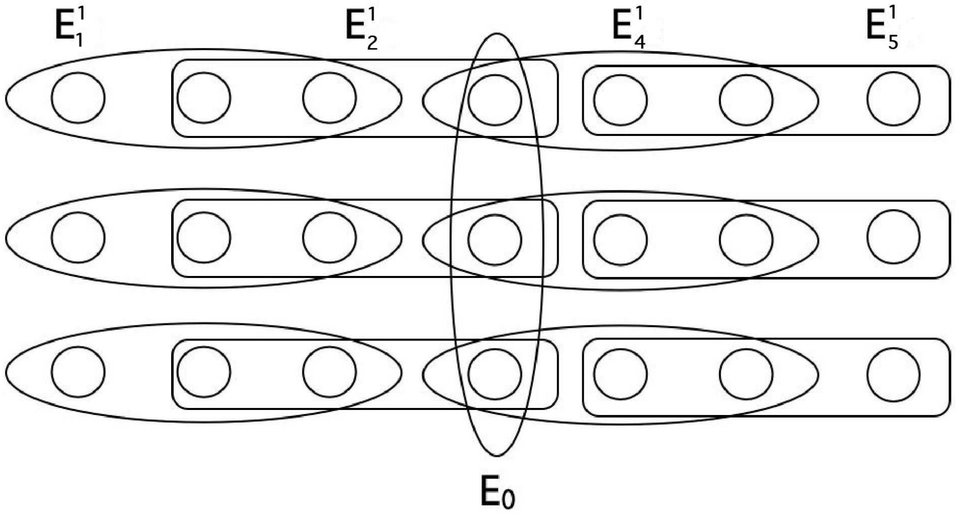

Figure 1: The hypergraph $\mathcal{F}$ for $k-l=3$

$T' \in K_{k-l+1}$ and $T'' \in K_2$. Since $|I_2 \cap I_{k-l+1}| \le l \le k-2$, the choice of T' nd T'' is always possible. Thus, the obtained matching M' has size $2(k-l)+2$, but it intersects only $2(k-l)+1$ edges of M (see Fig. 2).

Subcase 2b. If $I_2 \not\subset e^j_{k-l+1}$ then take as M' a matching M_2 of size $k-l$ in K_2 and a matching M_{k-l+2} of size $k-l$ in K_{k-l+2}, and supplement them by an edge, $T \in K_{k-l+1}$. The obtained matching M' has size $2(k-l)+1$, but it intersects only $2(k-l)$ edges of M. □

As a consequence of Facts 2 and 3, the number of sets $S \in \binom{X}{l}$ violating (3) is smaller than $2\gamma\binom{x}{l}$, and so, there is a set S not satisfying (2). This concludes the proof of Lemma 2. □

Remark 1. In order to close the gap between the conjectured threshold (1) and the bound we proved in this paper, whenever $1-(1-\frac{1}{k})^{k-l} \ge \frac{1}{2}$, one should try to find a $(k-l)$-partite, $(k-l)$-graph $\mathcal{F}$ with the following property: for any replacement of its edges $E \in \mathcal{F}$ with copies of (possibly different) $(k-l)$-partite, $(k-l)$-graphs Q_E such that, for each $E \in \mathcal{F}$, Q_E has

- k vertices in each partition class,
- more than $k^{k-1}-(k-1)^{k-l}$ edges, and

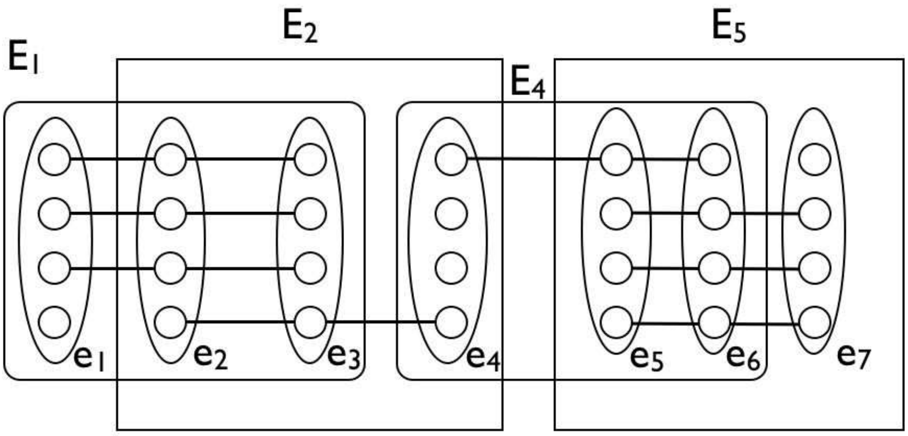

Figure 2: Illustration to Subcase 2a

- no matching of size $k - l + 1$,

the resulting hypergraph contains a matching of some size h which stretches over less than h partition classes. Then, the method applied in this paper would work. This is a finite problem and could, in principle, be solved by a computer search. However, the complexity for such an approach grows prohibitively fast with k.

4 Further improvement for $k = 4$, $l = 1$

As an encouragement toward the approach described in Remark 1, for $k = 4$ and $l = 1$ we show here how one can improve the coefficient $\frac{47}{64}$ of the bound in Theorem 2. We believe that similarly but with a significantly bigger effort one can get down to the conjectured 37/64.

Theorem 4. *For all $\epsilon > 0$, there is n_0 such that if H is a 4-graph on $n > n_0$ vertices, n divisible by 4, with*

$$\delta_1(H) \geq \left(\frac{42}{64} + \epsilon\right)\binom{n-1}{3},$$

then H contains a perfect matching.

Proof. The proof follows the lines and notation of the proof of Theorem 2, but we analyze the structure of $L_S(E)$ with more care. Since now $|S|=1$, in our notation we will identify S with its element s. In order to prove an analog of Lemma 2, for every $s\in X$ we now partition the family of triples of the edges of M as follows. We write

$$\binom{M}{3}=P(s)\cup A(s)\cup B(s),$$

where

- $P(s)=\left\{E\in\binom{M}{3}:L_s(E)\text{ has a perfect matching (of size 4)}\right\}$
- $A(s)=\left\{E\in\binom{M}{3}:|L_s(E)|\le 42\right\}$
- $B(s)=\left\{E\in\binom{M}{3}\setminus A(s):L_s(E))\text{ has an isolated vertex}\right\}$

We checked by computer (cf. Table 1 in Section 2) that 3-partite 3-graphs L with 4 vertices in each class, at least 43 edges, and without a perfect matching must have $\delta(L)=0$. Hence, the above partition of $\binom{M}{3}$ is complete. All we have to show is that there exists a vertex $s\in X$ with

$$\max(|P(s)|,|B(s)|)\le\frac{\gamma}{3}\binom{m}{3}. \tag{5}$$

We handle $P(s)$ exactly as in Fact 2. For $B(s)$ we look closer at the structure of $L_s(E)$. For a 3-partite $4\times4\times4$ 3-graph L with partition classes $V(L)=e\cup f\cup g$, we call a vertex $v\in V(L)$ *free* if there exists a 3-matching M in L such that $v\notin V(M)$; we call a pair of vertices $v,w\in V(L)$ *free* if there exists a 3-matching M in L such that $\{v,w\}\cap V(M)=\emptyset$. Note that if $|L|\ge 37$ then L contains at most one isolated vertex.

Fact 4. *For every $s\in X$ and every $E\in B(s)$, if $e\in E$ contains the isolate of $L_s(E)$ then all pairs of vertices v,w, where $v\in f$ and $w\in g$, are free. In particular, every $v\in f\cup g$ is free. Moreover, e contains at least two vertices of degrees at least 14.*

Proof. Let $u\in e$, $deg(u)=0$. Take any $v\in f$ and $w\in g$. The total number of edges containing at least one of these two vertices, but not containing u is at most $48-27=21$. Thus, $L_s(E)-\{u,v,w\}$ is a 3-partite $3\times3\times3$ 3-graph with at least $43-21=22\ge19$ edges, and so, by Fact 1, it has a perfect matching, implying that the pair v,w is free in $L_s(E)$. The sum of degrees of the three vertices of $e\setminus\{u\}$ equals at least 43, so the second statement follows. □

It remains to prove the following lemma.

Fact 5. *For at most γx vertices $s\in X$ we have $|B(s)|>\frac{1}{3}\gamma\binom{m}{3}$.*

Proof. Suppose that at least γx vertices $s\in X$ satisfy $|B(s)|>\frac{1}{3}\gamma\binom{m}{3}$. Fix one such s. Let $\mathcal{F}$ consist of 3 disjoint copies $\mathcal{P}^1,\mathcal{P}^2,\mathcal{P}^3$ of the path $\mathcal{P}_4$ described in the proof of Theorem 2, whose midpoints are connected by an edge E_0 (see Fig. 1). By Lemma 1, there are $\Theta(m^{21})$ copies of $\mathcal{F}$ in $B(s)$.

By averaging, there exist 10 vertices $s_1, \ldots, s_{10}$ and a copy $\mathcal{F}_0$ of $\mathcal{F}$ such that for every edge E of $\mathcal{F}_0$ we have $E \in B(s_j)$, and the $4\times4\times4$ 3-graphs $L(E) := L_{s_j}(E)$ are the same for all j. Let us denote the edges forming $\mathcal{F}_0$ by E_i^1, E_i^2, E_i^3, $i = 1, 2, 4, 5$, and E_0, where the superscript indicates which path they belong to. The vertices of these paths are denoted, correspondingly, by e_i^1, e_i^2, e_i^3. Thus, $E_0 = \{e_4^1, e_4^2, e_4^3\}$. For each $E \in \mathcal{F}_0$ let $i(E)$ be the isolated vertex in $L(E)$. To get a contradiction with the maximality of M, we have to find a matching M' in $\bigcup_{E\in\mathcal{F}_0} L(E)$ of some size $h \le 10$ which touches at most $h-1$ edges of M.

Case 1. If for all $j = 1, 2, 3$, $i(E_4^j) \notin e_4^j$ then construct M' by taking any edge T_0 of $L(E_0)$ plus three $(k-l)$-matchings $M^j \subset L(E_4^j)$, $j = 1, 2, 3$, disjoint from T_0. Since, by Fact 4, the sole vertex in $T_0 \cap e_4^j$ is free in $L(E_4^j)$, the existence of M^j follows, $j = 1, 2, 3$. Then $M' = M^1 \cup M^2 \cup M^3 \cup \{T_0\}$ is a 10-matching $T_0, T_1, \ldots, T_9$ in $\bigcup_{j=1}^3 L(E_4^j) \cup L(E_0)$ which intersects only 9 edges of M.

Case 2. There exists $j \in \{1, 2, 3\}$ such that $i(E_4^j) \in e_4^j$. Without loss of generality we assume that $j = 1$ and suppress the superscript ^{1} thereafter. We will use a shorthand notation $L_i := L(E_i)$. Consider two subcases with respect to $i(E_2)$.

Subcase 2a: $i(E_2) \in e_4$. Let $i(E_4) = u \in e_4$ and $i(E_2) = x \in e_4$, x and u possibly equal. Let $x_1 \ne u_1$ be two vertices of e_4 such that $deg_{L_2}(x_1) \ge 14$ and $deg_{L_2}(u_1) \ge 14$. Since e_5 and e_6 could be swapped around, we may assume that $i(E_5) \notin e_5$. There is a vertex $v_1 \in e_5$ such that $\{u_1, v_1, w_1\} \in L_4$ for all $w \in e_6$. Let M_5 be a 3-matching in L_5 which avoids v_1; it also avoids a vertex $w_1 \in e_6$. Similarly, there exists a 3-matching M_1 in L_1 and an edge $T'' = \{x_1, y_1, z_1\} \in L_2$ disjoint from M_1. Hence, altogether, $M_1 \cup M_5 \cup \{T', T''\}$ is an 8-matching in $L_1 \cup L_2 \cup L_4 \cup L_5$ intersecting only 7 edges of M.

Subcase 2b: $i(E_2) \in e_2 \cup e_3$. Let M_5 and T' be as in Subcase 2a, and let M_2 be a 3-matching in L_2 which avoids u_1. Then $M_2 \cup M_5 \cup \{T'\}$ is a 7-matching in $L_2 \cup L_4 \cup L_5$ intersecting only 6 edges of M. $\square$

This completes the proof of Theorem 4. $\square$

5 The proof of Proposition 1

Our proof is based on known constructions. Observe that for a Hamilton r-cycle C we have $|C| = \frac{n}{k-r}$, and, assuming that $k-r|n$, all vertex degrees in C are equal $\frac{k}{k-r}$. We consider three cases.

Case 1: $\frac{k}{k-r}$ is odd.

Let $H_1 = (V, E)$ where $V = A \cup B$, $\frac{1}{2}n - 1 \le |A| \le \frac{1}{2}n$, $|A|$ is odd, and E consists of all $e \in \binom{V}{k}$ such that $|e \cap V|$ is even. Note that $\delta_{k-1}(H_1) \ge \frac{1}{2}n - k$. Suppose that H_1 contains a Hamilton r-cycle C. Then, by double counting,

$$\sum_{e\in C} |e \cap A| = \sum_{v\in A} deg_C(v) = |A|\frac{k}{k-r}. \tag{6}$$

This is a contradiction, because the L-H-S is even, while the R-H-S is odd.

Case 2: $\frac{k}{k-r}$ is even and $\frac{n}{k-r}$ is odd.

Let $H_2 = (V, E)$ where $V = A \cup B$, $|A| = \lceil \frac{1}{2}n \rceil$, and E consists of all $e \in \binom{V}{k}$ such that $|e \cap V|$ is odd. Note that $\delta_{k-1}(H_2) \geq \frac{1}{2}n - k$. Suppose that H_2 contains a Hamilton r-cycle C. Then the L-H-S of (6) is odd, while the R-H-S is even, again, a contradiction.

Case 3: Both, $\frac{k}{k-r}$ and $\frac{n}{k-r}$ are even. Let $s \geq 2$ be the greatest common divisor of $\frac{k}{k-r}$ and $\frac{n}{k-r}$ (in fact, the highest common power of two would do). Set $r_s = k - s(k-r) = r - (s-1)(k-r)$ and note that every Hamilton r-cycle in H contains a Hamilton r_s-cycle in H, and consequently, $h_{r_s}(n, k) \leq h_r(k, n)$ (recall that for $r_s = 0$ this is a perfect matching). Finally, observe that the greatest common divisor of $\frac{k}{k-r_s}$ and $\frac{n}{k-r_s}$ equals one, and so, we are back in either Case 1 or Case 2 for r_s. Thus, either H_1 or H_2 shows that $h_{r_s}(n, k) \geq \frac{1}{2}n - k$, completing the proof. $\square$

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