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Measurable chromatic and independence numbers for ergodic graphs and group actions

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0 Introduction

We study in this paper some combinatorial invariants associated with ergodic actions of infinite, countable (discrete) groups.

Let (X, μ) be a standard probability space and Γ an infinite, countable group with a set of generators $S \subseteq \Gamma$. Given a free, measure-preserving, ergodic action a of Γ on (X, μ), we consider the graph $G(S, a) = (X, E(S, a))$, whose vertices are the points in X and where $x \neq y \in X$ are adjacent if there is a generator $s \in S$ taking one to the other. It is clear that the connected components of this graph are isomorphic to the Cayley graph $\text{Cay}(\Gamma, S)$ and thus parameters such as the chromatic number of $G(S, a)$ are identical to those of $\text{Cay}(\Gamma, S)$. This however requires selecting an element of each connected component and thus essentially depends upon a use of the Axiom of Choice. However, the situation is vastly different when one considers instead measurable colorings and the associated measurable chromatic numbers.

Let us introduce first the combinatorial invariants we will be interested in.

Consider a locally countable, Borel graph $G = (X, E)$ on a standard probability space (X, μ). We denote by E^* the associated Borel equivalence relation whose classes are the connected components of G. Given a property of equivalence relations $\mathcal{P}$, we say that G has property $\mathcal{P}$ if E^* has property $\mathcal{P}$. This explains what it means to say that G is (μ-)measure preserving, ergodic, hyperfinite, smooth, etc. In particular, the graphs $G(S, a)$ discussed before are measure preserving and ergodic.

Given such a graph $G = (X, E)$ its (μ-)*measurable chromatic number*, $\chi_\mu(G)$, is the smallest cardinality of a standard Borel space Y for which there is a (μ-)measurable coloring $c : X \to Y$ (i.e., $xEy \Rightarrow c(x) \neq c(y)$). Clearly $\chi_\mu(G) \in \{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$. It is well known (see, e.g., [12]) that there are acyclic such graphs G for which of course the usual chromatic number $\chi(G)$ is equal to 2 but $\chi_\mu(G) = 2^{\aleph_0}$.

In addition, we consider the *approximate* (μ-)*measurable chromatic number*, $\chi_\mu^{\mathrm{ap}}(G)$, which is defined as the smallest cardinality of a standard Borel space Y such that for each $\varepsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a measurable coloring $c : A \to Y$ of the induced subgraph $G|A$. Clearly $\chi_\mu^{\mathrm{ap}}(G) \leq \chi_\mu(G)$.

Finally, the (μ-)*independence number* of G, $i_\mu(G)$, is the supremum of the measures of Borel independent sets ($A \subseteq X$ is *independent* if no two elements of A are adjacent in G). Clearly $i_\mu(G) \in [0, 1]$. It is easy to check that $\chi_\mu^{\mathrm{ap}}(G) i_\mu(G) \geq 1$, so graphs with small independence number have large (approximate) chromatic number.

We discuss in §2 various examples of invariant, ergodic graphs G with small chromatic number $\chi(G)$ (e.g., acyclic) but for which $\chi_\mu^{\mathrm{ap}}(G)$ or $\chi_\mu(G)$ take various finite or infinite values, and others in which $i_\mu(G)$ takes any value in $[0, 1)$ (the value 1 can be easily seen to be impossible to realize in such a graph).

However for graphs of bounded degree, there are further restrictions (see 2.16).

Theorem 0.1. *Let (X, μ) be a standard probability space and $G = (X, E)$ an invariant, ergodic graph with degree bounded by $d \geq 2$. Then $\chi_\mu^{\mathrm{ap}}(G) \leq d$ and thus $i_\mu(G) \geq 1/d$.*

In §3, we consider the case of hyperfinite graphs. Using some techniques of Miller [18], we show (see 3.1, 3.8):

Theorem 0.2. *Let (X, μ) be a standard probability space and G a locally countable, acyclic, (μ-)hyperfinite graph. Then $\chi_\mu^{\mathrm{ap}}(G) \leq 2$ and thus $i_\mu(G) \geq 1/2$. If moreover G is locally finite, (μ-)hyperfinite but not necessarily acyclic, then $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$ and thus $i_\mu(G) \geq 1/\chi(G)$.*

In §4 we consider the graphs associated with group actions, as discussed in the beginning of this introduction. Let $\chi_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$, $i_\mu(S, a)$ be the parameters associated with $G(S, a)$. It is easy to see that $i_\mu(S, a) \leq 1/2$.

Let $a \prec b$ be the relation of weak containment among measure preserving actions of Γ on (X, μ); see [10]. We have $a \prec b$ iff a is in the closure, in the weak topology, of the conjugacy class of b. We now have the following monotonicity properties (see 4.2, 4.4).

Theorem 0.3. *Let Γ be a countable group and S a finite set of generators. Then*

$$a \prec b \Rightarrow i_\mu(S, a) \leq i_\mu(S, b), \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, b).$$

It follows that $i_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$ are invariants of weak equivalence, $a \sim b$, where $a \sim b \Leftrightarrow a \prec b$ and $b \prec a$. Now it is known, see, e.g., [10], 13.2, that any two free, measure-preserving, ergodic actions of an amenable group are weakly equivalent, thus for such Γ,

$$\begin{aligned} \chi_\mu^{\mathrm{ap}}(\Gamma, S) &= \chi_\mu^{\mathrm{ap}}(S, a) \\ i_\mu(\Gamma, S) &= i_\mu(S, a) \end{aligned}$$

are independent of a. Recall that $i_\mu(\Gamma, S) \leq 1/2$ and thus $\chi_\mu^{\mathrm{ap}}(\Gamma, S) \geq 2$. We show in 4.9, 4.10:

Theorem 0.4. *If Γ is a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators, then*

$$i_\mu(\Gamma, S) = 1/2 \Leftrightarrow \chi_\mu^{\mathrm{ap}}(\Gamma, S) = 2 \Leftrightarrow \mathrm{Cay}(\Gamma, S) \textit{ is bipartite.}$$

Note that $\mathrm{Cay}(\Gamma, S)$ is bipartite iff there is no odd length word in $S \cup S^{-1}$ equal to the identity in Γ. We show in 4.7 that (for any Γ, S) if $\mathrm{Cay}(\Gamma, S)$ is not bipartite, then $i_\mu(S, a) < 1/2$ and $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$. In fact in this case $i_\mu(S, a) \leq 1/2 - 1/(2g)$, where g is the odd girth (= length of shortest odd cycle) in $\mathrm{Cay}(\Gamma, S)$. On the other hand for Γ, S with bipartite $\mathrm{Cay}(\Gamma, S)$, we have the following characterization of amenability (see 4.12).

Theorem 0.5. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators with $\mathrm{Cay}(\Gamma, S)$ bipartite. Then the following are equivalent:*

(i) *Γ is amenable,*

(ii) *$i_\mu(S, a) = 1/2$, for every free, measure-preserving, ergodic action a,*

(iii) *$\chi_\mu^{\mathrm{ap}}(S, a) = 2$, for every free, measure-preserving, ergodic action a.*

We also have an analogous characterization of groups that have property (T) and the Haagerup Approximation Property HAP (see 4.13).

Theorem 0.6. *Let Γ be an infinite, countable group and $S \subseteq \Gamma$ a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *has property (T),*

(ii) $i_\mu(S, a) < 1/2$ *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$ *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$.

Also the following are equivalent:

(i*) Γ *does not have the (HAP),*

(ii*) $i_\mu(S, a) < 1/2$ *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iii*) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$ *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$.

We next consider the shift action s_Γ of the free group $\Gamma = \mathbb{F}_m$, with m generators $S = \{a_1, \dots, a_m\}$, on 2^Γ with the product measure. Using a result of Kesten [14] for the norm of averaging operators, we show the following result (see 4.15).

Theorem 0.7. *Let $\Gamma = \mathbb{F}_m$ be the free group with a free set of generators S and let s_Γ be its shift action on 2^Γ. Then*

$$\frac{1}{2m} \leq i_\mu(S, s_\Gamma) \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}},$$

and

$$2m \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m + \sqrt{2m-1}}{\sqrt{2m-1}}.$$

From this it also follows that for any $2 \leq n < \aleph_0$ there are acyclic, bounded degree, invariant, ergodic G with $\chi_\mu(G) = n$ (see 2.5). Without the requirements of bounded degree or invariance, such examples were first found by Laczkovich (see [12], Appendix).

The exact values of $i_\mu(S, s_\Gamma)$, $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma)$ in 0.7 are unknown. It should be noted that there is no known example of (Γ, S), with Γ amenable and S finite, for which $\chi_\mu(S, s_\Gamma) > \chi(\mathrm{Cay}(\Gamma, S)) + 1$. For instance, Gao-Jackson [4] have shown that for $\Gamma = \mathbb{Z}^m$, with S the usual set of generators, $\chi_\mu(S, s_\Gamma) \leq 4$ (while of course $\chi(\mathrm{Cay}(\Gamma, S)) = 2$).

Finally in §4 we discuss, for any (Γ, S), canonical finite graphs that "approximate" the infinite graph $G(S, s_\Gamma)$ associated with the shift of Γ on 2^Γ. Applying this to $\Gamma = \mathbb{F}_m$ and a free set of generators S produces a natural explicit family of finite graphs $G_{n,m,k}$ $(n, m, k \geq 1)$ which simultaneously have arbitrarily large odd girth $g_{\mathrm{odd}}(G)$ and arbitrarily small independence ratio $i(G)$ (= the ratio of the maximum size of an independent set to the total number of vertices), thus arbitrarily large chromatic numbers. Such explicit families appear to be of interest in finite graph theory. More precisely, we have (see 4.19):

Theorem 0.8. *There is an explicit family of graphs $G_{m,n,k}$ $(m, n, k \geq 1)$ of finite graphs such that for any m, k, if n is large enough (depending upon m, k), then*

$$\begin{aligned} g_{\mathrm{odd}}(G_{m,n,k}) &> k, \\ i(G_{m,n,k}) &\leq \frac{2\sqrt{2m-1}}{m+\sqrt{2m-1}}, \end{aligned}$$

and thus

$$\chi(G_{m,n,k}) \geq \frac{m+\sqrt{2m-1}}{2\sqrt{2m-1}}.$$

In the last section §5 we discuss a matching problem in the Borel and measurable contexts related to earlier work of Laczkovich [16] and Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [15].

Addendum. We have now received a preliminary draft of a paper by R. Lyons and F. Nazarov with title "Perfect matchings as IID factors on non-amenable groups" that has some overlap with results in our paper.

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1 Preliminaries

(**A**) A *graph* is a pair $G = (X, E)$, where X is a set whose elements we call *vertices* of G and $E \subseteq X^2$ satisfies: $(x, x) \notin E$ (i.e., there are no loops) and $(x, y) \in E \Leftrightarrow (y, x) \in E$ (i.e., the graph is symmetric). We often write xEy to denote $(x, y) \in E$, and we identify E with the set of unordered pairs $\{\{x, y\} : xEy\}$, which we call the *edges* of G. If xEy we say that x, y are *adjacent*.

Occasionally we will also consider *graphs with* (possible) *loops*, those in which $(x, x) \in E$ is allowed, for some $x \in X$, and *directed graphs*, those in which $E \subseteq X^2$ is not necessarily symmetric.

A *path* in G is a sequence $x_0, x_1, \ldots, x_n$, $n \geq 1$, of distinct vertices such that $x_i E x_{i+1}$, $0 \leq i < n$. The *length* of such a path is the number of edges it uses, so the length of $x_0, x_1, \ldots, x_n$ is n. Such a path is a *cycle* if $n \geq 2$ and, moreover, $x_n E x_0$; its length is $n + 1$.

We denote by E^* the smallest equivalence relation containing E. Its equivalence classes are the *connected components* of G, and thus two vertices x, y are *connected* exactly when $x = y$ or there is a path $x = x_0, x_1, \ldots, x_n = y$. If x, y are connected, we set $d_G(x, y)$ equal to the length of the shortest path from x to y, and call it the *distance* from x to y. The graph is *connected* if it has a unique connected component.

A graph is *acyclic* if it contains no cycles; we sometimes call such graphs *forests* and their connected components *trees*. One easily sees that if x, y are distinct vertices of a tree, then there is a unique path from x to y.

The *girth* of G, in symbols $g(G)$, is the length of the smallest cycle in G. By convention, we set $g(G) = \infty$ when G is a forest. The *odd girth* of G, $g_{\text{odd}}(G)$, is the length of the smallest odd cycle in G. Again $g_{\text{odd}}(G) = \infty$ if there are no odd cycles.

The *degree* of a vertex x, denoted $d(x)$, is the cardinality of the set $E_x = \{y \in X : xEy\}$. If $d(x) \leq \aleph_0$ for all vertices x, we say that G is *locally countable*, and if $d(x) < \aleph_0$ for all x, we say that G is *locally finite*. If $d(x) \leq d < \aleph_0$, for all x, we say that G has *bounded degree*.

Given $A \subseteq X$, we define the *induced subgraph* on A, written $G|A$, to be $(A, E \cap A^2)$. We say that $A \subseteq X$ is *independent* for G if $G|A$ is trivial, i.e., no two vertices in A are adjacent.

A graph $G = (X, E)$ is *bipartite* if there is a partition $X = X_1 \sqcup X_2$, with each X_i $(i = 1, 2)$ independent. It is well known that a graph is bipartite if it has no odd length cycles. Any acyclic graph is therefore bipartite.

The *chromatic number* of a graph G, in symbols $\chi(G)$, is the smallest cardinality of a set Y for which there is a map $c : X \to Y$ (a vertex coloring) such that $xEy \Rightarrow c(x) \neq c(y)$. Thus the graph is bipartite iff $\chi(G) \leq 2$.

(**B**) Let now X be a standard Borel space. By a *measure* on X we mean a finite Borel measure. If μ is a measure on X with $0 < \mu(X) < \infty$ we call the pair (X, μ) a *standard measure space*. If $\mu(X) = 1$, we call (X, μ) a *standard probability space* and μ a *probability measure*. *Unless otherwise indicated or clear from context (e.g., when X is finite), measures will be assumed to be non-atomic.*

(**C**) When $G = (X, E)$ is a graph and X a standard Borel space, we say that G is *Borel* if $E \subseteq X^2$ is Borel. In this situation we have that E^* is an analytic equivalence relation, but if we assume in addition that G is locally countable, then E^* is a countable (i.e., having all its classes countable) Borel equivalence relation. We will be primarily interested in locally countable, Borel graphs, and will thus borrow from the theory of countable Borel equivalence relations. For more details see, e.g., [11].

A countable Borel equivalence relation R on a standard measure space (X, μ) is *invariant* (relative to μ) if whenever $f : X \to X$ is a Borel automorphism with graph contained in R, $f_*\mu = \mu$ (where, as usual, $f_*\mu(A) = \mu(f^{-1}(A))$). For $A \subseteq X$, we define the *R-saturation* of A, written $[A]_R$, to be the set $\{x \in X : \exists y \in A(xRy)\}$. We say that R is *ergodic* (relative to μ) if for all Borel $A \subseteq X$, $[A]_R$ is either μ-null or μ-conull. When R can be written as an increasing union of finite (i.e., having all its classes finite) Borel equivalence relations on X we call it *hyperfinite*, and when R admits a Borel transversal (i.e., a set meeting each R-class in exactly one point), we say it is *smooth*. We say $G = (X, E)$ is *invariant* if E^* is; likewise we say that G is *ergodic*, *hyperfinite*, or *smooth*, if E^* is.

2 Chromatic and independence numbers

(**A**) A *coloring* of a locally countable, Borel graph $G = (X, E)$ on a standard Borel space X is a map $c : X \to Y$, where Y is a standard Borel space, such that

$$xEy \Rightarrow c(x) \neq c(y),$$

i.e., $\forall y \in Y(c^{-1}(\{y\})$ is independent). The *chromatic number* of G, in symbols

$$\chi(G),$$

is the smallest cardinality of a space Y as above for which there is a coloring $c : X \to Y$. Clearly $\chi(G) \in \{1, 2, \ldots, n, \ldots, \aleph_0\}$ (and $\chi(G) \geq 2$ unless $E = \emptyset$).

(**B**) If the coloring c as in (**A**) is Borel as a map from X into Y, we call c a *Borel coloring*. We define the *Borel chromatic number* of G, in symbols

$$\chi_B(G),$$

to be the smallest cardinality of a standard Borel space Y for which there is a Borel coloring $c : X \to Y$. Clearly $\chi_B(G) \in \{1, 2, 3, \ldots, n, \ldots, \aleph_0, 2^{\aleph_0}\}$ and

$$\chi(G) \leq \chi_B(G).$$

Example 2.1. In [12] and Miller [18], various examples of non-smooth $G = (X, E)$ are discussed, all of which are acyclic (so $\chi(G) = 2$), but $\chi_B(G)$ ranges over all values in $\{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$. It follows that for any $m, n \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$ with $m \leq n$ and $m < 2^{\aleph_0}$, there is such a G with $\chi(G) = m$ and $\chi_B(G) = n$ (just add a single connected component of chromatic number m to a graph G that has $\chi(G) = 2$ and $\chi_B(G) = n$).

(**C**) Suppose now that (X, μ) is a standard measure space (perhaps with atoms) and $G = (X, E)$ is a locally countable, Borel graph on X. We define the (μ-)*measurable chromatic number* of G, in symbols

$$\chi_\mu(G),$$

as the smallest cardinality of a standard Borel space Y for which there is a (μ-)measurable coloring $c : X \to Y$. Again,

$$\chi(G) \leq \chi_\mu(G) \leq \chi_B(G).$$

Example 2.2. There is an acyclic, locally countable, Borel graph G on a standard measure space (X, μ) with G invariant, ergodic and

$$2 = \chi(G) < \chi_\mu(G) = \chi_B(G) = 2^{\aleph_0}.$$

To see this, take a compact, metrizable group X that contains a dense subset $\{a_n\}_{n\in\mathbb{N}}$ which generates a free subgroup (e.g., $X = \mathrm{SO}_3(\mathbb{R})$) and let μ be the Haar measure on X. Consider the graph $G = (X, E)$, where $xGy \Leftrightarrow \exists n(x = a_n^{\pm 1}y)$. Then G is acyclic and invariant, ergodic. So $\chi(G) = 2$ but if $c : X \to \mathbb{N}$ is a measurable coloring, then for some n, $Y = c^{-1}(\{n\})$ has positive measure, so YY^{-1} contains an open neighborhood of 1. Then there are $x, y \in Y$, $k \in \mathbb{N}$ with $xy^{-1} = a_k$, thus xEy, a contradiction.

Example 2.3. There is an acyclic, locally countable, Borel graph G on a standard measure space (X, μ) with G invariant, ergodic and

$$2 = \chi(G) < 3 = \chi_\mu(G) < \chi_B(G) = 2^{\aleph_0}.$$

Take, for instance, the graph G_0 on $X = 2^{\mathbb{N}}$ defined in [12], where it is shown that G_0 is acyclic (thus $\chi(G_0) = 2$), but $\chi_B(G_0) = 2^{\aleph_0}$. Miller [18] showed that $\chi_\mu(G_0) = 3$, where μ is the usual product measure on $2^{\mathbb{N}}$, for which G_0 is invariant, ergodic.

Example 2.4. It is easy to construct an example of a locally countable, Borel graph G on a standard measure space (X, μ) with G invariant, ergodic, for which $\chi(G) = \chi_\mu(G) = \chi_B(G) = 2$. Take a countable, measure-preserving, ergodic equivalence relation R on (X, μ) and let $X = A \sqcup B$ be a Borel partition of X with A, B meeting each R-class. Let E be the bipartite graph with edges between all pairs of R-related points, one in A and the other in B. Clearly $\chi(G) = \chi_\mu(G) = \chi_B(G) = 2$ and G generates R.

Example 2.5. We will see in Section 4 examples of acyclic, bounded degree, Borel graphs $G = (X, E)$ on standard (X, μ) which are invariant, ergodic, and $\chi_\mu(G)$ is finite but arbitrarily large (although of course $\chi(G) = 2$). From this it follows that for each $2 \leq n < \aleph_0$, there is an acyclic, bounded degree $G = (X, E)$ on a standard measure space (X, μ) with G invariant, ergodic such that $\chi_\mu(G) = n$. To go from such a G that has $\chi_\mu(G) = k + 1 > 3$ to a $\overline{G}, \overline{\mu}$ that has $\chi_{\overline{\mu}}(\overline{G}) = k$, take a partition $A_0 \sqcup \cdots \sqcup A_k = X$ given by a measurable coloring of G, and assume without loss of generality that $\mu(A_0) < 1$. Let $X' = A_1 \sqcup \cdots \sqcup A_k$, $G' = G|X' = (X', E')$ be the induced subgraph on X' and let $\mu' = \mu|X'$. Clearly G' is acyclic, $\chi_{\mu'}(G') = k$, and G' is invariant (for μ'). Consider then the ergodic decomposition associated with $(E')^*$ (on X'). If all the pieces of this decomposition have (for the induced subgraph G') measurable chromatic number $\leq k - 1$ then, by measurable selection, we can find a μ'-conull set on which G' admits a $k-1$ measurable coloring, and since G' has chromatic number ≤ 2, it follows that $\chi_\mu(G') \leq k-1$, a contradiction. So there is a piece of the ergodic decomposition $(\overline{X}, \overline{\mu})$, so that if $\overline{G} = G'|\overline{X}$ is the induced subgraph, then $\overline{G}$ is acyclic, $\chi_{\overline{\mu}}(\overline{G}) = k$, and $\overline{G}$ is invariant, ergodic. Finally, $\overline{\mu}$ is non-atomic, else $\overline{X}$ would have to be finite, and so $\overline{G}$ would have chromatic (and so $\overline{\mu}$-measurable chromatic) number at most $2 < k$, a contradiction.

An analogous argument produces, for each $2 \leq n < \aleph_0$, an acyclic, bounded degree, Borel graph G with $\chi_B(G) = n$.

Example 2.6. There is an example of a locally countable, Borel G on a standard measure space (X, μ) with G invariant, ergodic for which $\chi(G) \leq 3$ and $\chi_\mu(G) = \aleph_0$. To see this, take, for each n, as in Example 2.5 an acyclic, locally countable, Borel graph $G_n = (X_n, E_n)$ on a standard probability space (X_n, μ_n) with G_n invariant, ergodic, and $\chi_{\mu_n}(G_n) > n$. Fix a standard probability space (X, μ) and a Borel partition $X = \bigsqcup_{n=1}^{\infty} A_n$ with $\mu(A_n) = 1/2^n$. Fix a Borel bijection $\varphi_n : X_n \to A_n$ sending μ_n to $\mu_{A_n} = \frac{\mu|A_n}{\mu(A_n)}$ and let $G'_n = (A_n, E'_n)$ be the image of G_n under this bijection. Find, for each n, two disjoint, Borel sets $C_n, D_n \subseteq A_n$ of positive measure which are independent for G'_n, and such that there are Borel isomorphisms $\varphi_n : C_n \to C_{n+1}$, for $n \geq 1$, n odd, and $\psi_n : D_n \to D_{n+1}$ for $n \geq 2$, n even. Let $G = (X, E)$ be the graph on X whose edges are those in $\bigcup_n E'_n$ together with the graphs of φ_n, ψ_n, and their inverses. Then G is invariant, ergodic and $\chi_\mu(G) = \aleph_0$. Finally, $\chi(G) \leq 3$. Indeed, fix the same colors a, b witnessing the 2-colorability of each G'_n. Then change the color of each element of $\bigcup_{n,\text{ odd}} C_n \cup \bigcup_{n,\text{ even}} D_n$ to some third color c. This gives a 3-coloring of G.

We do not know an example of G, X, μ as above for which G is acyclic (or even has $\chi(G) = 2$) but $\chi_\mu(G) = \aleph_0$. More generally, we do not know what are the possible values of $k, l, m \in \{2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$ with $k \leq l \leq m$ such that there is a locally countable, Borel graph G on a standard measure space (X, μ) which is invariant, ergodic, and

$$\chi(G) = k, \chi_\mu(G) = l, \chi_B(G) = m.$$

Remark 2.7. One can also define the (μ-)*almost everywhere measurable chromatic number* of G, in symbols $\chi_\mu^{\mathrm{ae}}(G)$, as the smallest cardinality of a standard Borel space Y for which there is a Borel set $A \subseteq X$ with $\mu(A) = 1$ and a Borel coloring $c : A \to Y$ of the induced subgraph $G|A = (A, E \cap A^2)$.

Clearly, $\chi_\mu^{\mathrm{ae}}(G) \leq \chi_\mu(G)$. However, if $\chi_\mu^{\mathrm{ae}}(G) \geq \chi(G)$, which will be the case for most graphs that we will be interested in, then $\chi_\mu^{\mathrm{ae}}(G) = \chi_\mu(G)$.

(**D**) Finally, when (X, μ) is a standard measure space and G is a locally countable, Borel graph on X, we define the *approximate* (μ-)*measurable chromatic number* of G, in symbols

$$\chi_\mu^{\mathrm{ap}}(G),$$

to be the smallest cardinality of a standard Borel space Y such that for every $\varepsilon > 0$ there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \varepsilon$ and a measurable coloring $c : A \to Y$ of the induced subgraph $G|A$. Again,

$$\chi_\mu^{\mathrm{ap}}(G) \leq \chi_\mu(G).$$

Example 2.8. There is an acyclic, locally countable, Borel graph $G = (X, E)$ on a standard measure space (X, μ) which is invariant, ergodic, and $\chi_\mu^{\mathrm{ap}}(G) < \chi_\mu(G)$. For instance, consider the shift S on $2^{\mathbb{Z}}$ and let $X \subseteq 2^{\mathbb{Z}}$ be its aperiodic part. Let μ be the restriction of the usual product measure to X (note that $\mu(X) = 1$). Let for $x, y \in X$, $xEy \Leftrightarrow x = S^{\pm 1}(y)$. Then by Rokhlin's Lemma (see, e.g., [11] 7, 7.5), $\chi_\mu^{\mathrm{ap}}(G) = 2$. On the other hand, $\chi_\mu(G) = 3$. Otherwise, there is a measurable partition $X = A \sqcup B$ into independent sets. Then $S(A) = B$ and $S(B) = A$, and so $\mu(A) = \mu(B) = 1/2$ and both A, B are S^2-invariant, which is impossible as S^2 is ergodic.

We have seen in Example 2.2 examples of acyclic, locally countable, Borel G on standard (X, μ) which are invariant, ergodic and have no independent sets of positive measure, thus $\chi_\mu^{\mathrm{ap}}(G) = 2^{\aleph_0}$. It is also easy to see that there is no such G, X, μ with $\chi_\mu^{\mathrm{ap}}(G) = 1$ (i.e., there cannot exist Borel independent sets whose measure is arbitrarily close to 1). Indeed, if $G = (X, E)$ is such that G is invariant, ergodic, then by the uniformization theorem for Borel sets with countable sections, there is a measure-preserving Borel bijection $\varphi : A \to B$ between Borel sets of positive measure such that $(x, \varphi(x)) \in E$, for all $x \in A$. If $\mu(A) = \mu(B) = \delta$ and $\varepsilon < \delta/2$, there can be no Borel independent set of measure bigger than $1 - \varepsilon$.

Example 2.9. There is an example of G,X,μ as in Example 2.6 with $\chi(G) \leq 3$ and $\chi_\mu^{\mathrm{ap}}(G) = \aleph_0$. The graphs $G = (X, E)$ on (X, μ) from Section 4 (mentioned previously in Example 2.5) which have arbitrarily large finite χ_μ actually have arbitrary large finite χ_μ^{ap}. Then, as in Example 2.6, this gives examples of G with $\chi(G) \leq 3$ and $\chi_\mu^{\mathrm{ap}}(G) = \aleph_0$.

Again, we do not know examples of such G with $\chi(G) = 2$ and $\chi^{\rm ap}_\mu(G) = \aleph_0$. Also, we do not know if there are such examples for which $\chi^{\rm ap}_\mu(G)$ takes an *arbitrary* value $3 \leq k < \aleph_0$. We will see in Proposition 4.15 examples where $\chi^{\rm ap}_\mu(G)$ is finite but ≥ 3.

The more general problem is again whether there is any other relationship between $\chi(G)$,$\chi_\mu(G)$,$\chi^{\rm ap}_\mu(G)$, beyond the obvious $\chi(G) \leq \chi_\mu(G)$, $\chi^{\rm ap}_\mu(G) \leq \chi_\mu(G)$ for locally countable (or locally finite), Borel graphs G on standard measure spaces (X, μ) which are invariant, ergodic.

(**E**) Let finally G be a locally countable, Borel graph on a standard probability space (X, μ). We define the *independence number* of G, in symbols

$$i_\mu(G),$$

by

$$i_\mu(G) = \sup\{\mu(Y) : Y \subseteq X \text{ is a Borel independent set}\}.$$

Clearly, we can replace "Borel" by "(μ-)measurable" in this definition. (If μ is not a probability measure we replace $\mu(Y)$ by $\mu(Y)/\mu(X)$ in the definition above.)

Obviously, $0 \leq i_\mu(G) \leq 1$ and $i_\mu(G) = 0$ means that there is no positive measure independent set. We have seen in Example 2.2 examples of such graphs (they clearly have $\chi_\mu(G) = 2^{\aleph_0}$). If $G = (X, E)$ is a locally countable, Borel graph on a standard probability space (X, μ) with G invariant, ergodic, then we have seen in (**D**) that $i_\mu(G) < 1$ (otherwise $\chi^{\rm ap}_\mu(G) = 1$).

Example 2.10. For each $0 < a < 1$ there is an acyclic, locally countable Borel graph G on a standard probability space (X, μ) which is invariant, ergodic and $i_\mu(G) = a$ with the supremum being attained.

To see this, first fix an acyclic $G_1 = (X_1, E_1)$ on (X_1, μ_1) with G_1 invariant, ergodic, $\mu_1(X_1) = 1 - a$, and $i_{\mu_1}(G_1) = 0$. Also fix $k > \frac{a}{1-a}$. Let X_2 be an uncountable standard Borel space disjoint from X_1 and partition it into k uncountable Borel sets: $X_2 = A_1 \sqcup \cdots \sqcup A_k$. Fix a Borel subset Y_1 of X_1, meeting each E_1^*-class, such that $\mu_1(Y_1) = \frac{a}{k} < 1 - a$. For each $1 \leq i \leq k$, let $f_i : Y_1 \to A_i$ be a Borel bijection. Use f_i to copy the measure $\mu_1|Y_1$ to A_i, say ν_i, and let $\mu_2 = \sum_{i=1}^n \nu_i$. Then $\mu_2(X_2) = k \cdot \frac{a}{k} = a$. Let $X = X_1 \sqcup X_2$, $\mu = \mu_1 + \mu_2$. Define the graph $G = (X, E)$ as follows: the edges of G are those in E_1 together with the graph of each f_i and its inverse. Clearly it is acyclic an it is easy to see that G is invariant, ergodic. Finally, X_2 is independent for G and if $A \subseteq X$ is Borel independent, then clearly $\mu(A \cap X_1) = 0$, so $\mu(B) \leq \mu(X_2) = a$. So $i_\mu(G) = a$ and the sup is attained.

Remark 2.11. When X is a finite set and μ is normalized counting measure, $i_\mu(G) = i(G)$ is usually called the *independence ratio* and $\alpha(G) =$ (the maximum cardinality of an independent subset of X) is called the independence number (thus $i(G) = \frac{\alpha(G)}{|X|}$). We will not use $\alpha(G)$ in this paper, so this should not cause any confusion.

We now have the following simple inequality:

Proposition 2.12. *Let G be a locally countable Borel graph on a standard probability space (X, μ) (perhaps with atoms). Then*

$$\chi_\mu^{\mathrm{ap}}(G) \geq \frac{1}{i_\mu(G)}.$$

Proof. This is clear if $i_\mu(G) = 0$. If $\chi_\mu^{\mathrm{ap}}(G) = k \in \mathbb{N}$, fix $\varepsilon > 0$ and independent, pairwise disjoint, Borel sets $A_1, \ldots A_k$ with $\mu\left(\bigcup_{i=1}^k A_i\right) > 1 - \varepsilon$. Then

$$k \cdot i_\mu(G) \geq \mu\left(\bigcup_{i=1}^k A_i\right) > 1 - \varepsilon,$$

so $k > \frac{1-\varepsilon}{i_\mu(G)}$, and we are done. □

(**F**) We will often be interested in locally finite graphs and, in particular, those of bounded degree, where G has *bounded degree* if

$$d(G) = \sup\{d(X) : x \in X\} < \aleph_0.$$

Proposition 2.13 ([12] 4.5, 4.6)**.** *If G is a locally finite, Borel graph, then $\chi_B(G) \leq \aleph_0$. If G is of bounded degree, then $\chi_B(G) \leq d(G) + 1$.*

Corollary 2.14. *Let (X, μ) be a standard probability space and $G = (X, E)$ a locally finite, Borel graph. Then $i_\mu(G) > 0$.*

Example 2.15. For each $0 < a < 1$, there is a bounded degree $G = (X, E)$ on a standard probability space (X, μ) which is invariant, ergodic, such that $i_\mu(G) = a$ and the supremum is attained.

To see this, let S be the shift on $2^{\mathbb{Z}}$ and let $X_1 \subset 2^{\mathbb{Z}}$ be its aperiodic part. Let μ_1 be the restriction to X_1 of the product measure on $2^{\mathbb{Z}}$. Let E_1 be the union of the graph of $S|X_1$ and its inverse. This is invariant, ergodic on (X_1, μ_1). Let $n > 3$ be such that $a \in \left[\frac{1}{n}, \frac{n-1}{n}\right]$, so that $a = \frac{1}{n}\alpha + \frac{n-1}{n}\beta$ for some

$\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Let $A \sqcup B = X_1$ be a Borel partition with $\mu_1(A) = \alpha$, $\mu_1(B) = \beta$. Let $X = X_1 \times \{1, \ldots, n\}$ and give X the product measure $\mu = \mu_1 \times \nu$, where ν is the normalized counting measure on $\{1, \ldots, n\}$. Let $G = (X, E)$ be the following graph on X:

$$\begin{aligned} E &= \{((x,1),(y,1)) : xE_1y\} \cup \\ &\quad \{((x,i),(x,j)) : x \in A, 1 \leq i \neq j \leq n\} \cup \\ &\quad \{((x,1),(x,j)) : x \in B, 2 \leq j \leq n\} \cup \\ &\quad \{((x,j),(x,1)) : x \in B, 2 \leq j \leq n\}. \end{aligned}$$

Clearly, $d(G) \leq n+1$. It is easy to see that $(x,i)E^*(y,j) \Leftrightarrow xEy$, and thus G is invariant, ergodic. We claim that $i_\mu(G) = a$ and the supremum is attained. First note that if $Y \subseteq X$ is independent, then for each $x \in A$ there is at most one $1 \leq i \leq n$ with $(x,i) \in Y$ and for each $x \in B$ there are at most $n-1$ $1 \leq j \leq n$ with $(x,j) \in Y$. Thus $\mu(Y) \leq \frac{1}{n}\mu(A) + \frac{n-1}{n}\mu(B) = a$. On the other hand, $Y = \{(x,2) : x \in A\} \cup \{(x,j) : x \in B \text{ and } j \in \{2, \ldots, n\}\}$ is independent and $\mu(Y) = \frac{1}{n}\mu(A) + \frac{n-1}{n}\mu(B) = a$, so $i_\mu(G) = a$ and the supremum is attained.

We do not know however if examples as in 2.15 with arbitrary $i_\mu(G) = a \in (0,1)$ can be found which are acyclic, even if we replace "bounded degree" by "locally finite." We also do not know if for any given integer $d > 2$, there is an upper bound $f(d) < 1$ for the independence number $i_\mu(G)$ of every invariant, ergodic G with $d(G) \leq d$ (even in the case of acyclic G). We will see in Section 3 that for $d = 2$, $f(d) = 1/2$ works. Note that in the examples of 2.15 to achieve $i_\mu(G) = a < 1$, we needed a graph G of degree $n+1$, where $n \geq \frac{1}{1-a}$.

Recall that if R is a countable Borel equivalence relation on X, then a Borel set $A \subseteq X$ is called *R-smooth* if $R|A$ admits a Borel selector. If $G = (X, E)$ is a locally countable, Borel graph on X, then we call a Borel set $A \subseteq X$ *G-smooth* if A is E^*-smooth.

Theorem 2.16. *Let (X, μ) be a standard probability space, $G = (X, E)$ a Borel graph on X with degree bounded by d, where $d \geq 2$. If every G-smooth set is null, then $\chi_\mu^{\mathrm{ap}}(G) \leq d$ and thus $i_\mu(G) \geq 1/d$.*

Proof. By induction on $d \geq 2$.

Consider first the case $d = 2$. By assumption, neglecting a null set, every E^* class (i.e., connected component of G) is infinite and every vertex has degree 2. Fix $\varepsilon > 0$. Then by the Marker Lemma (see, e.g., [11] 6.7) there is a Borel set A meeting every E^*-class with $\mu(A) < \varepsilon$. Note that each E^*-class looks like an undirected $\mathbb{Z}$-line. The set $Y = \{x : A \cap [x]_{E^*}$ is bounded in one of the two directions of $[x]_{E^*}\}$ is clearly G-smooth, so μ-null. So we may assume that $Y = \emptyset$. Then we Borel color $G|(X \setminus A)$ by two colors as follows:

Let $x \notin A$. Then there are unique $a \neq b$ in A such that x is in the (unique) path in G from a to b, but no other element of A is in this path. Fix a Borel ordering $<$ of X and let $f(x)$ be the $<$-smaller of a, b. Color x blue if the graph distance of x to $f(x)$ is even and otherwise color it green.

This completes the proof in the $d = 2$ case.

Assume now that the result holds for $d \geq 2$ and consider $d + 1$. By [12], 4.2, there is a maximal Borel independent set $A \subseteq X$, i.e., A is independent and for every $x \in X \setminus A$ there is $y \in A$ with $(x, y) \in E$. Let $X' = X \setminus A$, $G' = G|X' = (X', E')$, $\mu' = \frac{\mu|X'}{\mu(X')}$ (we can clearly assume that $\mu(X') > 0$). Then $d(G') \leq d$ and every G'-smooth set is μ'-null (as it is contained in a G-smooth set). Then $\chi^{\mathrm{ap}}_{\mu}(G') \leq d$, by the induction hypothesis. Using A as an additional color, it follows that $\chi^{\mathrm{ap}}_{\mu}(G) \leq d + 1$. $\square$

Note that the hypothesis that every G-smooth set is μ-null is satisfied if G is invariant, ergodic.

Remark 2.17. The $d = 2$ case of the above proof hinges upon the fact that the graph $G|(X \setminus A)$ is Borel 2-colorable. This is a special case of a more general phenomenon: whenever $G = (X, E)$ is a Borel graph with finite connected components, then $\chi(G) = \chi_B(G)$. To see this, simply choose a Borel transversal T of E^*. Since each $x \in T$ sees only finitely many ways (but at least one) of coloring its connected component using the colors $\{1, 2, \ldots, \chi(G)\}$, the required coloring is granted by the uniformization theorem for Borel sets with countable sections. In Section 3 we exploit this fact by using such graphs to approximate more complicated graphs.

For $d \geq 3$, the conclusion of Theorem 2.16 can be slightly improved.

Proposition 2.18. *Under the hypotheses of Theorem 2.16, if $d \geq 3$, then there is an independent Borel set of measure $\geq 1/d$.*

Proof. Let A be a maximal discrete Borel set. If $\mu(A) \geq 1/d$ we are done. Else, $1 - \mu(A) > 1 - 1/d$ and so we can choose $\varepsilon > 0$ small enough so that $(1 - \mu(A)) \left(\frac{1}{d-1} - \varepsilon\right) \geq \frac{1}{d}$. Since $G|(X \setminus A)$ has degree bounded by $d - 1 \geq 2$, by the previous result, there is an independent subset of $X \setminus A$ with measure at least

$$\mu(X \setminus A)\left(\frac{1}{d-1} - \varepsilon\right),$$

so of measure at least $1/d$. □

3 Hyperfinite graphs

(**A**) Recall that a countable Borel equivalence relation R on a standard Borel space X is called *hyperfinite* if it can be written as an increasing union $\bigcup_{n=1}^{\infty} F_n$ with each F_n a finite Borel equivalence relation. If instead R is on a standard measure space (X, μ), we say that R is *μ-hyperfinite* if there is a conull Borel set $A \subseteq X$ such that $R|A$ is hyperfinite. By Connes-Feldman-Weiss (see, e.g., [11], 10.1), measure-preserving actions of amenable groups give rise to μ-hyperfinite orbit equivalence relations. We will examine such actions in Section 4.

(**B**) We say that a locally countable, Borel graph $G = (X, E)$ on a standard measure space (X, μ) is *μ-hyperfinite* if the equivalence relation E^* is μ-hyperfinite. In Miller [18] it is shown that if G is μ-hyperfinite and acyclic, then $\chi_\mu(G) \leq 3$. A slight modification of these techniques allows us to compute $\chi_\mu^{\text{ap}}(G)$ for such graphs.

We say that a locally countable, Borel graph $G = (X, E)$ is *smooth* if E^* admits a Borel selector. Such a graph G is *directable* if there exists a Borel function $f : X \to X$ such that $xEy \Leftrightarrow y = f(x)$ or $x = f(y)$. Finally, such a graph is *essentially linear* if there is a Borel set $B \subseteq X$ such that every connected component of G contains exactly one connected component of $G|B$ and, moreover, $G|B$ is an acyclic graph which is regular of degree two (i.e., it is a forest of lines).

Theorem 3.1. *Let G be a locally countable, acyclic, μ-hyperfinite, Borel graph on a standard probability space (X, μ). Then $\chi_\mu^{\text{ap}}(G) \leq 2$, and thus $i_\mu(G) \geq 1/2$.*

Proof. Following [18] 3.1 and [9] 3.19, we may find pairwise disjoint, E^*-invariant Borel sets X_0, X_1, X_2 such that $\mu(X_0 \cup X_1 \cup X_2) = 1$, $G|X_0$ is smooth, $G|X_1$ is directable, and $G|X_2$ is essentially linear. We handle these three parts separately.

Fix a Borel transversal A of $E^*|X_0$, and color each point $x \in X_0$ by the parity of $d_G(x, T)$. Thus, $\chi_B(G|X_0) \leq 2$, and consequently $\chi_\mu^{\mathrm{ap}}(G|X_0) \leq 2$.

We next handle X_2. Fix $\varepsilon > 0$ and a Borel set B witnessing the essential linearity of $G|X_2$. By Theorem 2.16, we may find a Borel partition $B = B_0 \sqcup B_1 \sqcup B_2$, with $\mu(B_2) < \varepsilon$ and B_0,B_1 forming a 2-coloring of $G|(B \setminus B_2)$ (if $\mu(B) = 0$ we may take $B_2 = B$). We may extend this to a 2-coloring of $G|(X_2 \setminus B_2)$ in the obvious way: for each $x \in X_2$ set $b(x)$ to be the closest element of B to x, then color x by the parity of $d_G(x, b(x))$ if $b(x) \in B_0$ and by the parity of $d_G(x, b(x)) + 1$ if $b(x) \in B_1 \cup B_2$. Thus, $\chi_\mu^{\mathrm{ap}}(G|X_2) \leq 2$.

Finally, we handle X_1. Fix $\varepsilon > 0$ and a Borel function $f : X_1 \to X_1$ witnessing that $G|X_1$ is directable. Define a partial order on X_1 by $x \leq y \Leftrightarrow \exists n(y = f^n(x))$. The following generalization of the Marker Lemma ensures that we may find small sets cofinal in this partial order. For a relation R on X, we say that $A \subseteq X$ is an *R-complete section* if A meets every vertical section of R, i.e., for all x in X, $\exists y \in A\ (xRy)$.

Lemma 3.2 (Miller)**.** *Suppose that X is a Polish space and R is a transitive Borel binary relation on X whose vertical sections are all countably infinite. Then there are Borel R-complete sections $A_0 \supseteq A_1 \supseteq \cdots$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.*

Using the above lemma, we may find a Borel set $C \subseteq X_1$ with $\mu(C) < \varepsilon$ so that for all $x \in X_1$ there exists $y \in C$ with $x \leq y$. We may then color each $x \in X_1 \setminus C$ by the parity of the least n such that $f^n(x) \in C$, so $\chi_\mu^{\mathrm{ap}}(G|X_2) \leq 2$. $\square$

Proof of Lemma 3.2, provided by Miller. Fix an enumeration $B_0, B_1, \ldots$ of a countable family of Borel subsets of X which separates points, and for each $s \in 2^{<\mathbb{N}}$, define $B_s \subseteq X$ by

$$B_s = \left(\bigcap_{s(i)=0} X \setminus B_i \right) \cap \left(\bigcap_{s(i)=1} B_i \right).$$

For each $n \in \mathbb{N}$, define $S_n : X \to \mathcal{P}(2^n)$ by

$$S_n(x) = \{ s \in 2^n : \forall y \in R_x\ (|B_s \cap R_y| = \aleph_0) \}.$$

Sublemma 3.3. *Suppose that $x, y \in X$, $n \in \mathbb{N}$, $s \in 2^n$, and $i \in \{0, 1\}$. Then:*

1. $(x, y) \in R \Rightarrow S_n(x) \subseteq S_n(y)$;

2. $si \in S_{n+1}(x) \Rightarrow s \in S_n(x)$.

Proof. The first claim is a consequence of the transitivity of R, and the second is a trivial consequence of the definition of S_n. $\square$

For each $s \in 2^n$, define $C_s \subseteq X$ by

$$C_s = \{x \in X : \forall y \in R_x \ (s = \min_{\text{lex}} S_n(y))\},$$

and for each $n \in \mathbb{N}$, define $D_n \subseteq X$ by

$$D_n = \bigcup_{s \in 2^n} B_s \cap C_s.$$

We will show that the sets $D_0, D_1, \ldots$ are nearly as desired.

Sublemma 3.4. $\forall n \in \mathbb{N}\ (D_{n+1} \subseteq D_n)$.

Proof. Fix $n \in \mathbb{N}$ and suppose that $x \in D_{n+1}$. Then there exists $s \in 2^n$ and $i \in \{0, 1\}$ such that $x \in B_{si} \cap C_{si}$. In particular, it follows that $x \in B_s$, so to see that $x \in D_n$, it is enough to show that $x \in C_s$. Suppose, towards a contradiction, that there exists $y \in R_x$ such that $s \neq t$, where $t = \min_{\text{lex}} S_n(y)$. As $x \in C_{si}$, it follows that $si \in S_{n+1}(x)$. As (1) ensures that $S_n(x) \subseteq S_n(y)$ and (2) ensures that $s \in S_n(x)$, it follows that $s \in S_n(y)$, thus $t <_{\text{lex}} s$. As $t0 <_{\text{lex}} si$ and $si = \min_{\text{lex}} S_{n+1}(y)$, it follows that $t0 \notin S_{n+1}(y)$, so there exists $z \in R_y$ such that $|B_{t0} \cap R_z| < \aleph_0$. Similarly, since $t1 <_{\text{lex}} si$ and $si = \min_{\text{lex}} S_{n+1}(z)$, it follows that $t1 \notin S_{n+1}(z)$, so there exists $w \in R_z$ such that $|B_{t1} \cap R_w| < \aleph_0$. As the transitivity of R ensures that $R_w \subseteq R_z$, this implies that $|B_t \cap R_w| < \aleph_0$. As the transitivity of R implies also that $(y, w) \in R$, this contradicts our assumption that $t \in S_n(y)$. $\square$

While each D_n is an R-complete section, we will show something stronger:

Sublemma 3.5. $\forall x \in X\ \forall n \in \mathbb{N}\ (|D_n \cap R_x| = \aleph_0)$.

Proof. Fix an enumeration $\langle s_i \rangle_{i<2^n}$ of $\{0, 1\}^n$. For each $x \in X$, set $x_0 = x$, and given x_i, let x_{i+1} be any element of R_{x_i} such that $\min S_n(x_{i+1}) \neq \min S_n(x_i)$, if such an element exists. Otherwise, set $x_{i+1} = x_i$. Let $y = x_{2^n}$ and $s = \min S_n(y)$, and observe that $\forall z \in R_y\ (s = \min S_n(z))$, thus $y \in C_s$. As $s \in S_n(y)$, it follows that $|B_s \cap R_y| = \aleph_0$, and since $y \in C_s$, it follows that $B_s \cap R_y = B_s \cap C_s \cap R_y$, thus $|B_s \cap C_s \cap R_y| = \aleph_0$. As $B_s \cap C_s \subseteq D_n$ and the transitivity of R ensures that $R_y \subseteq R_x$, it follows that $|D_n \cap R_x| = \aleph_0$. $\square$

Unfortunately, it need not be the case that the set $D = \bigcap_{n \in \mathbb{N}} D_n$ is empty. However, this is not so far from the truth:

Sublemma 3.6. $\forall x, y \in D\ (x \neq y \Rightarrow (x, y) \notin R)$.

Proof. Suppose, towards a contradiction, that there are distinct points $x, y \in D$ such that $(x, y) \in R$. Fix $n \in \mathbb{N}$ and $s \in 2^n$ such that $x \in B_s$ and $y \notin B_s$. As $x \in D_n$, it follows that $S_n(x) = S_n(y) = s$, so $y \notin D_n$, thus $y \notin D$, the desired contradiction. □

Now define $A_n = D_n \setminus D$. Sublemma 3.3 implies that these sets are decreasing, and they clearly have empty intersection, so it only remains to check that each A_n is an R-complete section. Towards this end, fix $x \in X$, and observe that two applications of Sublemma 3.5 ensure that there are distinct points $y \in D_n \cap R_x$ and $z \in D_n \cap R_y$. Sublemma 3.6 then ensures that $y \notin A_n \Rightarrow y \in D \Rightarrow z \notin D \Rightarrow z \in A_n$, and the transitivity of R then implies that $A_n \cap R_x \neq \emptyset$. □

It is natural to ask whether the assumption of acyclicity in Theorem 3.1 can be weakened to $\chi(G) \leq 2$. This is not the case:

Example 3.7. There is a Borel graph G on a standard probability space (X, μ) which is locally countable, μ-hyperfinite, and satisfies both $\chi(G) = 2$ and $i_\mu(G) = 0$. To see this, set $X = 2^{\mathbb{N}}$, μ the usual product measure, and $xEy \Leftrightarrow x$ and y differ on exactly one coordinate. Then $E^* = E_0$, the equivalence relation of eventual agreement, which is hyperfinite. Certainly, $\chi(G) = 2$ as G contains no cycles of odd length. It remains only to see that $i_\mu(G) = 0$.

Suppose that $Y \subseteq X$ is a set of positive measure. Then, by Lebesgue density, there is a finite binary string s such that $\frac{\mu(Y \cap \mathcal{N}_s)}{\mu(\mathcal{N}_s)} > 1/2$, where $\mathcal{N}_s = \{x \in X : s \sqsubseteq x\}$. This implies that the set $\{x \in X : s0x \in Y \text{ and } s1x \in Y\}$ has positive measure, so Y cannot be independent for G.

(**C**) We now turn our attention to locally finite μ-hyperfinite graphs. In this context, counterexamples such as 3.7 do not arise:

Theorem 3.8. *Let G be a locally finite, μ-hyperfinite, Borel graph on a standard probability space (X, μ). Then $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$, and thus $i_\mu(G) \geq 1/\chi(G)$.*

Proof. Discarding a null set if necessary, we may assume that G is hyperfinite. Fix $\varepsilon > 0$ and finite Borel equivalence relations $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ witnessing the hyperfiniteness of G. For each n, define the set

$$X_n = \{x \in X : \forall y \in X(xEy \Rightarrow xF_ny)\}.$$

Then, since G is locally finite, $X = \bigcup_n X_n$. Choose n such that $\mu(X_n) > 1-\varepsilon$, and define $G' = G|X_n = (X_n, E')$. Since $E' \subseteq F_n$ we have $(E')^* \subseteq F_n$, and thus the connected components of G' are finite. Consequently (see Remark 2.17), $\chi_B(G') \leq \chi(G') \leq \chi(G)$, and therefore $\chi_\mu^{\mathrm{ap}}(G) \leq \chi(G)$. □

4 Graphs associated with group actions

(**A**) Consider a countable group Γ, which we will assume to be infinite, unless otherwise indicated. Let $A : \Gamma \times X \to X$ be a free Borel action of Γ on a standard Borel space. (Free means that $A(\gamma, x) \neq x$, $\forall \gamma \neq 1$, $\forall x \in X$.) We put $\gamma^A(x) = A(\gamma, x)$ and often write $\gamma \cdot x$ for $A(\gamma, x)$ if there is no danger of confusion.

If $S \subseteq \Gamma$ is a set of generators for Γ, where we always assume, unless otherwise indicated, that $1 \notin S$, we define the associated graph

$$G(S, A) = (X, E(S, A)),$$

by

$$(x, y) \in E(S, A) \Leftrightarrow \exists s \in S(y = s^{\pm 1} \cdot x).$$

Clearly this is a locally countable, Borel graph on X whose connected components are the Γ-orbits of the action A. Denote by $\chi(S, A)$, resp., $\chi_B(S, A)$ the associated with $G(S, A)$ chromatic, resp., Borel chromatic numbers. Let also $\mathrm{Cay}(\Gamma, S)$ be the (left) *Cayley graph* of Γ with respect to S, where γ, δ are connected by an edge iff there is $s \in S$ with $\delta = s^{\pm 1}\gamma$. If $x \in X$ then the map $\gamma \in \Gamma \mapsto \gamma \cdot x$ gives an isomorphism between $\mathrm{Cay}(\Gamma, S)$ and the connected component of x in $G(S, A)$. Thus

$$\chi(S, A) = \chi(\mathrm{Cay}(\Gamma, S)).$$

Let now (X, μ) be a standard probability space and let a be a free, measure-preserving action of Γ on (X, μ). This is an equivalence class of Borel actions of Γ on X that preserve the measure μ, where two actions $A(\gamma, x)$, $B(\gamma, x)$ are identified if $A(\gamma, x) = B(\gamma, x)$, a.e., $\forall \gamma$, and free means that $\forall \gamma \neq 1(a(\gamma, x) \neq x$, a.e.). We again denote, for each set S of generators of Γ, by $G(S, a) = (X, E(S, a))$ the associated graph on X and by $\chi_\mu(S, a)$, $\chi_\mu^{\mathrm{ap}}(S, a)$, $i_\mu(S, a)$ the corresponding numbers. It should be noted that $G(S, a)$ is only defined almost everywhere in the sense that if A, B are representatives for a, then there is a conull set Y which is invariant under both A and B on which A and B agree, thus Y is a set of connected components of both $G(S, A)$, $G(S, B)$ and $G(S, A)|Y = G(S, B)|Y$. It follows that the numbers χ_μ,χ_μ^{ap},i_μ are well-defined, i.e., depend only upon a (and S). Finally, we usually write E_a for $E^*(S, a)$, the equivalence relation induced by the action a. Clearly, $G(S, a)$ is measure preserving and is ergodic iff a is ergodic.

We first note the following obvious inequality

$$i_\mu(S, a) \leq 1/2.$$

This is because if $A \subseteq X$ is independent and $s \in S$, then $A \cap s^a(A) = \emptyset$ and $\mu(A) = \mu(s^a(A))$.

Moreover, if for every $\Gamma_0 \leq \Gamma$ of index at most 2, the action $a|\Gamma_0 \in A(\Gamma_0, X, \mu)$ is ergodic (e.g., if a is weakly mixing), then there can be no Borel independent A of measure exactly 1/2 (so if $i_\mu(S, a) = 1/2$, the supremum is not attained). Otherwise $s \cdot A = X \setminus A$ for every $s \in S^{\pm 1}$ and so A is Γ_0-invariant, where $\Gamma_0 = \{t_1 t_2 \cdots t_{2n} : n \geq 0, t_i \in S^{\pm 1}\}$ and $S^{\pm 1} = \{s^{\pm 1} : s \in S\}$. Since $[\Gamma, \Gamma_0] \leq 2$, this gives a contradiction to our assumption.

We denote by $\mathrm{FR}(\Gamma, X, \mu)$ the space of free, measure-preserving actions of Γ on (X, μ) and we equip it with the weak topology in which $\mathrm{FR}(\Gamma, X, \mu)$ is a Polish space (see [10], 10).

Theorem 4.1. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then the map*

$$a \mapsto i_\mu(S, a)$$

is lower semicontinuous in $\mathrm{FR}(\Gamma, X, \mu)$.

Proof. Note that for $r \in \mathbb{R}$,

$$r < i_\mu(S, a) \;\Leftrightarrow\; \exists \text{ Borel } A\ \exists \varepsilon > 0 \\ \left(\mu(A) > r + \varepsilon \text{ and } \forall t \in S^{\pm 1} \left(\mu(A \cap t^a(A)) < \frac{\varepsilon}{n}\right)\right),$$

where $|S^{\pm 1}| = n$. The direction from left to right is clear, because we can take A to be a Borel independent set of measure $r + \varepsilon$ for some $\varepsilon > 0$. Conversely, let A, ε satisfy the right-hand side. Then

$$B = A \setminus \bigcup_{t \in S^{\pm 1}} t^a(A)$$

is independent and $\mu(B) \geq \mu(A) - n \cdot \frac{\varepsilon}{n} > r$, so $i_\mu(S, a) > r$.

Since the map $a \mapsto \mu(A \cap \gamma^a(A))$ from $\mathrm{FR}(\Gamma, X, \mu)$ to $\mathbb{R}$ is continuous in the weak topology, for each $\gamma \in \Gamma$, $\{a \in \mathrm{FR}(\Gamma, X, \mu) : r < i_\mu(S, a)\}$ is open and the proof is complete. □

Recall that $a \in \mathrm{FR}(\Gamma, X, \mu)$ is *weakly contained* in $b \in \mathrm{FR}(\Gamma, X, \mu)$, in symbols $a \prec b$, if a is in the closure of the conjugacy class of b (see [10], 10, 10.1). So we have

Corollary 4.2. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then*

$$a \prec b \Rightarrow i_\mu(S, a) \leq i_\mu(S, b).$$

In particular, $i_\mu(S, a)$ is an invariant of weak equivalence, defined by $a \sim b \Leftrightarrow a \prec b$ and $b \prec a$.

Corollary 4.3. *Let Γ be a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators. Then $i_\mu(S, a)$ is constant, for all $a \in \mathrm{FR}(\Gamma, X, \mu)$.*

Proof. If $a, b \in \mathrm{FR}(\Gamma, X, \mu)$ are ergodic, then, by [10], 13.2, $a \sim b$, and so $i_\mu(S, a) = i_\mu(S, b)$. Thus $i_\mu(S, a)$ is constant for all free, ergodic a. Using the ergodic decomposition, this is true for all free a. □

We denote by

$$i_\mu(\Gamma, S)$$

the constant value as in Corollary 4.3.

Concerning χ_μ^{ap} we have the following result.

Theorem 4.4. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Then for any $a, b \in \mathrm{FR}(\Gamma, X, \mu)$,*

$$a \prec b \Rightarrow \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, b).$$

Proof. Assume $a \prec b$ and let $k = \chi_\mu^{\mathrm{ap}}(S,a)$, $n = |S^{\pm 1}|$. Fix $\varepsilon > 0$. Let then $A_1, \dots, A_k$ be Borel, pairwise disjoint, independent subsets of X with $\mu(A_1 \cup \cdots \cup A_k) > 1 - \frac{\varepsilon}{k+1}$. Since $a \prec b$, there are Borel, pairwise disjoint subsets $B_1, \dots, B_k$ of X with $\mu(B_1 \cup \cdots \cup B_k) > 1 - \frac{\varepsilon}{k+1}$ and $|\mu(s^a(A_i) \cap A_i) - \mu(s^b(B_i) \cap B_i)| < \frac{\varepsilon}{n(k+1)}$ for all $s \in S^{\pm 1}$, thus $\mu(s^b(B_i) \cap B_i) < \frac{\varepsilon}{n(k+1)}$ for all $s \in S^{\pm 1}$. If $\overline{B}_i = B_i \setminus \bigcup_{s \in S^{\pm 1}} s^b(B_i)$, then $\overline{B}_i$, $1 \leq i \leq k$ are Borel, pairwise disjoint, independents (for the action b) sets, and $\mu(\overline{B}_i) \geq \mu(B_i) - \frac{\varepsilon}{k+1}$, so $\mu(\overline{B}_1 \cup \cdots \cup \overline{B}_k) > 1 - \varepsilon$, therefore $k \geq \chi_\mu^{\mathrm{ap}}(S,b)$. $\square$

It follows that $a \mapsto \chi_\mu^{\mathrm{ap}}(S,a)$ also an invariant of weak equivalence. Thus we have as in Corollary 4.3:

Corollary 4.5. *Let Γ be a countable, amenable group and $S \subseteq \Gamma$ a finite set of generators. Then $\chi_\mu^{\mathrm{ap}}(S,a)$ is constant for all $a \in \mathrm{FR}(\Gamma, X, \mu)$.*

Denote by

$$\chi_\mu^{\mathrm{ap}}(\Gamma, S)$$

the constant value as in Corollary 4.5.

Next notice the following simple fact.

Proposition 4.6. *Let Γ be a countable group and $S \subseteq \Gamma$ a set of generators. Then the following are equivalent:*

(i) $\chi(\mathrm{Cay}(\Gamma, S)) = 2$ *(i.e.,* $\mathrm{Cay}(\Gamma, S)$ *is bipartite),*

(ii) *There is a homomorphism $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ that sends S to 1,*

(iii) $\{s_1 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$ *has index 2 in* Γ,

(iv) *For any $s_1, \dots, s_{2n+1} \in S^{\pm 1}$, $n \geq 0$, we have $s_1 \cdots s_{2n+1} \neq 1$.*

We now have:

Proposition 4.7. *Let Γ be a countable group and $S \subseteq \Gamma$ a set of generators. Let $g = g_{\mathrm{odd}}(\mathrm{Cay}(\Gamma, S))$ be the odd girth of the Cayley graph $\mathrm{Cay}(\Gamma, S)$. Then for any $a \in \mathrm{FR}(\Gamma, X, \mu)$, we have*

$$i_\mu(S,a) \leq 1/2 - 1/(2g).$$

Also if $\Gamma_0 = \{s_1 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$ and $a|\Gamma_0 \in \mathrm{FR}(\Gamma_0, X, \mu)$ is strongly ergodic, then $i_\mu(S,a) < 1/2$.

Proof. We can assume that $g < \infty$, i.e., that the Cayley graph is not bipartite. Let $A \subseteq X$ be a Borel independent set and let $\mu(A) = 1/2 - \varepsilon$. Then for any $s, t \in S^{\pm 1}$, it is easy to see that $\mu(A \triangle st \cdot A) \leq 4\varepsilon$. So, by induction, if $\gamma = s_1 \cdots s_{2n}$, where $s_i \in S^{\pm 1}$, then $\mu(\gamma \cdot A \triangle A) \leq 4n\varepsilon$. If $g = 2n+1$, then for some $s, s_1, \ldots, s_{2n} \in S^{\pm 1}$, we have that $s = s_1 \cdots s_{2n}$, so $\mu(s \cdot A \triangle A) \leq 4n\varepsilon$, thus $\varepsilon \geq 1/(4n+2) = 1/(2g)$, therefore $\mu(A) \leq 1/2 - 1/(2g)$.

In the case $a|\Gamma_0$ is strongly ergodic but $i_\mu(S, a) = 1/2$, there are Borel independent sets A_n with $\mu(A_n) \to 1/2$. Then for any finite $F \subseteq \Gamma_0$, $\varepsilon > 0$, and all large enough n, we have $\mu(\gamma \cdot A_n \triangle A_n) < \varepsilon$, $\forall \gamma \in F$, i.e., $a|\Gamma_0$ has non-trivial almost invariant sets, contradicting strong ergodicity. □

Thus in the context of 4.7, if $\text{Cay}(\Gamma, S)$ is not bipartite (so that $g < \infty$) or if $a|\Gamma_0$ is strongly ergodic, we have that $i_\mu(S, a) < 1/2$ and $\chi_\mu(S, a) \geq 3$.

Theorem 4.8. *Let Γ be a countable group and $S \subseteq \Gamma$ a set of generators. Then the following are equivalent:*

(i) $\text{Cay}(\Gamma, S)$ *is bipartite,*

(ii) *There is an action $a \in \text{FR}(\Gamma, X, \mu)$ with $i_\mu(S, a) = 1/2$,*

(iii) *There is an action $a \in \text{FR}(\Gamma, X, \mu)$ with $\chi_\mu^{\text{ap}}(S, a) = 2$,*

(iv) *There is an action $a \in \text{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) = 2$,*

(v) *There is an ergodic action $a \in \text{FR}(\Gamma, X, \mu)$ with $\chi_\mu(S, a) = 2$.*

Proof. (ii)⇒(i) follows from 4.7. Clearly (v) ⇒ (iv) ⇒ (iii) ⇒ (ii).

Conversely, assume (i) in order to prove (v). Let $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism that sends S to 1. Let $a_0 \in \text{FR}(\Gamma, X, \mu)$ be weakly mixing. Then there is no independent Borel set A with $\mu(A) = 1/2$, because in that case $s \cdot A = X \setminus A$ (modulo null sets) for any $s \in S^{\pm 1}$, so A is invariant under $\Gamma_0 = \{s_1 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$, which is an index 2 subgroup of Γ (again using Proposition 4.6(iii)) and thus acts ergodically on X, a contradiction.

Let now $a \in \text{FR}(\Gamma, X \times \{0, 1\}, \mu \times \nu)$, where $\nu(\{0\}) = \nu(\{1\}) = 1/2$, be defined by

$$\gamma \cdot (x, i) = (\gamma \cdot x, \varphi(\gamma) + i).$$

Clearly, $X \times \{0\}$,$X \times \{1\}$ gives a measurable 2-coloring of the graph $G(S, a)$, so $\chi_\mu(S, a) = 2$. It only remains to show that a is ergodic. If not, let

$A \subseteq X \times \{0,1\}$ be a Borel invariant set with $0 < \mu(A) < 1$. Define $f : X \to \{0,1,2\}$ by

$$f(x) = |\{i \in \{0,1\} : (x,i) \in A\}|.$$

Then f is invariant under a_0, so it is constant (μ-)a.e. If $f(x) = 0$, a.e., then A is null, while if $f(x) = 2$, a.e., then A is co-null, both contradictions. So $f(x) = 1$, a.e. Then $A_0 = \{x \in X : (x,0) \in A\}$ is an independent (for $G(S,a_0)$) set of measure 1/2, a contradiction. □

Using now the results in Section 3 we have:

Corollary 4.9. *Let* Γ *be a countable, amenable group and* $S \subseteq \Gamma$ *a finite set of generators. Then:*

(i) *If* $\mathrm{Cay}(\Gamma, S)$ *is bipartite, then* $i_\mu(\Gamma, S) = 1/2$,

(ii) *If* $\mathrm{Cay}(\Gamma, S)$ *is not bipartite, then* $i_\mu(\Gamma, S) \leq 1/2 - 1/(2g) < 1/2$, *where* g *is the odd girth of* $\mathrm{Cay}(\Gamma, S)$.

Proof. (i) follows from 3.8, and (ii) from 4.7. □

Corollary 4.10. *Let* Γ *be a countable, amenable group and* $S \subseteq \Gamma$ *a finite set of generators.*

(i) *If* $\mathrm{Cay}(\Gamma, S)$ *is bipartite, then* $\chi_\mu^{\mathrm{ap}}(\Gamma, S) = 2$,

(ii) *If* $\mathrm{Cay}(\Gamma, S)$ *is not bipartite, then* $\chi_\mu^{\mathrm{ap}}(\Gamma, S) \geq 3$.

Proof. (i) follows from Theorem 3.8, while (ii) follows from 4.9, using the inequality $\chi_\mu^{\mathrm{ap}}(\Gamma, S) \geq \frac{1}{i_\mu(\Gamma,S)}$. □

Remark 4.11. Note that a group Γ admits a set of generators S with $\mathrm{Cay}(\Gamma, S)$ bipartite iff $\mathbb{Z}/2\mathbb{Z}$ is a factor of Γ.

When Γ is not amenable $i_\mu(S,a)$ and $\chi_\mu^{\mathrm{ap}}(S,a)$ might not be constant (we will see examples below). However, Abért and Weiss [1] showed that among all $a \in \mathrm{FR}(\Gamma, X, \mu)$, there is a minimum one in the sense of weak containment, namely, the shift action s_Γ of Γ on 2^Γ (with the usual product measure), and earlier Hjorth (unpublished) and (independently) Glasner-Thouvenot-Weiss [5] showed that there is a maximum one, denoted by a_∞ (see also [10], 10.7). Similarly there is a free, ergodic action which is maximum in the sense of weak containment among all the free, ergodic actions, denoted by a_∞^{erg}. Then

$$s_\Gamma \prec a_\infty^{\mathrm{erg}} \prec a_\infty.$$

Therefore for any finite generating set $S \subseteq \Gamma$, we have for any $a \in \mathrm{FR}(\Gamma, X, \mu)$,

$$i_\mu(S, s_\Gamma) \leq i_\mu(S, a) \leq i_\mu(S, a_\infty),$$

and

$$\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, a_\infty),$$

and for any ergodic $a \in \mathrm{FR}(\Gamma, X, \mu)$,

$$i_\mu(S, s_\Gamma) \leq i_\mu(S, a) \leq i_\mu(S, a_\infty^{\mathrm{erg}}) \leq i_\mu(S, a_\infty),$$

and

$$\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, a) \geq \chi_\mu^{\mathrm{ap}}(S, a_\infty^{\mathrm{erg}}) \geq \chi_\mu^{\mathrm{ap}}(S, a_\infty).$$

We next have the following characterization of amenability (in certain situations).

Theorem 4.12. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *is amenable,*

(ii) $i_\mu(S, a) = 1/2$ *for any* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) = 2$ *for any* $a \in \mathrm{FR}(\Gamma, X, \mu)$,

(iv) $i_\mu(S, s_\Gamma) = 1/2$,

(v) $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) = 2$.

In particular, if Γ is a finitely generated group having $\mathbb{Z}/2\mathbb{Z}$ as a factor, then the following are equivalent:

(a) Γ *is amenable,*

(b) *There is a finite generating set $S \subseteq \Gamma$ such that* $i_\mu(S, s_\Gamma) = 1/2$,

(c) *As in* (ii) *with* $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma) = 2$.

Proof. If Γ is amenable, then (ii) – (v) hold by Corollaries 4.9 and 4.10. If Γ is not amenable, then $s_\Gamma | \Gamma_0$ is strongly ergodic for any index 2 subgroup of Γ (see, e.g., [7], Appendix A, A4.1), so the negations of (ii) – (v) follow from Proposition 4.7. ∎

We also have an analogous characterization of groups that have property (T) and the HAP.

Theorem 4.13. *Let* Γ *be an infinite, countable group and* $S \subseteq \Gamma$ *a finite set of generators such that* $\mathrm{Cay}(\Gamma, S)$ *is bipartite. Then the following are equivalent:*

(i) Γ *has property (T),*

(ii) $i_\mu(S, a) < 1/2$ *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$*,*

(iii) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$ *for every weakly mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$*.*

Also the following are equivalent:

(i*) Γ *does not have the HAP,*

(ii*) $i_\mu(S, a) < 1/2$ *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$*,*

(iii*) $\chi_\mu^{\mathrm{ap}}(S, a) \geq 3$ *for every mixing* $a \in \mathrm{FR}(\Gamma, X, \mu)$*.*

Proof. Suppose first that Γ has property (T). If $\Gamma_0 = \{s_1 s_2 \cdots s_{2n} : n \geq 0, s_i \in S^{\pm 1}\}$, then Γ_0 has index 2 in Γ and thus Γ_0 itself has property (T). Moreover, if $a \in \mathrm{FR}(\Gamma, X, \mu)$ is weakly mixing, then $a|\Gamma_0 \in \mathrm{FR}(\Gamma_0, X, \mu)$ is ergodic, so strongly ergodic (see, e.g., [10] 11.2), thus $i_\mu(S, a) < 1/2$ by Proposition 4.7. So (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Assume now that Γ does not have property (T). By Theorem 4.8 there is $b \in \mathrm{FR}(\Gamma, X, \mu)$ with $i_\mu(S, b) = 1/2$ and $\chi_\mu^{\mathrm{ap}}(S, b) = 2$. By a result of Kerr-Pichot [13] (see also [10], 12.9), there is a weakly mixing $a \in \mathrm{FR}(\Gamma, X, \mu)$ with $b \prec a$, so $i_\mu(S, b) \leq i_\mu(S, a)$, thus $i_\mu(S, a) = 1/2$, and $\chi_\mu^{\mathrm{ap}}(S, b) \geq \chi_\mu^{\mathrm{ap}}(S, a)$, therefore $\chi_\mu^{\mathrm{ap}}(S, a) = 2$.

If now Γ does not have the HAP, then Γ_0 as above does not have the HAP, and $a|\Gamma_0 \in \mathrm{FR}(\Gamma_0, X, \mu)$ is mixing, so (see, e.g., [10] 11.1) it is strongly ergodic, thus $i_\mu(S, a) < 1/2$ as before. So (i*) $\Rightarrow$ (ii*) $\Rightarrow$ (iii*).

Conversely, if Γ has the HAP, then we can repeat the argument above (for the case that Γ does not have property (T)) using the result of Hjorth [6] (see also [10], 12.11) to replace in this argument weakly mixing by mixing. □

(**B**) For any unitary representation $\pi : \Gamma \to U(H)$ of a countable group Γ on a Hilbert space H, and a finite set of generators $S \subseteq \Gamma$, one defines the *averaging operator* $T_{S,\pi}$ by

$$T_{S,\pi}(f) = \frac{1}{|S^{\pm 1}|} \sum_{s \in S^{\pm 1}} \pi(s)(f).$$

Clearly $T_{S,\pi}$ is a self-adjoint operator and $\|T_{s,\pi}\| \leq 1$. It is easy to check that if π, ρ are unitary representations and π is weakly contained in ρ (see, e.g, Bekka-de la Harpe-Valette [2], Appendix F), which is denoted by $\pi \prec \rho$, then $\|T_{s,\pi}\| \leq \|T_{s,\rho}\|$, i.e.,

$$\pi \prec \rho \Rightarrow \|T_{s,\pi}\| \leq \|T_{s,\rho}\|.$$

When Γ is amenable, Kesten [14] showed that $\|T_{S,\lambda_\Gamma}\| = 1$, where λ_Γ is the (left) regular representation of Γ.

For each $a \in \mathrm{FR}(\Gamma, X, \mu)$, consider the corresponding Koopman unitary representation κ_0^a on $L_0^2(X,\mu) = \{f \in L^2(X,\mu) : \int f \, d\mu = 0\} = \mathbb{C}^\perp$ (where $\mathbb{C}$ is identified with the subspace of constant functions in $L^2(X,\mu)$). Then for a finite generating set $S \subseteq \Gamma$, let

$$T_{S,a} = T_{S,\kappa_0^a}.$$

There is a well-known connection between norms of averaging operators and independence ratios in the case of finite graphs, due to Hoffman [8] (see, e.g., Davidoff-Sarnak-Valette [3], 1.5.3), and a version of this carries over to our context.

Theorem 4.14. *Let Γ be a countable group and $S \subseteq \Gamma$ a finite set of generators. Let $a \in \mathrm{FR}(\Gamma, X, \mu)$ and let $T_{S,a}$ the corresponding averaging operator. If $\nu = \|T_{S,a}\|$, then*

$$i_\mu(S,a) \leq \frac{\nu}{1+\nu},$$

and thus

$$\chi_\mu^{\mathrm{ap}}(S,a) \geq \frac{1+\nu}{\nu}.$$

Proof. Let $A \subseteq X$ be independent and let $T = T_{S,a}$, $f = \chi_A - \mu(A)$. Then $f \in L_0^2(X,\mu)$ and

$$\|f\|^2 = \mu(A)(1-\mu(A)).$$

Also,

$$\begin{aligned}
\langle T(f), f\rangle &= \int T(f)(x)f(x)\ d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|}\sum_{s\in S^{\pm 1}}\int f(s\cdot x)f(x)\ d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|}\sum_{s\in S^{\pm 1}}\int (\chi_{s^{-1}\cdot A}(x)-\mu(A))(\chi_A(x)-\mu(A))\ d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|}\sum_{s\in S^{\pm 1}}\int (-\mu(A)(\chi_A(x)+\chi_{s^{-1}\cdot A}(x))+\mu(A)^2)\ d\mu(x) \\
&= \frac{1}{|S^{\pm 1}|}\sum_{s\in S^{\pm 1}}(-2\mu(A)^2+\mu(A)^2) \\
&= -\mu(A)^2.
\end{aligned}$$

Since $|\langle T(f), f\rangle| \leq \|T\|\cdot\|f\|^2$, letting $\alpha=\mu(A)$, we have

$$\alpha^2 \leq \nu\cdot\alpha(1-\alpha),$$

so

$$\alpha \leq \frac{\nu}{1+\nu}.$$

□

Since for $a, b \in \mathrm{FR}(\Gamma, X, \mu)$,

$$a \prec b \Rightarrow \kappa_0^a \prec \kappa_0^b$$

(see [10], 10.5), it follows that

$$a \prec b \Rightarrow \|T_{s,a}\| \leq \|T_{s,b}\|$$

(in fact it is not hard to see that $a \mapsto \|T_{s,a}\|$ is lower semicontinuous), thus, since $s_\Gamma \prec a$, $\forall a \in \mathrm{FR}(\Gamma, X, \mu)$, $\|T_{S,s_\Gamma}\|$ is minimum among all such $\|T_{s,a}\|$. Now it is well known (see, e.g., [2], E.4.5) that $\kappa_0^{s_\Gamma} \sim \lambda_\Gamma$ and thus $\|T_{S,s_\Gamma}\| = \|T_{S,\lambda_\Gamma}\|$.

Suppose now that $S=\{\gamma_1,\dots,\gamma_m\}$. Kesten [14] has shown that

$$\|T_{S,\lambda_\Gamma}\| \geq \frac{\sqrt{2m-1}}{m}$$

and if S is a free set of generators, so that $\Gamma = \mathbb{F}_m$, then

$$\|T_{S,\lambda_\Gamma}\| = \frac{\sqrt{2m-1}}{m}.$$

Also, if $S = \{\gamma_1, \dots, \gamma_m, \delta_1, \dots, \delta_m\}$, where $\gamma_1, \dots, \gamma_m$ are free and $\delta_1, \dots, \delta_n$ are free satisfying $\delta_i^2 = 1$, $i = 1, \dots, n$, so that $\mathrm{Cay}(\Gamma, S)$ is still acyclic and $\Gamma = \mathbb{F}_m * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (n times), then again

$$\|T_{S,\lambda_\Gamma}\| = \frac{2\sqrt{(2m+n)-1}}{2m+n}.$$

We then have:

Theorem 4.15. *Let $\Gamma = \mathbb{F}_m$ be the free group with a free set S of m generators, and s_Γ its shift action on 2^Γ, with the product measure μ. Then*

$$\frac{1}{2m} \leq i_\mu(S, s_\Gamma) \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}}$$

and

$$2m \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) \geq \frac{m + \sqrt{2m-1}}{\sqrt{2m-1}},$$

while

$$2m + 1 \geq \chi_B(S, s_\Gamma)$$

(where we view in the last inequality s_Γ as the shift action restricted to its free part).

Proof. This follows from Theorems 2.16, 4.14, Proposition 2.13, and the preceding paragraphs. □

This, in particular, gives examples of invariant, ergodic, Borel graphs of bounded degree which are acyclic but the approximate chromatic number and thus the measurable and Borel chromatic numbers are finite but tend towards ∞.

An analogous result to Theorem 4.15 holds when $\Gamma = \mathbb{F}_m * \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (n times).

We do not know what are the exact values of $i_\mu(S, s_\Gamma)$, $\chi_\mu^{\mathrm{ap}}(S, s_\Gamma)$ for $\Gamma = \mathbb{F}_m$ and S a free set of generators (similarly for $\chi_B(S, s_\Gamma)$, $\chi_\mu(S, s_\Gamma)$).

Concerning Borel chromatic numbers of shifts, denote below by s_Γ the restriction of the shift action of Γ on 2^Γ to its free part, and recall that for a generating set $S \subseteq \Gamma$, $\chi_B(G(S, s_\Gamma))$ denotes the Borel chromatic number associated with s_Γ. If $\Gamma = \mathbb{Z}^m$, with S the usual set of m generators, then Gao-Jackson [4] showed that $\chi_B(G(S, s_\Gamma)) \in \{3, 4\}$, while of course $\chi(G(S, s_\Gamma)) = \chi(\mathrm{Cay}(\Gamma, S)) = 2$. In Theorem 4.15 we have seen that for $\Gamma = \mathbb{F}_m$ and a free set of generators S, $\chi(G(S, s_\Gamma)) = \chi(\mathrm{Cay}(\Gamma, S)) = 2$ but $\chi_B(G(S, s_\Gamma)) \to \infty$ as $m \to \infty$.

Is it true that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that if Γ is amenable and S is any finite generating set, then $\chi_B(G(S, s_\Gamma)) \leq f(\chi(\mathrm{Cay}(\Gamma, S)))$ (analogous to what Theorem 3.8 indicates for χ_μ^{ap})? We do not know a counterexample even for $f(n) = n + 1$.

On the other hand, if Γ is finitely generated with $\mathbb{F}_2 \leq \Gamma$ and $\mathbb{Z}/2\mathbb{Z}$ is a factor of Γ, then for each $\varepsilon > 0$, there is a finite generating set $S \subseteq \Gamma$ with $\chi(\mathrm{Cay}(\Gamma, S)) = 2$, but $i_\mu(S, s_\Gamma) < \varepsilon$ and so $\chi_B(S, s_\Gamma) \geq \chi_\mu(S, s_\Gamma) \geq \chi_\mu^{\mathrm{ap}}(S, s_\Gamma) > 1/\varepsilon$. Indeed, choose m large enough so that $\frac{\sqrt{2m-1}}{m+\sqrt{2m-1}} < \varepsilon$ and let $\varphi : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be a surjective homomorphism. Then $\Gamma_0 = \ker(\varphi)$ contains a free subgroup $\Delta = \langle a_1, \ldots, a_m \rangle$ with m free generators. Let $S_0 \supset \{a_1, \cdots, a_m\}$ be a finite set of generators for Γ_0 and let $a \notin \Gamma_0$. Put $S = \{a\} \cup \{aS_0\}$. Clearly S generates Γ and there are no odd cycles in $\mathrm{Cay}(\Gamma, S)$, so $\chi(\mathrm{Cay}(\Gamma, S)) = 2$. However, if A is an independent set in the graph associated with s_Γ, then it is independent for the graph associated with the action $s_\Gamma|\Delta$. One can again see that $\kappa_0^{s_\Gamma|\Delta} \sim \lambda_\Delta$, so if $s_\Delta = \{a_1, \ldots, a_m\}$,

$$i_\mu(S, s_\Gamma) \leq i_\mu(S_\Delta, s_\Gamma|\Delta) \leq \|T_{S_\Delta, \lambda_\Delta}\| \leq \frac{\sqrt{2m-1}}{m + \sqrt{2m-1}} < \varepsilon.$$

(**C**) We will next see some connections with finite graphs.

Let Γ be a countable group and fix a sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq \Gamma$ of finite, non-empty subsets of Γ with $\bigcup_n F_n = \Gamma$. Consider the space 2^Γ with the product topology. If $p \in 2^{F_n}$, let

$$\mathcal{N}_p = \{f \in 2^\Gamma : f|F_n = p\}.$$

Then $\{\mathcal{N}_p\}_{n \geq 1, p \in 2^{F_n}}$ is a clopen basis for the topology of 2^Γ. Let now $S \subseteq \Gamma$ be a set of generators for Γ. Consider the finite graph with loops $G_{n,S} = (2^{F_n}, E_{n,S})$, where

$$pE_{n,S}q \Leftrightarrow \exists s \in S(s^{\pm 1} \cdot \mathcal{N}_p \cap \mathcal{N}_q \neq \emptyset).$$

Here $\gamma \cdot f$ ($\gamma \in \Gamma$, $f \in 2^\Gamma$) refers to the shift action of Γ on 2^Γ. Thus, there is a loop from p to p iff $\exists s \in S(s \cdot \mathcal{N}_p \cap \mathcal{N}_p \neq \emptyset)$. We may view each $G_{n,S}$ as a finite approximation to the graph associated with the shift action of Γ. Below recall that for a finite graph with loops $G = (X, E)$ its *independence ratio* $i(G)$ is defined as the ratio of the largest size of an independent set divided by $|X|$. When we have a graph with loops we define an independent set to be one for which there are no edges between two (not necessarily distinct) elements of A (thus, A cannot contain any vertex incident with a loop).

Theorem 4.16. *For the graphs $G_{n,S}$ as above, $i(G_{n,S}) \leq i(G_{n+1,S})$, $\forall n \geq 1$, and*

$$\lim_{n\to\infty} i(G_{n,S}) = i_\mu(S, s_\Gamma),$$

where μ is the usual product measure on 2^Γ.

Proof. Let $A \subseteq 2^{F_n}$ be an independent set for $G_{n,S}$, i.e., for $p, q \in A$ (not necessarily distinct), $s \cdot \mathcal{N}_p \cap \mathcal{N}_q = \emptyset$, $\forall s \in S^{\pm 1}$. This is the same thing as saying that

$$s \cdot \left(\bigcup_{p \in A} \mathcal{N}_p \right) \cap \bigcup_{p \in A} \mathcal{N}_p = \emptyset, \forall s \in S^{\pm 1}.$$

Let $A' \subseteq 2^{F_{n+1}}$ be defined by

$$q \in A' \Leftrightarrow q|F_n \in A.$$

Then $\frac{|A|}{|2^{F_n}|} = \frac{|A'|}{|2^{F_{n+1}}|}$ and for $p \in A$, $\mathcal{N}_p = \bigcup_{q \in A', q|F_n = p} \mathcal{N}_q$, so $\bigcup_{p \in A} \mathcal{N}_p = \bigcup_{p \in A} \bigcup_{q \in A', q|F_n = p} \mathcal{N}_q = \bigcup_{q \in A'} \mathcal{N}_q$, so $s \cdot \left(\bigcup_{q \in A'} \mathcal{N}_q \right) \cap \bigcup_{q \in A'} \mathcal{N}_q = \emptyset$, $\forall s \in S^{\pm 1}$, i.e., A' is independent for $G_{n+1,S}$. Thus $i(G_{n,S}) \leq i(G_{n+1,S})$.

Also if A is independent for $G_{n,S}$ and $\hat{A} = \bigcup_{p \in A} \mathcal{N}_p$, then $\hat{A}$ is independent for $G(S, s_\Gamma)$ and $\frac{|A|}{|2^{F_n}|} = \mu(\hat{A})$, thus $i(G_{n,S}) \leq i_\mu(S, s_\Gamma)$.

Assume now that $\alpha < i_\mu(S, s_\Gamma)$ and let $B \subseteq 2^\Gamma$ be an independent Borel set for $G(S, s_\Gamma)$ with $\mu(B) > \alpha$. Let $\varepsilon > 0$ and let $K \subseteq 2^\Gamma$ be compact with $K \subseteq B$, $\mu(B \setminus K) < \varepsilon$. Then K is also independent, so $s \cdot K \cap K = \emptyset$, $\forall s \in S^{\pm 1}$. Since the shift action is continuous, there is an open set $U \supseteq K$ with $\mu(U \setminus K) < \varepsilon$ such that $s \cdot U \cap U = \emptyset$, $\forall s \in S^{\pm 1}$, i.e., U is also independent. By compactness, let now n be large enough and $A \subseteq 2^{F_n}$ be such that $K \subseteq \hat{A} \subseteq U$ (where, as before, $\hat{A} = \bigcup_{p \in A} \mathcal{N}_p$). Thus A is independent in $G_{n,S}$ and so $\alpha - \varepsilon \leq \mu(\hat{A}) = \frac{|A|}{|2^{F_n}|} \leq i(G_{n,s})$. Letting $\varepsilon \to 0$ we have that

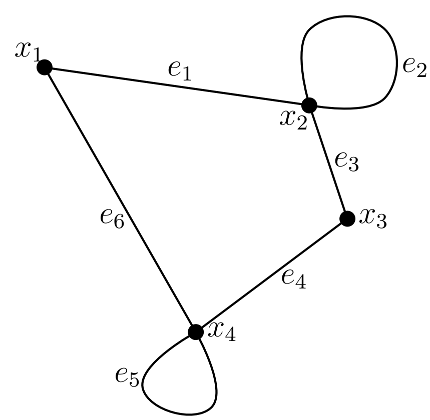
$$\alpha \leq \lim_{n\to\infty} i(G_{n,s}),$$

so

$$\lim_{n\to\infty} i(G_{n,S}) = i_\mu(S, s_\Gamma).$$

□

For the next result, if $G = (X, E)$ is a graph with loops, by a *cycle* we understand a sequence of distinct elements $x_1, \ldots, x_n$ of X and distinct edges $e_1, \ldots, e_m$ such that each e_i is either an edge connecting some x_j, x_{j+1}, $1 \leq k < n$, or x_n, x_0, or else a loop incident with some x_j, and moreover there is an edge e_i from each x_j to x_{j+1}, $1 \leq j < n$, and from x_n to x_0. The *length* of this cycle is the number m of edges. For example



is a cycle of length 6.

Theorem 4.17. *If* $\mathrm{Cay}(\Gamma, S)$ *is bipartite, then if*

$$X_{S,n,k} = \{p \in 2^{F_n} : p \text{ belongs to an odd cycle of length} \leq k \text{ in } G_{n,S}\},$$

we have $\frac{|X_{S,n,k}|}{|2^{F_n}|} \to 0$.

Proof. If $\hat{X}_{S,n,k} = \bigcup\{\mathcal{N}_p : p \in X_{S,n,k}\}$, we will show that

$$\mu(\hat{X}_{S,n,k}) = \frac{|X_{S,n,k}|}{|2^{F_n}|} \to 0.$$

Let $X \subseteq 2^\Gamma$ be the free part of the shift action of Γ on 2^Γ, so that $\mu(X) = 1$. It is enough to show that

$$\bigcap_{n \geq 1} \bigcup_{m \geq n} \hat{X}_{S,m,k} \subseteq 2^\Gamma \setminus X.$$

(Then $\lim_{n\to\infty} \mu\left(\bigcup_{m\geq n} \hat{X}_{S,m,k}\right) = 0$, so $\lim_{n\to\infty} \mu(\hat{X}_{S,n,k}) = 0$.)

Fix $x \in \bigcap_n \bigcup_{m\geq n} \hat{X}_{S,m,k}$. Then $x \in \hat{X}_{S,n_i,k}$, where $1 < n_1 < n_2 < \cdots$, so $x \in \mathcal{N}_{p_i}$, where $p_i \in X_{S,n_i,k}$. Then p_i belongs to some odd cycle of length $\leq k$, so, by going to a subsequence of (n_i), we may assume that every p_i belongs to a $(2l+1)$ cycle for some l with $2l+1 \leq k$. Then there are $p_i^0 = p_i, p_i^1, \ldots, p_i^{2l}$ in $X_{S,n_i,k}$ and $s_i^0, s_i^1, \ldots, s_i^{2l}$ in $S^{\pm 1}$ with

$$\begin{aligned} s_i^0 \cdot \mathcal{N}_{p_i^0} \cap \mathcal{N}_{p_i^1} &\neq \emptyset, \\ s_i^1 \cdot \mathcal{N}_{p_i^1} \cap \mathcal{N}_{p_i^2} &\neq \emptyset, \\ &\vdots \\ s_i^{2l-1} \cdot \mathcal{N}_{p_i^{2l-1}} \cap \mathcal{N}_{p_i^{2l}} &\neq \emptyset, \\ s_i^{2l} \cdot \mathcal{N}_{p_i^{2l}} \cap \mathcal{N}_{p_i^0} &\neq \emptyset. \end{aligned}$$

By again passing to a subsequence of (n_i), we may assume that $s_i^0 = s^0, s_i^1 = s^1, \ldots, s_i^{2l} = s^{2l}$ do not depend upon i. Thus

$$\begin{aligned} s^0 \cdot \mathcal{N}_{p_i^0} \cap \mathcal{N}_{p_i^1} &\neq \emptyset, \\ &\vdots \\ s^{2l} \cdot \mathcal{N}_{p_i^{2l}} \cap \mathcal{N}_{p_i^0} &\neq \emptyset. \end{aligned}$$

By once again going to a subsequence of (n_i), we may assume that there are $x^0 = x, x^1, \ldots, x^{2l} \in 2^\Gamma$ such that $x^k|n_i = p_i^k|n_i$ for all $k \leq 2l$ and all i. Thus for each i,

$$\begin{aligned} s^0 \cdot \mathcal{N}_{x^0|n_i} \cap \mathcal{N}_{x^1|n_i} &\neq \emptyset, \\ &\vdots \\ s^{2l} \cdot \mathcal{N}_{x^{2l}|n_i} \cap \mathcal{N}_{x^0|n_i} &\neq \emptyset, \end{aligned}$$

therefore by the continuity of the shift action again,

$$s^0 \cdot x^0 = x^1, s^1 \cdot x^1 = x^2, \ldots, s^{2l} \cdot x^{2l} = x^0,$$

i.e., $s^{2l}s^{2l-1}\cdots s^1 s^0 \cdot x = x$. Since $s^{2l}s^{2l-1}\cdots s^1 s^0 \neq 1$, we have $x \in 2^\Gamma \setminus X$. $\square$

Remark 4.18. One can actually calculate quantitative upper bound estimates for $\frac{|X_{S,n,k}|}{|2^{F_n}|}$ in the preceding theorem.

Let

$$G_{n,S,k} = G_{n,S}|(2^{F_n} \setminus X_{S,n,k})$$

be the induced graph on $2^{F_n} \setminus X_{S,n,k}$. Then for n large enough (depending upon S, k), $2^{F_n} \setminus X_{X,n,k} \neq \emptyset$ and $G_{n,S,k}$ is an ordinary graph, i.e., has no loops. Moreover, the odd girth of $G_{n,S,k}$ is bigger than k, i.e.,

$$g_{\text{odd}}(G_{n,S,k}) > k.$$

Furthermore, if $\delta_{S,n,k} = \frac{|X_{S,n,k}|}{|2^{F_n}|}$,

$$\begin{aligned} i(G_{n,S,k}) &\leq \frac{1}{1-\delta_{S,n,k}} i(G_{n,S}) \\ &\leq \frac{1}{1-\delta_{S,n,k}} i_\mu(S, s_\Gamma). \end{aligned}$$

Let now $\Gamma = \mathbb{F}_m$ with free generating set $S_m = \{a_1, \ldots, a_m\}$, and let

$$G_{m,n,k} = G_{S_m,n,k}, \delta_{m,n,k} = \delta_{S_m,n,k}.$$

Then

$$i(G_{m,n,k}) \leq \frac{1}{1-\delta_{m,n,k}} \cdot \frac{\sqrt{2m-1}}{m+\sqrt{2m-1}},$$

and $\delta_{m,n,k} \to 0$ as $n \to \infty$. Thus we have a new family of explicitly given (finite) graphs with large odd girth and small independence ratio, thus large chromatic number. For example,

Theorem 4.19. *Given m, k, for all large enough n (depending upon m, k),*

$$\begin{aligned} g_{\text{odd}}(G_{m,n,k}) &> k, \\ i(G_{m,n,k}) &\leq \frac{2\sqrt{2m-1}}{m+\sqrt{2m-1}}, \end{aligned}$$

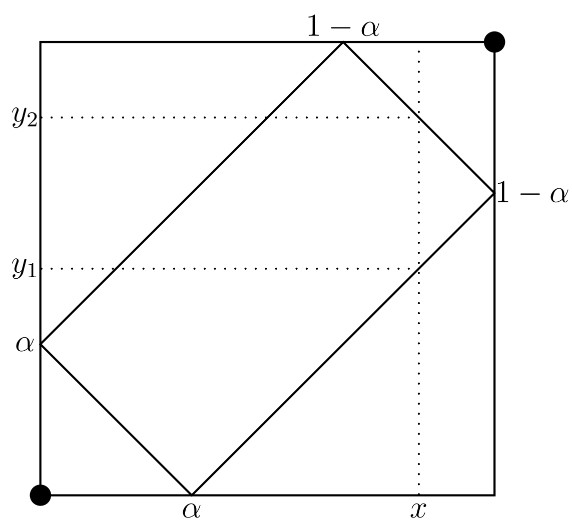
and thus

$$\chi(G_{m,n,k}) \geq \frac{m+\sqrt{2m-1}}{2\sqrt{2m-1}}.$$

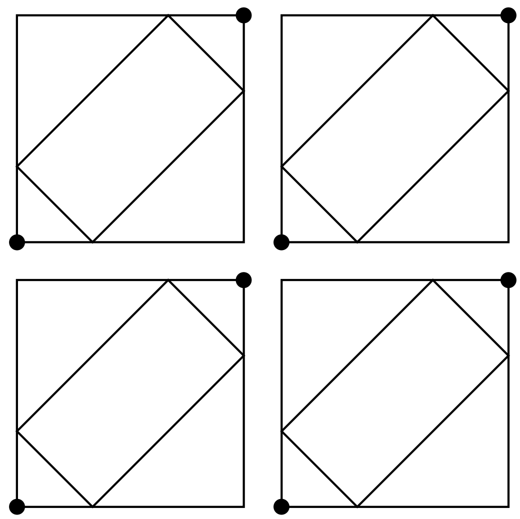
5 A matching problem

Consider a Borel bipartite graph $G = (X, E)$, i.e., $X = X_1 \sqcup X_2$ is a Borel partition and if $(x, y) \in E$ then one of x, y is in X_1 and the other is in X_2. If $d(x) = k < \aleph_0$ for every $x \in X$, then by a theorem of König (a special case of Hall's Theorem), G admits a *matching*, i.e., a bijection $\varphi : X_1 \to X_2$ such that $(x, \varphi(x)) \in E$, $\forall x \in X$. The question was raised (see, e.g., Miller [17]) whether there is a Borel version of that theorem, more precisely, whether there is a Borel matching.

Laczkovich [16] provided the following counterexample for $k = 2$. Fix an irrational $0 < \alpha < 1$ and consider the set R consisting of the following rectangle inscribed in the unit square, together with the indicated two corner points.

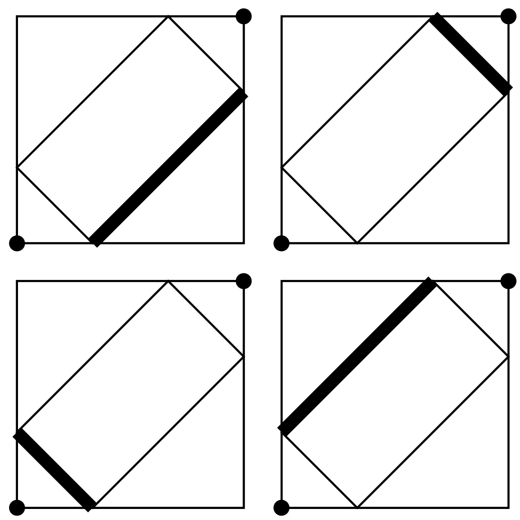


We take X_1, X_2 to be two disjoint copies of $[0, 1]$ and for $x \in X_1$ its two neighbors $y_1, y_2 \in X_2$ are such that $(x, y_i) \in R$. The two neighbors of any $y \in X_2$ are defined analogously. Clearly this is a Borel graph in which every vertex has degree 2, but Laczkovich showed that it does not have a Borel matching. In the paper Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [15], the authors argue that the following graph



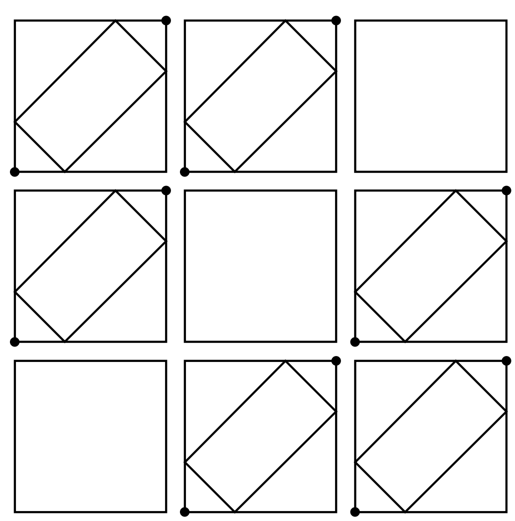
which consists of 4 “copies” of the preceding graph (actually, the authors discard finitely many connected components rather than adding dots at the corners, but it is clear that one graph has admits Borel matching if and only if the other does), and in which every vertex has degree 4 provides a counterexample for $k = 4$ (and similarly for all even k). They also raised the question of whether there is a counterexample for $k = 3$.

Lyons (private communication) showed that the above example actually does not work, as it has a Borel matching. A simpler argument is as follows:



The boldface lines provide the matching.

However, it turns out that there is a way to modify this construction to find counterexamples for every even k. For example, for $k = 4$, the idea is to construct a "Sudoku" version which is illustrated in the following picture:



Let us give a detailed argument. Fix a Borel bipartite graph $(X_1 \sqcup X_2, E)$ with degree $k = 2$ possessing no Borel matching. Define from this a new graph $(\overline{X}_1 \sqcup \overline{X}_2, \overline{E})$ as follows: $\overline{X}_1 = X_1 \times \{1, 2, 3\}$, $\overline{X}_2 = X_2 \times \{1, 2, 3\}$, and $(x, i)\overline{E}(y, j) \Leftrightarrow (i \neq j \text{ and } xEy)$. This has degree $k = 4$ and it is enough to show that if there is a Borel injection $\overline{f} : \overline{X}_1 \to \overline{X}_2$ such that $(x, i)\overline{E}\ \overline{f}(x, i)$, then there is a Borel injection $f : X_1 \to X_2$ with $xEf(X)$ (and similarly if we switch the roles of $\overline{X}_1, \overline{X}_2$). Granting this, if there is a Borel matching for $(\overline{X}_1 \sqcup \overline{X}_2, \overline{E})$, there are two Borel injections, from X_1 to X_2 and vice versa, whose graphs are contained in E, so by a Schröder-Bernstein argument, there is a Borel matching for $(X_1 \sqcup X_2, E)$, a contradiction.

So fix $\overline{f}$ as above, which we will use to define f. Given $x \in X$, consider $\overline{f}(x, 1) = (u, a)$, $\overline{f}(x, 2) = (v, b)$, and $\overline{f}(x, 3) = (w, c)$. Then xEu, xEv, and xEw. Since $(X_1 \sqcup X_2, E)$ has degree 2, at least two of u, v, w are equal. So there is a unique $y \in X_2$ such that for at least two distinct $i, j \leq 3$, we have $\overline{f}(x, i) = (y, k)$, $\overline{f}(x, j) = (y, l)$ (for some necessarily distinct k,l). Put $f(x) = y$; we claim that this works. To see this, take $x \neq x'$. If $f(x) = f(x') = y$, then let $i \neq j$ be such that $\overline{f}(x, i) = (y, k)$, $\overline{f}(x, j) = (y, l)$ and $i' \neq j'$ such that $\overline{f}(x', i') = (y, k')$, $\overline{f}(x', j') = (y, l')$. As before, $k \neq l$ and $k' \neq l'$. It follows that one of k, l is equal to one of k', l', contradicting the injectivity of $\overline{f}$.

The same proof works for degree $k = 6$ by dropping from the definition of $\overline{E}$ the condition $i \neq j$ (i.e., in the preceding picture inscribing the rectangle into all nine of the small squares). In general, for degrees $k = 4n$ and $k = 4n + 2$ $(n \geq 1)$ one uses the same argument with the $(2n + 1) \times (2n + 1)$ square.

As far as we know, the case $k = 3$ is open. We sketch below an alternative approach to the $k = 2$ case which adapts naturally to the $k = 3$ case, relating the question of whether a bipartite graph has no Borel matching to the calculation of the independence number associated with the shift action of an appropriate group. This was actually for us a motivation for looking at the independence number of such graphs.

Let $m \geq 2$ and $A = \{1, a, a^2, \ldots, a^{m-1}\}$ and $B = \{1, b, b^2, \ldots, b^{m-1}\}$ be two copies of the cyclic group of order m. Let $\Gamma_m = A * B$ and consider the shift action of Γ_m on 2^{Γ_m}, and let $Y \subseteq 2^{\Gamma_m}$ be its free part. Let $X_1 = Y/A$, the set of A-orbits under the shift action, and $X_2 = Y/B$. Then X_1 and X_2 are standard Borel spaces and let $X = X_1 \sqcup X_2$. Define the bipartite graph $G_m = (X, E)$ by

$$pEq \Leftrightarrow p \cap q \neq \emptyset.$$

If $p \in X_1$, $q \in X_2$ and $p \cap q \neq \emptyset$, then for some $y \in p \cap q$, $p = A \cdot y$ and $q = B \cdot y$. Since the action of Γ on Y is free, clearly $p \cap q = \{y\}$. Thus there is a canonical bijective correspondence between Y and E, namely

$$y \mapsto \{A \cdot y, B \cdot y\}$$

(we view E here as a set of unordered pairs). Clearly each vertex in G_m has degree exactly m.

Suppose now that $f : X_1 \to X_2$ is a Borel matching for G. By the above identification, f can be viewed as a Borel subset $M \subseteq Y$ and the condition of being a matching corresponds exactly to the assertion that M meets every A-orbit in exactly one point, and likewise meets every B-orbit. That is, M is a common transversal for the A- and B-orbits.

The set $S = (A \cup B) \setminus \{1\} \subseteq \Gamma_m$ is a set of generators for Γ_m and the above condition for M implies that M is a Borel independent set for the graph $G(S, s_{\Gamma_m})$. Moreover it is clear that for the product measure μ on 2^{Γ_m}, $\mu(M) = 1/m$. Thus, in particular, if there is a Borel matching in $G_m = (X, E)$, then $i_\mu(G(S, s_{\Gamma_m})) \geq 1/m$.

On the other hand, if $i_\mu(G(S, s_{\Gamma_m})) \geq 1/m$ and the supremum is attained, say by a Borel independent set C, then C must meet almost every A-orbit and almost ever B-orbit in exactly one point. It follows that the existence of an almost everywhere Borel matching in G_m is equivalent to the statement that $i_\mu(G(S, s_{\Gamma_m})) = 1/m$ and the supremum is attained.

If $m = 2$ this is impossible: since the action s_{Γ_m} is weakly mixing, there can be no independent set of measure $1/2$. Thus there is no Borel matching in the graph G_2, providing an alternate proof of Laczkovich's theorem.

We do not know whether there is a Borel matching for G_3. If either $i_\mu(G(S, s_{\Gamma_3})) < 1/3$ or else $i_\mu(G(S, s_{\Gamma_3}) = 1/3$ but the supremum is not attained, then there would be no Borel matching for G_3.

More generally, we do not know if there is a Borel matching for any G_m, when $m \geq 3$.

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