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Woodin's axiom $(*)$, bounded forcing axioms, and precipitous ideals on ω_1

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Abstract

If the Bounded Proper Forcing Axiom BPFA holds, then Mouse Reflection holds at $\aleph_2$ with respect to all mouse operators which do not go beyond $M_1^{\#}$. This yields that if Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ holds, then BPFA implies its own strengthening BMM^{++}, the Bounded Proper Forcing Axiom with a predicate for the nonstationary ideal on ω_1. We also discuss stronger Mouse Reflection principles which follow from universally Baire versions of BPFA and consequences of BMM plus "NS_{ω_1} is precipitous" which yield inner models with Woodin cardinals.

0 Introduction.

Let Γ be a class of forcings, e.g. the class of all c.c.c., proper, semi–proper, or stationary set preserving forcings. The *bounded forcing axiom for* Γ says that

$$(H_{\omega_2};\in) \prec_{\Sigma_1} ((H_{\omega_2})^{V^{\mathbb{P}}};\in)$$

whenever $\mathbb{P} \in \Gamma$. The bounded forcing axiom for c.c.c., proper, semi–proper, and stationary set preserving forcings is called MA_{ω_1} ("Martin's axiom"), BPFA (the "Bounded Proper Forcing Axiom"), BSPFA (the "Bounded Semi–Proper Forcing Axiom"), and BMM ("Bounded Martin's Maximum"), respectively. (Cf. [5] and [1].) This paper will be concerned with BPFA and BMM.

We may strengthen these statements by adding a predicate for the non–stationary ideal on ω_1, NS_{ω_1}, and/or for a given universally Baire set A of reals.

The statement that

$$(H_{\omega_2};\in,\mathsf{NS}_{\omega_1}) \prec_{\Sigma_1} ((H_{\omega_2})^{V^{\mathbb{P}}};\in,(\mathsf{NS}_{\omega_1})^{V^{\mathbb{P}}})$$

whenever $\mathbb{P} \in \Gamma$ is called the "++ version" of the bounded forcing axiom for Γ. This paper will be concerned with BMM^{++}, the ++ version of BMM (cf. [18, Definition 10.91]).

If A is a universally Baire set of reals, as being witnessed by the (class sized) trees T and U with $A = p[T]$, and if $\mathbb{P}$ is any forcing, then we may denote by A^* the projection of T as computed in $V^{\mathbb{P}}$. The set A^* depends on $\mathbb{P}$ and on the generic filter, but it does not depend on the choice of the trees T and U witnessing the universally Baireness of A (cf. [4]). By the "universally Baire version" of the

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bounded forcing axiom for Γ we shall mean the statement that for every universally Baire set A of reals and for every $\mathbb{P} \in \Gamma$,

$$(H_{\omega_2}; \in, A) \prec_{\Sigma_1} ((H_{\omega_2})^{V^{\mathbb{P}}}; \in, A^*).$$

(Cf. [18, Definition 10.122].) This paper will be concerned with what we'll abbreviate by $\mathsf{BPFA}^{\mathrm{uB}}$ and by which we mean the universally Baire version of BPFA.

We shall study mouse reflection principles under the hypothesis of bounded forcing axioms. A mouse reflection principle says that if an initial segment of V is closed under a given mouse operator $X \mapsto \mathcal{M}(X)$, then so is some longer initial segment of V. (Cf. Definition 1.1 for a precise definition of what we mean by a "mouse operator.") A typical example would be the statement that if H_{ω_2} is closed under the mouse operator $X \mapsto X^{\#}$, then V is closed under $X \mapsto X^{\#}$. Woodin [18, Theorem 10.108] essentially showed that if Bounded Martin's Maximum BMM^{++} holds and if H_{ω_2} is closed under the mouse operator $X \mapsto M_1^{\#}(X)$, then V is closed under $X \mapsto M_1^{\#}(X)$. (This gives that the model of BMM^{++} constructed in the proof of [18, Theorem 10.99] starts from an optimal large cardinal hypothesis.) We here show the following theorem.[2] (Cf. Definition 1.1 on what it means that a mouse operator "does not go beyond $M_1^{\#}$.")

Theorem 0.1 *Assume* BPFA *to hold. Let J be a mouse operator which does not go beyond $M_1^{\#}$, and suppose H_{ω_2} to be closed under J. Then V is closed under J.*

In the light of [18, Theorem 10.99], this readily implies the following theorem (to be shown in section 3). Recall that Woodin's axiom $(*)$ is the conjunction of the following two statements (cf. [18, Definition 5.1]):

- AD, the Axiom of Determinacy, holds in $L(\mathbb{R})$, and
- $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$–generic extension of $L(\mathbb{R})$.

Theorem 0.2 *Assume Woodin's axiom $(*)$ to hold. Then the following statements are equivalent.*

(1) The Bounded Proper Forcing Axiom BPFA *holds.*

(2) Bounded Martin's Maximum BMM^{++} *holds.*

(3) V is closed under $X \mapsto M_1^{\#}(X)$.

The only further ingredient beyond Theorem 0.1 which is necessary to now derive Theorem 0.2 is the following lemma (to be shown in section 2) which might be part of the folklore. (Cf. Definition 1.1 on our concept of a "nice" mouse operator in $L(\mathbb{R})$.)

Lemma 0.3 *Assume Woodin's axiom $(*)$ to hold. Then H_{ω_2} is closed under any nice mouse operator in $L(\mathbb{R})$, and in particular H_{ω_2} is closed under the mouse operator $X \mapsto M_1^{\#}(X)$.*

[2]We thank Boban Velickovic for asking us whether under BPFA the closure of H_{ω_2} under $X \mapsto X^{\#}$ implies the closure of V under $X \mapsto X^{\#}$. We also thank Andrés Caicedo and Martin Zeman for very helpful comments on an earlier version of this paper.

As a consequence of Theorem 0.2, BPFA does not hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$. In contrast to Theorem 0.2, Lemma 4.1 (of section 4) will show that there is a model of $\mathsf{AD}^{L(\mathbb{R})}$ plus BPFA which is not even closed under $X \mapsto X^{\#}$.

To derive a mouse reflection theorem which goes beyond Theorem 0.1 we (provably, by Theorem 0.2) need more than BPFA. It turns out that $\mathsf{BPFA}^{\mathrm{uB}}$ is the right axiom:

Theorem 0.4 *Assume $\mathsf{BPFA}^{\mathrm{uB}}$. Suppose J to be a correct mouse operator with the relativization and extension property such that J is total on V. Then $(\aleph_2, \infty)$–mouse reflection holds with respect to mouse operators which do not go beyond $X \mapsto M_1^J(X)$.*

Here, M_1^J is the "Woodin-in-J" mouse operator, cf. Definition 1.1. Theorem 0.4 produces the following result:

Theorem 0.5 *Assume $\mathsf{BPFA}^{\mathrm{uB}}$ and there is a precipitous ideal on ω_1. Then Projective Determinacy holds.*

As $\mathsf{BPFA}^{\mathrm{uB}}$ by itself is not stronger than BPFA, the hypothesis on the existence of a precipitous ideal on ω_1 cannot be removed in Theorem 0.5.

It is currently open whether BMM implies that NS_{ω_1}, the non–stationary ideal on ω_1, must be precipitous. The theory BMM plus "NS_{ω_1} is saturated" has many nice consequences, cf. [18], [2], and [3], including $\boldsymbol{\delta}^1_2 = \aleph_2$.

The authors of [9] ask whether their forcing which uses a precipitous ideal on ω_1 to increase $\boldsymbol{\delta}^1_2$ can be iterated. An affirmative answer to this question in the absence of inner models with Woodin cardinals would be particularly interesting. The same question could be asked concerning the forcing of [2]. The paper [3] has a negative result in this direction: it says that the forcings of [9] and [2] are semi–proper iff and only if all stationary set preserving forcings are semi–proper, cf. [3, Theorem 5.7].

In this paper, we shall answer the question of [9] in the negative in a strong sense. We shall prove that in the absence of an inner model with a Woodin cardinal, once ω_2 of V is collapsed to ω_1 by a forcing which preserves ω_1, then in the extension there is no forcing whatsoever which does not collapse ω_1 and which resurrects a precipitous ideal on ω_1. See Theorem 5.2. Therefore, the forcings of [9] and [2] don't even exist in further forcing extension after having forced with either of them once.

Theorem 5.2 has the following consequences.

Theorem 0.6 *The following theories are equiconsistet.*

(1) ZFC plus there is a precipitous ideal on ω_1 and for every $f\colon \omega_1 \to \omega_1$ there is a canonical function $f_\alpha\colon \omega_1 \to \omega_1$, $\alpha < \omega_2$, such that $\{\xi < \omega_1 \colon f(\xi) \le f_\alpha(\xi)\}$ is stationary.

(2) ZFC plus there is a Woodin cardinal.

Theorem 0.7 *Suppose that there is a precipitous ideal on ω_1 and $\boldsymbol{\delta}^1_2 = \aleph_2$. There is then an inner model with a Woodin cardinal.*

Woodin has shown that the conclusion of Theorem 0.7 can be forced over a model of ZFC in which there are $\delta^* < \delta$ such that δ is a Woodin cardinal, δ^* is Woodin in $L(V_{\delta^*})$, and $V_{\delta^*} \prec V_\delta$. (Cf. [18, Theorem 3.25]. In fact, NS_{ω_1} will be saturated in the extension.).

By the above results, all the consequences of BMM plus "NS_{ω_1} is precipitous" which are discussed in [3] (except for ϕ_{AC}) imply, in the presence of a precipitous ideal on ω_1, that there is an inner model with a Woodin cardinal. The methods presented in this paper may indeed be used to show that BPFA plus "NS_{ω_1} is precipitous" yields an inner model with two Woodin cardinals.

1 Preliminaries.

This section defines the concept of "mouse reflection" as it will be used in the present paper. We refer the reader to [19] on inner model theory. We will use standard notation throughout. We use the phrase "mouse" here as being defined in [8, Definition 1.1]: a *mouse* is a premouse such that the transitive collapses of all of its countable (sufficiently elementary) substructures are ω_1+1 iterable. If X is a set of ordinals, then an X-*mouse* is an X-premouse such that the transitive collapses of all of its countable (sufficiently elementary) substructures are ω_1+1 iterable.

Definition 1.1 *Let $\varphi \equiv \varphi(v_0, v_1)$ be a Σ_1–formula in the language of boldface premice, and let $z \in \mathbb{R}$. Let X be a set of ordinals which codes z. An X–premouse $\mathcal{M}$ is called φ,z–*small *iff*

$$\mathcal{M} \models \neg\, \varphi(z, X).$$

The mouse operator given by φ, z *is the unique partial map $X \mapsto J(X) = J_{\varphi,z}(X)$ which assigns to any set X of ordinals the unique X–mouse $J(X)$ such that $J(X)$ is sound above X, $J(X)$ is not φ,z–small, but every proper initial segment of $J(X)$ is φ,z–small, if it exists (otherwise $J(X) = J_{\varphi,z}(X)$ remains undefined). A* mouse operator *is a partial map $X \mapsto J(X)$ for which there is some Σ_1–sentence φ and some $z \in \mathbb{R}$ such that $X \mapsto J(X)$ is the mouse operator given by φ, z.*

Let J be a mouse operator which is given by φ, z. Then M_1^J is the unique mouse operator such that for all sets X of ordinals, $M_1^J(X)$ is the least X–mouse such that $M_1^J(X)$ is sound above X, and there is some δ such that $M_1^J(X) \models$ "δ is a Woodin cardinal" and $M_1^J(X) = J(M_1^J(X)||\delta)$, if it exists (otherwise $M_1^J(X)$ remains undefined). If the Σ_1– formula ψ is such that $M_1^J = J_{\psi,z}$, then we say that the mouse operator $X \mapsto J^*(X)$ does not go beyond $M_1^J(X)$ iff all the proper initial segments of $J^*(X)$ are ψ, z–small.

We say that $X \mapsto J^*(X)$ does not go beyond $X \mapsto M_1^\#(X)$ *iff J^* does not go beyond M_1^J, where $J(X) = X^\#$ for all sets X of ordinals.*

A mouse operator $X \mapsto J(X)$ is said to be in $L(\mathbb{R})$ iff it is total on (sets of ordinals in) H_{ω_1}, every $J(X)$ is tame, and the function which assigns to a bounded subset X of ω_1 the unique ω_1 iteration strategy for $J(X)$ with respect to stacks of normal iteration trees on $J(X)$ is an element in $L(\mathbb{R})$.

Notice that if $\mathcal{M}_\varphi(X)$ exists, then $\rho_1(\mathcal{M}_\varphi(X)) \leq \sup(X)$.

Examples of mouse operators we shall be concerned with are $X \mapsto X^\#$ and, more generally, $X \mapsto M_n^\#(X)$ for $n < \omega$. If $n > 0$, then $X \mapsto M_n^\#(X)$ is "nice" in the sense of the following ad hoc definition. The concept of "niceness" will play a role in Lemma 2.1.

Definition 1.2 *Let $X \mapsto J(X)$ be a mouse operator. We call $X \mapsto J(X)$* nice *iff for all X such that $J(X)$ exists, $J(X)$ has at least two measurable cardinals, and if λ is the second smallest measurable cardinal of $J(X)$ and if Y is a set of ordinals in $J(X)^{\mathbb{P}}$, where $\mathbb{P} \in J(X)|\lambda$ is a poset, then $J(Y)$ exists and is Σ_1–definable over $J(X)^{\mathbb{P}}$ from the parameter Y.*

Definition 1.3 *Let κ, λ be infinite cardinals with $\kappa < \lambda$. We also allow $\lambda = \infty$. Then (κ, λ)–*mouse reflection *says that whenever $X \mapsto J(X)$ is a mouse operator which is total on all bounded subsets of κ, then $X \mapsto J(X)$ is total on all bounded subsets of λ.*

In this language, [18, Theorems 9.78 and 9.84] study $(\aleph_2, \aleph_3)$–mouse reflection; cf. also [16]. We shall be concerned with $(\aleph_2, \infty)$–mouse reflection in what follows.

Definition 1.4 *Let J be a mouse operator which is given by $\varphi(-, z)$. We say that J has the* relativization and extension property *iff there is a formula $\psi_J(v_0, v_1, v_2, v_3)$ such that in every extension $V[G]$, where G is $\mathbb{P}$–generic over V for some forcing $\mathbb{P} \in V$, the following holds true. Let H be transitive, $z \in H$, let g ($\in V[G]$) be $\mathbb{Q}$–generic over $J(H)$, and let $x \in H[g]$. Moreover, let P be a transitive model with $J(H)[g] \in P$. Then $J(x) \in P$ and $J(x)$ is the unique object $\mathcal{M}$ such that $P \models \psi_J(\mathcal{M}, z, x, J(H)[g])$.*

Lemma 1.5 *Let J be a mouse operator which has the relativization and extension property. Then*

$$\{(x, y) \colon x \geq_T z \wedge y \text{ codes } J(x)\}$$

is universally Baire.

Proof. Let θ be an uncountable regular cardinal. Let T be a tree of height ω which searches for x, y, $\mathcal{N}$, H, π, P such that

- x, $y \in \mathbb{R}$, where $x \geq_T z$,
- y codes an x–premouse $\mathcal{M}$,
- $\mathcal{N}$ is an H–premouse, $\pi \colon \mathcal{N} \to J(H_\theta)$, and $\pi(H) = H_\theta$,
- there is some $\mathbb{Q} \in H$ and some $\mathbb{Q}$–generic filter g over $\mathcal{N}$ such that $x \in \mathcal{N}[g]$,
- P is a countable transitive model of ZFC^- such that $P \models \psi_J(\mathcal{M}, z, x, \mathcal{N}[g])$.

We claim that T projects to the set of all (x, y) such that $x \geq_T z$ and y codes $J(x)$ in any extension $V[G]$, where G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$. In order to verify this, let us first assume that G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$ and that $x \geq_T z$ and y codes $J(x)$ in $V[G]$. Then (x, y) is in the projection of T, as being witnessed by $\mathcal{N}$, H, π, P, where for some σ and P^*, $\pi = \sigma \restriction \mathcal{N} \colon \mathcal{N} \to J(H_\theta)$, $\sigma \colon P^* \to H_{(2^{<\theta})^+}[G]$, P^* is countable and transitive, $H = \pi^{-1}(H_\theta)$, $\mathcal{N} = \sigma^{-1}(J(H_\theta))$, and $P = \sigma^{-1}(H_\theta)$. We may simply put $\mathbb{Q} = \sigma^{-1}(\mathbb{P})$ and $g = \sigma^{-1}(G)$. On the other hand, if (x, y) is in the projection of T, as being witnessed by $\mathcal{N}$, H, π, P, $\mathbb{Q}$ and g, then $\mathcal{N}[g]$ inherits the iterability from $\mathcal{N}$, which is in turn iterable as being certified by $\pi \colon \mathcal{N} \to J(H_\theta)$. Therefore, y must code $J(x)$. We have used heavily that J has the relativization and extension property.

We may now also construct a tree U such that U searches for x, y, y', Q, where (x, y') is in the projection of T and Q is a countable transitive model with y, $y' \in Q$ such that Q knows that y' is not isomorphic to y. Then U projects to the complement of $p[T]$ in any extension $V[G]$, where G is $\mathbb{P}$–generic over V for some $\mathbb{P} \in H_\theta$. $\square$

The following lemma is part of the folklore.

Lemma 1.6 *Let J be a mouse operator which is total on V. Let X be any set of ordinals. Then either $M_1^J(X)$ exists, or else for every X–premouse $\mathcal{M}$ with no definable Woodin cardinal the following holds true: if every $\bar{\mathcal{M}}$ is ω_1–iterable, where $\bar{\mathcal{M}}$ is countable and embeds into $\mathcal{M}$, then $\mathcal{M}$ is fully iterable.*

Definition 1.7 *Let J be a mouse operator. We say that J is* correct *iff there is a formula*[3] *$\psi(v_0)$ such that for all x, $y \in \mathbb{R}$ and for every M_1^J–small x–premouse $\mathcal{M}$ which is coded by y,*

$$\mathcal{M} \text{ is } \omega_1\text{–iterable } \Leftrightarrow J(y) \models \psi(y).$$

The following is an easy application of the extender algebra.

Lemma 1.8 *Let J be a correct mouse operator which has the relativization and extension property. Then M_1^J is also correct and has the relativization and extension property.*

PROOF. We show that M_1^J is correct. Let $\mathcal{M}$ be an x–premouse which is M_1^J–small. Let us suppose that $\mathcal{M}$ is not iterable. There is then an iteration tree $\mathcal{T}$ on $\mathcal{M}$ which is guided by strict initial of M_1^J which either has a last ill–founded model or else has no cofinal branch with a $\mathcal{Q}$–structure which is isomorphic to a certain strict initial segment of $M_1^J(\mathcal{T})$. Let us assume w.l.o.g. that there is some $\lambda < \mathrm{lh}(\mathcal{T})$ such that the $\mathcal{Q}$–structure for $\mathcal{T} \restriction \lambda$ is *not* given by a strict initial segment of $J(\mathcal{T})$ (but still by a strict initial segment of $M_1^J(\mathcal{T} \restriction \lambda)$. Then $\mathcal{T} \restriction [\lambda, \mathrm{lh}(\mathcal{T}))$ is above $\mathcal{M}_\lambda^\mathcal{T}$ and is guided by strict initial segments of J.

Let us make $\mathcal{T} \restriction \lambda$ generic over a simple iterate $\mathcal{P}$ of $M_1^J(\mathcal{M})$, say $\mathcal{T} \restriction \lambda \in \mathcal{P}^*[g]$. By the relativization and extension property of J, we may define $J(\mathcal{M}_\lambda^\mathcal{T})$ over $\mathcal{P}^*[g]$. By the correctness of $J(\mathcal{M}_\lambda^\mathcal{T})$, it can verify that $\mathcal{M}_\lambda^\mathcal{T}$ is not iterable. It is therefore forced over $M_1^J(\mathcal{M})$ by the extender algebra at its least Woodin cardinal that $\mathcal{M}$ is not iterable.

The same argument shows that if $\mathcal{M}$ is iterable, then this is verified by $M_1^J(\mathcal{M})$. $\square$

As $X \mapsto X^\#$ is certainly a correct mouse operator (by the proof of Lemma 3.1) which alo has the relativization and extension property, Lemma 1.8 immediately gives:

Lemma 1.9 *For every $n < \omega$, the mouse operator $X \mapsto M_n^\#(X)$ is correct and has the relativization and extension property.*

[3]typically, the formula will say "v_0 is iterable"

2 $(*)$ and the closure under mouse operators.

We now first prove Lemma 0.3 which we slightly reformulate as follows.

Lemma 2.1 *Assume $(*)$ to hold. Let $X \mapsto \mathcal{M}(X)$ be a nice mouse operator in $L(\mathbb{R})$. Then $X \mapsto \mathcal{M}(X)$ is total on (sets of ordinals in) H_{ω_2}.*

PROOF. For $x \in \mathbb{R}$, we let κ_x denote the least measurable cardinal of $\mathcal{M}(x)$, we let U_x denote the unique measure on κ_x in $\mathcal{M}(x)$, and we let $\mathbb{P}_x \in \mathcal{M}(x)^{\mathrm{Col}(\omega,<\kappa_x)}$ denote the standard c.c.c. forcing for producing Martin's Axiom MA_{ω_1}. Let us consider the set D of all $p \in \mathbb{P}_{\max}$ such that if $p = (M, I, a)$, then there is some $x \in \mathbb{R}$ such that

$$M = \mathcal{M}(x)^{\mathrm{Col}(\omega,<\kappa_x)*\mathbb{P}_x},$$

and I is the precipitous ideal of M induced by U_x, i.e.,

$$I = \{X \in \mathcal{P}(\kappa_x) \cap M \colon \exists Y \in U_x \ X \cap Y = \emptyset\}.$$

Standard $\mathbb{P}_{\max}$ arguments show that $D \in L(\mathbb{R})$ and D is dense in $\mathbb{P}_{\max}$ (cf. [18, Lemma 4.36]).

Now let $A \subset \omega_1$. By $(*)$, we may assume without loss of generality that A is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$. Let G_A be the $\mathbb{P}_{\max}$–generic filter which is given by A. As $D \in L(\mathbb{R})$ is dense in $\mathbb{P}_{\max}$, we may pick $p \in D \cap G_A$. Let

$$((M_\alpha, I_\alpha, a_\alpha), (\pi_{\bar{\alpha},\alpha} \colon \bar{\alpha} \leq \alpha \leq \omega_1))$$

be the generic iteration of $p = (M_0, I_0, a_0)$ given by G_A, so that $a_{\omega_1} = A$. Let $M_0 = \mathcal{M}(x)^{\mathrm{Col}(\omega,<\kappa_x)*\mathbb{P}_x}$, where $x \in \mathbb{R}$. For all $\alpha \leq \omega_1$, the map $\pi_{0,\alpha} \restriction \mathcal{M}(x)$ is the map obtained by iterating U_x and its images α times, and

$$M_\alpha = \pi_{0,\alpha}(\mathcal{M}(x))^{\mathrm{Col}(\omega,<\pi_{0,\alpha}(\kappa_x))*\pi_{0,\alpha}(\mathbb{P}_x)}.$$

The reason is that the forcing $\mathrm{Col}(\omega,<\kappa_x) * \mathbb{P}_x$ has the κ_x-c.c. from the point of view of $\mathcal{M}(x)$.

We claim that M_{ω_1} witnesses that $\mathcal{M}(A)$ exists. We first need to see that M_{ω_1} is a mouse, i.e., that transitive collapses of its countable (Σ_1 elementary) substructures are $\omega_1 + 1$ iterable. Let

$$\sigma \colon \mathcal{P} \to M_{\omega_1}$$

be Σ_1 elementary, where $\mathcal{P}$ is countable and transitive. Then there is some $\alpha < \omega_1$ and some

$$\sigma' \colon \mathcal{P} \to M_\alpha$$

such that $\pi_{\alpha,\omega_1} \circ \sigma' = \sigma$. But $M_\alpha = \pi_{0,\alpha}(\mathcal{M}(x))^{\mathrm{Col}(\omega,<\pi_{0,\alpha}(\kappa_x))*\pi_{0,\alpha}(\mathbb{P}_x)}$, where $\pi_{0,\alpha}(\mathcal{M}(x))$ is the α^{th} iterate of $\mathcal{M}(x)$ obtained by hitting the measure U_x and its images α times. This clearly implies that M_α (and hence $\mathcal{P}$) is $\omega_1 + 1$ iterable.

But $a_{\omega_1} = A$ and $X \mapsto \mathcal{M}(X)$ is nice. Therefore, $\mathcal{M}(A)$ exists and is Σ_1–definable over M_{ω_1} from the parameter A. $\square$ (Lemma 2.1)

3 Mouse reflection at $\aleph_2$.

We need to verify that the "mousehood" of a premouse of size at most $\aleph_1$ is a $\Sigma_1^{H_{\omega_2}}$ property provided that there be no inner model with a Woodin cardinal.

Lemma 3.1 *Suppose that there is no inner model with a Woodin cardinal. Let U be a transitive model of* ZFC^- *(i.e., of* ZFC *without the power set axiom) with $\omega_1 \subset U$. Let $\mathcal{M} \in U$ be a premouse (possibly of uncountable size) and with no Woodin cardinal. Then*

$$\mathcal{M} \textit{ is a mouse } \iff U \models \mathcal{M} \textit{ is a mouse.}$$

Proof. Let $\bar{\mathcal{M}}$ be a countable premouse such that there is a sufficiently elementary embedding of $\bar{\mathcal{M}}$ into $\mathcal{M}$. Because of our hypotheses, $\bar{\mathcal{M}}$ is countably (and hence fully) iterable with respect to normal trees if any only if the following holds true: if $\mathcal{T}$ is any countable putative[4] normal iteration tree on $\bar{\mathcal{M}}$, then either $\mathcal{T}$ has successor length and its last model is well–founded or else $\mathcal{T}$ has limit length and there is a maximal (and hence cofinal) branch b through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to some $J_\alpha(\mathcal{M}(\mathcal{T}))$, call it $\mathcal{Q}$, such that if $n < \omega$ is least with $\rho_n(\mathcal{Q}) \leq \delta(\mathcal{T})$, then either there is a $r\Sigma_{n+1}^{\mathcal{Q}}$–witness to the fact that $\delta(\mathcal{T})$ is not a Woodin cardinal or $\rho_\omega(\mathcal{Q}) < \delta(\mathcal{T})$. Here, $\mathcal{M}(\mathcal{T})$ is the common part model of $\mathcal{T}$ and $\delta(\mathcal{T})$ is its height; the model $\mathcal{Q}$ would be called the $\mathcal{Q}$–structure for $\mathcal{T}$.

Now in U there is a tree S of height ω searching for

(1) a countable premouse $\bar{\mathcal{M}}$ together with a sufficiently elementary embedding of $\bar{\mathcal{M}}$ into $\mathcal{M}$,
(2) a countable putative normal iteration tree $\mathcal{T}$ on $\bar{\mathcal{M}}$, and either
($3a$) a proof that $\mathcal{T}$ has a last ill–founded model, or else
($3b$) a proof that $\mathcal{T}$ has limit length but no cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to the $\mathcal{Q}$–structure $\mathcal{Q}$ for $\mathcal{T}$ such that $\mathcal{Q} = J_\alpha(\mathcal{M}(\mathcal{T}))$ for some α.

We may let S prove ($3b$) by having S search for a countable transitive model U' of ZFC^- such that $\{\bar{\mathcal{M}}, \mathcal{T}\} \subset U'$, $J_\alpha(\mathcal{M}(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is not a Woodin cardinal" for some $\alpha < U' \cap \mathrm{OR}$, and $U' \models$ "$\mathcal{M}$ and $\mathrm{lh}(\mathcal{M})$ are countable, and $\mathcal{T}$ has no cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to the $\mathcal{Q}$–structure for $\mathcal{T}$." Notice that the statement that $\mathcal{T}$ has a cofinal branch b such that $\mathcal{M}_b^{\mathcal{T}}$ has an initial segment which is isomorphic to the $\mathcal{Q}$–structure for $\mathcal{T}$ is absolute between V and U' by $\boldsymbol{\Sigma}_1^1$ absoluteness, so that this works.

But now $\mathcal{M}$ is not a mouse in V if and only if S is ill–founded in V if and only if S is ill–founded in U if and only if $\mathcal{M}$ is not a mouse in U. $\square$ (Lemma 3.1)

We now show Theorem 0.1 which we reformulate as follows. Its proof uses a key idea of Stevo Todorčević to phrase a Σ_2 statement in a Σ_1 way under favorable circumstances (cf. [17]).

[4] i.e., we do not demand that if $\mathcal{T}$ has successor length, then the last model of $\mathcal{T}$ be well–founded

Theorem 3.2 *Assume* BPFA. *Then* $(\aleph_2, \mathrm{OR})$*–mouse reflection holds with respect to mouse operators which do not go beyond* $X \mapsto M_1^\#(X)$*.*

Proof. Let $X \subset \kappa$, where $\kappa \geq \aleph_2$ is a cardinal. Let $S(X)$ denote the stack of all X–mice which are sound above κ and project to κ or below κ, i.e., $\mathcal{P} \lhd S(X)$ iff there is some X–mouse $\mathcal{Q} \trianglerighteq \mathcal{P}$ such that $\mathcal{Q}$ is sound above κ and $\rho_\omega(\mathcal{Q}) \leq \kappa$. Then $S(X)$ is itself an X–mouse, $S(X) \models \mathsf{ZFC}^-$, and κ is the largest cardinal of $S(X)$.

Let us now suppose that $\mathcal{M}(X)$ does not exist and work towards a contradiction. Let us suppose that the mouse operator $\mathcal{M}$ is defined in terms of the Σ_1–sentence φ. By our hypothesis, $S(X)$ is φ–small which implies that $S(X)$ is 1–small.

We first claim that $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) \geq \aleph_2$. In order to prove this, let

$$\pi \colon \bar{S} \to S(X)$$

be fully elementary, where $\mathrm{Card}(\bar{S}) = \aleph_1$. Setting $\bar{X} = \pi^{-1}(X)$ and $\bar{\kappa} = \pi^{-1}(\kappa)$, $\bar{S}$ is a φ–small $\bar{X}$–mouse with largest cardinal $\bar{\kappa}$. Because $\mathcal{M}(\bar{X})$ exists, we may let $\mathcal{Q} \trianglelefteq \mathcal{M}(\bar{X})$ be least such that $\bar{S} \lhd \mathcal{Q}$ and $\rho_\omega(\mathcal{Q}) \leq \bar{\kappa}$. Let $n < \omega$ be such that $\rho_{n+1}(\mathcal{Q}) \leq \bar{\kappa} < \rho_n(\mathcal{Q})$, and let

$$\mathcal{Q}^* = \mathrm{ult}_n(\mathcal{Q}; E_\pi),$$

where E_π is the extender derived from π. We may and shall assume that π was chosen in such a way that $\mathcal{Q}^*$ is well–founded (i.e., transitive) and in fact is an X–mouse (cf. [10]). Now if $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) < \aleph_2$, then we may and shall also assume that $\mathrm{ran}(\pi) \cap \mathrm{OR}$ is cofinal in $S(X) \cap \mathrm{OR}$. We would then have $\mathcal{Q}^* \rhd S(X)$, $\mathcal{Q}^*$ is an X–mouse which is sound above κ and $\rho_{n+1}(\mathcal{Q}) \leq \kappa$, which contradicts the definition of $S(X)$. We must therefore have $\mathrm{cf}^V(S(X) \cap \mathrm{OR}) \geq \aleph_2$.

Let us now define a tree $T = T_{S(X)}$, derived from $S(X)$, as follows. We put $\mathcal{Q} \in T$ iff $\mathcal{Q} \lhd S(X)$, and setting $\lambda^\mathcal{Q} = \kappa^{+\mathcal{Q}}$ we have that $\mathcal{Q}|\lambda^\mathcal{Q} \prec_{\Sigma_\omega} S(X)$ and $\rho_\omega(\mathcal{Q}) \leq \kappa$. If $\mathcal{Q} \in T$, then we shall write $n(\mathcal{Q})$ for the unique $n < \omega$ with $\rho_{n+1}(\mathcal{Q}) \leq \kappa < \rho_n(\mathcal{Q})$. If $\bar{\mathcal{Q}}$, $\mathcal{Q} \in T$, then we write $\bar{\mathcal{Q}} \leq_T \mathcal{Q}$ iff $n(\bar{\mathcal{Q}}) = n(\mathcal{Q})$, call it n, and there is a weakly $r\Sigma_n$ elementary embedding

$$\sigma \colon \bar{\mathcal{Q}} \to \mathcal{Q}$$

such that $\sigma \restriction \lambda^{\bar{\mathcal{Q}}} = \mathrm{id}$, $\sigma(p_n(\bar{\mathcal{Q}})) = p_n(\mathcal{Q})$, and if $\lambda^{\bar{\mathcal{Q}}} < \bar{\mathcal{Q}} \cap \mathrm{OR}$, then $\sigma(\lambda^{\bar{\mathcal{Q}}}) = \lambda^\mathcal{Q}$. If $\bar{\mathcal{Q}}$, $\mathcal{Q} \in T$, then we shall write $\sigma_{\bar{\mathcal{Q}},\mathcal{Q}}$ for the unique map σ as above. The elements of T and the maps between them are thus as in the usual construction of $\Box_\kappa$ inside $S(X)$ (cf. [12]).

Let us write $\lambda = S(X) \cap \mathrm{OR}$. In $V^{\mathrm{Col}(\omega_1,\lambda)}$, T can be shrinked a little bit so as to produce a tree of height and size $\aleph_1$. Namely, letting $C \in V^{\mathrm{Col}(\omega,\lambda)}$ be a club subset of λ of order type ω_1, we may let $\mathcal{Q} \in T^* = T^*_\mathcal{S}$ iff $\mathcal{Q} \in T$ and $\lambda^\mathcal{Q} = \kappa^{+\mathcal{Q}} \in C$, and we let $\leq_{T^*} = \leq_T \restriction T^*$. We claim that T^* does not have a branch of length ω_1 in $V^{\mathrm{Col}(\omega_1,\lambda)}$. Suppose not, and let b be a branch through T^* of length ω_1. It is then easy to verify that

$$\mathrm{dir\ lim}\ (\mathcal{Q}, \sigma_{\bar{\mathcal{Q}},\mathcal{Q}} \colon \bar{\mathcal{Q}} \leq_{T^*} \mathcal{Q} \in b)$$

is (isomorphic to) an X–mouse, call it $\mathcal{Q}^*$, with $\kappa^{+\mathcal{Q}^*} = \lambda$ and which is sound above κ and $\rho_\omega(\mathcal{Q}^*) \leq \kappa$. However, there can be only one such X–mouse of the same height as $\mathcal{Q}^*$ in $V^{\mathrm{Col}(\omega_1,\lambda)}$, so that in fact $\mathcal{Q}^* \in V$. As $\mathcal{Q}^*$ is certainly 1–small, we must have that $\mathcal{Q}^*$ is an X–mouse in V as well, so that we have a contradiction with the definition of $S(X)$. Hence in fact T^* does not have a branch of length ω_1 in $V^{\mathrm{Col}(\omega_1,\lambda)}$.

Now let $\mathbb{P}$ denote the natural forcing for specializing T^*, i.e., for partitioning T^* into countably many antichains (cf. [7, p. 274f.]). The following statement is then true in $V^{\mathrm{Col}(\omega_1,\lambda)*\mathbb{P}}$, as being witnessed by κ, X, and $S(X)$. "There is some $\bar\kappa < \aleph_2$ and some $\bar X \subset \bar\kappa$ and there is some φ–small $\bar X$–mouse $\mathcal{S}$ such that $\mathcal{S} \cap \mathrm{OR} < \aleph_2$, $\mathcal{S}$ is a model of ZFC^- with largest cardinal $\bar\kappa$ such that the tree $T^*_{\mathcal{S}}$ which is derived from $\mathcal{S}$ is a tree of height and size $\aleph_1$ which does not have any branches of length ω_1." (Here, by the "tree $T^*_{\mathcal{S}}$ derived from $\mathcal{S}$" we mean a tree which is derived from $\mathcal{S}$ in exactly the same manner as $T^* = T^*_{S(X)}$ was derived from $S(X)$ above.) Now $\mathrm{Col}(\omega_1,\lambda) * \mathbb{P}$ is ω–closed $*$ c.c.c. and hence proper. Moreover, by Lemma 3.1, the fact that $\mathcal{S}$ is a φ–small $\bar X$–mouse can be expressed in a Σ_1 fashion over H_{ω_2}, because φ–small premice are 1–small. Therefore, by BPFA this statement holds true in V, as being witnessed by κ', X', and $\mathcal{S}'$, say.

But because $\mathcal{M}(X')$ exists, there is some least $\mathcal{Q} \trianglelefteq \mathcal{M}(X')$ such that $\mathcal{Q} \triangleright \mathcal{S}'$ and $\rho_\omega(\mathcal{Q}) \leq \kappa'$. The usual $\Box_{\kappa'}$–type arguments then show that there is a cofinal branch through the tree $T_{\mathcal{S}'}$ which is derived from $\mathcal{S}'$. Contradiction! $\Box$ (Lemma 3.2)

Proof of Theorem 0.2. Let us assume $(*)$. By Lemmas 0.3 and 2.1, BPFA implies that V is closed under $X \mapsto M_1^\#(X)$. This gives (1) $\Longrightarrow$ (3). By [18, Theorem 10.99], if V is closed under $M_1^\#$, then BMM^{++} holds. This gives (3) $\Longrightarrow$ (2). The implication (2) $\Longrightarrow$ (1) is trivial. $\Box$ (Corollary 0.2)

The following result provides the obvious generalization of Lemma 3.2).

Lemma 3.3 *Assume* $\mathsf{BPFA}^{\mathrm{uB}}$. *Suppose J to be a correct mouse operator with the relativization and extension property such that J is total on V. Then $(\aleph_2,\infty)$–mouse reflection holds with respect to mouse operators which do not go beyond $X \mapsto M_1^J(X)$.*

Proof. We imitate the argument for Lemma 3.2. Suppose there to be some $X \subset \kappa$, where $\kappa \geq \aleph_2$, such that $M_1^J(X)$ does not exist. As in the proof of Lemma 3.2, the following will be true in an ω–closed $*$*c.c.c.* forcing extension of V. "There is some $\bar\kappa < \aleph_2$ and some $\bar X \subset \bar\kappa$ and there is some φ–small $\bar X$–mouse $\mathcal{S}$ such that $\mathcal{S} \cap \mathrm{OR} < \aleph_2$, $\mathcal{S}$ is a model of ZFC^- with largest cardinal $\bar\kappa$ such that the tree $T^*_{\mathcal{S}}$ which is derived from $\mathcal{S}$ is a tree of height and size $\aleph_1$ which does not have any branches of length ω_1." We may obviously assume w.l.o.g. that $\bar\kappa = \omega_1^V$ here.

We may now phrase the truth which is expressed by the above statement as follows. "There are H, $\bar X$, $\mathcal{S}$ such that H is a transitive model (of size $\aleph_1$) with $\omega_1 \in H$ such that $\mathbb{R} \cap H$ is closed under $x \mapsto J(x)$, $\bar X \subset \omega_1^V$, $\mathcal{S} \in H$ is an $\bar X$–premouse with largest cardinal ω_1^V, $\mathrm{cf}(\mathcal{S} \cap \mathrm{OR}) = \omega_1$, for all $\gamma < \mathcal{S} \cap \mathrm{OR}$ with $\rho_\omega(\mathcal{S}||\gamma = \omega_1^V$, if $\pi \colon \bar{\mathcal{S}} \to \mathcal{S}||\gamma$ is in H, then $J(\bar{\mathcal{S}}) \models \psi(\bar{\mathcal{S}})$, and the tree $T^*_{\mathcal{S}}$ which is derived from $\mathcal{S}$ is a tree of height and size $\aleph_1$ which does not have any branches of length ω_1." Here, ψ is as in Definition 1.7.

By $\mathsf{BPFA}^{\mathrm{uB}}$, this statement holds true in V. In order to finish the proof of Lemma 3.3, it suffices to prove that if H, $\bar{X}$, $\mathcal{S}$ are as given by this statement, then $\mathcal{S}$ is a $\bar{X}$–mouse. Let $\gamma < \mathcal{S} \cap \mathrm{OR}$ be such that $\rho_\omega(\mathcal{S}||\gamma) = \omega_1$, and let $\pi \colon \bar{\mathcal{S}} \to \mathcal{S}$ be such that $\bar{\mathcal{S}}$ is countable. We need to verify that $\bar{\mathcal{S}}$ is ω_1–iterable. It is easy to see that $\bar{\mathcal{S}}$ is then in H, because, setting $\alpha = \mathrm{crit}(\pi)$,

$$\mathrm{ran}(\pi) = \mathrm{Hull}^{\mathcal{S}||\gamma}(\alpha \cup \{p\}),$$

where p is the standard parameter of $\mathcal{S}||\gamma$ and the hull is the appropriate fine structural one.

But now we have that $J(\bar{\mathcal{S}} \models \psi(\bar{\mathcal{S}})$ as being given by the statement, and therefore $\bar{\mathcal{S}}$ is ω_1–iterable by the correctness of J. $\square$

4 BPFA and $\mathsf{AD}^{L(\mathbb{R})}$.

In order to show that in the presence of just $\mathsf{AD}^{L(\mathbb{R})}$ (rather than Woodin's $(*)$) BPFA doesn't imply any form of $(\aleph_2, \infty)$–mouse reflection, we need the concepts of remarkable and reflecting cardinals which are introduced in [13] and [5], respectively.

Lemma 4.1 *Let $\kappa < \lambda$, where κ is a remarkable limit of Woodin cardinals and λ is reflecting. Let $V = L[A]$, where $A \subset \kappa$. There is then a set–generic extension $V^{\mathbb{Q}}$ of V such that*

$$V^{\mathbb{Q}} \models \mathsf{BPFA} \ + \ \mathsf{AD}^{L(\mathbb{R})}$$

and $\kappa = \omega_1^{V^{\mathbb{Q}}}$.

If $V^{\mathbb{Q}}$ is as in Lemma 4.1, then in $V^{\mathbb{Q}}$ there is a subset of ω_1 (for instance, A) which does not have a sharp. The hypothesis of Lemma 4.1 is consistent by [13, Lemma 1.7].

Proof of Lemma 4.1. If $\mathbb{P} \in V^{\mathrm{Col}(\omega,<\kappa)}$ is proper in $V^{\mathrm{Col}(\omega,<\kappa)}$, then

$$L(\mathbb{R})^{V^{\mathrm{Col}(\omega,<\kappa)*\mathbb{P}}} \equiv L(\mathbb{R})^{V^{\mathrm{Col}(\omega,<\kappa)}},$$

i.e., these two models have the same first order theory (cf. [13, Theorem 2.4]). As AD holds in the $L(\mathbb{R})$ of $V^{\mathrm{Col}(\omega,<\kappa)}$, due to the fact that κ is a limit of Woodin cardinals, AD therefore holds in the $L(\mathbb{R})$ of $V^{\mathrm{Col}(\omega,<\kappa)*\mathbb{P}}$ as well.

We may now let $\mathbb{P} \in V^{\mathrm{Col}(\omega,<\kappa)}$ be the Goldstern–Shelah poset for forcing BPFA, exploiting the fact that λ is still reflecting in $V^{\mathrm{Col}(\omega,<\kappa)}$, and we may set $\mathbb{Q} = \mathrm{Col}(\omega, <\kappa) * \mathbb{P}$. $\square$ (Lemma 4.1)

The hypothesis that κ be a limit of Woodin cardinals is not really necessary as part of the hypothesis of Lemma 4.1, of course. It would just be enough to assume for V_κ to be closed under $X \mapsto M_\omega^\#(X)$ or even slightly less.

5 Precipitous ideals.

Let us now turn to precipitous ideals. In order to prove the main result of this section, Theorem 5.2, we need an abstract criterion for the iterability of a certain phalanx, which is provided by the following lemma.

Lemma 5.1 *Let $\mathcal{M}$ and $\mathcal{N}$ be fully iterable premice, and let $j\colon \mathcal{M} \to \mathcal{N}$ be a non–trivial elementary embedding. Let $\mathcal{T}$ and $\mathcal{U}$ denote the iteration trees on $\mathcal{M}$ and $\mathcal{N}$, respectively, arising from the comparison of $\mathcal{M}$ and $\mathcal{N}$. Let us suppose that there is no drop along the main branch of $\mathcal{U}$, and that the final models $\mathcal{M}^{\mathcal{T}}_\infty$ of $\mathcal{T}$ and $\mathcal{M}^{\mathcal{U}}_\infty$ of $\mathcal{U}$ are the same. Let us write $\mathcal{Q} = \mathcal{M}^{\mathcal{T}}_\infty = \mathcal{M}^{\mathcal{U}}_\infty$. By the Dodd–Jensen Lemma, there is no drop along the main branch of $\mathcal{T}$ anyway, and we may let $\pi^{\mathcal{T}}_\infty$ and $\pi^{\mathcal{U}}_\infty$ denote the iteration maps from the main branches of $\mathcal{T}$ and $\mathcal{U}$, respectively.*

Let us assume that $\mathrm{crit}(\pi^{\mathcal{T}}_\infty) \geq \mathrm{crit}(j)$ and $\mathrm{crit}(\pi^{\mathcal{T}}_\infty) \geq \mathrm{crit}(j)$. Set $\kappa = \mathrm{crit}(j)$. Then, setting $\lambda = \kappa^{+\mathcal{M}}$, we have that $\lambda = \kappa^{+\mathcal{N}}$, $\mathcal{M}|\lambda = \mathcal{N}|\lambda$ and both $\mathcal{T}$ and $\mathcal{U}$ only use extenders with indices larger than λ.

Let F be the $(\kappa, j(\kappa))$–extender on $\mathcal{M}$ derived from j, i.e., $X \in F_a$ iff $a \in j(X)$, where $a \in [j(\kappa)]^{<\omega}$ and $X \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap \mathcal{M}$. Then F is an extender over $\mathcal{N}$ as well, and in fact the phalanx

$$(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$$

is fully iterable.

Proof. We shall produce an iterate $\mathcal{Q}^*$ of $\mathcal{N}$ obtained by using only extenders with index above $j(\lambda)$ and an embedding

$$\ell\colon \mathrm{Ult}(\mathcal{N}; F) \to \mathcal{Q}^*$$

with $\ell \restriction j(\kappa) = \mathrm{id}$. The phalanx

$$(\mathcal{N}, \mathcal{Q}^*, j(\kappa))$$

is certainly iterable, as every iteration of it may be construed as the continuation of an iteration of $\mathcal{N}$, and therefore the phalanx

$$(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$$

is also iterable.

The construction to follow is summarized by figure 1.

Let us first copy the iteration $\mathcal{T}$ onto $\mathcal{N}$ via the map j, producing an iteration tree $\mathcal{T}^j$ on $\mathcal{N}$. As usual, we shall have that if the j_α's are the copy maps,

$$j_\alpha\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{M}^{\mathcal{T}^j}_\alpha,$$

where $\alpha < \mathrm{lh}(\mathcal{T})$, then

$$j_\beta \restriction \mathrm{lh}(E^{\mathcal{T}}_\alpha) = j_\alpha \restriction \mathrm{lh}(E^{\mathcal{T}}_\alpha)$$

whenever $\alpha \leq \beta < \mathrm{lh}(\mathcal{T})$. As $\nu(E^{\mathcal{T}}_\alpha) > \lambda$ for all $\alpha < \mathrm{lh}(\mathcal{T})$, i.e., all indices of extenders used in $\mathcal{T}$ are larger than λ, this agreement implies that for all $a \in [j(\kappa)]^{<\omega}$, for all $X \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap \mathcal{M}$, and for all α, $\beta < \mathrm{lh}(\mathcal{T})$,

$$a \in j_\beta(X) \text{ iff } a \in j_\alpha(X).$$

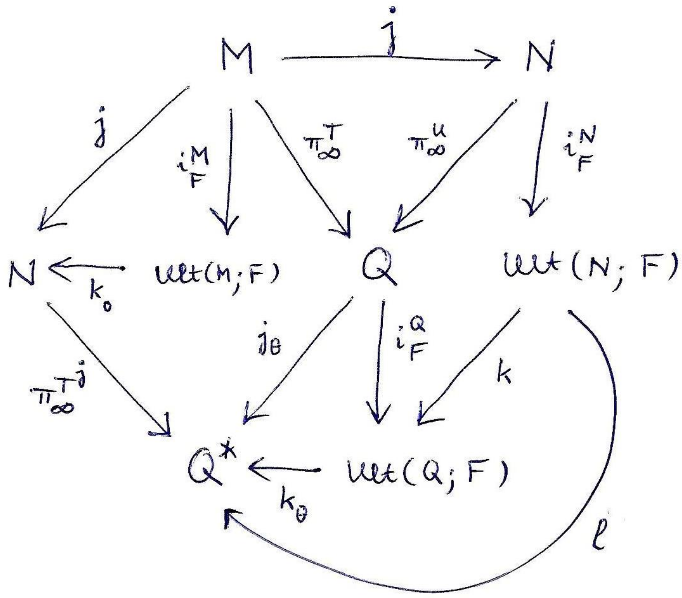

Figure 1: The iterability of $(\mathcal{N}, \mathrm{Ult}(\mathcal{N}; F), j(\kappa))$.

But $j_0 = j$, so that in fact the $(\kappa, j(\kappa))$–extender derived from any j_α, $\alpha < \mathrm{lh}(\mathcal{T})$, is just F.

In other words, we may factor any of the maps $j_\alpha \colon \mathcal{M}_\alpha^\mathcal{T} \to \mathcal{M}_\alpha^{\mathcal{T}^j}$ as

$$j_\alpha = k_\alpha \circ i_F^{\mathcal{M}_\alpha^\mathcal{T}},$$

where

$$i_F^{\mathcal{M}_\alpha^\mathcal{T}} : \mathcal{M}_\alpha^\mathcal{T} \to \mathrm{Ult}(\mathcal{M}_\alpha^\mathcal{T}; F)$$

is the ultrapower map and

$$k_\alpha \colon \mathrm{Ult}(\mathcal{M}_\alpha^\mathcal{T}; F) \to \mathcal{M}_\alpha^{\mathcal{T}^j}$$

is the factor map which is defined as

$$i_F^{\mathcal{M}_\alpha^\mathcal{T}}(f)(a) \mapsto j_\alpha(f)(a),$$

where $a \in [j(\kappa)]^{<\omega}$ and $f \in \mathcal{M}_\alpha^\mathcal{T}$ is appropriate. Notice that

$$\mathrm{crit}(k_\alpha) \geq j_\alpha(\kappa) = j(\kappa)$$

for every $\alpha < \mathrm{lh}(\mathcal{T})$.

Let $\theta + 1 = \mathrm{lh}(\mathcal{T})$, so that $\mathcal{Q} = \mathcal{M}_\theta^\mathcal{T}$.

Let

$$i_F^\mathcal{N} : \mathcal{N} \to \mathrm{Ult}(\mathcal{N}; F)$$

and

$$i_F^{\mathcal{Q}} \colon \mathcal{Q} \to \mathrm{Ult}(\mathcal{Q}; F)$$

be the ultrapower maps. We may define

$$i \colon \mathrm{Ult}(\mathcal{N}; F) \to \mathrm{Ult}(\mathcal{Q}; F)$$

by

$$i_F^{\mathcal{N}}(f)(a) \mapsto j_F^{\mathcal{Q}} \circ \pi_\infty^{\mathcal{U}}(f)(a),$$

where $a \in [j(\kappa)]^{<\omega}$ and $f \in \mathcal{N}$ is appropriate. Notice that $\mathrm{crit}(i) \geq j(\kappa)$.

Now consider

$$\ell = j_\theta \circ i \colon \mathrm{Ult}(\mathcal{N}; F) \to \mathcal{Q}^*.$$

We have that $\mathrm{crit}(\ell) \geq j(\kappa)$, and $\mathcal{Q}^*$ is an iterate of $\mathcal{N}$ obtained by an iteration which uses only extenders with index larger than $j(\lambda)$. That is, $\mathcal{Q}^*$ and ℓ are as desired. □

Theorem 5.2 *Suppose that there is no inner model with a Woodin cardinal, and let K denote the core model. Assume κ to be such that there is a precipitous ideal on κ. Then $\kappa^{+K} = \kappa^{+V}$.*

Proof. In order to not get involved into issues which only hide the key idea, let us pretend that there be a large cardinal, Ω, up to which K may be defined. K will be fully iterable.

Let I be a precipitous ideal on κ, let G be $(\mathcal{P}(\kappa) \setminus I)/I$–generic over V, and let, in $V[G]$,

$$j \colon V \to M \subset V[G]$$

be the generic elementary embedding produced by the ultrapower given by G. Here, M is transitive. Let us assume that $\kappa^{+K} < \kappa^{+V}$ and work towards a contradiction. Let us write $\lambda = \kappa^{+K}$.

Let $f \colon \kappa \to \mathcal{P}(\kappa) \cap K$ be bijective, $f \in V$. Then $f \in M$, as $f(\xi) = j(f)(\xi) \cap \kappa$ for all $\xi < \kappa$, and therefore $j \restriction \mathcal{P}(\kappa) \in M$, as $(j \restriction \mathcal{P}(\kappa))(x) = y$ iff there is some $\xi < \kappa$ with $f(\xi) = x$ and $j(f)(\xi) = y$. (This is "the old Kunen argument.") This implies that in fact $j \restriction K|\lambda \in M$.

Let us denote by F the $(\kappa, j(\kappa))$–extender on K derived from j. We have seen that $F \in M$. We know that K is still the core model of $V[G]$ (and is still fully iterable there). Let K^M denote the core model from the point of view of M. K^M is fully iterable inside M; by our hypothesis that there be no inner model with a Woodin cardinal, this implies that K^M is also fully iterable in $V[G]$. By the Dodd–Jensen Lemma, K^M is a universal weasel.

Claim 1. Let $\mathcal{T}, \mathcal{U}$ denote the iteration trees arising from the comparison of K with K^M, let $\mathcal{Q}$ be the common coiterate of K and K^M, and let $\pi_\infty^{\mathcal{T}} \colon K \to Q$ and $\pi_\infty^{\mathcal{U}} \colon K^M \to Q$ be the embeddings given by the cofinal branches through $\mathcal{T}$ and $\mathcal{U}$, respectively. Then $\mathrm{crit}(\pi_\infty^{\mathcal{T}}) = \kappa$ and $\mathrm{crit}(\pi_\infty^{\mathcal{U}}) \geq \kappa$.

Proof of Claim 1: Standard arguments (playing with the hull and definability property) give that $\pi_\infty^{\mathcal{U}} \circ j = \pi_\infty^{\mathcal{T}}$, so that $\mathrm{crit}(\pi_\infty^{\mathcal{U}}) < \kappa$ iff $\mathrm{crit}(\pi_\infty^{\mathcal{T}}) < \kappa$. But then if $\mathrm{crit}(\pi^{\mathcal{U}}_\infty) < \kappa$, then the first extenders used along the main branches of $\mathcal{T}$, $\mathcal{U}$, resp., would be compatible (which is impossible). We get that $\mathrm{crit}(\pi^{\mathcal{U}}_\infty) \geq \kappa$, and $\mathrm{crit}(\pi^{\mathcal{T}}_\infty) = \kappa$ (as $\mathrm{crit}(j) = \kappa$). $\square$

Claim 1 implies that $\lambda = \kappa^{+K}$ and $K|\lambda = K^M|\lambda$. In particular, we may take the ultrapower $\mathrm{Ult}(K^M; F)$ of K^M by F. By our next Claim, $\mathrm{Ult}(K^M; F)$ is well–founded, an in fact more is true.

Claim 2. The phalanx $(K^M, \mathrm{Ult}(K^M; F), j(\kappa))$ is iterable (in $V[G]$ as well as in M).

Proof of Claim 2: This readily follows from Lemma 5.1 above, by setting $\mathcal{M} = K$ and $\mathcal{N} = K^M$. We may apply 5.1 by Lemma 1. $\square$

Notice that $\mathrm{Ult}(K^M; F)|j(\lambda) = K^M|j(\lambda)$.

We may now derive a contradiction by using standard arguments. Let us work in M, and let $\mathcal{T}$ and $\mathcal{U}$ denote the iteration trees on K^M and $(K^M, \mathrm{Ult}(K^M; F), j(\kappa))$, respectively, arising from their comparison. Let Q be the common coiterate. We cannot have that Q is above K^M on the phalanx–side of the comparison. Let $\pi^{\mathcal{T}}_\infty \colon K^M \to Q$ and $\pi^{\mathcal{U}}_\infty \colon \mathrm{Ult}(K^M; F) \to Q$ be the embeddings given by the cofinal branches through $\mathcal{T}$ and $\mathcal{U}$, respectively. Let us write $i_F \colon K^M \to \mathrm{Ult}(K^M; F)$ for the ultrapower map.

We have that $\pi^{\mathcal{U}}_\infty \circ i_F = \pi^{\mathcal{T}}_\infty$, $\mathrm{crit}(\pi^{\mathcal{T}}_\infty) = \mathrm{crit}(i_F) = \kappa$, and $\mathrm{crit}(\pi^{\mathcal{U}}_\infty) \geq j(\kappa)$. The first extender used along the main branch of $\mathcal{T}$ is therefore compatible with F, and in fact cannot be shorter than F. Therefore, F must be on the sequence of K^M, which is nonsense, as K^M does not have superstrong cardinals. $\square$

Proof of Theorem 0.6: The implication (2) $\Longrightarrow$ (1) is due to Shelah who showed that if there is a Woodin cardinal, then there is a forcing extension in which there is a saturated ideal on ω_1.

Let us now prove (1) $\Longrightarrow$ (2). Suppose (1) holds, but (2) fails. Let K denote the core model. By Theorem 5.2, $\omega_1^{+K} = \omega_2$. It is easy to see that ω_1 must be a limit cardinal in K, so that ω_1 is inaccessible in K. (It is in fact measurable in K.)

Let us define $f \colon \omega_1 \to \omega_1$ by $f(\xi) = \xi^{+K}$ for $\xi < \omega_1$. Suppose that there is some $\alpha < \omega_2$ such that $S = \{\xi < \omega_1 \colon f(\xi) < f_\alpha(\xi)\}$ is stationary. We may pick

$$\sigma \colon (H; \in, \bar{K}) \to (H_{\omega_2}; \in, K|\omega_2)$$

such that H is countable and transitive, $\xi = \mathrm{crit}(\sigma) \in S$, $\sigma(\xi) = \omega_1$, and, setting $\tau = f_\alpha(\xi)$, $\tau \in H$ and $\sigma(\tau) = \alpha$. By the Condensation Lemma, $\bar{K}|\tau = K|\tau$. So $\tau \leq f(\xi)$ by the definition of f. I.e., $f_\alpha(\xi) \leq f(\xi)$. Contradiction! $\square$

Proof of Theorem 0.7. Let us suppose that there is no inner model with a Woodin cardinal, and let again K denote the core model. Let us first verify the following.

Claim 1. If $(\omega_1^V)^{+K} = \omega_2$, then $K|\omega_1^V$ is universal with respect to countable mice.

PROOF. Let $\mathcal{M}$ be a countable mouse. Let us assume that $\mathcal{M}$ does not have a definable Woodin cardinal. As $K|\omega_2^V$ is universal with respect to countable mice (cf. [11]), there must in fact be some $\delta < \omega_2^V$ such that $K||\delta$ wins the comparison against $\mathcal{M}$. Say $\rho_1(K||\delta) = \omega_1^V$. Let $\mathcal{T}$ and $\mathcal{U}$ denote the normal iteration trees on $\mathcal{M}$ and $K||\delta$, respectively, arising from the comparison of $\mathcal{M}$ with $K||\delta$. Notice that both $\mathcal{M}$ and $K||\delta$ have unique iteration strategies.

Let $f : \omega_1^V \to K||\delta$ be bijective, where $f \in K$. Let us pick

$$\pi : H \to H_\theta$$

such that H is countable and transitive, θ is large enough, and

$$\{\mathcal{M}, K||\delta, \mathcal{T}, \mathcal{U}, f\} \subset \mathrm{ran}(\pi).$$

Set $\bar{K} = \pi^{-1}(K||\delta)$, $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$, and $\bar{\mathcal{U}} = \pi^{-1}(\mathcal{U})$. By our hypotheses, the iteration trees $\bar{\mathcal{T}}$ and $\bar{\mathcal{U}}$ are according to the unique iteration strategies for $\mathcal{M}$ and $\bar{K}$, respectively, and they witness that $\bar{K}$ ins the comparison against $\mathcal{M}$.

But $\bar{K}$ is the transitive collapse of $\mathrm{ran}(f \restriction \mathrm{crit}(\pi))$, and therefore $\bar{K} \in K$ and has size $< \omega_1^V$ in K. Inside K, $K|\omega_1^V$ is certainly universal with respect to mice of size $< \omega_1^V$, and therefore the fact that $\bar{K} \in K$ wins the comparison against $\mathcal{M}$ implies that $K|\omega_1$ wins the comparison against $\mathcal{M}$, too. $\square$

In the light of Theorem 5.2, the proof of the following Claim, which is shown in [6], finishes the proof of Theorem 0.7.

Claim 2. Suppose that $x^\#$ exists for every $x \in \mathbb{R}$, and $\boldsymbol{\delta}^1_2 = \aleph_2$. Then $K|\omega_1^V$ is not universal with respect to countable mice, and in fact that the mouse order on the set of all countable mice has length ω_2.

PROOF. Jensen has shown that the hypothesis of this Claim implies that $x^\dagger$ exists for every real x.

Let us fix $x \in \mathbb{R}$ for a while, and let $\kappa = \kappa_x < \Omega = \Omega_x$ denote the two measurable cardinals of $x^\dagger$. Let K_x denote the core model of $x^\dagger$ of height Ω. By absoluteness, K_x is a mouse in V. Let

$$(\mathcal{N}^x_i, \pi^x_{ij} : i \leq j \leq \omega_1)$$

denote the linear iteration of $\mathcal{N}^x_0 = x^\dagger$ obtained by iterating the unique measure on κ and its images ω_1 times. By [14], $\pi^x_{ii+1} \restriction \pi^x_{0i}(K_x)$ is an iteration of $\pi^x_{0i}(K_x)$, and there is hence a (not necessarily normal) iteration tree $\mathcal{T}$ on K_x of length $\omega_1 + 1$ such that

$$\mathcal{M}^{\mathcal{T}}_{\omega_1} = \pi^x_{0\omega_1}(K_x).$$

By [15],

$$\kappa^{+x^\dagger} = \kappa^{+K_x},$$

so that

$$\omega_1^{+\mathcal{N}^x_{\omega_1}} = \omega_1^{+\pi^x_{0\omega_1}(K_x)}.$$

Now by $\boldsymbol{\delta}^1_2 = \aleph_2$,

$$\sup(\{\omega_1^{\mathcal{N}^x_{\omega_1}} : x \in \mathbb{R}\}) = \aleph_2,$$

and therefore the supremum of all $\mathcal{P} \cap \mathrm{OR}$ such that there is some countable mouse $\mathcal{M}$ and some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is equal to $\aleph_2$. On the other hand, a boundedness argument shows that for a fixed countable mouse $\mathcal{M}$, the supremum of all $\mathcal{P} \cap \mathrm{OR}$ such that there is some iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length $\omega_1 + 1$ such that $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\omega_1}$ is smaller that $\omega_1^{+L[\mathcal{M}]}$ (cf. [18]).

This shows that the mouse order on the set of all countable mice has length ω_2. This readily implies that $K|\omega_1$ cannot be universal with respect to countable mice, as otherwise $\{K|\delta \colon \delta < \omega_1\}$ would be cofinal in the mouse order on the set of all countable mice. $\square$

$\square$

The proof of Theorem 0.5 is easy via the methods which we presented here.

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