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REPORT No. 7, 2009/2010, fall

ISSN 1103-467X
ISRN IML-R- -7-09/10- -SE+fall

AN IMPROPER ARITHMETICALLY CLOSED BOREL SUBALGEBRA OF $\mathcal{P}(\omega)$ MOD FIN

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Abstract. We show the existence of a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ that satisfies the following three conditons.

- $\mathcal{A}$ is Borel (when $\mathcal{P}(\omega)$ is identified with 2^{ω}).
- $\mathcal{A}$ is arithmetically closed (i.e., $\mathcal{A}$ is closed under the Turing jump, and Turing reducibility).
- The forcing notion $(\mathcal{A}, \subseteq)$ modulo the ideal FIN of finite sets collapses the continuum to $\aleph_0$.

1. INTRODUCTION

For a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{P}_{\mathcal{A}}$ be the partial order obtained by reducing $(\mathcal{A}, \subseteq)$ modulo the ideal FIN of finite sets. Gitman [G-1] made an advance towards the *Scott set problem*[1] by showing that, assuming the proper forcing axiom (PFA), if $\mathcal{A}$ is arithmetically closed[2] and $\mathbb{P}_{\mathcal{A}}$ is a proper notion of forcing, then there is a model of Peano arithmetic whose standard system is $\mathcal{A}$. Gitman [G-2] also investigated proper posets of the form $\mathbb{P}_{\mathcal{A}}$ and showed that the existence of proper uncountable arithmetically closed algebras $\mathcal{A} \neq \mathcal{P}(\omega)$ is consistent with ZFC+PFA. These results naturally motivate the question whether there is an arithmetically closed $\mathcal{A}$ for which $\mathbb{P}_{\mathcal{A}}$ is not proper. This question was answered in the affirmative by Enayat [E, Theorem D], using a highly nonconstructive reasoning that establishes the existence of an arithmetically closed $\mathcal{A}$ of power $\aleph_1$ such that $\mathbb{P}_{\mathcal{A}}$ collapses $\aleph_1$ (and is therefore not proper). The nonconstructive feature of the proof prompted Question II(b) of [E], which asked whether $\mathbb{P}_{\mathcal{A}}$ is a proper poset if $\mathcal{A}$ is both arithmetically closed and *Borel* (when $\mathcal{P}(\omega)$ is identified with the Cantor set).

The main result of this paper[3], Theorem A below, provides a strong negative answer to the above question.

Theorem A. *There is an arithmetically closed Borel subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is equivalent to* Levy$(\aleph_0, 2^{\aleph_0})$.

Date: December 8, 2009.

[1]The Scott set problem [KS, Question 1] asks whether every Scott set $\mathcal{A}$ can be realized as the standard system of a model of Peano arithmetic (a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Scott set if $\mathcal{A}$ is closed Turing reducibility, and every infinite subtree of ${}^{<\omega}2$ that is coded in $\mathcal{A}$ has an infinite branch that is coded in $\mathcal{A}$). It is known that the answer to the Scott set problem is positive when $|\mathcal{A}| \leq \aleph_1$, and when $\mathcal{A} = \mathcal{P}(\omega)$.

[2]A subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed if $\mathcal{A}$ is closed under (1) Turing jump and (2) Turing reducibility. Note that if $\mathcal{A}$ is arithmetically closed, then $\mathcal{A}$ is a Scott set, but not vice versa.

[3]This is [En-Sh 936] in Shelah's master list of publications.

Theorem A is established in Section 3 using a rich toolkit from set theory and model theory. For this reason, Section 2 is devoted to the description of the machinery employed in the proof of Theorem A.

Acknowledgments. The final version of this paper was prepared while both authors participated in the special program at the Mittag-Leffler Institute devoted to mathematical logic during September 2009. Enayat's visit was partially supported by a grant from the European Scince Foundation, via the activity *New Frontiers of Infinity: Mathematical, Philosophical and Computational Prospects.*

2. PRELIMINARIES

2.1. FORCING

Given an infinite cardinal κ, $\text{Levy}(\aleph_0, \kappa)$ is the usual partial order that collapses κ to $\aleph_0$, i.e., $\text{Levy}(\aleph_0, \kappa) = ({}^{<\omega}\kappa, \subseteq)$. The following result provides a structural characterization of $\text{Levy}(\aleph_0, \kappa)$.[4]

2.1.1. Theorem (McAloon [K, Theorem 14.17]). *The following conditions are equivalent for a partial order $\mathbb{P}$ of infinite cardinality κ.*

(a) $\mathbb{P}$ *is equivalent*[5] *to* $\text{Levy}(\aleph_0, \kappa)$.
(b) $\mathbb{P}$ *is* $(\aleph_0, \kappa)$ *nowhere distributive, i.e., there is a family* $\{I_n : n \in \omega\}$ *of maximal antichains of* $\mathbb{P}$ *such that for every* $p \in \mathbb{P}$, *there is some* $n < \omega$ *such that there are* κ *elements of* I_n *that are compatible with* p.

2.1.2. Corollary [J, Lemma 26.7]**.** *The following conditions are equivalent for a partial order* $\mathbb{P}$ *of cardinality* $\kappa \geq \aleph_0$.

(a) $\mathbb{P}$ *is equivalent to* $\text{Levy}(\aleph_0, \kappa)$.
(b) $\mathbb{V}^{\mathbb{P}} \models$ "*there is a surjective function* $f : \omega \to \kappa$".

The next result shows that one can use standard techniques to build a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is not proper.

2.1.3. Proposition[6]**.** *There is a family* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *of cardinality* $\aleph_1$ *such that* $\mathbb{P}_{\mathcal{A}}$ *is equivalent to* $\text{Levy}(\aleph_0, \aleph_1)$.

Proof: By a theorem of Parovičenko [Pa] (see also [K, 5.28 and 5.29]) every Boolean algebra of cardinality $\leq \aleph_1$ can be embedded into $\mathcal{P}(\omega)$ mod FIN. On the other hand $\text{Levy}(\aleph_0, \aleph_1)$ can be densely embedded into a Boolean algebra of power $\aleph_1$ since each s in $\text{Levy}(\aleph_0, \aleph_1)$ determines a basic clopen X_s set in ${}^{\omega}\omega_1$, and the Boolean algebra $\mathbb{B}$ of clopen sets generated by the family $\{X_s : s \in \text{Levy}(\aleph_0, \aleph_1)\}$ is of size $\aleph_1$. So by Parovičenko's theorem there is an embedding f of $\mathbb{B}$ into $\mathcal{P}(\omega)$ mod FIN. Let $\mathcal{A} := \{X \subseteq \omega : [X]$ is in the range of $f\}$. Since $\mathbb{P}_{\mathcal{A}}$ is isomorphic to $\mathbb{B}$, and $\mathbb{B}$ is equivalent to $\text{Levy}(\aleph_0, \aleph_1)$, $\mathbb{P}_{\mathcal{A}}$ collapses $\aleph_1$. Therefore, by Corollary 2.1.2, $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{Levy}(\aleph_0, \aleph_1)$. $\square$

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[4] See [BS, Cor.1.15] for a characterization of other collapsing algebras.

[5] For separative partial orders $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1$ is *equivalent* to $\mathbb{P}_2$ if $B(\mathbb{P}_1) \cong B(\mathbb{P}_2)$, where $B(\mathbb{P})$ is the complete Boolean algebra consisting of regular cuts subsets of $\mathbb{P}$ [J, Thm 14.10].

[6] Thanks to K.P. Hart and Ken Kunen for (independently) drawing our attention to this consequence of Parovičenko's theorem.

2.1.4. Remark[7]**.** Zapletal [Z, Lemma 2.3.1] used Woodin's Σ^2_1-absoluteness theorem [L, Thm 3.2.1] to show that in the presence of the continuum hypothesis and large cardinals (more precisely: a measurable Woodin cardinal), a projective partial order $\mathbb{P}$ preserves $\aleph_1$ iff $\mathbb{P}$ is proper. Note that if $\mathcal{A}$ is Borel, then $\mathbb{P}_{\mathcal{A}}$ is projective.

2.2. INFINITE COMBINATORICS

2.2.1. Definition. Suppose $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

(a) $\mathcal{A}$ is *almost disjoint* (AD) if the intersection of any two distinct members of $\mathcal{A}$ is finite.

(b) An AD family $\mathcal{A}$ is *maximal almost disjoint* (MAD) if $\mathcal{A}$ has no proper extension to another AD family.

(c) A MAD family $\mathcal{A}$ has *true cardinality* $\mathfrak{c}$ (where $\mathfrak{c} = 2^{\aleph_0}$) if for all $X \subseteq \omega$, $\{A \in \mathcal{A} : X \cap A \text{ is infinite}\}$ is either finite or of cardinality $\mathfrak{c}$.

(d) A MAD family $\mathcal{A}$ is *completely separable* if every set $X \subseteq \omega$ either contains a member of $\mathcal{A}$ or is a subset of the union of a finite subfamily of $\mathcal{A}$.

There is a close relationship between (c) and (d) above. It is easy to see that a completely separable MAD family has true cardinality $\mathfrak{c}$. Moreover, the existence of a MAD family of cardinality $\mathfrak{c}$ implies the existence of a completely separable MAD family. Hechler [H, Theorem 8.2, Lemma 9.2] showed that Martin's axiom (MA) implies the existence of a completely separable family.[8] A similar proof yields the following result.

2.2.2. Theorem. *The following statement* (#) *is provable within* $\mathsf{ZFC} + \mathsf{MA}$.

(#) *For every increasing sequence* $\overline{n} = \langle n_i : i < \omega\rangle$ *with* $\lim \langle n_{i+1} - n_i : i < \omega\rangle = \infty$ *there is a MAD family* $\mathcal{A} = \mathcal{A}_{\overline{n}}$ *that satisfies the following two conditions:*

(1) $\mathcal{A} \subseteq \{u \subseteq \omega : \forall i \in \omega \ \ |u \cap [n_i,\ n_{i+1})| = 1\}$, *and*

(2) *If* $X \subseteq \omega$ *and* $\limsup \langle |X \cap [n_i,\ n_{i+1}]| : i < \omega\rangle = \infty$, *then* $\{B \in \mathcal{A} : B \cap X$ *is infinite*$\}$ *is either finite or of cardinality* $2^{\aleph_0}$.

2.3. TREE INDISCERNIBLES

2.3.1. Definition. Suppose $\mathcal{M}$ is a model with signature $\tau_{\mathcal{M}}$. An indexed family $\{a_\eta : \eta \in {}^{\omega}2\}$ of pairwise distinct elements of $\mathcal{M}$ is said to be a family of *tree indiscernibles in* $\mathcal{M}$ if for every $\varphi(x_0, \cdots, x_{m-1}) \in \mathbf{L}(\tau_{\mathcal{M}})$, there is some $n_\varphi < \omega$, such that for all natural numbers $n > n_\varphi$ and all infinite sequences $\eta_0, \cdots, \eta_{m-1} \in {}^{\omega}2$, $\nu_0 \cdots, \nu_{m-1} \in {}^{\omega}2$ the following implication is true

$$\left(\bigwedge_{i<m} \eta_i \restriction n = \nu_i \restriction n\right) \wedge \left(\bigwedge_{i<j<m} \eta_i \restriction n \neq \eta_j \restriction n\right) \Longrightarrow$$
$$\mathcal{M} \models \varphi[a_{\eta_0}, \cdots, a_{\eta_{m-1}}] \leftrightarrow \varphi[a_{\nu_0}, \cdots, a_{\nu_{m-1}}].$$

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[7]We owe this remark to Paul Larson.

[8]The question of the existence of a completely separable MAD family in ZFC was posed by Erdős and Shelah [ES] and remains open. Shelah [Sh-7] has recently showed that (1) the existence of such families can be established within $\mathsf{ZFC} + 2^{\aleph_0} < \aleph_\omega$, and (2) the nonexistence of such families has very high large cardinal strength. See also [HS] for further open questions regarding completely separable families.

Tree indiscernibles were invented by Shelah[9] ([Sh-1], [Sh-2]) to prove certain 2-cardinal theorems, including $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$. More recently, Shelah [Sh-4] further developed the machinery of tree indiscernibles in his work on Borel structures. In particular, he isolated a cardinal $\lambda_{\omega_1}(\aleph_0)$ that satisfies the following three properties (note that the third property implies that $\lambda_{\omega_1}(\aleph_0)$ is the Hanf number of $\mathsf{L}_{\omega_1\omega}$ below the continuum when $\lambda_{\omega_1}(\aleph_0) < 2^{\aleph_0}$).

- $\lambda_{\omega_1}(\aleph_0) \leq \beth_{\omega_1}$ [Sh-4, Def. 1.1, Conclusion 1.8].
- $\lambda_{\omega_1}(\aleph_0)$ is preserved in c.c.c. extensions [Sh-4, Claim 1.10].
- If a sentence $\psi \in \mathsf{L}_{\omega_1,\omega}$ has a model $\mathcal{M}_0$ with $\left|R^{\mathcal{M}_0}\right| \geq \lambda_{\omega_1}(\aleph_0)$ (where R is a distinguished unary predicate of $\mathcal{M}_0$), then ψ has a model $\mathcal{M}$ with Skolem functions that is generated by a family of tree indiscernibles in $R^{\mathcal{M}}$ (in particular $\left|R^{\mathcal{M}}\right| = 2^{\aleph_0}$) [Sh-4, Claim 2.1].

The above three facts immediately imply the following result.

2.3.2. Theorem. *Suppose* $\mathbf{V}$ *satisfies* $\aleph_{\omega_1} = \beth_{\omega_1}$ *and* $\mathbb{P}$ *is a c.c.c. notion of forcing. Then the following statement* ($\blacktriangle$) *holds in* $\mathbf{V}^{\mathbb{P}}$.

($\blacktriangle$) *If a sentence* ψ *in* $\mathsf{L}_{\omega_1,\omega}$ *has a model* $\mathcal{M}_0$ *with* $\left|R^{\mathcal{M}_0}\right| \geq \aleph_{\omega_1}$, *then there is a countable first order theory* T_1 *with Skolem functions such that* $\tau(T) \supseteq \tau(\psi)$, *and* $T + \psi$ *has a model* $\mathcal{M}$ *that is generated from a family of tree indiscernibles in* $R^{\mathcal{M}}$.

The next result shows that for a given $\mathsf{L}_{\omega_1,\omega}$ sentence ψ the existence of a model of ψ that is generated by tree indiscernibles is absolute.[10]

2.3.3. Theorem. *Let* ψ *be a sentence of* $\mathsf{L}_{\omega_1,\omega}$ *and consider the following statement* S.

$S :=$ *"there is a Skolemized model* $\mathcal{M}$ *with a countable signature* $\tau(\mathcal{M}) \supseteq \tau(\psi)$ *such that* $\mathcal{M}$ *satisfies* ψ *and* $\mathcal{M}$ *is generated from a family of tree indiscernibles".*

Then for any partial order $\mathbb{P}$, *if* $\mathbf{V}^{\mathbb{P}} \vDash S$, *then* $\mathbf{V} \vDash S$.

Proof: It is well known [B, Theorem 6.18] that there is countable Skolemized first order theory T_ψ in a countable signature τ^+ and a countable set Γ_ψ of 1-types of τ^+ such that (1) every model $\mathcal{M}$ of ψ has an expansion to a model $\mathcal{M}^+$ of T_ψ which omits the types in Γ_ψ, and (2) every model of T_ψ that omits the types in Γ_ψ satisfies ψ. Suppose ψ has a model $\mathcal{M}$ generated from a family $\{a_\eta : \eta \in {}^\omega 2\}$ of tree indiscernibles in $\mathbf{V}^{\mathbb{P}}$. Then in $\mathbf{V}^{\mathbb{P}}$ we can form the multi-sorted structure $(\mathcal{M}^+, \mathcal{N}, f)$, where $\mathcal{N}$ is the standard model for second order number theory $(\omega, \mathcal{P}(\omega))$ (which is itself a two-sorted structure) and $f : \mathcal{P}(\omega) \to M$ by $f(A) = a_{\chi_A}$ (where χ_A is the characteristic function of A). In particular, the signature τ^* appropriate to $(\mathcal{M}^+, \mathcal{N}, f)$ has a sort U_M for the universe of $\mathcal{M}^+$, a sort $U_{\mathcal{P}(\omega)}$ for $\mathcal{P}(\omega)$, and a

[9] Shelah also employed tree indiscernibles in his work on classification theory [Sh-3, VII, Sec.4] to show that for all $\lambda \geq \max\{|T|, \aleph_1\}$ T has 2^λ nonisomorphic models of cardinality λ for every complete theory T that is not superstable ([Pi] includes an expository account). Tree indiscernibles were also discovered by Paris and Mills ([PM], [KS, Theorem 3.5.3]) in the context of nonstandard models of Peano arithmetic to show, e.g., the existence of model $\mathcal{M}$ of PA with a nonstandard integer m in $\mathcal{M}$ such that the set of $\mathcal{M}$-predecessors of m is externally countable but the set of $\mathcal{M}$-predecessors of 2^m is of power $2^{\aleph_0}$ (this result is also an immediate corollary of [Sh-1, Theorem 1]).

[10] This result is stated for generic extensions, but the proof shows that this absoluteness result is true for any two ω-models $\mathbf{V}$ and $\mathbf{W}$ of $\mathsf{ZF} + \mathsf{DC}$ with $\mathcal{P}^{\mathbf{V}}(\omega) \subseteq \mathcal{P}^{\mathbf{W}}(\omega)$.

sort U_ω for ω. Let θ be the conjunction of the following sentences $\theta_1, \cdots, \theta_4$ of $\mathsf{L}_{\omega_1,\omega}(\tau^*)$. Note that θ_4 is the only finitary sentence in the list.

- θ_1 expresses: ψ holds in U_M".
- θ_2 expresses: the axioms of second order arithmetic[11] (Z_2) hold in $(U_{\mathcal{P}(\omega)}, U_\omega)$.
- θ_3 expresses: U_ω is an ω-model.
- θ_4 expresses: f is an injection from $\mathcal{P}(\omega)$ into M.

Consider the subset B of $({}^\omega 2)^2$ that consists of elements of the form (r, s), where r codes a countable model $(\mathcal{M}_r^+, \mathcal{N}_r, f_r)$ of θ such that $\mathcal{M}_r^+$ omits the types in Γ_ψ, and s codes a function $g_s : \omega \to \omega$ that witnesses the fact that the image of f_r forms a family of tree indiscernibles in the sense of $\mathcal{N}_r$, i.e., g_s has the property that for every formula $\varphi = \varphi(x_0, \cdots, x_{m-1}) \in \mathbf{L}_{\omega,\omega}(\tau_\psi))$, if $n > g_s(\ulcorner\varphi\urcorner)$, then for all $x_0, \cdots, x_{m-1} \in U_{\mathcal{P}(\omega)}$, and for all $y_0 \cdots, y_{m-1} \in U_{\mathcal{P}(\omega)}$ the following implication is true (in what follows, φ^{U_M} is the relativization of φ to U_M)

$$\left(\bigwedge_{i<m} \chi_{x_i} \restriction n = \chi_{y_i} \restriction n\right) \wedge \left(\bigwedge_{i<j<m} \chi_{x_i} \restriction n \neq \chi_{x_j} \restriction n\right) \to$$
$$\varphi^{U_M}[f(x_0), \cdots, f(x_{m-1})] \leftrightarrow \varphi^{U_M}[f(y_0), \cdots, f(y_{m-1})].$$

It is easy to see that B is a Borel set with a Borel code c in $\mathbf{V}$. Also, by downward Löwenheim-Skolem theorem for $\mathsf{L}_{\omega_1,\omega}$ sentences,

$$\mathbf{V}^{\mathbb{P}} \models \text{“the Borel set coded by } c \text{ is not empty”}.$$

On the other hand, the statement “the Borel set coded by c is empty” is provably equivalent (in $\mathsf{ZF} + \mathsf{DC}$) to a Π^1_1-statement [J, Lemma 25.45] and therefore by Mostowski Π^1_1-absoluteness theorem [J, Theorem 25.4], the Borel set coded by c is nonempty in the real world $\mathbf{V}$. This shows that in $\mathbf{V}$ there is a countable model $(\mathcal{M}_0, \mathcal{N}_0, f_0)$ of ψ, and a function $g_0 : \omega \to \omega$ that witnesses the fact that the image of f forms a family of tree indiscernibles in the sense of $\mathcal{N}_0$ (in particular, $\mathcal{N}_0$ is an ω-model of second order arithmetic).

The countable model $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$ together with g_0 provides us with a blueprint to produce a model of ψ of cardinality continuum that is generated by tree indiscernibles. To do so, add new constants $\{c_\eta : \eta \in {}^\omega 2\}$ to the vocabulary τ^+ of $\mathcal{M}_0^+$. Consider the theory Σ defined as follows. Given $\varphi(x_0, \cdots, x_{m-1}) \in \mathbf{L}_{\omega,\omega}(\tau_{\mathcal{M}})$, fix any $n > g_s(\ulcorner\varphi\urcorner)$, and find $\nu_0 \cdots, \nu_{m-1} \in {}^\omega 2$ such that each ν_i is coded in $\mathcal{N}_0$ (i.e., there is a some A_i in $U_{\mathcal{P}(\omega)}$ of $\mathcal{N}_0$ such that $\chi_{A_i} = \nu_i$) and $\eta_i \restriction n = \nu_i \restriction n$ for each $i < m$. Then put $\varphi[c_{\eta_0}, \cdots, c_{\eta_{m-1}}]$ or its negation in Σ depending on whether $\mathcal{M}_0^+$ satisfies $\varphi[f_0(A_0), \cdots, f_{m-1}(A_{m-1})]$ or its negation. Since $\mathcal{M}_0^+$ is Skolemized, Σ uniquely determines an elementary extension $\mathcal{M}_2^+$ of $\mathcal{M}_0^+$ that is generated by tree indiscernibles. In order to arrange an elementary extension of $\mathcal{M}_0^+$ which is generated by tree indiscernibles that also satisfies ψ we need to thin $\mathcal{M}_2^+$ as follows. Since $\mathcal{M}_0^+$ omits every type in Γ_ψ and g_0 provides a witness to the tree indiscernibility of the range of f_0, we can easily construct a perfect subtree Δ of ${}^\omega 2$ such

[11] We just need a workable theory of finite and infinite sequences, so it is more than sufficient to use RCA_0 or ACA_0 instead of Z_2 here.

that the submodel $\mathcal{M}_1^+$ of $\mathcal{M}_2^+$ generated by $\{c_\eta : \eta \in \Delta\}$ omits every type in Γ_ψ. Therefore $\mathcal{M}_1^+$ is our desired model of ψ which is generated by indiscernibles. $\square$

2.4. BOREL STRUCTURES

Recall that a model $\mathcal{M}$ is said to be *totally Borel* if the universe of $\mathcal{M}$ is a Borel subset of $\mathbb{R}$, and every subset of X that is parametrically definable in $\mathcal{M}$ is a Borel set. It is known that every countable theory has an uncountable totally Borel model. This result was established by H. Friedman [St-1] and also (later, but independently) by Malitz-Mycielski-Reinhardt [MMR]. The following results are included for those readers favoring a shorter but less self-contained proof of Theorem A.

2.4.1. Theorem (Steinhorn [St-2]). *If $\mathcal{M}$ is a model generated by tree indiscernibles, then $\mathcal{M}$ is isomorphic to a totally Borel model.*

2.4.2. Theorem (Harrington-Shelah [HMS]) *No analytic linear order contains an uncountable well-ordered set. In particular, the cofinality of every Borel linear order is $\aleph_0$.*

3. PROOF OF THEOREM A

Before presenting the full technical details of the proof, let us describe a high-level summary of the three stages of the argument.

- **Stage 1 Outline.** Start with the constructible universe $\mathbf{L}$ and a regular cardinal $\kappa > (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$. Then force $\mathsf{MA} + 2^{\aleph_0} = \kappa$ with the usual c.c.c. partial order $\mathbb{Q}$ of cardinality κ. By Theorem 2.2.2, in $\mathbf{L}^{\mathbb{Q}}$ there is a MAD family satisfying (#). In $\mathbf{L}^{\mathbb{Q}}$, use Theorem 2.3.2 to get hold of an ω-standard model $\mathcal{M}'$ of ZFC^- + "$2^{\aleph_0}$ is the last cardinal" (here ZFC^- is ZFC without the powerset axiom) that is generated by tree indiscernibles and in which (#) holds.
- **Stage 2 Outline.** By Theorem 2.4.1 $\mathbf{L}^{\mathbb{Q}}$ believes that $\mathcal{M}'$ is a totally Borel model. Combined with Theorem 2.3.3 this shows that there is also a totally Borel model $\mathcal{M}$ in $\mathbf{V}$ that shares the salient features of $\mathcal{M}'$. In particular, $\mathcal{M}$ is an ω-standard model of ZFC^- that is generated by tree indiscernibles in which (#) of Theorem 2.2.2 holds. The family $\mathcal{A}$ of Theorem A is the set of reals of $\mathcal{M}$. This family $\mathcal{A}$ is both Borel and arithmetically closed.
- **Stage 3 Outline.** Let $\mathfrak{b}^{\mathcal{M}}$ be the *bounding number* $\mathfrak{b}$ as computed in $\mathcal{M}$. By Theorem 2.4.2 $\mathsf{cf}(\mathfrak{b}^{\mathcal{M}}) = \aleph_0$. Coupled with the fact that (#) is true in $\mathcal{M}$, one can then verify that $\mathbb{P}_{\mathcal{A}}$ is $(\aleph_0, 2^{\aleph_0})$ nowhere distributive. By Theorem 2.1.1, this completes the proof of Theorem A.

We now proceed to flesh out the above outline.

Stage 1. Let $\mu = (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$, and fix a regular cardinal $\kappa > \mu$. By GCH in $\mathbf{L}$, $\kappa = \kappa^{<\kappa}$ holds in $\mathbf{L}$. Let $\mathbb{Q}$ be the usual notion of forcing $\mathsf{MA} + 2^{\aleph_0} = \kappa$ [J, Theorem 16.13]. Note that $\mathbb{Q} \in \mathcal{H}^{\mathbf{L}}(\kappa^+)$, where $\mathcal{H}(\kappa^+)$ is the collection of sets whose transitive closure has cardinality at most κ, and $\mathcal{H}^{\mathbf{L}}(\kappa)$ is $\mathcal{H}(\kappa)$ as calculated in $\mathbf{L}$. In the forcing extension $\mathbf{L}^{\mathbb{Q}}$ let $\mathcal{M}_0$ be an expansion of the structure $(\mathcal{H}(\kappa^+), \in)$ by Skolem functions, a well-ordering of $\mathcal{H}(\kappa)$, and individual constants c_n and c_ω, where $c_n^{\mathcal{M}_0} = n$, and $c_\omega^{\mathcal{M}_0} = \omega$. Let $\tau = \tau_{\mathcal{M}_0}$ = the signature of $\mathcal{M}$. We may assume that $\tau \in \mathbf{L}$ and τ is countable in $\mathbf{L}$, but note that $\mathsf{Th}(\mathcal{M}_0)$ need not be in $\mathbf{L}$. Of course $\mathcal{M}_0$ is a model of $\mathsf{ZFC}^- +$ "$2^{\aleph_0}$ is the last cardinal".

Since $\kappa > \mu$ we may invoke Theorem 2.3.2 to obtain a model $\mathcal{M}'$ in $\mathbf{L}^{\mathbb{Q}}$ that satisfies the following five conditions:

(a) $\mathcal{M}'$ is a τ-model of $\mathsf{Th}(\mathcal{M}_0)$.
(b) $\mathcal{M}'$ is an ω-model, i.e., $\mathcal{M}$ omits $\{x \in c_\omega\} \cup \{x \neq c_n : n < \omega\}$.
(c) There is a family $\{a_\eta : \eta \in {}^\omega 2\}$ of tree indiscernibles in $\mathcal{M}$.
(d) For each $\eta \in {}^\omega 2$, $\mathcal{M}' \models$ "$a_\eta \subseteq c_\omega$" (i.e., each a_η is a real in the sense of $\mathcal{M}'$).
(e) $\mathcal{M}'$ is the Skolem hull of $\{a_\eta : \eta \in {}^\omega 2\}$.

Stage 2. Let $T = \{\varphi \in \mathbf{L}_{\omega,\omega}(\tau) : \ 1 \Vdash_{\mathbb{Q}} \mathcal{M}' \models \varphi\}$. Note that since $\mathcal{M}'$ is actually a $\mathbb{Q}$-name, $T \in \mathbf{L}$. By Theorem 2.3.3 there is a τ-model $\mathcal{M}$ of T in $\mathbf{V}$ and a family of tree indiscernibles $\langle a_\eta : \eta \in 2^\omega\rangle$ that satisfy conditions (b), (c), (d) and (e) above. We may assume that the model $\mathcal{M}$ is in "reduced form", i.e., the well-founded part of $\mathcal{M}$ is transitive. In particular, $\omega^{\mathcal{M}} = \omega$, and if $\mathcal{M} \models b \subseteq c_\omega$, then $b \in \mathcal{P}(\omega)$. Let $\mathcal{A} = \{b : \mathcal{M} \models b \subseteq c_\omega\}$. Obviously $\mathcal{A}$ is arithmetically closed[12]. By Theorem 2.4.1 $\mathcal{A}$ is also Borel. This fact can also be established directly as follows. For any $\tau_{\mathcal{M}}$-term $\sigma = \sigma(x_0, \cdots, x_{m-1})$, $m < \omega$, $n^* < \omega$, and pairwise distinct $\nu_0, \cdots, \nu_{m-1} \in {}^{n^*}2$, let $\overline{\nu} = \langle \nu_i : i < m\rangle$, and define $\mathcal{A}_{\sigma,\overline{\nu}}$ via

$$\mathcal{A}_{\sigma,\overline{\nu}} := \{\sigma^{\mathcal{M}}(a_{\eta_0}, \cdots, a_{\eta_{m-1}}) : \bigwedge_{i<m} \nu_i \lhd \eta_i \in {}^\omega 2\}.$$

It is sufficient to prove that $\mathcal{A}_{\sigma,v}$ is Borel for any $(\sigma, \overline{\nu})$ since $\mathcal{A}$ is the union of the countable family of sets of the form $\mathcal{A}_{\sigma,\overline{v}}$. We can find an increasing $f : \omega \to \omega \backslash n^*$ and $\langle g_n : n < \omega\rangle$ such that

(α) g_n is a function from ${}^m\left({}^{f(n)}2\right)$ to $\{0,1\}$.

(β) If $\eta_0, \cdots, \eta_{m-1} \in {}^\omega 2$ and $\bigwedge_{i<m} \nu_i \lhd \eta_i \in {}^\omega 2$ and $n < \omega$, then (using tree indiscernibility)

$$n \in \sigma^{\mathcal{M}}(a_{\eta_0}, \cdots, a_{\eta_{m-1}}) \Leftrightarrow g_n(\eta_0, \cdots, \eta_{m-1}) = 1.$$

By König's lemma, for each $A \subseteq \omega$, we have:

(γ) $A \in \mathcal{A}_{\sigma,\overline{\nu}}$ iff for every n there are $\rho_0, \cdots, \rho_{m-1} \in {}^{f(n)}2$ such that

$$k < n \Rightarrow (k \in A \equiv g_k(\rho_0 \restriction f(k), \cdots, \rho_{m-1} \restriction f(k)) = 1).$$

This shows that each $\mathcal{A}_{\sigma,\overline{\nu}}$ is Borel.

Stage 3: In $\mathcal{M}$ we can define the *bounding number*[13] $\mathfrak{b}$. Let $\overline{f} = \langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be an unbounded family. We may assume that each f_α is increasing. Clearly $(\mathfrak{b}, \in)^{\mathcal{M}}$ is a linear ordering as viewed in the real world.

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[12]Indeed $\mathcal{A}$ is even *hyperarithmetically* closed. This follows from the fact that any ω-model of $\Sigma^1_1-\mathsf{AC}_0$ contains all hyperarithmetical sets [Si, Lemma VIII.4.15] ($\mathcal{M}$ satisfies the axiom of choice, so the standard model of second order arithmetic in the sense of $\mathcal{M}$ satisfies $\Sigma^1_n-\mathsf{AC}_0$ for all $n < \omega$).

[13]Recall that $\mathfrak{b}$ is defined as the least cardinality of an *unbounded* family $\mathcal{F} \subseteq {}^\omega\omega$. Here $\mathcal{F}$ is an unbounded family if for all $g \in {}^\omega\omega$ there is some $f \in {}^\omega\omega$ such that $g(n) < f(n)$ for infinitely many $n \in \omega$). See [J, Sec. 26] for more information.

Claim 3.1. *The external cofinality of* $(\mathfrak{b}, \in)^{\mathcal{M}}$ *is* $\aleph_0$.

Proof: This follows from Theorems 2.4.1 and 2.4.2. Alternatively, one can argue directly as follows. Suppose to the contrary. Then for some regular uncountable cardinal κ, there is an increasing and unbounded subset $\{b_\alpha : \alpha < \kappa\}$ of $(\mathfrak{b}, \in)^{\mathcal{M}}$. Each b_α can be written as

$$b_\alpha = \sigma_\alpha(a_{\eta_0^\alpha}, \cdots, a_{\eta^\alpha_{m_\alpha - 1}}),$$

where, without loss of generality, $\sigma_\alpha = \sigma$, $m_\alpha = m$, and $\eta_0^\alpha <_{\mathsf{lex}} \eta_1^\alpha <_{\mathsf{lex}} \cdots$, and $\left\{\{\eta_0^\alpha, \cdots, \eta^\alpha_{m_\alpha - 1}\} : \alpha < \kappa\right\}$ forms a Δ-system [J, Theorem 9.18]. So, we may assume that for some m,

$$l < m \Rightarrow \eta_l^\alpha = \eta_l^0,$$

and

$$\eta^{\alpha_1}_{l_1} = \eta^{\alpha_2}_{l_2} \Rightarrow (l_1 = l_2 < m) \vee (\alpha_1, l_1) = (\alpha_2, l_2).$$

We can easily construct a *countable* $Y \subseteq \kappa$ such that if $\alpha < \kappa$ and $k < \omega$, then for some $\beta \in Y$ we have

$$\bigwedge_{l<m} \eta_l^\alpha \restriction k = \eta_l^\beta \restriction k.$$

We claim that $\{b_\alpha : \alpha \in Y\}$ is cofinal in $(\mathfrak{b}, \in)^{\mathcal{M}}$. To see this, let $\alpha < \kappa$, and choose k that satisfies the following two conditions (a) and (b).

(a) $\langle \eta_l^\alpha : l < n\rangle \frown \langle \eta_l^{\alpha+1} : l \in [m, n]\rangle$ has no repetition.

(b) If $\eta_l \in {}^\omega 2$ for all $l < n$, $\nu_l \in {}^\omega 2$ for all $l \in [m, n]$, and $(\eta_l \restriction k = \eta_l^\alpha \restriction k) \wedge \left(\nu_l \restriction k = \eta_l^{\alpha+1} \restriction k\right)$, then

$$\mathcal{M} \models \sigma(\cdots, a_{\eta_l}, \cdots) < \sigma(a_{\eta_0} \cdots, a_{\eta_{m-1}}, a_{v_m}, \cdots).$$

Lastly, choose $\beta \in Y$ such that

$$\bigwedge_{l<m} \eta_l^\beta \restriction k = \eta_l^{\beta+1} \restriction k.$$

Hence

$$\left(b_\alpha <^{\mathcal{M}} b_{\alpha+1}\right) \Leftrightarrow \left(b_\beta <^{\mathcal{M}} b_{\beta+1}\right),$$

This completes the proof. □ (Claim 3.1)

We now complete the proof of Theorem A by showing that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\mathrm{L{\scriptsize EVY}}(\aleph_0, 2^{\aleph_0})$. By Theorem 2.1.1, it suffices to prove the following claim.

Claim 3.2. *There is a family* $\{I_n : n \in \omega\}$ *of maximal antichains in* $\mathbb{P}_{\mathcal{A}}$ *such that for every* $p \in \mathbb{P}_{\mathcal{A}}$*, there is some* $n < \omega$ *such that* $\{q \in I_n : p \textit{ and } q \textit{ are compatible}\}$ *has cardinality* $2^{\aleph_0}$.

Proof: By Claim 3.1 we may fix a countable family of functions $F = \{f_n : n < \omega\} \subseteq {}^\omega\omega$ with $X \in \mathcal{M}$, such that $\mathcal{M}$ satisfies "there is no $g \in {}^\omega\omega$ such that every member of X is eventually dominated by g". We may assume that $f_n(i+1) - f_n(i) > i$ for all n and $i < \omega$. Since $\mathcal{M}$ satisfies (#), given the increasing sequence $\langle f_n(i) : i < \omega\rangle$ there is for some $I_n \in \mathcal{M}$ such that $\mathcal{M}$ satisfies following statement (∗)

(∗) I_n is MAD and for each $A \in I_n$ and $|A \cap [f_n(i),\ f_n(i+1))| = 1$ for all $i < \omega$.

We shall prove that $\langle I_n : n < \omega\rangle$ exemplifies condition (b) of Theorem 2.1.1.

Given $p = [B] \in \mathbb{P}_{\mathcal{A}}$, we may assume that B is infinite. It is routine to construct a function $g \in \mathcal{M}$ by recursion such that $g(0) = 0$, and

$$\forall n\ (g(n) < g(n+1) \wedge B \cap [g(n),\ g(n+1)) \text{ has at least } g(n) \text{ elements}).$$

Now, by the choice of F there is some $f_n \in F$ such that the following set Y is infinite:

$$Y := \{i : \exists k\, (f_n(i) < g(k) < g(k+1) < f_n(i+1))\}\,.$$

Let $\mathcal{B} := \{A \in I_n : A \cap B \text{ is infinite}\}$. Note that $\mathcal{B} \in \mathcal{M}$. Therefore by (#) $\mathcal{M}$ satisfies "$\mathcal{B}$ is finite or has cardinality $2^{\aleph_0}$". But $\mathcal{B}$ cannot be finite, since each $A \in \mathcal{B}$ has at most one element in each interval $[f_n(i), f_n(i+1))$, whereas A has more than $f_n(i)$ members for infinitely many i's, and therefore so does $\mathcal{B}$. Hence $\{A : \mathcal{M} \vDash A \in \mathcal{B}\}$ has cardinality $2^{\aleph_0}$ in the sense of $\mathcal{M}$, which finishes the proof since $\mathcal{M}$ has continuum-many reals. □ (Claim 3.2)

References

[BS] B. Balcar and P. Simon, *Disjoint Refinement*, in **Handbook of Boolean algebras**. vol. 2, Edited by J. Donald Monk and Robert Bonnet, North-Holland Publishing Co., Amsterdam, 1989.

[B] J. T. Baldwin, **Categoricity**, University Lecture Series, vol. 50, American Mathematical Society, Providence, RI, 2009.

[E] A. Enayat, A *standard model of arithmetic with no conservative elementary end extension*, **Annals of Pure and Applied Logic,** vol. 156, Issues 2-3 (2008), pp. 308-318.

[ES] P. Erdős and S. Shelah, *Separability properties of almost disjoint families of sets*, **Israel J. Math.** vol.12 (1972), pp. 207-214.

[G-1] V. Gitman, *Can the proper forcing axiom give new solutions to the Scott set problem?* **Journal of Symbolic Logic**, vol. 73 (2008), pp. 845-860.

[G-2] ___________, *Proper and piecewise proper families of reals*, **Mathematical Logic Quarterly,** vol. 55 (2009), pp. 542-550.

[HMS] L. Harrington, D. Marker, and S. Shelah, *Borel orderings,* **Trans. Amer. Math. Soc**. vol. 310 (1988), no. 1, 293–302.

[H] S. H. Hechler, *Classifying almost disjoint families with applications to $\beta N - N$*, **Israel J. Math.**, vol. 10 (1971), pp. 413-432.

[HS] M. Hrušak and P. Simon, *Completely separable MAD families*, in **Open problems in topology II** (edited by Elliott Pearl),. Elsevier B. V., Amsterdam, 2007, pp.179-184.

[J] T. Jech, **Set Theory**, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[K] S. Koppelberg, *Algebraic Theory,* in vol. 1 of **Handbook of Boolean Algebras**, edited by J.D. Monk and R. Bonnet, North-Holland Publishing Co., Amsterdam, 1989.

[KS] R. Kossak and J.H. Schmerl, **The Structure of Models of Peano Arithmetic**, Oxford Logic Guides, vol. 50. Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2006.

[L] Paul Larson, **The Stationary Tower** (Notes on a course by W. Hugh Woodin), University Lecture Series, 32. American Mathematical Society, Providence, RI, 2004.

[MMR] J. Malitz, J. Mycielski & W. Reinhardt, *The Axiom of Choice, the Löwenheim-Skolem theorem and Borel models*, **Fundamenta Mathematicae** vol. 137 (1991), pp. 53-58. [Erratum: Fund. Math. 140 (1992), 197.]

[PM] J. Paris and G. Mills, *Closure properties of countable nonstandard integers.* **Fundamenta Mathematicae** vol. 103 (1979), no. 3, pp. 205–215.

[Pa] I. I. Parovičenko, *A universal bicompact of weight $\aleph$*, **Soviet Mathematics Doklady** 4 (1963), pp. 592 (Russian original: Ob odnom universalnom bikompakte vesa $\aleph$, Doklady Akademii Nauk SSSR 150 (1963) pp. 36-39).

[Pi] A. Pillay, *Number of models of theories with many types*, **Study Group on Stable Theories** (Bruno Poizat, ed.), Second year: 1978/79 (French), Exp. No. 9, 10 pp., Secrétariat Math., Paris, 1981.

[Sh-1] S. Shelah, *A two-cardinal theorem*, **Proc. Amer. Math. Soc**. vol. 48 (1975), pp. 207–213.

[Sh-2] ———, *A two-cardinal theorem and a combinatorial theorem*, **Proc. Amer. Math. Soc**. 62 (1977). pp. 134–136.

[Sh-3] ———, **Classification theory and the number of nonisomorphic models** (second edition), Studies in Logic and the Foundations of Mathematics, vol. 92. North-Holland Publishing Co., Amsterdam, 1990.

[Sh-4] ———, *Borel sets with large squares*, **Fundamenta Mathematica** vol. 159 (1999), no. 1, pp. 1–50.

[Sh-5] ———, *Models of expansions of* $\mathbb{N}$ *with no end extensions* (2009), arXiv:0808.2960.

[Sh-6] ———, *Models of PA: Standard systems without minimal ultrafilters* (2009), arXiv:0901.1499.

[Sh-7] ———, *MAD Families and SANE Player* (2009), arXiv:0904.0816.

[Si] S. Simpson, **Subsystems of second order arithmetic**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

[St-1] C. Steinhorn, *Borel structures and measure and category logics*, in **Model-theoretic logics**, Perspect. Math. Logic., Springer-Verlag, New York, 1985, pp. 579-596.

[St-2] ———, *Borel structures for first-order and extended logics,* in **Harvey Friedman's research on the foundations of mathematics**, edited by L. A. Harrington, M. D. Morley, A. Sčedrov and S. G. Simpson, Studies in Logic and the Foundations of Mathematics, vol. 117. North-Holland Publishing Co., Amsterdam, 1985, pp. 161–178.

[Z] J. Zapletal, **Descriptive set theory and definable forcing**. Mem. Amer. Math. Soc. 167 (2004), no. 793.

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