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## **On cofinal submodels and first interstices**

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REPORT No. 8, 2009/2010, fall

ISSN 1103-467X

ISRN IML-R- -8-09/10- -SE+fall

# ON COFINAL SUBMODELS AND FIRST INTERSTICES

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*A relatively neglected aspect of the study of nonstandard models of arithmetic is the study of their cofinal extensions. These extensions certainly do not present themselves to the intuition as readily as do their more popular cousins the end extensions; but they are not exactly shrouded in mystery or unnatural objects of study either. They are equal partners with end extensions in the construction of general extensions of models; they offer both special advantages and disadvantages worthy of our interest; and, occasionally, they are useful in understanding the generally more simply behaved end extensions. Cofinal extensions deserve more attention than they have traditionally received.*

*Craig Smoryński [8]*

Smoryński's remark was justified in the 1980s, and its validity has not diminished since then. The study of elementary end extensions of models of PA is much more advanced than the study of cofinal ones. In this note we will heed Smoryński's advice and we will consider isomorphism types and the first-order theories of pairs of the form  $(N, M)$ , where  $N \models \text{PA}$  and  $M$  is an elementary cofinal submodel of  $N$ . We will review what is known about such pairs, we will prove some partial results, and we will pose open problems. Much in this note is based on Smoryński's three papers listed in the references. We are aiming at a more systematic study of some problems left open in those papers.

The complete treatment of the general case of pairs  $(N, M)$ , even in the countable case, seems far out of reach. For example, there are many open basic questions concerning the interstructure lattices of elementary submodels  $\text{Lt}(N/M) = \{K : M \preceq K \preceq N\}$ . We do not even now if there is a finite lattice which cannot be realized as an interstructure lattice with  $M$  cofinal in  $N$ . We will consider a much more tractable case of recursively saturated  $N$  and its recursively saturated cofinal elementary submodels. By the well-known theorem of Gaifman the assumption of elementarity of the extension is redundant. If  $M$  is cofinal in  $N$  and both models are models of PA, then  $M \prec N$ .

## 1. SOURCES OF DIVERSITY

By  $E(M)$  we will denote the set of all elementary submodels of a model  $M$ . If  $M \models \text{PA}$  and  $A \subseteq M$ , then  $\text{Scl}(A)$  is the smallest  $K$  in  $E(M)$  which contains  $A$ , and we call it the *Skolem closure* of  $A$ . We will also consider Skolem closures those expansions of  $M$  which

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Work done during *Mathematical Logic: Set theory and Model Theory* program in the fall of 2009 at the Institut Mittag-Leffler (Djursholm, Sweden).

satisfy the induction schema. In such a case we will be more specific and, for a given expansion  $(M, X)$ , the Skolem closure of  $A$  in  $(M, X)$  will be denoted by  $\text{Scl}^{(M, X)}(A)$ . For  $a \in M$ ,  $\text{Scl}(a)$  is  $\text{Scl}(\{a\})$ .

Recall that a model  $M \models \text{PA}$  if *short* is  $\text{Scl}(a)$  is cofinal in  $M$  for some  $a \in M$ . Otherwise  $M$  is *tall*. The notions of sup and inf are used in the context of the natural ordering  $<$  of  $M$ . In particular, if  $M$  is short, then  $M = \sup(\text{Scl}(a))$ , for some  $a \in M$ .

*Throughout the paper  $N$  will be a fixed countable recursively saturated model of PA.*

If every complete type realized in a countable model  $M \models \text{PA}$  is also realized in  $N$ , then  $M$  can be elementarily embedded into  $N$ . As we will see, this implies that  $\text{E}(N)$  contains uncountably many nonisomorphic models. Choice for elementary cuts is more limited. If  $M \in \text{E}(N)$  is a cut, then either  $M$  is tall, and in this case  $M$  is isomorphic to  $N$ , or  $M$  is short, and then  $M$  is one of the countably many nonisomorphic elementary cuts of the form  $\sup(\text{Scl}(a))$ . The fact that every countable recursively saturated model of PA has infinitely many pairwise nonisomorphic short elementary cuts was proved independently by Kotlarski and Smoryński (see [8]), but it also follows easily from earlier results of Gaifman [2] which imply that every recursively saturated model of PA realizes infinitely many pairwise independent minimal types (see Chapter 3 of [7]). Minimal types are strongly indiscernible, which means that if  $\bar{a} = \langle a_0, \dots, a_n \rangle$  and  $\bar{b} = \langle b_0, \dots, b_n \rangle$  are increasing tuples of elements realizing the same minimal type in a model  $M$ , then for every  $c \in M$  such that  $\text{Scl}(c) < \min\{a_0, b_0\}$ ,  $\text{tp}(\bar{a}, c) = \text{tp}(\bar{b}, c)$ . This property will be used several times.

How do we know that there are continuum many nonisomorphic models which are elementarily embeddable into  $N$ ? There are many arguments providing different kinds of diversity. We will use two in the following two propositions. If  $M$  is a model, then by  $\text{Tp}(M)$  we will denote the set  $\{\text{tp}(a) : a \in M\}$ .

**Proposition 1.1.** *There is  $\mathcal{I} \subseteq \text{E}(N)$  such that  $|\mathcal{I}| = 2^{\aleph_0}$ , for all  $M, M' \in \mathcal{I}$ ,  $\text{Tp}(M) = \text{Tp}(M')$ , and if  $M \neq M'$  then  $M \not\cong M'$ .*

*Proof.* Let  $p$  and  $q$  be independent minimal types realized in  $N$ . For each  $X \subseteq \omega$  let  $M_X = \text{Scl}\{a_i^X : i < \omega\}$ , where  $\langle a_i^X : i < \omega \rangle$  is an increasing sequence such that for all  $i < \omega$ , if  $i \in X$  then  $a_i^X$  realizes  $p$ , and otherwise it realizes  $q$ . Let  $\mathcal{I} = \{M_X : X \subseteq \omega\}$ . It follows from general results of Gaifman [2], that  $M_X$  is isomorphic to  $M_Y$  iff  $X = Y$ . Also, since every element of  $M_X$  is of the form  $t(\bar{a}, \bar{b})$ , where  $t$  is a Skolem term,  $\bar{a}$  is an increasing sequence of elements realizing  $p$  and  $\bar{b}$  is an increasing sequence of elements realizing  $q$ , it is easy to see that strong indiscernibility implies that if both  $X$  and  $Y$  are infinite and coinfinite, then  $M_X$  and  $M_Y$  realize exactly the same types.  $\square$

We also have the other extreme:

**Proposition 1.2.** *There is  $\mathcal{J} \subseteq \text{E}(N)$  such that  $|\mathcal{J}| = 2^{\aleph_0}$ , and for all  $M, M' \in \mathcal{J}$  if  $M \neq M'$ , then  $\text{Tp}(M) \neq \text{Tp}(M')$ .*

*Proof.* Using standard arguments one can show that for every Scott set  $\mathfrak{X} \subseteq \text{SSy}(N)$ ,  $N$  has a recursively saturated elementary submodel  $M$  such that  $\text{SSy}(M) = \mathfrak{X}$ . The result now follows from an unpublished result of Stephen Simpson which says that every Scott

set has continuum many Scott subsets, and the fact that every recursively saturated model  $M \models \text{PA}$  is  $\text{SSy}(M)$ -saturated.  $\square$

While all tall elementary cuts of  $N$  are isomorphic to  $N$ , there are continuum many nonisomorphic, and in fact even non elementarily equivalent, structures of the form  $(N, M)$ , where  $M$  is an elementary cut of  $N$ . This can be shown in various ways (see [9] and [4]). As a corollary we can note the following proposition.

**Proposition 1.3.** *Let  $M$  be a tall model which is elementarily embeddable into  $N$ . Then there are continuum many complete theories of structures of the form  $(N, K)$ , where  $K \prec N$  and  $K \cong M$ .*

*Proof.* Let  $M$  be a tall model and suppose that  $K \in \text{E}(N)$  is isomorphic to  $M$ . Then  $K' = \text{sup}(K)$  is recursively saturated. By the results mentioned above, there are continuum many theories of structures of the form  $(N, K'')$ , where  $K''$  is an elementary cut in  $N$  and  $K' \cong K''$ . Since  $K'$  is definable in  $(N, K)$  the result follows.  $\square$

If  $M \in \text{E}(N)$  is short, then we have two cases. Either  $M$  is finitely generated, or not. If  $M$  is finitely generated, then by homogeneity of  $N$ , for all  $M' \in \text{E}(N)$ , if  $M' \cong M$  then  $(N, M) \cong (N, M')$ . If  $M$  is not finitely generated then the situation is more complex and is similar to the case of cofinal submodels which will be discussed later.

Before we move to the next section, we pose one question concerning elementary cuts. In the known constructions of large families of pairwise nonisomorphic pairs  $(N, M)$  with  $M \in \text{E}(N)$ ,  $M$  is either not semiregular in  $N$ , or  $M$  is strong in  $N$ . For  $I \subseteq_{\text{end}} N$ ,  $\text{Cod}(N/I) = \{I \cap X : X \text{ Def}(N)\}$ , where  $\text{Def}(N)$  is the set of subsets of  $N$  which are definable with parameters. Then, a cut  $I$  is *semiregular* iff  $(N, \text{Cod}(N/I))$  is a model of the  $\Sigma_1$  induction schema,  $I\Sigma_1$ , and  $I$  is *regular* iff  $(N, \text{Cod}(N/I))$  is a model of the  $\Sigma_2$  collection schema,  $B\Sigma_2$ , where both schemas are in the language of  $\text{PA}$  with relation symbols for all sets in  $\text{Cod}(N/I)$ <sup>1</sup>.

**Question 1.4.** *Is there a large family of pairwise nonisomorphic pairs  $(N, M)$  such that  $M$  semiregular but not regular in  $N$ ?*

The meaning of “large” in the question above is left vague. We do not even know how to get two nonisomorphic such pairs.

## 2. FIRST INTERSTICES OF FINITELY GENERATED MODELS

If  $M \in \text{E}(N)$  is bounded in  $N$ , then  $\text{sup}(M)$  is an elementary cut and since it is definable in  $(N, M)$ , it is a useful isomorphism invariant for the pair  $(N, M)$ . For  $M$  which are cofinal in  $N$  we have to look for something else. Smoryński proved that there are continuum many theories of pairs  $(N, M)$  where  $M$  is cofinal in  $N$  by considering the greatest common initial segment of  $M$  and  $N$ , denoted  $\text{GCIS}(N, M)$ . Smoryński’s proof is based on the fact that  $\text{GCIS}(N, M)$  can be any cut of  $N$  as long as it is closed under addition and multiplication and the fact that there are continuum many complete theories of such cuts.

<sup>1</sup>These are not the original definitions given by Kirby and Paris, but their convenient equivalent versions.

Since  $\text{GCIS}(N, M)$  is definable in  $N$ , this gives rise to continuum many theories  $\text{Th}(M, N)$ . One of our goals is to see how diverse is the collection of pairs  $(N, M)$  with a fixed greatest common initial segment. Since  $\text{GCIS}$  is fixed, we need other isomorphism invariants. The idea is to look at the bottom end of  $M$ .

**Definition 2.1.** For  $M \prec N$  let the first interstice of  $M$ , denoted  $\Omega(M)$ , be the set  $\{x \in N : \text{Scl}(0) < x < (M \setminus \text{Scl}(0))\}$ . For  $a \in N$  we define  $\Omega(a)$  to be  $\Omega(\text{Scl}(a))$ .

Terminology and notation of first interstices are partially in accordance with the definitions introduced by Bamber and Kotlarski [1] when applied to structures of the form  $(M, a)_{a \in I}$ . Instead of the first interstice of  $M$  we can also consider interstices determined by cuts in  $N$  other than  $\text{sup}(\text{Scl}(0))$ . For any cut  $I \subseteq_{\text{end}} N$  we define  $\Omega(M, I)$  to be the set  $\{x \in N : I < x < (M \setminus I)\}$ . In particular  $\Omega(M) = \Omega(M, \text{sup}(\text{Scl}(0)))$ . In this note we will consider the interesting case of interstices  $\Omega(M, I)$ , where  $I = \text{sup}(I')$  for some  $I' \prec_{\text{end}} M$ . Every nonempty interstice  $\Omega(M, I)$  determines the cut  $\Omega_+(M, I) = \text{sup}(\Omega(M, I))$ . Our goal now is to study the diversity among isomorphism types and complete theories of pairs  $(N, \Omega(M))$  and  $(N, \Omega(M, \text{GCIS}(N, M)))$ , for  $M \in \mathbf{E}(N)$ .

**Proposition 2.2.** If  $I \prec_{\text{end}} M$  and  $\Omega(M, I)$  is nonempty, then  $\Omega_+(M, I) \prec_{\text{end}} N$ .

*Proof.* Suppose that for an  $a \in \Omega_+(M, I)$  and a Skolem term  $t(x)$  we have  $t(a) > b$  for some  $b \in (M \setminus I)$ . Then  $\min\{x : t(x) > b\} \in I$ , contradiction.  $\square$

We are primarily interested in first interstices of models which are not finitely generated, but let us first take a quick look at finitely generated models.

There are examples of  $a > \text{Scl}(0)$  such that  $\Omega(a)$  is empty. Let  $\mathbb{N}$  be the standard model. Suppose that  $N \equiv \mathbb{N}$  and  $N$  is not arithmetically saturated. Then, since  $\mathbb{N}$  is not strong in  $N$ , there is an  $a \in M$  such that the set  $\{(a)_n : n < \omega \text{ and } (a)_n > \mathbb{N}\}$  is downward cofinal in  $N \setminus \mathbb{N}$ . It follows that  $\Omega(a) = \emptyset$ .

Recall that for  $a \in N$ , the gap of  $a$ , denoted  $\text{gap}(a)$ , is the set theoretic difference  $\text{sup}(\text{Scl}(a)) \setminus \text{sup}(\{b : \text{Scl}(b) < a\})$ . Notice that  $\text{gap}(0) = \text{sup}(\text{Scl}(0))$ . We call  $\text{gap}(0)$  the least gap. If  $a > \text{gap}(0)$ , we will say that  $\text{gap}(a)$  is *proper*.

We call a cut  $M \in \mathbf{E}(N)$  *co-short* if  $N \setminus M$  has a least gap. Notice that a co-short model must be tall, but there are continuum many tall cuts which are not co-short.

If  $a > \text{Scl}(0)$  and  $\text{Scl}(a)$  has a least gap  $\gamma$ , then  $\Omega(a)$  is nonempty and  $\Omega_+(a) = \text{inf}(\gamma)$  is co-short. However, not every co-short elementary cut is of the form  $\Omega_+(a)$ . It is shown in [6] that there are  $a, b \in N$  such that  $b < \text{gap}(a)$  and for all  $c \in \text{gap}(a)$ ,  $b \in \text{Scl}(c)$ . It follows that  $\text{inf}(\text{gap}(a))$  is a co-short cut which is not of the form  $\Omega_+(d)$  for any  $d$ . The next proposition shows that with the exception of  $b = 0$ , short cuts  $\text{sup}(\text{Scl}(b))$  can never be of the form  $\text{inf}\{\text{Scl}(a) \setminus \text{sup}(\text{Scl}(0))\}$ .

**Proposition 2.3.** If  $\text{Scl}(0) < b < \Omega(a)$ , then  $\Omega_+(a) \neq \text{sup}(\text{Scl}(b))$ .

*Proof.* Let  $\langle t_n(x) : n < \omega \rangle$  be a recursive enumeration of all Skolem terms in one variable. Consider the type  $p(x, a, b)$ :

$$\{t_n(b) < x < t_m(a) : m, n < \omega \text{ and } b < t_m(a)\}.$$

Since  $b < \Omega(a)$ , it is easy to see that  $p(x, a, b)$  is finitely realizable in  $N$ , and since it is recursive in  $\text{tp}(a, b)$ , it is realized in  $N$ , which finishes the proof.  $\square$

If  $N$  is arithmetically saturated, we can strengthen Proposition 2.3 by including the following variant for the case of  $b = 0$ . In fact, we will show that this property characterizes arithmetic saturation.

**Proposition 2.4.**  *$N$  is arithmetically saturated if and only if for all  $a > \text{Scl}(0)$ ,  $\Omega(a) \neq \emptyset$ .*

*Proof.* The proof the  $\Rightarrow$  direction is the same as that of Proposition 2.3 once we notice that the type

$$\{t_n(0) < x < t_m(a) : m, n < \omega \text{ and } \text{Scl}(0) < t_m(a)\}$$

is arithmetic in  $\text{tp}(a)$ .

The proof of the opposite direction is due to Jim Schmerl. If  $N$  is not arithmetically saturated, then there is a  $b \in N$  such that  $\inf\{(b)_n : n < \omega\} = \mathbb{N}$ . By recursive saturation there is  $c \in N$  such that  $\langle (c)_n : n < \omega \rangle$  is an increasing sequence cofinal in  $\text{Scl}(0)$ . Let  $a = \langle b, c \rangle$ . We will show that  $\inf\{\text{Scl}(a) \setminus (\text{Scl}(0))\} = \sup(\text{Scl}(0))$ . Let  $n_0 > \mathbb{N}$  be such that  $\langle (c)_n : n < n_0 \rangle$  is increasing. We can assume that for all  $n < \omega$ ,  $(b)_n < n_0$ . Then for  $X = \{(c)_{(b)_n} : n < \omega\}$ , we have  $X \subseteq \text{Scl}(a)$  and  $\inf\{x \in X : x > \text{Scl}(0)\} = \text{Scl}(0)$ , which finishes the proof.  $\square$

The two propositions above tell us what first interstices of finitely generated models in  $E(N)$  cannot be. It is a bit harder to see what they can be.

If  $a \in N$  is such that  $\text{Scl}(a)$  has a least proper gap  $\gamma$ , then  $\Omega_+(a) = \inf(\gamma)$ . The case of  $\text{Scl}(a)$  with no least proper gap is more interesting. It is not obvious that such an  $a$  exists. The fact that there are  $a$  such that  $\text{Scl}(a)$  has no least proper gap follows directly from Theorem 2.1.12 of [7]. The theorem says that every countable model  $M \models \text{PA}$  has a finitely generated elementary end extension (with a stronger property of superminimality). We can apply this theorem to a countable model  $M$  which has no least proper gap to get its elementary end extension of the form  $\text{Scl}(a)$ . Then we can extend  $\text{Scl}(a)$  to a countable recursively saturated model. One can modify this argument to get such  $M$  and  $a$  such that  $\text{tp}(a) \in \mathfrak{X}$ , for any Scott set  $\mathfrak{X}$ , which proves that there is such an  $a$  in every recursively saturated model of PA.

**Proposition 2.5.** *Suppose  $N$  is arithmetically saturated. Then for all  $a, b \in N$  such that  $\text{Scl}(a)$  and  $\text{Scl}(b)$  have no least proper gaps,  $(N, \Omega_+(a)) \cong (N, \Omega_+(b))$ .*

*Proof.* Let  $\langle t_n(x) : n < \omega \rangle$  be a recursive enumeration of all Skolem terms in one variable with  $t_0(x) = x$ . Since  $N$  is arithmetically saturated, there is  $c \in N$  such that  $\{(c)_n : n < \omega\} = \{x \in \text{Scl}(a) : x > \text{Scl}(0)\}$ .

Let  $a \in N$  be such that  $\text{Scl}(a)$  has no least proper gap. We define  $f : \mathbb{N}^2 \rightarrow N$  as follows

$$f(m, n) = \begin{cases} \min\{k : t_k(t_m(a)) \geq t_n(a)\} & \text{if there is such } k, \\ a & \text{otherwise.} \end{cases}$$

Since  $N$  is arithmetically saturated,  $f$  is coded in  $N$ . Now we define  $a_0 = t_0(a_0) = a$ . If  $a_n$  is defined and is equal to  $t_{k_n}(a)$ , to define  $a_{n+1}$  we first get

$$k_{n+1} = \min\{k : t_k(a) < \min\{a_n, (c)_n\} \wedge f(k, k_n) = a\}.$$

Since  $\text{Scl}(a)$  has no least proper gap,  $k_{n+1}$  is well defined. We let  $a_{n+1} = t_{k_{n+1}}(a)$ . By recursive saturation, the sequence  $\langle a_n : n < \omega \rangle$  is coded in  $N$ . Notice that for all  $n < \omega$ ,  $a_n \in \text{Scl}(a)$  and  $\text{gap}(a_{n+1}) < a_n$ . Let  $\langle b_n : n < \omega \rangle$  be similarly defined descending sequence in  $\text{Scl}(b)$ . By a theorem analogous to Theorem 2.4 of [10], we have  $(N, \inf\{a_n : n < \omega\}) \cong (N, \inf\{b_n : n < \omega\})$  (see the proof of Proposition 10.2.3 in [7]). Since  $\Omega_+(a) = \inf\{a_n : n < \omega\}$  and  $\Omega_+(b) = \inf\{b_n : n < \omega\}$ , this finishes the proof.  $\square$

It is shown in [5] that for countable recursively saturated model  $N$  of PA the following statements are equivalent.

- (1)  $N$  is arithmetically saturated.
- (2) For any two tall models  $M, K \in \mathbf{E}(N)$ , if there exist increasing coded sequences  $\langle a_n : n < \omega \rangle$  and  $\langle b_n : n < \omega \rangle$  such that  $M = \sup\{a_n : n < \omega\}$  and  $K = \sup\{b_n : n < \omega\}$ , then  $(N, M) \cong (N, K)$ .

I have been trying use similar arguments to show that the condition in Proposition 2.5 characterizes arithmetic saturation. No success so far, so let us formulate this as a question.

**Question 2.6.** *Suppose that for all  $a, b \in N$  such that  $\text{Scl}(a)$  and  $\text{Scl}(b)$  have no least proper gaps  $(N, \Omega_+(a)) \cong (N, \Omega_+(b))$ . Is  $N$  arithmetically saturated?*

### 3. FIRST INTERSTICES OF MODELS WHICH ARE NOT FINITELY GENERATED

We are looking for isomorphism invariants for pairs of the form  $(N, M)$  where  $M$  is a cofinal elementary submodel of  $N$ , but cofinality is not an important feature. If  $M \in \mathbf{E}(N)$  is tall, then  $M$  is cofinal in  $\sup(M)$ , and  $\sup(M)$  is isomorphic to  $N$ , so we can consider any elementary embedding of  $M$  into  $N$  and later, if needed, transform it into a cofinal one. We are also interested in another not very well understood case of pairs  $(N, M)$  where  $M \in \mathbf{E}(N)$  is short, but not finitely generated. In both cases it is desirable to have a description the spectrum of possible cuts of the form  $\Omega_+(M)$ , or  $\Omega_+(M, \text{GCIS}(N, M))$ .

**Proposition 3.1.** *Let  $I \prec_{\text{end}} N$  be either*

- *short, or*
- *tall and not co-short, or*
- *co-short with a minimal type realized in the first gap above it.*

*Then there is  $M \in \mathbf{E}(N)$  which is cofinal in  $N$  and  $\Omega_+(M, \text{GCIS}(N, M)) = I$ .*

*Proof.* Let  $p$  be a minimal type realized in  $M$  and in the first gap above  $I$  in the third case, and let  $p^N$  be the set of realizations of  $p$  in  $N$ . Let  $M = \text{Scl}[\sup(\text{Scl}(0)) \cup (p^N \setminus I)]$ . Since  $p$  is strongly indiscernible,  $M \cap [I \setminus \sup(\text{Scl}(0))] = \emptyset$ , and for any  $I$  satisfying the assumptions of the proposition,  $I = \inf[p^N \cap (N \setminus I)]$ , and the result follows.  $\square$

By the results of [6] already mentioned in the remarks before Proposition 2.3, some restriction in third case of Proposition 3.1 is necessary. Still we have an open problem:

**Question 3.2.** *Characterize all co-short  $I$  cuts for which there is  $M \prec N$  such that  $\Omega_+(M) = I$ .*

The next goal is to improve Proposition 3.1 by putting additional requirements on the isomorphism type of  $M$ . Can  $M$  be recursively saturated? Can it be isomorphic to  $M$ ? If  $I \in \mathbf{E}(N)$  is a strong cut, then  $N$  has a countable cofinal elementary extension  $N'$  such that  $I = \text{GCIS}(N, N')$  and  $\Omega^{N'}(N, I) \prec_{\text{end}} N'$  (this is a result of Kirby [3]). Since  $N$  is isomorphic to  $N'$  this gives us examples of first interstices of cofinal submodels in  $\mathbf{E}(N)$  which are isomorphic to  $N$ . Kirby proved that  $N'$  can be constructed in such a way that  $\Omega^{N'}(N, I)$  is again a strong elementary cut, but beyond that there is not much more information.

**Proposition 3.3.** *Let  $I \in \mathbf{E}(N)$  be such that  $(N, I)$  is recursively saturated. Then there is a recursively saturated model  $M \in \mathbf{E}(N)$  which is cofinal in  $N$ ,  $M \cong N$  and  $\Omega(M, \text{GCIS}(N, M)) = I$ .*

*Proof.* Since  $(N, I)$  is resplendent, the result follows directly from Proposition 3.1, nevertheless we will prove it in a bit more complicated manner in preparation for the proof of the next result. Let  $S$  be a partial inductive satisfaction class such that  $(N, S)$  is resplendent and  $(I, S \cap I) \prec (N, S)$ . Let  $p$  be a minimal type of  $\text{Th}(N, S)$  realized in  $(N, S)$  and let  $P$  be the set of its realizations in  $(N, S)$ . Let  $J = \sup(\text{Scl}^{(N, S)}(0))$ , and let  $M = \text{Scl}^{(N, S)}(J \cup [(N \setminus I) \cap P])$ . Since  $(N, I)$  is recursively saturated,  $I$  is not short in  $(N, S)$ ; hence  $J < I$ . By strong indiscernibility of  $p$  we can conclude that  $J = \text{GCIS}(M, N)$ , and  $M \cap (I \setminus J) = \emptyset$ . Finally, since  $I$  is not co-short in  $(N, S)$ ,  $I = \inf[(N \setminus I) \cap P]$ , and this implies that  $\Omega_+(M, J) = I$ . Also, since  $M$  is recursively saturated and  $\text{SSy}(M) = \text{SSy}(N)$ ,  $M$  is isomorphic to  $N$ .  $\square$

We will modify the proof of Proposition 3.3 to get a stronger result improving slightly Smoryński's Theorem 1.3 [9].

**Theorem 3.4.** *There is an elementary cut  $J$  for which there are infinitely many theories  $\text{Th}(N, M)$ , where  $M \in \mathbf{E}(N)$  is cofinal in  $N$ ,  $\text{GCIS}(N, M) = J$  and  $M \cong N$ .*

*Proof.* There is a recursive list  $\langle \Phi_n : n < \omega \rangle$  of first-order properties of elementary cuts, such that for all distinct  $n$  and  $m$  and all cuts  $M \in \mathbf{E}(N)$ ,  $(N, M) \models (\Phi_n \rightarrow \neg \Phi_m)$ . For example  $\Phi_n$  can be a sentence expressing that  $(M, \text{Cod}(N/M))$  satisfies all  $\Sigma_n$ -induction axioms with set parameters from  $\text{Cod}(N/M)$  plus a negation of a particular instance of the  $\Sigma_{n+1}$  collection axiom. By chronic resplendency of  $N$  we get  $X, S \subseteq N$  such that  $(N, X, S)$  is resplendent and for each  $n < \omega$  the following conditions hold:

- (1)  $(X)_n \prec_{\text{end}} (X)_{n+1} \prec_{\text{end}} N$
- (2)  $(N, (X)_n) \models \Phi_n$ .
- (3)  $S$  is a partial inductive satisfaction such that for all  $n < \omega$ ,
 
$$((X)_n, S \cap (X)_n) \prec (N, S).$$

As in Proposition 3.3, let  $p$  be a minimal type of  $\text{Th}(N, S)$  realized in  $(N, S)$ . Let  $P$  be the set of realizations of  $p$  in  $(N, S)$ . Let  $J = \sup(\text{Scl}^{(N, S)}(0))$ . Condition (3) implies that



$J < (X)_0$ . Finally, for each  $n < \omega$  we define  $M_n$  to be  $\text{Scl}^{(N,S)}(J \cup (P \setminus (X)_n))$ . As in the proof of Proposition 3.3 one can show that for all distinct  $n < \omega$ ,  $\text{GCIS}(N, M_n) = J$ , and  $\Omega_+(M_n, J) = (X)_n$ . By condition (2),  $\text{Th}(N, M_m) \neq \text{Th}(N, M_n)$  for all  $m \neq n$ .  $\square$

**Question 3.5.** *Can we replace “infinitely many” by “uncountably many” in Theorem 3.4?*

All pairs  $(N, M)$  in the last theorem are recursively saturated. We finish with two other constructions which are slight modifications of two basic constructions from [10]. Both constructions yield pairs  $(N, M)$  which are not recursively saturated.

**Proposition 3.6.** *Let  $\langle a_n : n < \omega \rangle$  be a sequence which is coded in  $N$  and such that for each  $n < \omega$ ,  $\text{Scl}(a_n) < a_{n+1}$ . Let  $K = \sup\{a_n : n < \omega\}$ . Then there is an  $M \in \text{E}(N)$  such that  $M$  is cofinal in  $N$ ,  $M \cong N$ , and  $\Omega_+(M, J) = K$ , for some  $J \prec_{\text{end}} K$ .*

*Proof.* Let  $S$  be a partial inductive satisfaction class such that  $(N, S)$  is recursively saturated. Let  $\langle b_n : n < \omega \rangle$  be a coded sequence such that for all  $n < \omega$ ,  $\text{Scl}^{(N,S)}(b_n) < b_{n+1}$  with  $b_0 > \text{Scl}^{(N,S)}(0)$ . Apply the proof of Proposition 3.1 to the structure  $(N, S)$  to get  $M$  and  $J$  such that  $\Omega_+(M, J) = K' = \sup\{b_n : n < \omega\}$ . Both  $K$  and  $K'$  are suprema of coded increasing sequences of gaps, hence, by the previously quoted Theorem 2.4 of [10],  $(N, K) \cong (N, K')$ , and the result follows.  $\square$

**Theorem 3.7.** *There is  $J \prec_{\text{end}} N$  and continuum many nonisomorphic pairs  $(N, M)$  such that  $M \in \text{E}(N)$  is cofinal in  $N$ ,  $\text{GCIS}(N, M) = J$ , and  $M \cong N$ .*

*Proof.* Let  $S$  be a partial inductive satisfaction class such that  $(N, S)$  is recursively saturated and let  $p$  be a minimal minimal type of  $\text{Th}(N, S)$  realized in  $N$ . We will follow the proof of Theorem 3.6 of [10]. Let  $<_\eta$  be a definable ordering of  $\mathbb{N}$  of the order type of the rationals. By recursive saturation, there is a coded sequence  $\langle a_r : r \in \mathbb{N} \rangle$  such that for all  $r, s \in \mathbb{N}$ ,  $\text{gap}^{(N,S)}(a_r) < a_s$  iff  $r <_\eta s$ . In addition we require that all  $a_r$  realize  $p$ .

Let  $C$  be a Dedekind cut in  $(\mathbb{N}, <_\eta)$ , and let  $K[C] = \sup\{a_r : r <_\eta C\}$ . It is easy to see that  $K[C]$  is a tall elementary cut of  $N$ . Hence  $K[C] \cong N$ . Let  $a \in N$  code  $\langle a_r : r \in \mathbb{N} \rangle$ . One can show that for each  $C$ ,  $C$  is uniformly defined in  $(K, K[C], a)$  (see [10]). This implies that there are continuum many nonisomorphic pairs of the form  $(N, K[C])$ .

As before, we define  $J = \sup(\text{Scl}^{(N,S)}(0))$ , and for a Dedekind cut  $C$ ,  $M[C] = \text{Scl}^{(N,S)}(J \cup \{a_s : C <_\eta a_s\})$ . To finish the proof, notice that  $M[C]$  is isomorphic to  $N$  by an isomorphism fixing  $K[C]$  pointwise.  $\square$

**Question 3.8.** *Can “There is  $J \prec_{\text{end}} N \dots$ ” in Theorem 3.7 be replaced by “For all tall  $J \prec_{\text{end}} N$ , there are...”?*

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