THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Interpreting Groups and Fields in Simple, Finitary AECs

T. Hyttinen and M. Kesälä

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Interpreting Groups and Fields in Simple, Finitary AECs

Tapani Hyttinen and Meeri Kesälä*

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Abstract

We prove the following theorem for simple and superstable finitary AECs:

We assume that p is a regular Lascar strong type over a finite set A and Q is an A-invariant set such that for any independent n-sequence $a_1, ..., a_n$ of realizations of p and every finite $C \subset Q$ we have that $\dim(a_1, ..., a_n/C) = n$ but for some independent $n+1$ sequence $a_1, ..., a_{n+1}$ there is a finite set $C \subset Q$ such that the subsequence $a_1, ..., a_n$ dominates $a_1, ..., a_{n+1}$ over $A \cup C$. We conclude that then the monster model interprets a group G which acts on the geometry $\mathbf{P}/E$ obtained on the set of realizations of p. Furthermore, either the monster model interprets a non-classical group or $n \in \{1, 2, 3\}$ and

- If $n = 1$, then G is abelian and acts regularly on $\mathbf{P}/E$.
- If $n = 2$, the action of G on $\mathbf{P}/E$ is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebrically closed field K.
- If $n = 3$, the action of G on $\mathbf{P}/E$ is isomorphic to the action of $PGL_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.

1 Introduction

In the paper [7] Hyttinen, Lessmann and Shelah partially generalize a famous theorem of geometric stability theory by Hrushovski to some non-elementary frameworks, that is, to *homogeneous model theory* and to atomic ω-stable *excellent classes*. This showed that it is possible to do geometric stability theory without compactness and the stability-theoretic machinery developed for these non-elementary frameworks is adequate and exploitable. In this paper we want to investigate further this approach: how much of this work can be carried out without any trace of compactness in a form of homogeneity, but only with an appropriate independence calculus, which is available in *simple finitary AECs*. This is a natural question, since we should be able to work with only *geometric* tools. Furthermore, we want to prove the theorem in the context of *regular* types, when Hyttinen, Lessmann, Shelah [7] only work with *quasiminimal* types. Then our result is analogous to the first order result due to Hrushovski [1]. For the history of geometric stability theory and this particular problem, see the introduction of Hyttinen, Lessmann and Shelah [7]. Our main results is the following.

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Theorem 1.1 *Assume that* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *is a simple, superstable finitary AEC and let* $\mathfrak{M}$ *be the monster model for* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. *Assume that* A *is a finite set,* p *is an unbounded regular Lascar strong type over* A *and* $\mathbf{Q}$ *is an* A*-invariant subset of* $\mathfrak{M}$. *Assume that there exists an integer* $0 < n < \omega$ *such that*

1. *For any independent sequence* $(a_1, ..., a_n)$ *of realizations of* p *and any finite subset* C *of* $\mathbf{Q}$ *we have*
$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$
2. *For some independent sequence* $a_1, ..., a_{n+1}$ *of realizations of* p *there is* C *a finite subset of* $\mathbf{Q}$ *such that* $(a_1, ..., a_n)$ *dominates* $(a_1, ..., a_{n+1})$ *over* $A \cup C$.

Then $\mathfrak{M}$ *interprets a group* G *which acts on the geometry* $\mathbf{P}/E$ *induced on the set* $\mathbf{P}$ *of realizations of* p. *Furthermore, either* $\mathfrak{M}$ *interprets a nonclassical group or* $n \in \{1, 2, 3\}$ *and*

- *If* $n = 1$, *then* G *is abelian and acts regularly on* $\mathbf{P}/E$.
- *If* $n = 2$, *the action of* G *on* $\mathbf{P}/E$ *is isomorphic to the affine action of* $K^+ \rtimes K^*$ *on the algebrically closed field* K.
- *If* $n = 3$, *the action of* G *on* $\mathbf{P}/E$ *is isomorphic to the action of* $PGL_2(K)$ *on the projective line* $\mathbb{P}^1(K)$ *of the algebraically closed field* K.

We explain further the chosen framework. Abstract elementary classes are a standard framework for extending first order model theory. As explained, we want to work with regular types and we want to reduce the homogeneity assumptions in [7]. The price to pay is to assume that *simplicity, finite character* and the assumption about *domination* for finite sequences. At first we explain the choice of the framework of simple, finitary AECs. In these classes, the only homogeneity assumption is the amalgamation property. This is much less than for example in excellent classes, since we cannot conclude tameness. However, we assume *finite character* for the elementary substructure relation. It connects the Galois types of finite sequences of a model to the elementary submodel relation and gives many tools to manage types over finite and countable sets. This assumption holds for classes definable in $L_{\infty,\omega}$ and there is a fundamental connection between this property and definability, see Kueker [8]. Since geometry studies relations between finite sequences of elements, the finite character property guarantees that these relations do capture at least some properties of the class.

Another assumption is simplicity. In non-elementary classes, it is not guaranteed that even categoricity in all uncountable cardinals would imply the existence of a well-behaved notion of independence, see Hyttinen and Kesälä [4]. Since geometric stability theory studies the applications of an independence calculus, it seems reasonable to assume simplicity to guarantee that we have such a calculus. Notice also that a type p over A being regular guarantees that the type is also simple, i.e. for any set $B \supseteq A$ and a realizing p, the type of a over B is free from the empty set. The properties of types and the independence calculus needed are listed in section 2 and the rest of the paper relies only on these properties, not the details of the definition of the framework.

Our third assumption in need of further explaining is item 2. in the assumption of the main Theorem. Instead of just assuming that we find a finite subset C of $\mathbf{Q}$ such that $\dim(a_1, ..., a_{n+1}/C) = n$, we assume that C witnesses the n-subsequence $a_1, ..., a_n$ *dominating* the sequence $a_1, ..., a_{n+1}$. This stronger assumption is needed in order to analyse the structure of the interpreted group G, and we don't know how to prove the theorem without the assumption. To be precise, domination is needed to show that if $g \in \mathbf{G}$ is *generic* over X, then it is 'free' of X, see the proof of Proposition 5.11. When p is quasiminimal, we get domination from just $\dim(a_1, ..., a_{n+1}/C) = n$, hence this stronger form is not needed in [7]. In the first order theorem by Hrushovski [1], the sets **P** and **Q** are modified using tools available in $\mathfrak{M}^{eq}$ to get this and more. We have not yet developed the machinery of $\mathfrak{M}^{eq}$ for simple finitary AECs, although that could very likely be done. But now, as also in [7], we work in the original context.

2 The framework

A finitary abstract elementary class was introduced in Hyttinen, Kesälä [2], but there the definition was slightly less general than in the consequent papers Hyttinen, Kesälä [5], [4] and [6]. A finitary AEC is an abstract elementary class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ with a countable Löwenheim-Skolem number, amalgamation, joint embedding, arbitrarily large models and *finite character:*[1] For any two models $N, M \in \mathbb{K}$ with $N \subseteq M$, we have that

$$N \preccurlyeq_{\mathbb{K}} M \text{ iff}$$

$$\text{for every finite sequence } \bar{a} \in N \text{ there is a } \mathbb{K}\text{-embedding } f : N \to M \text{ fixing } \bar{a}.$$

We work inside $\mathfrak{M}$, which is the the κ-universal and κ-model homogeneous monster model of the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. We say that a subset $A \subset \mathfrak{M}$ is *bounded*, if $|A| < \kappa$. We assume that κ is sufficiently large.

We can define a notion of a weak type tp^{w} and Lascar splitting and then deduce a notion of independence $\downarrow$ with built- in extension as follows:

$$A \downarrow_B C$$

if for every finite sequence $\bar{a} \in A$ there is a finite set $E \subseteq B$ such that for every extension $D \supseteq B \cup C$ there is $\bar{a}'$ realizing the weak type $\mathrm{tp}^{\mathrm{w}}(\bar{a}/B \cup C)$ such that $\mathrm{tp}^{\mathrm{w}}(\bar{a}'/D)$ does not Lascar- split over E. Then we say that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is *simple*, if for every sequence and every *finite* set A,

$$\bar{a} \downarrow_A A.$$

In this paper work in the context of simple, finitary AECs $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ which are superstable in the following sense

Assumption 2.1 (Superstability) *The class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is weakly stable in some cardinal and there is no finite tuple $\bar{a}$ and an increasing sequence of finite sets A_i, $i < \omega$ such that*

- $\bar{a} \not\downarrow_{A_i} A_{i+1}$ *for each $i < \omega$ and*
- $\bigcup_{i<\omega} A_i$ *is a model.*

This notion of superstability is implied by $\aleph_0$-stability with respect to weak types (See Corollary 3.28 of [5]) and therefore also from categoricity in any uncountable cardinal. It also follows from a weaker form of categoricity, so called *a-categoricity* in a suitable cardinal, see [5]. Both implications use simplicity.

2.1 Lascar types and independence

We recall the notion of a *Lascar strong type* and *Lascar type.* Two finite tuples $\bar{a}$ and $\bar{b}$ have the same Lascar strong type over a bounded set C, written $\mathrm{Lstp}(\bar{a}/C) = \mathrm{Lstp}(\bar{b}/C)$ if $E(\bar{a}, \bar{b})$ holds for any C-invariant equivalence relation E with a bounded number of classes. An automorphism which preserves all Lascar strong types over A is called a *Strong* automorphism. The group of these automorphisms is denoted by $\mathrm{Saut}(\mathfrak{M}/A)$, it is a normal

[1]This formulation of finite character is due to Kueker [8].

subgroup of $\mathrm{Aut}(\mathfrak{M}/A)$ and we can show that $\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A)$ if and only if there is $f \in \mathrm{Saut}(\mathfrak{M}/A)$ mapping $\bar{a}$ to $\bar{b}$.

Two tuples $\bar{a}$ and $\bar{b}$ have the same Lascar type over C, written $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$ if they have the same Lascar strong type over every *finite* subset C_0 of C, or equivalently, we have strong automorphisms $f \in \mathrm{Saut}(\mathfrak{M}/C_0)$ mapping $\bar{a}$ to $\bar{b}$ for any finite subset $C_0 \subseteq C$. Clearly if C is finite, $\mathrm{Lstp}(\bar{a}/C)$ equals $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C)$. For details about Lascar types in finitary AECs, see Hyttinen and Kesälä [6].

The following theorem is proved in [6]. We list also a stronger form of superstability, which will be used in the paper, although it is a straightforward application of the properties local character and finite character. A similar list of properties is stated in [5], but with an additional assumption called the 'Tarski-Vaught property'. In [6] the authors notice that this assumption is not needed.

Theorem 2.2 *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is simple and superstable. Let A, B, C and D be bounded subsets of the Monster model. Then the relation $\downarrow$ has the following properties.*

1. ***Invariance:*** *If $A \downarrow_C B$ and f is an automorphism of the monster model, then $f(A) \downarrow_{f(C)} f(B)$.*
2. ***Monotonicity:*** *If $A \downarrow_B D$ and $B \subset C \subseteq D$ then $A \downarrow_C D$ and $A \downarrow_B C$.*
3. ***Transitivity:*** *Let $B \subseteq C \subseteq D$. If $A \downarrow_B C$ and $A \downarrow_C D$, then $A \downarrow_B D$.*
4. ***Symmetry:*** *$A \downarrow_C B$ if and only if $B \downarrow_C A$.*
5. ***Extension:*** *For any $\bar{a}$ and $C \subseteq B$ there is $\bar{b}$ such that $\mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C)$ and $\bar{b} \downarrow_C B$.*
6. ***Finite character:*** *$A \downarrow_C B$ if and only if $\bar{a} \downarrow_C \bar{b}$ for every finite $\bar{a} \in A$ and $\bar{b} \in B$.*
7. ***Local character:*** *For any finite $\bar{a}$ and any B there exists a finite $E \subseteq B$ such that $\bar{a} \downarrow_E B$.*
8. ***Reflexivity:*** *If the weak type $\mathrm{tp}^{\mathrm{w}}(\bar{a}/A)$ is not bounded, then $\bar{a} \not\downarrow_A \bar{a}$.*
9. ***Stationarity:*** *If $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$, $\bar{a} \downarrow_C B$ and $\bar{b} \downarrow_C B$, then $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/B) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/B)$.*
10. ***Superstability:*** *For any increasing sequence of finite sets A_i, $i < \omega$, and any finite sequence $\bar{a}$, there is $n < \omega$ with $\bar{a} \downarrow_{A_n} A_{n+1}$.*

We remark the following property given by superstability.

Lemma 2.3 *Let Q be some, possibly unbounded, set and let $\bar{a}$ be some finite tuple. There is a finite set $D \subseteq Q$ with*

$$\bar{a} \downarrow_D C$$

for any subset $C \subseteq Q$.

Proof: Assume there does not exist such D. We define an increasing sequence of finite sets $A_i \subseteq Q$, $i < \omega$ such that $\bar{a} \not\downarrow_{A_i} A_{i+1}$. This will contradict superstability. First, define $A_0 = \emptyset$. Assume we have defined A_i. However, the set A_i cannot be as required in lemma, and hence there is some $C \subseteq Q$ with $\bar{a} \not\downarrow_{A_i} C$. By finite character of $\downarrow$ we may assume C is finite, and hence take $A_{i+1} = C \cup A_i$. □

We also recall the following facts, which are proved in [6].

Fact 2.4 *The supremum for the number of Lascar strong types over any finite set is bounded.*

Fact 2.5 *Let* $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ *be simple and superstable.*

Let C *be a countable set and let* $\bar{a}$, $\bar{b}$ *be finite tuples such that* $\mathrm{Lstp}^{\mathrm{w}}(\bar{a}/C) = \mathrm{Lstp}^{\mathrm{w}}(\bar{b}/C)$. *Then there is* $f \in \mathrm{Aut}(\mathfrak{M}/C)$ *such that* $f(\bar{a}) = \bar{b}$.

Furthermore, if p_i, $i < \omega$, *are countably many Lascar types over subsets* $D_i \subseteq C$, *we can choose* f *such that* $f(p_i) = p_i$ *for all* $i < \omega$.

3 Regular types

For the rest of this paper let $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ be a simple, superstable, finitary AEC. From now on we will not use finite character or other details of the definition of the class $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. Essentially we need a class of structures with a monster model and a notion of independence as in section 2. We also need the notion of a Lascar strong type (or other notion of type) with a related notion of a Strong automorphism and the properties listed in section 2, especially we need stationarity and results comparable to Fact 2.4 and Fact 2.5.

We fix a finite set A.

We assume that p is some unbounded Lascar strong type over A. That is, the set

$$\mathbf{P} = \{a \in \mathfrak{M} : \mathrm{Lstp}(a/A) = p\}$$

is unbounded. As notation, we write a, b, c etc to denote realizations of p, that is, elements in $\mathbf{P}$. The notation $\bar{a}, \bar{b}, \bar{c}$ refers to finite sequences of realizations of p. We note that $\mathbf{P}$ in general is not invariant undel automorphisms fixing A poitwise. However if an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/A)$ maps *some* element $a \in \mathbf{P}$ to $\mathbf{P}$, then f fixes $\mathbf{P}$ setwise, since $\mathrm{Lstp}(b/A) = p$ implies $\mathrm{Lstp}(f(b)/A) = \mathrm{Lstp}(f(a)/A) = p$.

When C is a bounded subset of $\mathfrak{M}$ and p' is a type, we define the following operator on the realizations of p':

$$\mathrm{cl_C}(B) = \{a \models p' : \ a \not\downarrow_{A \cup C} B\}.$$

Furthermore, we assume that the type p is *regular*, that is, the closure operator $\mathrm{cl_A}(-) = \mathrm{cl}(-)$ defines a *pregeometry* on $\mathbf{P}$. Hence we assume:

Assumption 3.1 (Regularity) *For any subsets* $B \subseteq B' \subset \mathbf{P}$ *and elements* $a, b \in \mathbf{P}$

(i) $B \subseteq \mathrm{cl}(B) \subseteq \mathrm{cl}(B')$,

(ii) $\mathrm{cl}(\mathrm{cl}(B)) = \mathrm{cl}(B)$,

(iii) Exchange: *if* $a \in \mathrm{cl}(B \cup \{b\}) \setminus \mathrm{cl}(B)$, *then* $b \in \mathrm{cl}(B \cup \{a\})$,

(iv) Finite character: *if* $a \in \mathrm{cl}(B)$, *then* $a \in \mathrm{cl}(B_0)$ *for some finite subset* B_0 *of* B.

We prove that this definition of regularity is equivalent to the more traditional one based on *orthogonality.* This equivalence is proved exactly as the same result with forking in stable first order theories (see for example Pillay [9]), but we prove it as an exercise.

Lemma 3.2 *Let* p *be a Lascar strong type over a finite set* A. *The following are equivalent:*

1. $\mathrm{cl_A}(-)$ *defines a pregeometry on the realizations of* p.

2. $\mathrm{cl_C}(-)$ *defines a pregeometry on the realizations of* p' *for any set* C *containing* A, *where* p' *be the free extension of* p *to* C.

3. *Let C contain A and let p' be a free extension of p to C. For any b realizing p' such that $b \not\downarrow_C B$, the types p' and $\mathrm{Lstp^w}(b/B)$ are orthogonal, that is, for any $D \supseteq C \cup B$ and a, b' satisfying the free extensions of p' and $\mathrm{Lstp^w}(b/B)$ to D respectively, we have*

$$a \downarrow_D b'.$$

Proof: Clearly 2. implies 1. We show that 3. implies 2. First we show the following claim: Assume that B, D contain realizations of p' and $a \models p'$ such that $a \not\downarrow_C D$ and for all $d \in D$, $d \not\downarrow_C B$. Then $a \not\downarrow_C B$.

To prove the claim, we assume the contrary that $a \downarrow_C B$. By regularity of p' we get for each $d \in D$ that $a \downarrow_{C \cup B} d$ and by transitivity, $a \downarrow_C B \cup d$. We show by induction on n that this holds for every finite $d_0,,, d_n \subseteq D$: on the $n+1$th step, we use regularity and induction to show that $a \downarrow_{C \cup B \cup d_0,...,d_n} d_{n+1}$, and then get $a \downarrow_C B \cup d_0, ..., d_n, d_{n+1}$ by transitivity. Hence finite character of $\downarrow$ gives that $a \downarrow_C D$, a contradiction.

Now we use this claim to show that $\mathrm{cl_C}(\mathrm{cl_C}(B)) \subseteq \mathrm{cl_C}(B)$ with taking $D = \mathrm{cl_C}(B)$. If a realizes p' and a is in $\mathrm{cl_C}(\mathrm{cl_C}(B))$, we have that $a \not\downarrow_C \mathrm{cl_C}(B)$ and for all $d \in \mathrm{cl_C}(B)$, $d \not\downarrow_C B$. Hence $a \not\downarrow_C B$, by the previous claim, that is, $a \in \mathrm{cl_C}(B)$.

Then $(i), (iii)$ and (iv) follow from the properties of $\downarrow$. Since p' is unbounded, we get that $b \not\downarrow_C b$ for each b realizing p'. This and monotonicity imply (i). Item (iv) is given by finite character of $\downarrow$. To prove Exchange, let $a \downarrow_C B$ and $a \not\downarrow_C B \cup b$. By transitivity, $a \not\downarrow_{C \cup B} b$ and furthermore by symmetry, $b \not\downarrow_{C \cup B} a$. Monotonicity gives that $b \not\downarrow_C B \cup a$.

Then we show that 1. implies 3. First we prove the implication in the case where $C = A$ and for finite sets B and D. Let a, b realize extensions of p to $D \supseteq A \cup B$ such that $a \downarrow_A D$ and $b \not\downarrow_A B$. We want to show that

$$a \downarrow_D b.$$

We assume the contrary, that $a \not\downarrow_D b$. By Lemma 2.3 there is $\bar{e} \in \mathbf{P}$ such that

$$D \downarrow_{A \cup \bar{e}} B'$$

for any subset B' of $\mathbf{P}$.

We may assume that

$$\bar{e}, D \downarrow_A a:$$

by extension there is $\bar{e}'$ realizing $\mathrm{Lstp^w}(\bar{e}/A \cup D)$ such that $\bar{e}' \downarrow_{A \cup D} a$. Then by transitivity and symmetry $\bar{e}', D \downarrow_A a$. Furthermore, since $A \cup D$ is finite, there is $f \in \mathrm{Saut}(\mathfrak{M}/A \cup D)$ with $f(\bar{e}) = \bar{e}'$. Then since f fixes $\mathbf{P}$ setwise, we can take $\bar{e}'$ as $\bar{e}$.

Then since $\bar{b} \downarrow_{A \cup \bar{e}} D$ by symmetry, transitivity implies that

$$b \not\downarrow_A \bar{e}.$$

Furthermore, we claim that

$$a \not\downarrow_A \bar{e}, b.$$

If not, then $a \downarrow_{A \cup \bar{e}} b$. The definition of $\bar{e}$ implies that $D \downarrow_{A \cup, a} b$. Then by symmetry and transitivity, $a \cup D \downarrow_{A \cup \bar{e}} b$ and furthermore by monotonicity and transitivity, $a \downarrow_D b, \bar{e}$, which is a contradiction.

Hence we have that $a \in \mathrm{cl_A}(b, \bar{e})$ and $b \in \mathrm{cl_A}(\bar{e})$. Then by (i) and (ii) of the definition of a pregeometry,

$$a \in \mathrm{cl_A}(\bar{e}, b) \subseteq \mathrm{cl_A}(\mathrm{cl_A}(\bar{e})) = \mathrm{cl_A}(\bar{e}).$$

Hence $a \not\downarrow_A \bar{e}$, a contradiction.

Then finally we prove 3. for arbitrary C, B and D. Assume that p' is a free extension of p to C, let a, b realize p' and $D \supset C \cup B$ where $a \downarrow_C D$ and $b \not\downarrow_C B$. We want to show that $a \downarrow_D b$.

Since p' is a free extension of p we get by transitivity that $a \downarrow_A D$. By monotonicity $b \not\downarrow_A C \cup B$ and by finite character there is finite $B_0 \subset C \cup B$ containing A such that $b \not\downarrow_A B_0$. Then by the previous claim, for arbitrary finite $D_0 \subset D$ containing $A \cup B_0$, $a \downarrow_{D_0} b$. Furthermore by transitivity, $a \downarrow_A D_0 \cup b$. Since D_0 was arbitrary, finite character implies that $a \downarrow_D b$, and hence we have shown the claim. □

By the previous result, Assumption 3.1 implies that for any $C \subset \mathfrak{M}$ and p' a free extension of p to C, the operator $\mathrm{cl_C}(-)$ defines a pregeometry on the realizations of p'. Hence we can define a notion of dimension $dim(-/C)$ on the realizations of the free extension of p to $C \supseteq A$. There a sequence $a_1, ..., a_p$ is C-independent of a set B, if

$$a_i \downarrow_C B \cup \{a_0, ..., a_{i-1}, a_{i+1}, ..., a_p\} \text{ for each } i \in \{1, ..., p\}.$$

Equivalently,

$$a_i \downarrow_A C \cup B \cup \{a_0, ..., a_{i-1}, a_{i+1}, ..., a_p\} \text{ for each } i \in \{1, ..., p\}.$$

By independence calculus, it follows that

$$a_0, ..., a_p \downarrow_A C \cup B.$$

We write *independent* for A-independent.

Now we give our geometric assumption for the sets $\mathbf{P}$ and $\mathbf{Q}$, where p is regular in the sense of Assumption 3.1.

Assumption 3.3 *Assume that A is finite, $\mathbf{Q}$ is an A-invariant set and that p is a regular unbounded Lascar strong type over A. Let $n < \omega$. Assume that*

1. *For any independent sequence $(a_1, ..., a_n)$ of realizations of p and any finite subset C of $\mathbf{Q}$ we have*
$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$
2. *For some independent sequence $(a_1, ..., a_{n+1})$ of realizations of p there is a finite subset C of $\mathbf{Q}$ such that*
$$\dim(a_1, ..., a_{n+1}/A) > \dim(a_1, ..., a_{n+1}/A \cup C).$$

We should interpret item 1. so that for any element a realizing p and any (finite) set $C \subseteq \mathbf{Q}$, $a \downarrow_A C$. Hence that gives that the dimension $dim(-/A \cup C)$ is well-defined on $\mathbf{P}$. We note that item 2. of the assumption actually implies that the set $\mathbf{Q}$ is unbounded. One property of our independence relation is that if $\mathrm{tp^w}(c/A)$ is bounded, then $c \downarrow_A B$ for any subset B of the monster model.

Furthermore, we make $\mathbf{P}$ into a geometry $\mathbf{P}/E$ by considering the A-invariant equivalence relation

$$E(x, y), \text{ defined by } \mathrm{cl_A}(x) = \mathrm{cl_A}(y).$$

Then $\mathbf{P}/E$ is a geometry with universe consisting of elements $\mathrm{cl_A}(x)$, $x \in \mathbf{P}$. We use the notation cl_A also for the canonical closure operator on $\mathbf{P}/E$, that is

$$cl_A(\{\mathrm{cl_A}(x) : x \in X\}) = \{\mathrm{cl_A}(y) : y \in \mathrm{cl_A}(X)\} = \{\mathrm{cl_A}(y) : y \models p \text{ and } y \not\downarrow_A X\}.$$

Any sequence $a_1, ..., a_k \in \mathbf{P}$ is independent of $X \subset \mathbf{P}$ if and only if $\mathrm{cl_A}(a_1), ..., \mathrm{cl_A}(a_k)$ in $\mathbf{P}/E$ is independent of $\{\mathrm{cl_A}(x) : x \in X\}$.

Since p is unbounded both $\mathbf{P}$ and $\mathbf{P}/E$ have infinite dimension. Also by simplicity, $\mathrm{cl_A}(\emptyset) = \emptyset$ in $\mathbf{P}$.

4 The group G of permutations of $\mathbf{P}/E$

Let E be the equivalence relation on $\mathbf{P}$ with

$$E(x,y) \text{ iff } \mathrm{cl}_{\mathrm{A}}(x) = \mathrm{cl}_{\mathrm{A}}(y).$$

We define $\mathbf{G}$, the group of permutations of $\mathbf{P}/E$ as follows.

Definition 4.1 *Let $\mathbf{G}$ be the the group of permutations g of $\mathbf{P}/E$ such that for each countable $C \subset \mathbf{Q}$ and finite $X \subset \mathbf{P}$ there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ fixing $\mathbf{P}$ setwise such that $\sigma(a)/E = g(a/E)$ for each $a \in X$.*

Then we will show that this group n-*acts* on P/E. We define:

Definition 4.2 *An action of G on a pregeometry P is an n-action if*

1. *The action has rank n: Whenever the tuples $\bar{x}$ and $\bar{y}$ are two n-tuples of elements of P such that $\dim(\bar{x}\bar{y}) = 2n$, then there is $g \in G$ such that $g(\bar{x}) = \bar{y}$. However, for some $(n+1)$-tuples $\bar{x}$, $\bar{y}$ with $\dim(\bar{x}\bar{y}) = 2n+2$, there is no $g \in G$ such that $g(\bar{x}) = \bar{y}$.*
2. *The action is $(n+1)$-determined: Whenever the action of $g, h \in G$ agree on a $(n+1)$-dimensional subset X of P, then $g = h$.*

4.1 Interpreting an n-action

First we use Fact 2.5 to show that our action has rank n.

Lemma 4.3 *Let $a_1, ..., a_p$ be a finite sequence in $\mathbf{P}$ and $C \subset \mathbf{Q}$ with $\dim(a_1, ..., a_p/A \cup C) = p$. Then for any $k \le p$ and $i_1 < ... < i_k, j_1 < ... < j_k \in \{1, ..., p\}$ we have that*

$$\mathrm{Lstp}^{\mathrm{w}}(a_{i_1}, ..., a_{i_k}/A \cup C) = \mathrm{Lstp}^{\mathrm{w}}(a_{j_1}, ..., a_{j_k}/A \cup C).$$

Furthermore, if C is countable, for a given countable collection $\mathscr{S}$ of types over subsets of $A \cup C$ there is an automorphism $f \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving $\mathscr{S}$ and mapping $a_{i_1}, ..., a_{i_k}$ to $a_{j_1}, ..., a_{j_k}$.

Proof: By Fact 2.5 it is enough to prove the first claim. Furthermore, we may assume that $j_1, ..., j_k = 1, ..., k$. We prove the claim by induction on k. If $k = 1$, we get the claim by stationarity of weak Lascar strong types, since for each $i \in \{1, ..., p\}$, $a_i \models p$ and $a_i \downarrow_A C$ by Assumption 3.3. Assume we have shown the claim for k.

To prove the claim for $k+1$, let $C_0 \subseteq C$ be finite. By induction,

$$\mathrm{Lstp}(a_{i_1}, ..., a_{i_k}/A \cup C_0) = \mathrm{Lstp}(a_1, ..., a_k/A \cup C_0).$$

Hence there is a strong automorphism $f \in \mathrm{Saut}(\mathfrak{M}/A \cup C_0)$ mapping $a_{i_0}, ..., a_{i_k}$ to $a_1, ..., a_k$. Using the fact that $\dim(a_1, ..., a_p/A \cup C) = p$ and invariance, we get that

$$a_{k+1} \downarrow_A C_0 \cup a_1, ..., a_k \text{ and}$$

$$f(a_{i_{k+1}}) \downarrow_A C_0 \cup a_1, ..., a_k.$$

Since both $f(a_{i_{k+1}})$ and a_{k+1} realize p, we can use stationarity to conclude that

$$\mathrm{Lstp}(f(a_{i_{k+1}})/A \cup C_0 \cup a_1, ..., a_k) = \mathrm{Lstp}(a_{k+1}/A \cup C_0 \cup a_1, ..., a_k).$$

Furthermore, we get that

$$\mathrm{Lstp}(a_{i_1}, ..., a_{i_{k+1}}/A \cup C_0) = \mathrm{Lstp}(a_1, ..., a_k, f(a_{i_{k+1}})/A \cup C_0)\, \mathrm{Lstp}(a_1, ..., a_{k+1}/A \cup C_0).$$

Since the same holds for all finite $C_0 \subseteq C$, the claim follows. □

Lemma 4.4 *Let $a_1, ..., a_n \in \mathbf{P}$ and $b_1, ..., b_n \in \mathbf{P}$ be two independent sequences and let $C \subset \mathbf{Q}$ be countable and let $\mathscr{S}$ be a countable collection of types over subsets of $A \cup C$. Then there exists $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving $\mathscr{S}$ and mapping a_i to b_i for each $i \in \{1, ..., n\}$.*

Futhermore, if C is finite, we can take $f \in \mathrm{Saut}(\mathfrak{M}/A \cup C)$.

Proof: By Assumption 3.3, we have that

$$\dim(a_1, ..., a_n/A \cup C_0) = \dim(b_1, ..., b_n/A \cup C_0) = n$$

for any finite subset C_0 of C. Hence by finite character, the sequences $a_1, ..., a_n$ and $b_1, ..., b_n$ are independent over C. By using a third sequence if necessary, we may assume that

$$\dim(a_1, ..., a_n, b_1, ..., b_n/A \cup C) = 2n.$$

The previous Lemma implies the claim. □

As in [7], we define a notion of a *good pair* in order to show $n+1$-determinacy. However, since we have neither $\aleph_0$-stability or strong minimality, we have to define a different notion.

Definition 4.5 (Good pair) *We say that (X, C) is a good pair, if $X \subset \mathbf{P}$ is countable and infinite-dimensional and $C \subset \mathbf{Q}$ is countable and the following holds:*

For any $n+1$-tuple $\bar{a} \in X$ there is Morley-sequence $(C_i)_{i<\omega} \subseteq C$ of finite sets witnessing the dimension of $\bar{a}$ over $\mathbf{Q}$, that is

1. *Each $C_i \subset \mathbf{Q}$ is finite,*
2. $\mathrm{Lstp}(C_i/A \cup \bar{a}) = \mathrm{Lstp}(C_0/A \cup \bar{a})$,
3. $C_i \downarrow_{A \cup \bar{a}} \bigcup_{j<i} C_j$ *and*
4. $\dim(\bar{a}/A \cup C_i) = n$ *for each $i < \omega$*

Clearly by Assumption 3.3 and simplicity, for any countable $X' \subset \mathbf{P}$ there is a good pair (X, C) such that X contains X'.

Lemma 4.6 *Let (X, C) be a good pair. Suppose that $(a_1, ..., a_{n+1}) \subseteq X$ are independent and $\sigma(a_i)/E = a_i$, for $i = 1, ..., n+1$, for some $\sigma \in \mathrm{Aut}(\mathbf{P}/A \cup C)$. Then $\sigma(c/E) = c/E$ for any $c \in X$.*

Proof: We first prove the lemma for $c \in X$ with $c \downarrow_A a_1, ..., a_{n+1}$. First we claim that

$$\sigma(c) \not\downarrow_{\{a_1,...,a_{n+1}\} \setminus \{a_i\}} c, \text{ for each } i = 1, ..., n+1.$$

We only prove that

$$\sigma(c) \not\downarrow_{\{a_1,...,a_n\}} c.$$

Assume, for a contradiction, that this fails. Now $c, a_1, ..., a_n$ is an independent $n+1$-tuple in X, and hence by the definition of a good pair there is a Morley-sequence $(C_i)_{i<\omega} \subseteq C$ witnessing the dimension of $c, a_1, ..., a_n$ in $\mathbf{Q}$.

By extension, there is $e \in \mathbf{P}$ realizing $\mathrm{Lstp}(\sigma(c)/a_1, ..., a_n, c \cup A)$ such that

$$e \downarrow_{a_1,...,a_n \cup A} c \bigcup_{i<\omega} C_i.$$

We claim that there is finite $C' \subseteq C$ such that

$$\mathrm{Lstp}(\sigma(c), C'/c, a_1, ..., a_n \cup A) = \mathrm{Lstp}(e, C_0/c, a_1, ..., a_n \cup A).$$

To prove the claim, we first show that there is $p < \omega$ such that

$$\sigma(c) \downarrow_{a_1,...,a_n,c\cup A} C_p.$$

By superstability, there is some $i < \omega$ such that $\sigma(c) \downarrow_{a_1,...,a_n,c\cup A\cup C_i} C_{i+1}$. Then using symmetry, the fact that $C_{i+1} \downarrow_{a_1,...,a_n,c\cup A} C_i$ and transitivity, the get that

$$C_{i+1} \downarrow_{a_1,...,a_n,c\cup A} C_i \cup \sigma(c).$$

Hence we can choose C_{i+1} as C_p by monotonicity and symmetry. Now we can also take C_p as C', since by symmetry and stationarity of Lascar strong types,

$$\mathrm{Lstp}(\sigma(c), C_p/a_1, ..., a_n, c, A) = \mathrm{Lstp}(e, C_p/a_1, ..., a_n, c, A) = \mathrm{Lstp}(e, C_0/a_1, ..., a_n, c, A).$$

Now let $f \in \mathrm{Saut}(\mathfrak{M}/A \cup a_1, ..., a_n, c)$ map (e, C_0) to $(\sigma(c), C')$. By invariance,

$$\dim(c, a_1, ..., a_n/A \cup C') = n.$$

Since $\dim(a_1, ..., a_n/A \cup C') = n$ by Assumption 3.3, we must have that

$$c \in \mathrm{cl}_{A\cup C'}(a_1, ..., a_n). \tag{4.1}$$

Furthermore, $e \downarrow_{a_1,...,a_n,\cup A} c \cup C_0$ implies

$$\sigma(c) \downarrow_{a_1,...,a_n\cup A} c \cup C'.$$

Furthermore, $\dim(c, a_1, ..., a_n/A) = n+1$ implies $\sigma(c) \downarrow_A a_1, ..., a_n$, and hence by transitivity,

$$\sigma(c) \downarrow_A (a_1, ..., a_n \cup C').$$

This is, $\sigma(c) \notin \mathrm{cl}_{A\cup C'}(a_1, ..., a_n)$. But we have that σ fixes each a_i/E and hence $\mathrm{cl}_{A\cup C'}(a_1, ..., a_n) = \mathrm{cl}_{A\cup C'}(\sigma(a_1), ..., \sigma(a_n))$, giving

$$\sigma(c) \notin \mathrm{cl}_{A\cup C'}(\sigma(a_1), ..., \sigma(a_n)).$$

But then 4.1 implies that

$$\sigma(c) \in \mathrm{cl}_{A\cup C'}(\sigma(a_1), ..., \sigma(a_n)),$$

a contradiction.

Then we show that $\sigma(c/E) = c/E$. Again we assume the contrary, that

$$c \downarrow_A \sigma(c).$$

The previous claim and symmetry give that $c \in \mathrm{cl_A}(\sigma(c), a_1, ..., a_n)$. By exchange, there is $i \in \{1, ..., n\}$ such that

$$a_i \in \mathrm{cl_A}(c \cup \sigma(c) \cup \{a_1, ..., a_n\} \setminus \{a_i\}).$$

By the previous claim, $\sigma(c) \in \mathrm{cl_A}(c \cup \{a_1, ..., a_{n+1}\} \setminus \{a_i\})$ and we get that

$$\dim(c, \sigma(c), a_1, ..., a_{n+1}/A) = n + 1.$$

But we assumed $c, \sigma(c) \notin \mathrm{cl_A}(a_1, ..., a_{n+1})$, a contradiction.

We still need to prove the lemma for $c \in X$ with $c \not\downarrow_A a_1, ..., a_n$. For this, let $b_1, ..., b_n \in X$ be independent of $(c, a_1, ..., a_n)$. These can be found in X since X is infinite-dimensional. By the first case, we must have that

$$\sigma(b_i/E) = b_i/E \text{ for each } i = 1, ..., n.$$

Now $c \not\downarrow_A b_1, ..., b_n$ and we get $\sigma(c/E) = c/E$ by the first case. □

We deduce the next proposition.

Proposition 4.7 *Let* $a_1, ..., a_{n+1} \in \mathbf{P}$ *be independent. Let* $c \in \mathbf{P}$. *The exists a countable* $C_c \subset \mathbf{Q}$ *such that if* $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_c)$ *fix* $\mathbf{P}$ *setwise and*

$$\sigma(a_i)/E = \tau(a_i)/E, \text{ for each } i = 1, ..., n+1,$$

then $\sigma(c)/E = \tau(c)/E$.

Proof: Let (X, C) be a good pair with X containing $a_1, ..., a_{n+1}, b_1, ..., b_{n+1}, c$. We let C_c be C. Then, for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_c)$ fixing $\mathbf{P}$ setwise with $\sigma(a_i)/E = \tau(a_i)/E$, for each $i = 1, ..., n$, we have that $\tau^{-1} \circ \sigma(a_i)/E = a_i/E$ for each $i = 1, ..., n+1$. Hence by the previous lemma, we have that $\tau^{-1} \circ \sigma(c)/E = c/E$. This implies that $\sigma(c)/E = \tau(c)$. □

Proposition 4.8 *The action of* $\mathbf{G}$ *on* $\mathbf{P}/E$ *is an* n*-action.*

Proof: The $(n+1)$-determinacy of the action of $\mathbf{G}$ on $\mathbf{P}$ follows from the previous proposition. Now we have to show that the action has rank n.

First we prove the following claim: Assume that $\bar{a} = a_1, ..., a_n$ and $\bar{b} = b_1, ..., b_n$ are two independent sequences and let $c \downarrow_A \bar{a}\bar{b}$. Then there is $d \in \mathbf{P}$ such that for each countable $C \subset \mathbf{Q}$ there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C)$ preserving p with $\sigma(c) = d$ and $\sigma(a_i) = b_i$ for each $i = 1, ..., n$.

By Lemma 2.3 there is a finite set $D \subseteq A \cup \mathbf{Q}$ such that $\bar{a}, c \downarrow_D C$ for any set $C \subseteq A \cup \mathbf{Q}$. By Lemma 4.4 there is a strong automorphism $f \in \mathrm{Saut}(\mathfrak{M}/A \cup D)$ such that $f(\bar{a}) = \bar{b}$. We take $d = f(c)$ and claim this is a s required. Let $C \subset \mathbf{Q}$ be countable. By the choice of D, we have that $\bar{a}, c \downarrow_{A \cup D} C$, and since $f^{-1}(C) \subseteq A \cup \mathbf{Q}$, also $\bar{a}, c \downarrow_{A \cup D} f^{-1}(C)$. Invariance gives that $\bar{b}, d \downarrow_{A \cup D} C$. Now the claim follows by stationarity of weak Lascar strong types and Fact 2.5.

We can now show the action has rank n. Assume that $\bar{a}$ and $\bar{b}$ are independent n-tuples of realizations of p. We must find $g \in \mathbf{G}$ such that $g(\bar{a}/E) = \bar{b}/E$. Let c be in $\mathbf{P}$ be such that $c \downarrow_A \bar{a}\bar{b}$ and choose d as in the previous claim. We now define the following function $g : \mathbf{P}/E \to \mathbf{P}/E$. For each $e \in \mathbf{P}$, choose C_e as in Proposition 4.7, i.e. for any $\sigma, \tau \in \mathrm{Aut}(\mathfrak{M}/A \cup C_e)$ fixing $\mathbf{P}$ setwise such that $\sigma(\bar{a}/E) = \bar{b}/E = \tau(\bar{a}/E)$ and $\sigma(c)/E = d/E = \tau(c)/E$, we have $\sigma(e)/E = \tau(e)/E$.

By the choice of d, there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C_e)$ preserving p and sending the $n+1$-tuple $(\bar{a}, c)$ to the $n+1$-tuple $(\bar{b}, d)$. Define

$$g(e/E) = \sigma(e)/E.$$

The choice of C_e guarantees that it is well-defined.

We can also see that g is a permutation of P/E: We see that $g(e)$ does not depend on the choice of the set C_e. Let C_e and D_e be given by Proposition 4.7, and $\tau \in \mathrm{Aut}(\mathfrak{M}/C_e)$ and $\tau' \in \mathrm{Aut}(\mathfrak{M}/D_e)$ are as in the definition of g. Again by the choice of d there is $\sigma \in \mathrm{Aut}(\mathfrak{M}/C_e \cup D_e)$ mapping $(\bar{a}, c)$ to $(\bar{b}, d)$. Then by the choice of C_e and D_e, $\tau'(e)/E = \sigma(e)/E = \tau(e)/E$. Furthermore, studying the argument in Proposition 4.7, if $\tau \in \mathrm{Aut}(\mathfrak{M}/C_e)$ maps $(\bar{a}, c)$ to $(\bar{b}, d)$, we can choose C_e as $C_{\tau(e)}$. Then we see that $g \circ g(e/E) = e/E$ for e outside the E-classes of $(\bar{a}, c, \bar{b}, d)$ and hence g is bijective.

Further, suppose countable $C \subset \mathbf{Q}$ and finite $X \subset \mathbf{P}$ are given. By the choice of d, there is

$$\sigma \in \mathrm{Aut}(\mathfrak{M}/A \cup C \cup \bigcup_{e \in X} C_e)$$

preserving p and sending $(\bar{a}, c)$ to $(\bar{b}, d)$. By definition, we have $\sigma(e)/E = g(e/E)$. This implies that $g \in \mathbf{G}$. Since this fails for independent $n+1$-tuples by Assumption 3.3, the action of $\mathbf{G}$ on $\mathbf{P}$ has rank n. □

Definition 4.9 *A group* $(G, \cdot)$ *is* interpretable *in* $\mathfrak{M}$ *if there is a (bounded) subset* $B \subseteq \mathfrak{M}$ *and an unbounded set* $U \subseteq \mathfrak{M}^k$ *(for some* $k < \omega$*), an equivalence relation* E *on* U*, and a binary relation* $*$ *on* U/E *which are* B*-invariant and such that* $(G, \cdot)$ *is isomorphic to* $(U/E, *)$*.*

As in Hyttinen, Lessmann and Shelah [7], we can now prove:

Proposition 4.10 *The group* $\mathbf{G}$ *is interpretable in* $\mathfrak{M}$ *(over a finite set).*

Proof: This follows from the $(n+1)$-determinacy of the group action. Fix $\bar{a}$ an independent $(n+1)$-tuple of elements of P/E. Let $B = A \cup \bar{a}$.

We let $U/E \subseteq P^{(n+1)}/E$ consist of those $b \in P^{(n+1)}/E$ such that $g(\bar{a}) = \bar{b}$ for some $g \in \mathbf{G}$.

We show that U/E is B-invariant: Let $\tau \in \mathrm{Aut}(\mathfrak{M}/B)$. Since τ fixes $A \cup \bar{a}$ pointwise, it fixes $\mathbf{P}/E$ setwise. Also τ induces an automorphism of $\mathbf{G}$, where $\tau(g) \in G$ maps $\bar{a}$ to $\tau(\bar{b})$.

We now define $\bar{b}_1 * \bar{b}_2 = \bar{b}_3$ on U/E, if whenever $g_l \in \mathbf{G}$ such that $g_l(\bar{a}) = \bar{b}_l$, then $g_1 \circ g_2 = g_3$. This is well-defined by $(n+1)$-determinacy and the definition of U/E. Furthermore, the binary function $*$ is B-invariant. Also $(n+1)$-determinacy implies that the map $g \mapsto g(\bar{a})$ defines an isomorphism between $(\mathbf{G}, \circ)$ and $(U/E, *)$. □

5 Stationarity and unique generics

Following Hyttinen, Lessmann and Shelah [7], we choose a group Σ of automorphisms of the group action and show that the group $(\mathbf{G}, \circ)$ (Σ, n)-acts on a pregeometry (P, cl). That is, the group $\mathbf{G}$ n-acts on the universe P of the pregeometry in a way which respects the closure operator and which is ω-homogeneous with respect to Σ: for any finite $X \subseteq \mathbf{P}/E$ and $x, y \notin \mathrm{cl_A}(X)$ there is $\tau \in \Sigma$ fixing X poitwise and mapping x to y. Although [7] studies an arbitrary infinite-dimensional pregeometry (P, cl) with $\mathrm{cl}(\emptyset) = \emptyset$, we will only study the geometry $(\mathbf{P}/E, \mathrm{cl_A})$.

Let $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ be strong automorphism. Then τ induces an automorphism τ' of the group action as follows: τ' maps the equivalence class a/E in $\mathbf{P}/E$ to the class $\tau(a)/E = \tau(a/E)$ and for $g \in \mathbf{G}$, $\tau'(g)(a/E) = \tau(g(\tau^{-1}(a/E)))$. It is easy to verify that

$$\tau' : \mathbf{G} \to \mathbf{G}$$

is an automorphism of $\mathbf{G}$ and preserves the action.

We let Σ to be the group of automorphisms of the action induced by Strong automorphisms of the Monster model over the finite set A:

$$\Sigma = \{\tau' : \tau \in \mathrm{Saut}(\mathfrak{M}/A)\}.$$

We denote by Σ_X the subgroup consisting of those $\tau \in \Sigma$ which fix $X \subset \mathbf{P}/E$ pointwise.

Then we remark that the n-action defined in the previous section is ω-homogeneous with respect to this Σ.

Lemma 5.1 *If* $X \subseteq \mathbf{P}/E$ *is finite and* $x, y \in \mathbf{P}/E$ *are outside* $\mathrm{cl_A}(X)$*, then there is an strong automorphism* $\tau \in \Sigma$ *of the group action sending* x *to* y *which is the identity on* X*.*

Proof: Choose a, b elements and $\bar{d}$ a finite subset of $\mathbf{P}$ such that $x = a/E$, $y = b/E$ and $X = \bar{d}/E$. That is, $a, b, \bar{d}$ are chosen as representatives of the E-classes of x, y, X. Then $a \downarrow_A \bar{d}$ and $b \downarrow_A \bar{d}$. By stationarity, we have that $\mathrm{Lstp^w}(a/A \cup \bar{d}) = \mathrm{Lstp^w}(b/A \cup \bar{d})$. Since

$A \cup \bar{d} \subseteq \mathfrak{M}$ is finite, we get a strong automorphism $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{d})$ mapping a to b. Then, since τ preserves all E-classes, τ maps x to y and maps each element of $X \subseteq \mathbf{P}/E$ to itself. We can take $\tau' \in \Sigma$ to be the automorphism of the group action induced by τ. □

Definition 5.2 *We say that $g \in \mathbf{G}$ is* generic *over $X \subseteq \mathbf{P}/E$, if there exists an independent n-tuple $\bar{x}$ of $\mathbf{P}$ such that*

$$\dim(\bar{x}g(\bar{x})/X) = 2n.$$

Since $\mathbf{P}/E$ has infinite dimension and the action has rank n, for a given finite set $X \subset \mathbf{P}/E$, there is $g \in \mathbf{G}$ generic over X.

For $\tau \in \Sigma_X$, g is generic over X if and only if $\tau(g)$ is generic over X. Hence we can talk about *generic types* over X, which are orbits of generic elements $g \in \mathbf{G}$ under automorphisms in Σ_X, written $\mathrm{tp}(g/X)$.

Remark 5.3 *For any independent $n+1$-tuple $\bar{x}$ in $\mathbf{P}/E$ and any $g \in \mathbf{G}$, always $\dim(\bar{x}g(\bar{x})/A) \leq 2n+1$.*

Proof: Assume to the contrary, that $\dim(\bar{x}g(\bar{x})/A) = 2n+2$. Since the action has rank n, there are some $n+1$-tuples $\bar{x}'$ and $\bar{y}'$ with $\dim(\bar{x}'\bar{y}'/A) = 2n+2$ such that there do not exist $h \in \mathbf{G}$ with $h(\bar{x}') = \bar{y}'$. Since $\bar{x}g(\bar{x})$ and $\bar{x}'\bar{y}'$ are two independent tuples of the same length, by ω-homogeneity there is $\sigma \in \Sigma$ mapping $\bar{x}g(\bar{x})$ to $\bar{x}'\bar{y}'$. Then $\sigma'(g) \in \mathbf{G}$ and

$$\sigma'(g)(\bar{x}') = \sigma'(g)(\sigma(\bar{x})) = \sigma(g(\bar{x})) = \bar{y}',$$

a contradiction. □

We can now define stationarity of $\mathbf{G}$ with respect to Σ. Notice that the extra condition on the number of types follows from Fact 2.4.

Definition 5.4 *We say that $\mathbf{G}$ is* stationary *if whenever $g, h \in \mathbf{G}$ with $\mathrm{tp}(g/\emptyset) = \mathrm{tp}(h/\emptyset)$ and $X \subset \mathbf{P}/E$ is finite and both g and h are generic over X, then $\mathrm{tp}(g/X) = \mathrm{tp}(h/Y)$. Furthermore, we assume that the number of types over each finite set is bounded.*

The following is a strengthening of stationarity.

Definition 5.5 *We say that a subgroup G of $\mathbf{G}$ has* unique generics *if for all finite $X \subset \mathbf{P}/E$ and $g, h \in G$ generic over X we have $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.*

In [7] the following fact is proved for any group $(G, \cdot)$ (Σ, n)-acting an an infinite-dimensional pregeometry (P, cl) as Proposition 2.8. The proof also refers to Lemma 3.2 of Hyttinen [3].

Fact 5.6 *The* connected component *$\mathbf{G}^0$ is the intersection of all invariant, normal subgroups with bounded index.*

If $\mathbf{G}$ is stationary, then $\mathbf{G}^0$ is a normal invariant subgroup of $\mathbf{G}$ of bounded index and $\mathbf{G}^0$ (Σ^0, n)-acts on the pregeometry $(\mathbf{P}/E, \mathrm{cl}_A)$ by restriction, where Σ^0 is obtained from Σ by restriction to $\mathbf{G}^0$. Also the stationarity of $\mathbf{G}$ implies that $\mathbf{G}^0$ has unique generics.

In [7], stationarity of Lascar strong types is used to show stationarity for $\mathbf{G}$. The proof also uses quasiminimality of p. For regular types we can do something similar, but we need the additional assumption 5.7. This assumption is analogous to a condition holding in Hrushovski [1], where $\mathbf{P}$ and $\mathbf{Q}$ are slightly modified using the techniques available with $\mathfrak{M}^{eq}$. This assumption is a strengthening of Assumption 3.3(2).

Assumption 5.7 *For some independent sequence $a_1, ..., a_{n+1}$ of realizations of p there is finite $C \subset \mathbf{Q}$ such that $(a_1, ..., a_n)$ dominates $(a_1, ..., a_{n+1})$ over $A \cup C$, written*

$$(a_1, ..., a_n)\mathrm{D}\ _{CA}(a_1, ..., a_{n+1}).$$

That is, whenever $\bar{d}$ is some finite tuple in the monster model, $\bar{d} \downarrow_{A\cup C} a_1, ..., a_n$ implies $\bar{d} \downarrow_{A\cup C} a_1, ..., a_{n+1}$.

We remark that equivalently the same holds for *all* independent sequences $a_1, ..., a_{n+1}$ of realizations of p.

We also have be careful when we want to apply results about Lascar strong types in $\mathbf{P}$ to $\mathbf{P}/E$, since for an element $a \in \mathbf{P}$, the closure $\mathrm{cl_A}(a)$ can be unbounded. For a generic element $g \in \mathbf{G}$, we introduce a concept of *generic witnesses* in $\mathbf{P}$. Especially, we use Assumption 5.7 to get *domination*.

Definition 5.8 *Assume that $g \in \mathbf{G}$ is generic over finite $X \subset \mathbf{P}/E$, where $\bar{d} \in \mathbf{P}$ such that $X = \bar{d}/E$. We say that two $(n+1)$-tuples $\bar{a} = a_1, ..., a_{n+1}$ and $\bar{b} = b_1, ..., b_{n+1}$ are* generic witnesses *for g over $\bar{d}$, if*

1. $g(\bar{a}/E) = \bar{b}/E$ *and*
2. $\dim(\bar{a}, b_1, ..., b_n/A \cup \bar{d}) = 2n+1$.
3. *There are $n+1$-tuples $\bar{a}'$, $\bar{b}'$ such that $g(\bar{a}'/E) = \bar{b}'/E$, $\dim(\bar{a}', \bar{b}'/A \cup \bar{d}) = 2n+1$ and $\bar{a} \downarrow_A \bar{a}'\bar{b}'\bar{d}$.*
4. **Domination:** *The $2n+1$-tuple $a_1, ..., a_{n+1}b_1, ..., b_n$ dominates $\bar{a}\bar{b}$ over A.*

Note that if $\bar{a}$, $\bar{b}$ are generic witnesses for g over $\bar{d}$ and $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$, then $\tau(\bar{a})$ and $\tau(\bar{b})$ are generic witnesses for $\tau'(g)$ over $\tau(\bar{d})$.

Lemma 5.9 *Let $g \in \mathbf{G}$ be generic over finite $X \subseteq \mathbf{P}/E$, where $\bar{d} \in \mathbf{P}$ such that $X = \bar{d}/E$. There are $\bar{a}$ and $\bar{b}$ such that they are generic witnesses for g over $\bar{d}$.*

Proof: By the definition of genericity, there are $n+1$-tuples $\bar{a}'$ and $\bar{b}' = b'_1, ..., b'_{n+1}$ such that $g(\bar{a}'/E) = \bar{b}'/E$ and $dim(\bar{a}', b'_1, ..., b_n/A \cup \bar{d}) = 2n+1$.

By extension, there is $\bar{a}$ realizing $\mathrm{Lstp}(\bar{a}'/A \cup \bar{d})$ such that

$$\bar{a} \downarrow_{A\cup\bar{d}} \bar{a}'\bar{b}'.$$

Then by transitivity, also $\bar{a} \downarrow_A \bar{a}'\bar{b}'\bar{d}$. Furthermore, we get that $dim(\bar{a}, \bar{a}', b'_1, ..., b'_n/A \cup \bar{d}) = 3n+2$.

By Assumption 5.7, there is a finite set $C' \subset \mathbf{Q}$ such that

$$a_1, ..., a_n\mathrm{D}_{A\cup C'}a_1, ..., a_{n+1}.$$

Futhermore by extension there are finite sets C_i realizing $\mathrm{Lstp}(C'/A \cup \bar{a})$ such that

$$C_i \downarrow_{A\cup\bar{a}} \bigcup_{j<i} C_j \cup \bar{d}.$$

Then we choose $\bar{b}$ such that $g(\bar{a}/E) = \bar{b}/E$ and there exists $\tau \in \mathrm{Aut}(\mathfrak{M}/A \cup \bigcup_{i<\omega} C_i)$ mapping $\bar{a}$ to $\bar{b}$. This is possible by the definition of $\mathbf{G}$. Then for each $i < \omega$, $b_1, ..., b_n$ dominates $\bar{b}$ over $A \cup C_i$.

Also we must have that $\dim(\bar{a}, b_1, ..., b_n/A\cup\bar{d}) = 2n+1$. This holds, since by Remark 5.3, for each $i \in 1, ..., n+1$, $b_i \in \mathrm{cl_A}(a_i, a'_1, ..., a'_n, b'_1, ..., b'_n)$ and $b'_i \in \mathrm{cl_A}(a'_i, a_1, ..., a_n, b_1, ..., b_n)$ and hence

$$3n+2 = \dim(\bar{a}, \bar{a}', b'_1, ..., b'_n/A \cup \bar{d}) = \dim(\bar{a}, \bar{a}', b'_1, ..., b'_n, b_1, ..., b_n/A \cup \bar{d})$$

$$= \dim(\bar{a}, \bar{a}', b_1, ..., b_n/A \cup \bar{d}).$$

Now it is left to show the domination. For this, let $\bar{d}'$ be arbitrary such that

$$\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n.$$

We need to show that $\bar{d}' \downarrow_A \bar{a}\bar{b}$.

Since $\dim(\bar{a}, b_1, ..., b_n/A) = 2n + 1$ implies that $\bar{a} \downarrow_A b_1, ..., b_n$, we get by transitivity that

$$\bar{a} \downarrow_A b_1, ..., b_n, \bar{d}'. \tag{5.2}$$

By superstability, there is some $i < \omega$ such that

$$C_{i+1} \downarrow_{A \cup \bar{a} \cup C_i} \bar{b}\bar{d}'.$$

We denote $C = C_{i+1}$. Since $C_{i+1} \downarrow_{A \cup \bar{a}} C_i$, we get by transitivity that

$$C \downarrow_{A \cup \bar{a}} \bar{b}\bar{d}'. \tag{5.3}$$

Since $\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n$, 5.3, symmetry and transitivity imply that

$$\bar{d}' \downarrow_A \bar{a}, b_1, ..., b_n, C. \tag{5.4}$$

furthermore, 5.2,5.3 and transitivity imply that $C\bar{a} \downarrow_A b_1, ..., b_n, \bar{d}'$ and hence by 5.4, $C\bar{a}\bar{d}' \downarrow_A \bar{b}_n$. Furthermore by monotonicity, $\bar{a}\bar{d}' \downarrow_{A \cup C} b_1, ..., b_n$ and then since $b_1, ..., b_n$ dominates $\bar{b}$ over $(A \cup C)$,

$$\bar{a}\bar{d}' \downarrow_{A \cup C} \bar{b}. \tag{5.5}$$

Then 5.4, 5.5 and transitivity give that $\bar{d}' \downarrow_A \bar{a}, \bar{b}, C$. This proves the claim. □

To prove stationarity, we need one more lemma about Lascar strong types.

Lemma 5.10 *Assume that $\bar{a}, \bar{b}$ and $\bar{c}, \bar{d}$ are both witnesses for a generic $g \in \mathbf{G}$ over $\bar{d}$ in* $\mathbf{P}$*. Then there are $\bar{c}'\bar{d}'$, generic witnesses for g over $\bar{d}$ such that $\bar{c}'/E = \bar{c}/E$, $\bar{d}'/E = \bar{d}/E$ and $\bar{c}', \bar{d}'$ realizes the Lascar strong type* $\mathrm{Lstp}(\bar{a}, \bar{b}/A \cup \bar{d})$.

Proof: Since $\bar{a}$ and $\bar{b}$ are generic witnesses, there are $n+1$-tuples $\bar{e}$, $\bar{f}$ such that $g(\bar{e}/E) = \bar{f}/E$ and $\bar{a} \downarrow_A \bar{e}, \bar{f}, \bar{d}$. Similarly, there are such $n+1$-tuples $\bar{e}', \bar{f}'$ for $\bar{c}$.

First, let $\bar{a}'$ realize $\mathrm{Lstp}(\bar{a}/A \cup \bar{e} \cup \bar{f} \cup \bar{d})$ such that $\bar{a}' \downarrow_{A \cup \bar{e} \cup \bar{f} \cup \bar{d}} \bar{e}' \cup \bar{f}'$. By transitivity, $\bar{a}' \downarrow_A \bar{e}', \bar{f}', \bar{d}$. As independent $n+1$-tuples, $\bar{a}'$ and $\bar{c}$ realize the same Lascar strong type over A. Then by stationarity, $\bar{a}'$ and $\bar{c}$ realize the same Lascar strong type over $A \cup \bar{e}' \cup \bar{f}' \cup \bar{d}$.

We get two strong automorphisms $\tau_1 \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e} \cup \bar{f} \cup \bar{d})$ and $\tau_2 \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e}' \cup \bar{f}' \cup \bar{d})$ such that $\tau_1(\bar{a}) = \bar{a}'$ and $\tau_2(\bar{a}') = \bar{c}$. By $n+1$-determinacy we get that $\tau_1'(g) = g$ and $\tau_2'(g) = g$.

We write $\sigma = \tau_2 \circ \tau_1 \in \mathrm{Saut}(\mathfrak{M}/A)$ and $\bar{d}' = \sigma(\bar{b})$. Then $\bar{c}, \bar{d}' = \sigma(\bar{a}, \bar{b})$ realize $\mathrm{Lstp}(\bar{a}, \bar{b}/A)$ and are generic witnesses for $\sigma(g) = g$ over $\bar{d}$. Hence $\bar{c}, \bar{d}'$ are as needed for the claim. □

Finally we prove stationarity.

Proposition 5.11 $\mathbf{G}$ *is stationary with respect to* Σ.

Proof: First, notice that the number of Lascar strong types of $2n+2$-sequences over A is bounded by Fact 2.4. Since by $n+1$-determinacy the type of any $g \in \mathbf{G}$ is determined by the Lascar strong type of any $2n+2$-tuple $\bar{a}, \bar{b}$ such that $g(\bar{a}/E) = \bar{b}/E$, we get that the number of types $\mathrm{tp}(g/A)$ for $g \in \mathbf{G}$ is bounded.

Now assume that both g and h in $\mathbf{G}$ are generic over some finite $X \subset \mathbf{P}/E$ such that $\mathrm{tp}(g/A) = \mathrm{tp}(h/A)$. We want to show that $\mathrm{tp}(g/X) = \mathrm{tp}(h/X)$.

Let $\bar{e} \in \mathbf{P}$ be finite such that $\bar{e}/E = X$. By lemma 5.9 there are generic witnesses $\bar{a}, \bar{b}$ for g over $\bar{e}$ and generic witnesses $\bar{c}, \bar{d}$ for h over $\bar{e}$.

Since $\mathrm{tp}(g/A) = \mathrm{tp}(h/A)$, there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ such that $\tau(g) = h$. We have that $\tau(\bar{a}), \tau(\bar{b})$ are generic witnesses for h over $\emptyset$. Then by Lemma 5.10 there are $\bar{c}'$, $\bar{d}'$ generic witnesses for h over $\emptyset$ realizing $\mathrm{Lstp}(\tau(\bar{a}), \tau(\bar{b})/A) = \mathrm{Lstp}(\bar{a}, \bar{b}/A)$ such that $\bar{c}'/E = \bar{c}/E$ and $\bar{d}'/E = \bar{d}/E$.

We claim that $\bar{c}'\bar{d}' \downarrow_A \bar{e}$. By domination, it is enough to show that $\bar{c}', d_1', ..., d_n' \downarrow_A \bar{e}$. But hence $\bar{c}' \subset \mathrm{cl_A}(\bar{c})$ and $d_1', ..., d_n' \subset \mathrm{cl_A}(d_1, ..., d_n)$ and vice versa, we have that

$$\dim(\bar{c}', d_1', ..., d_n'/A \cup \bar{e}) = \dim(\bar{c}, \bar{c}', d_1, ..., d_n, d_1', ..., d_n'/A \cup \bar{e}) = \dim(\bar{c}, d_1, ..., d_n/A \cup \bar{e}) = 2n+1.$$

Since then $\bar{c}', d_1', ..., d_n'$ is independent over $\bar{e}$, we get the claim.

Similarly by domination, $\bar{a}, \bar{b} \downarrow_A \bar{e}$. Now $\bar{c}'\bar{d}' \downarrow_A \bar{e}$, $\bar{a}, \bar{b} \downarrow_A \bar{e}$ and the sequences $\bar{c}'\bar{d}'$ and $\bar{a}, \bar{b}$ realize the same Lascar strong type over A. By stationarity, they realize the same Lascar strong type over $A \cup \bar{e}$. Hence there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup \bar{e})$ mapping $\bar{a}\bar{b}$ to $\bar{c}'\bar{d}'$. This τ also fixes $X \subset \mathbf{P}/E$ pointwise. By $n+1$-determinacy, $\tau'(g) = h$, and hence we are done with the proof. □

The following corollary follows from Fact 5.6

Corollary 5.12 *The connected component $\mathbf{G}^0$ has unique generics with respect to Σ.*

5.1 Localization and hereditarily unique generics

The following definitions of a localised group action and hereditarily unique generics are from [7] and are the same for any group G (Σ, n)-acting on a pregeometry (P, cl), where G is ω-homogeneous with respect to some group Σ of automorphism of the group action.

When $B \subseteq P$ is an independent set of size k with $k < n$, we can form a new ω-homogeneous group action by *localizing* at B: The group G_B is the poitwise stabilizer of B, $G_B = \{g \in G : g \restriction B = Id\}$, which is a subgroup of G. The pregeometry P_B is obtained from P by considering the new closure operator

$$\mathrm{cl}_B(X) = \mathrm{cl}(B \cup X) \setminus \mathrm{cl}(B)$$

on the set $P \setminus \mathrm{cl}(B)$; then G_B acts on P_B by restriction; and let Σ_B be the group of automorphisms in σ fixing B pointwise. Then the group G_B $(\Sigma_B, n-k)$-acts on the pregeometry P_B.

If P is a *geometry*, it is not necessarily true that P_B would be a geometry, since for the elements $b \in P \setminus \mathrm{cl}(B)$, the closure $\mathrm{cl}(B \cup \{b\})$ is not necessarily contained in $\mathrm{cl}(b) \cup \mathrm{cl}(B)$. However, ω-homogeneity (now with respect to Σ_B), infinite-dimensionality and empty closure of the empty set are inherited.

Definition 5.13 *Assume that a group G (Σ, n)-acts on a pregeometry (P, cl). We say that G* admits hereditarily unique generics *if G has unique generics and for any independent k-set $B \subseteq P$ with $k < n$ there is a normal subgroup G' of G_B such that G' $(\Sigma', n-k)$-acts on the pregeometry P_B (for some subgroup Σ' of Σ), which has unique generics with respect to Σ.*

We claim that $\mathbf{G}^0$ admits hereditarily unique generics. For any independent k-tuple $\bar{x}$ in $\mathbf{P}/E$, we should consider the $(\Sigma_{\bar{x}}, n-k)$-action $(\mathbf{G}^0)_{\bar{x}}$ on $(\mathbf{P}/E)_{\bar{x}}$, where the connected component is defined with $\Sigma_{\bar{x}}$ and hence is $\Sigma_{\bar{x}}$-invariant. To prove that this action has unique generics it is enough to show that any g generic in $(\mathbf{G}^0)_{\bar{x}}$ is also generic in $\mathbf{G}^0$. Then, since $\mathbf{G}^0$ has unique generics, for any two such generics g, h there is $\sigma \in \Sigma$ mapping g to h. Note that by definition it is enough that $(\mathbf{G}^0)_{\bar{x}}$ has unique generics with respect to Σ.

To simplify notation we write $(\mathbf{G}^0)_{\bar{a}}$ for $(\mathbf{G}^0)_{\bar{x}}$, where $\bar{x} = \bar{a}/E$.

Proposition 5.14 *Let $\bar{a} = a_1, ..., a_k$ be an independent k-tuple for $0 < k < n$. Assume that g generic in $(\mathbf{G}^0)_{\bar{a}}$. Then it is also generic in $\mathbf{G}^0$.*

Proof: Since g is generic in $(\mathbf{G}^0)_{\bar{a}}$, there are $a_{k+1}, ..., a_n$, $b_{k+1}, ..., b_n$ in $\mathbf{P}$ such that $g(a_i/E) = b_i/E$ for each $i \in \{k+1, ..., n\}$ and

$$\dim(a_{k+1}, ..., a_n, b_{k+1}, ..., b_n/A \cup \bar{a}) = 2(n-k).$$

Denote $\bar{a}' = a_1, ..., a_k, a_{k+1}, ..., a_n$ and $\bar{b} = b_1, ..., b_n = a_1, ..., a_k, b_{k+1}, ..., b_n$. Then we have that $g(\bar{a}'/E) = \bar{b}/E$ and

$$\dim(\bar{a}'\bar{b}/A) = n + (n-k).$$

We choose a_{n+1}, b_{n+1} such that

$$a_{n+1} \downarrow_A \bar{a}'\bar{b}',$$

and $g(a_{n+1}/E) = b_{n+1}$. Then we choose a_{n+2}, b_{n+2} respectively such that

$$a_{n+2} \downarrow_A \bar{a}', \bar{b}, a_{n+1}, b_{n+1}$$

and $g(a_{n+2}/E) = b_{n+2}/E$. It follows that

$$\dim(\bar{a}', a_{n+1}, a_{n+2}, \bar{b}, b_{n+1}, b_{n+2}/A) = n + (n-k) + 2.$$

Denote $\bar{c} = a_2, ..., a_{n+2}, b_2, ..., b_{n+2}$. We claim that $\dim(\bar{c}/A) = n + (n-k) + 2$. Since $a_1 = b_1$, it is enough to show that

$$a_1 \not\downarrow_A \bar{c}.$$

We assume to the contrary, that $a_1 \downarrow_A \bar{c}$. The by extension we can choose d_p realizing $\mathrm{Lstp}(a_1/A \cup \bar{c})$ for $p = 1, ..., n-k+1$ such that $d_p \downarrow_A \bar{c}, a_1, d_1, ..., d_{p-1}$. Since the $2(n+1)$-sequence $\bar{c}$ determines g, we must have that $g(d_p/E) = d_p/E$ for each $p = 1, ..., n-k+1$. But then g fixes the $n+1$- sequence $d_1/E, ..., d_{n-k+1}/E, a_1/E, ..., a_k/E$ and hence by $n+1$-determinacy we must have that $g = \mathrm{Id}_{(\mathbf{P}/E)}$. On the other hand $g(a_{k+1}/E) = b_{k+1}/E \neq a_{k+1}/E$. This contradiction proves the claim, that is

$$\dim(a_2, ..., a_{n+2}, b_2, ..., b_{n+2}/A) = n + (n-k) + 2.$$

Furthermore, for each $m \in \{1, ..., k-1\}$ we choose a_{n+2+m} and b_{n+2+m} such that $g(a_{n+2+m}/E) = b_{n+2+m}$ and

$$a_{n+2+m} \downarrow_A \bar{a}'\bar{b}, a_{n+1}, ..., a_{n+1+m}, b_{n+1}, ..., b_{n+1+m}.$$

As in the previous claim, we conclude that

$$\dim(a_{2+m}, ..., a_{n+2+m}, b_{2+m}, ..., b_{n+2+m}/A) = n + (n-k) + 2 + m.$$

Then finally when $m = k-1$ we get that

$$\dim(a_{k+1}, ..., a_{n+k+1}, b_{k+1}, ..., b_{n+k+1}/A) = 2n + 1.$$

Now we have shown that g is generic in $\mathbf{G}^0$. □

We get the following corollary.

Corollary 5.15 $\mathbf{G}^0$ *admits hereditarily unique generics.*

We mention another corollary.

Corollary 5.16 *Assume that* $\bar{x} \in \mathbf{P}/E$ *is an independent* k*-tuple for* $k < n$*. Let* $g \in (\mathbf{G}^0)_{\bar{x}}$*. Then* g *is generic in* $\mathbf{G}^0$ *if and only if* g *is generic in* $(\mathbf{G}^0)_{\bar{x}}$

Proof: The other direction follows from Proposition 5.14. Then assume that g is generic in $\mathbf{G}^0$ and fixes $\bar{x}$. Hence there is an independent n-sequence $\bar{y} = y_1, ..., y_n$ such that $\dim(\bar{y}, g(\bar{y})/A) = 2n$. Since at most k elements of the independent sequence $\bar{y}, g(\bar{y})$ can belong to $\mathrm{cl}(\bar{x})$, we may assume that $y_{k+1}, ..., y_n, g(y_{k+1}), ..., g(y_n)$ are outside $\mathrm{cl}(\bar{x})$. Since $\dim(y_{k+1}, ..., y_n, g(y_{k+1}), ..., g(y_n)/A) = 2(n-k)$, these elements witness that g is generic in $(\mathbf{G}^0)_{\bar{x}}$. □

Hereditarily unique generics gives us either a non-classical group or n-determinacy and furthermore that $n \in \{1, 2, 3\}$. These are Definitions 1.1 and 1.11 of Hyttinen, Lessman and Shelah [7] and Facts 2.10 and 2.12 of [7] referring to Theorem 2.7 and Lemma 2.8 of Hyttinen [3]. It is a open question whether non-classical groups exist.

Definition 5.17 *We say that a group* G carries an ω-homogeneous pregeometry *if there exists a closure operator* cl *on the subsets of* G *satisfying the axioms of a pregeometry with* $\dim(G) = |G|$*, and such that whenever* $A \subseteq G$ *is finite and* $a, b \notin \mathrm{cl}(A)$*, then there is an automorphism of* G*, preserving* cl *and fixing* A *pointwise and sending* a *to* b*.*

We say that a group G *is* non-classical *if it is nonabelian and carries an* ω*-homogeneous pregeometry.*

In the following facts we assume that the pregeometry (P, cl) is infinite-dimensional and that $\mathrm{cl}(\emptyset) = \emptyset$.

Fact 5.18 *Assume that* G (Σ, n)*-acts on a pregeometry* (P, cl)*. Assume that* G *admits hereditarily unique generics. Then either* $(G_B)^0$ *is non-classical, for some independent* $(n-1)$*-subset* $B \subseteq P$ *or the action of* G *on* P *is* n*-determined.*

Fact 5.19 *Assume that the* (Σ, n)*-action of* G *on a pregeometry* (P, cl) *is* n*-determined. Then* $n \in \{1, 2, 3\}$.

We prove a small Lemma which will be used several times in the proof of the main theorem. A similar Lemma is used to prove Fact 5.18, but the proof if simpler due to 1-determinacy.

Lemma 5.20 *Assume that* G $(\Sigma, 1)$*-acts on an infinite-dimensional pregeomerty* (P, cl)*, where* $\mathrm{cl}(\emptyset) = \emptyset$*. Assume that the action is 1-determined. Then* G *admits an* ω*-homogeneous pregeometry.*

Proof: We define a closure operator cl on the subsets of G as follows: for $g \in G$ and $g_1, ..., g_k \in G$ we let

$$g \in \mathrm{cl}(g_1, ..., g_k),$$

if for some element $y \in P$ and $x \in P \setminus \mathrm{cl}(y, g(y), g_1(y), ..., g_k(y))$ we have that

$$g(x) \in \mathrm{cl}(x, g_1(x), ..., g_k(x)).$$

We note that then the same holds for all such x and y: let $x' \notin \mathrm{cl}(y', g(y'), g_1(y'), ..., g_k(y'))$. Let z be such that

$$z \notin \mathrm{cl}(y, g(y), g_1(y), ..., g_k(y), y', g(y'), g_1(y'), ..., g_k(y')).$$

Then since the action is ω-homogeneous with respect to Σ, there are $\tau, \tau' \in \Sigma$ such that $\tau(x) = z, \tau'(z) = x'$,

$$\tau \restriction \{y, g(y), g_1(y), ..., g_k(y)\} = \mathrm{Id} \text{ and}$$

$$\tau' \restriction \{y', g(y'), g_1(y'), ..., g_k(y')\} = \mathrm{Id}.$$

But then by 1-determinacy, $\tau'(g) = \tau(g) = g$ and $\tau'(g_i) = \tau(g_i) = g_i$ for each $i \in \{1, ..., k\}$. Hence $g(x) \in \mathrm{cl}(x, g_1(x), ..., g_k(x))$ if and only if $g(x') \in \mathrm{cl}(x', g_1(x'), ..., g_k(x'))$ by applying $\tau \circ \tau'$.

For an arbitrary subset $A \subseteq G$ we define that $g \in \mathrm{cl}(A)$ if there are $k < \omega$ and $g_1, ..., g_k \in A$ such that $g \in \mathrm{cl}(g_1, ..., g_k)$. It is not difficult to check that this induces a pregeometry on G with the same infinite dimension as P. Notice however, that even though the closure of the empty set is empty in P by assumption, the induced closure on G contains the identity element of G.

Also since the action is ω-homogeneous with respect to Σ, the induced pregeometry in ω-homogeneous with respect to Σ: suppose that $g, h \notin \mathrm{cl}(A)$ for some finite subset $A \subseteq G$. Then for some element $y \in P$ define $A(y) = \{f(y) : f \in A\}$ and let

$$x \in P \setminus \mathrm{cl}(y, g(y), h(y), A(y)\}.$$

Then by the definition of closure,

$$g(x), h(x) \notin \mathrm{cl}(x, A(x)).$$

There is $\tau \in \Sigma_{\{x, A(x)\}}$ mapping $g(x)$ to $h(x)$. Again by 1-determinacy, $\tau(g) = h$ and $\tau(f) = f$ for each $f \in A$. $\square$

6 The main result

We want to use Theorem 2.32 of Hyttinen, Lessmann and Shelah [7] to conclude the main result of this paper. There is one more obstacle we have to be aware of. In [7] it is assumed that $\dim(P) > 2^{|\mathrm{cl}(B)|}$ for any finite subset B of the pregeometry P. There it is a minor assumption, since in the strongly minimal case the closure $\mathrm{cl}(B)$ of a finite set B is bounded. Here we cannot assume such thing. However, we are able to copy the proofs of [7] only replacing the parts where this assumption is used and conclude our main result. More specifically, this assumption is used in Lemmas 2.17 and 2.28 of [7], which are needed to prove Proposition 2.29. We will reprove these, but only in our context, not in the context of a group acting on an arbitrary pregeometry. Note that the assumption $\dim(P) > 2^{|\mathrm{cl}(B)|}$ is not used in the paper Hyttinen [3].

6.1 The pregeometry $(\mathbf{P}/E)_x$ is a geometry

In this section we prove the following proposition, which replaces Proposition 2.29 of [7].

Proposition 6.1 *Assume that G^0 $(\Sigma, 3)$-acts on the geometry $\mathbf{P}/E$. Let x be an element in $\mathbf{P}/E$. Then the pregeometry $(\mathbf{P}/E)_x$ is a geometry.*

First we prove the following Lemma, replacing Lemma 2.17 of [7].

Lemma 6.2 *Assume that G^0 $(\Sigma, 3)$-acts on $\mathbf{P}/E$ and the action is n-determined. Let x, y be independent elements in $\mathbf{P}/E$ and $g \in (G^0)_x$ generic such that that $g(y) = y$. Then g fixes $\mathrm{cl}(a, b)$ pointwise in $\mathbf{P}/E$.*

Proof: Note that it is impossible for g to fix anything pointwise in the pregeometry $\mathbf{P}$, since g is only defined for the equivalence classes in $\mathbf{P}/E$, not for the elements in $\mathbf{P}$. Note that it is equivalent to say that $x, y \in \mathbf{P}/E$ are independent and that y is an element in $(\mathbf{P}/E)_x$.

Notice also that by Corollary 5.16, if $g \in G^0$ fixes x it is equivalent to say that g is generic in G^0 or g is generic in $(G^0)_x$.

Since g is generic (in $(\mathbf{G}^0)_{x,y}$), there is z independent of x and y such that

$$\dim(x, y, z, g(z)/A) = 4.$$

Now it suffices to find *some* generic $g' \in G^0$ such that $g' \restriction cl(x, y) = \mathrm{Id}$. Since then there is z' such that $\dim(x, y, z', g(z')/A) = 4$ and hence there is a strong automorphism $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ mapping $x, y, z', g(z')$ to $x, y, z, g(z)$ by Lemma 4.4. But now by 3-determinacy, $\tau'(g') = g$. Then since g' fixes $\mathrm{cl}(x, y)$ pointwise, also g fixes pointwise the set $\tau(\mathrm{cl}(x, y)) = \mathrm{cl}(x, y)$.

Let us write $x = a/E, y = b/E, z = c/E$ and $g(z) = d/E$, where a, b, c and d are elements in $\mathbf{P}$. We have that $\dim(a, b, c, d/A) = 4$. Although $\mathrm{cl}(a, b)$ might be unbounded, by Lemma 2.3 there exists a finite $D \subseteq \mathrm{cl}(a, b)$ such that

$$c, d \downarrow_{A \cup D \cup a, b} \mathrm{cl}(a, b).$$

By extension, there are c', d' realizing $\mathrm{Lstp}(c, d/A \cup D \cup a, b)$ such that

$$c', d' \downarrow_{A \cup D \cup a, b} c, d,$$

and furthermore a strong automorphism $\tau \in \mathrm{Saut}(\mathfrak{M}/A \cup D \cup a, b)$ mapping c, d to c', d'. Denote $h = \tau(g)$. Then $h(x, y) = x, y$ and $h(c'/E) = d'/E$. Since $\mathbf{G}^0$ is Σ-invariant, we have that $h \in \mathbf{G}^0$.

Furthermore, since τ fixes $\mathrm{cl}(a, b)$ as a set, we have that

$$c'd' \downarrow_{A \cup D \cup a, b} \mathrm{cl}(a, b).$$

By stationarity, for each finite $\bar{e} \in \mathrm{cl}(a, b)$ there is $\tau_{\bar{e}} \in \mathrm{Saut}(\mathfrak{M}/A \cup, a, b, \bar{e})$ mapping c, d to c', d'. By 3-determinacy, $\tau_{\bar{e}}(g) = h$.

Now let $a_1 \in \mathrm{cl}(a, b)$ and a_2 be such that $g(a_1/E) = a_2/E$. Then since g fixes $x, y = a, b/E$, also a_2 is in $\mathrm{cl}(a, b)$. Now let $\bar{e}$ in $\mathrm{cl}(a, b)$ contain a_1 and a_2. Then $h(a_1/E) = \tau_{\bar{e}}(g)((a_1/E)) = \tau_{\bar{e}}(g(\tau_{\bar{e}}^{-1}(a_1/E))) = a_2/E = g(a_1/E)$. This implies that

$$g \restriction \mathrm{cl}(x, y) = h \restriction \mathrm{cl}(x, y).$$

Hence $h^{-1} \circ g \in \mathbf{G}^0$ fixes $\mathrm{cl}(x, y)$. We need to show that $h^{-1} \circ g$ is generic. It is enough to show that it is generic in $(G^0)_{x,y}$.

Let us write $z' = c'/E$. Then $h(z') = d'/E$ and

$$\dim(z, g(z), z', h(z')/A \cup x, y) = 4.$$

Let $e \in \mathbf{P}/E$ be independent of $z, g(z), z', h(z'), x$ and y. By 3-determinacy, any $\tau \in \Sigma$ fixing $x, y, z, g(z), e$ must fix g^{-1} and hence also $g^{-1}(e)$. This implies that $g^{-1}(e) \in \mathrm{cl}_{\mathrm{A}}(x, y, z, g(z), e)$. Similarly, $h^{-1}(e) \in \mathrm{cl}_{\mathrm{A}}(x, y, z', h(z'), e)$. Hence

$$\dim(z, g(z), z', h(z'), e, g^{-1}(e), h^{-1}(e)/A \cup x, y) = \dim(z, g(z), z', h(z'), e/A \cup x, y) = 5.$$

By the same argument $g(z) \in \mathrm{cl}_{\mathrm{A}}(x, y, z, e, g^{-1}(e))$ and $h(z') \in \mathrm{cl}_{\mathrm{A}}(x, y, z', e, h^{-1}(e))$ and hence

$$\dim(z, z', e, h^{-1}(e), g^{-1}(e)/A \cup x, y) = 5.$$

Thus $\dim(h^{-1}(e), g^{-1}(e)/A\cup x, y) = 2$, where $(h^{-1}\circ g)(g^{-1}(e)) = h^{-1}(e)$. This proves that $h^{-1}\circ g$ is generic in $(G^0)_{x,y}$. $\square$

The proof of Lemma 2.28 of [7] uses again that $\mathrm{cl}(A)$ is bounded for a finite set A, but this is not really needed. We reprove a part of Lemma 2.28.

Lemma 6.3 *Assume that the $(\Sigma, 2)$-action of $(G^0)_x$ on the pregeometry $\mathbf{P}/E_x$ is 2-determined. Let $y, z \in \mathbf{P}/E_x$ be independent and $f \in (G^0)_x$ be such that for all $g \in (\mathbf{G}^0)_x$, $gfg^{-1}(y) \in \mathrm{cl}_x(y)$ and $gfg^{-1}(z) \in \mathrm{cl}_x(z)$.*

Then there are $k, l \in (\mathbf{G}^0)_x$ such that $kfk^{-1} = lfl^{-1}$ and

$$\dim(y, z, k(y), k(z), l(y), l(z)/A\cup x) = 6.$$

Proof: By simplicity and extension, there are independent $y' = a/E$ and $z' = b/E$ such that

$$a, b \downarrow_A x, y, z, f(y), f(z)$$

and hence $\dim(y', z', y, z/A\cup x) = 4$. Here we abuse the notation to mean that a, b are free of some representatives of the equivalence classes of x, y etc in $\mathbf{P}$. Since $(G^0)_x$ has rank 2, there is $k \in (G^0)_x$ such that $k(y, z) = y', z'$. Now since $kfk^{-1}(y) \in \mathrm{cl}_x(y)$ and $kfk^{-1}(z) \in \mathrm{cl}_x(z)$, we have that

$$a, b \downarrow_A x, y, z, f(y), f(z), kfk^{-1}(y), kfk^{-1}(z).$$

Then let $y'' = c/E, z'' = d/E$ be such that $\dim(y, z, y', z', y'', z''/A\cup x) = 6$, $c, d \models \mathrm{Lstp}(a, b/A)$ and

$$c, d \downarrow_A x, y, z, f(y), f(z), kfk^{-1}(y), kfk^{-1}(z).$$

Hence by stationarity, there is $\tau \in \mathrm{Saut}(\mathfrak{M}/A)$ fixing $x, y, z, f(y), f(z), kfk^{-1}(y), kfk^{-1}(z)$ and mapping y', z' to y'', z''. Then by 2-determinacy, $\tau'(f) = f$ and $\tau'(kfk^{-1}) = kfk^{-1}$. We choose $l = \tau'(k)$. Then $l^{-1} = \tau'(k^{-1})$.

Now we claim that these k, l will do. We have that $l(x, y, z) = \tau(k(\tau(x, y, z))) = x, y'', z''$ and hence $l \in (\mathbf{G}^0)_x$ and $\dim(y, z, k(y), k(z), l(y), l(z)/A\cup x) = 6$. Furthermore, for any element $w \in \mathbf{P}/E$,

$$kfk^{-1}(w) = \tau'(kfk^{-1})(w) = \tau'(k)\tau'(f)\tau'(k^{-1})(w) = lfl^{-1}(w),$$

since $\tau(kfk^{-1})\tau^{-1}(w) = (\tau k\tau^{-1})(\tau f\tau^{-1})(\tau k^{-1}\tau^{-1})(w)$. $\square$

With these Lemmas the proof of Proposition 6.1 is identical to the proof of Proposition 2.29 of [7]. Note that this implies that the pregeometry cl_{A} on $\mathbf{P}$ is 2-trivial: since for any pair $x, y \in \mathbf{P}/E$, $\mathrm{cl}_{\mathrm{A}}(x, y) = \mathrm{cl}_{\mathrm{A}}(x) \cup \mathrm{cl}_x(y) = \{x, y\}$, we get that for any $a, b \in \mathbf{P}$, $\mathrm{cl}_{\mathrm{A}}(a, b) = \mathrm{cl}_{\mathrm{A}}(a) \cup \mathrm{cl}_{\mathrm{A}}(b)$.

6.2 The main result

Now our main result follows as Theorem 2.32 of Hyttinen, Lessmann and Shelah [7]. We recall the main ingredients, but the proofs are identical. We define

$$I = \{g \in G : g^2 = 1\} \text{ and}$$

$$N_x = \{g \in G : \text{ the set } \{h(x) : h \in I, gh \notin I\} \text{ has bounded dimension}\}.$$

Several properties of N_x are shown in [7]. We list here those that are needed for the proof of our main theorem.

Fact 6.4 *Assume that a group G $(\Sigma, 2)$-acts on an infinite-dimensional geometry (P, cl) with $\mathrm{cl}(\emptyset) = \emptyset$. Then for each $x \in P$, $N_x \subseteq G$ is an invariant normal subgroup and $G = N_x \rtimes G_x$. Also the group N_x $(\Sigma', 1)$-acts on (P, cl), where the action is n-determined and Σ' is obtained from Σ by restriction.*

Furthermore, if G_x and N_x are abelian, then P can be given the structure of an algebraically closed field $(K, +, \times, 0, 1)$ and the action of $\mathbf{G}$ on P is isomorphic to the affine action of $K^+ \rtimes K^*$, $x \mapsto l + kx$, on K. Moreover, the field structure on P and the isomorphism of the group action are invariant once the identities of the field $0, 1$ are chosen.

The proof is as the proof of Propositions 2.27 and 2.31 of [7].

Theorem 6.5 *Assume that $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$ is a simple, superstable finitary AEC and let $\mathfrak{M}$ be the monster model for $(\mathbb{K}, \preccurlyeq_{\mathbb{K}})$. Assume that A is a finite set, p is an unbounded and regular Lascar strong type over A and $\mathbf{Q}$ is an A-invariant subset of $\mathfrak{M}$. Assume that there exists an integer $0 < n < \omega$ such that*

1. *For any independent sequence $(a_1, ..., a_n)$ of realizations of p and any finite subset C of $\mathbf{Q}$ we have*
$$\dim(a_1, ..., a_n/A) = \dim(a_1, ..., a_n/A \cup C).$$
2. *For some independent sequence $a_1, ..., a_{n+1}$ of realizations of p there is C a finite subset of $\mathbf{Q}$ such that $(a_1, ..., a_n)$ dominates $(a_1, ..., a_{n+1})$ over $A \cup C$.*

Then $\mathfrak{M}$ interprets a group G which acts on the geometry $\mathbf{P}/E$ induced on the set $\mathbf{P}$ of realizations of p. Furthermore, either $\mathfrak{M}$ interprets a nonclassical group or $n \in \{1, 2, 3\}$ and

- *If $n = 1$, then G is abelian and acts regularly on $\mathbf{P}/E$.*
- *If $n = 2$, the action of G on $\mathbf{P}/E$ is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebrically closed field K.*
- *If $n = 3$, the action of G on $\mathbf{P}/E$ is isomorphic to the action of $PGL_2(K)$ on the projective line $\mathbb{P}^1(K)$ of the algebraically closed field K.*

Proof: The group $\mathbf{G}$ is interpretable in $\mathfrak{M}$ by proposition 4.10. This group acts on the geometry $\mathbf{P}/E$; the action has rank n and is $n+1$-determined. Futhermore, $\mathbf{G}^0$ admits hereditarily unique generics with respect to the set of automorphisms induced by strong automorphisms of $\mathfrak{M}$. $\mathbf{G}^0$ is an invariant subgroup of of $\mathbf{G}$ and therefore interpretable. But, $\mathbf{G}^0$ (Σ^0, n)-acts on the geometry $\mathbf{P}/E$ (Σ^0 is simply obtained from Σ by restriction) and has hereditarily unique generics. Hence, we let $G = \mathbf{G}^0$. We also write only Σ for Σ^0.

Assume that $\mathfrak{M}$ does not interpret a nonclassical group. Then the action of G on $\mathbf{P}/E$ is n-determined by Fact 5.18, since groups of the form $((G^0)_B)^0$ are interpretable in $\mathfrak{M}$. Furthermore, then $n \in \{1, 2, 3\}$ by Fact 5.19.

Let $n = 1$. Since the $(\Sigma, 1)$-action of G on $\mathbf{P}/E$ is 1-determined, it is regular. Moreover, G carries a homogeneous pregeometry by Lemma 5.20. Since it cannot be nonclassical, it must be abelian.

Let $n = 2$. By Fact 6.4, N_x $(\Sigma', 1)$ acts on $\mathbf{P}/E$, where the action is 1-determined. Also G_x acts on the pregeometry $(\mathbf{P}/E)_x$ with an 1-determined action. The groups N_x and G_x are interpretable in $\mathfrak{M}$ and hence it follows from Lemma 5.20 that N_x and G_x must be abelian. Now the result follows from Fact 6.4.

Let $n = 3$. Choose a point $y \in \mathbf{P}/E$ and call it ∞. Then the $(\Sigma_\infty, 2)$-action of G_∞ on $(\mathbf{P}/E)_\infty$ is 2-determined. By Proposition 6.1, $(\mathbf{P}/E)_\infty$ is a geometry.

Choose $x \in (\mathbf{P}/E)_\infty$ and call it 0. Call $N_{\infty,0}$ the group N_x defined for $(\mathbf{P}/E)_\infty$ and let $G_{\infty,0}$ the group of elements in G_∞ fixing also 0. Then the 1-actions of $G_{\infty,0}$ on $(\mathbf{P}/E)_{\infty,0}$

and $N_{\infty,0}$ on $(\mathbf{P}/E)_\infty$ are 1-determined and the groups are interpretable in $\mathfrak{M}$. Again they must be abelian by Lemma 5.20.

By Proposition 6.4, the action of $G_\infty = N_{\infty,0} \rtimes G_{\infty,0}$ on $(\mathbf{P}/E)_\infty$ is isomorphic to the affine action of $K^+ \rtimes K^*$ on the algebraically closed field K (notice that $0 \in (\mathbf{P}/E)_\infty$ chosen above is the 0 of the field). Let $1 \in (\mathbf{P}/E)_\infty$ be the identity element for the multiplicative structure of the field K. Since $(\mathbf{P}/E)_\infty$ is a geometry, the set $\{0, 1, \infty\} \subset \mathbf{P}/E$ is 3-dimensional.

Since the $(\Sigma, 3)$-action of G on $\mathbf{P}/E$ is 3-determined, there is a unique $\alpha \in G$ such that $\alpha(0) = \infty$, $\alpha(\infty) = 0$ and $\alpha(1) = 1$. Notice that $\alpha^2 = 1$ by 3-determinacy. Exactly as in the proof of Theorem 2.32 in [7] we see that conjugation by α induces an idempotent automorphism τ of $G_{\infty,0}$, which is not the identity and furthermore, $\tau(g) = g^{-1}$ for each $g \in G_{\infty,0}$.

We can now complete the proof as in [7]: the geometry $\mathbf{P}/E$ is isomorphic to the projective line $\mathbb{P}^1(K)$, with ∞ being the point at infinity. Given $x \in K^*$, choose $h \in G_{\infty,0}$ such that $h1 = x$. Then $\alpha x = \alpha h x = h^{-1}\alpha 1 = h^{-1}1 = x^{-1}$. Also α permutes 0 and ∞, so α acts like an inversion on $\mathbb{P}^1(K)$. It follows that G contains the group of automorphisms of $\mathbb{P}^1(K)$ generated by the affine transformations and inversion. Hence $PLG_2(K)$ embeds in G. The actions of $PLG_2(K)$ and G are both sharply 3-transitive: any three elements $x, y, z \in \mathbf{P}/E$ are independent, since $\mathbf{P}/E$ is 2-trivial (see the note in the end of section 6.1), and hence there is exactly one $g \in G$ mapping a triple x, y, z to another triple x', y', z'. Hence the embedding of $PLG_2(K)$ on G is surjective.

The projective line structure and the isomorphism of the group action are invariant over the points $0, 1, \infty \in \mathbf{P}/E$. □

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Tapani Hyttinen
tapani.hyttinen@helsinki.fi
Meeri Kesälä
meeri.kesala@helsinki.fi
Department of Mathematics and Statistics
University of Helsinki
P.O. Box 68 (Gustaf Hällströmin katu 2b)
FI-00014 University of Helsinki
Finland