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TUKEY TYPES OF ULTRAFILTERS

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Abstract. We investigate the structure of the Tukey types of ultrafilters on countable sets partially ordered by reverse inclusion. A canonization of cofinal maps from a p-point into another ultrafilter is obtained. This is used in particular to study the Tukey types of p-points and selective ultrafilters. Results fall into three main categories: comparison to a basis element for selective ultrafilters, embeddings of chains and antichains into the Tukey types, and Tukey types generated by block-basic ultrafilters on FIN.

1. Introduction

Let D and E be partial orderings. We say that a function $f : E \rightarrow D$ is *cofinal* if the image of each cofinal subset of E is cofinal in D. We say that D is *Tukey reducible* to E, and write $D \leq_T E$, if there is a cofinal map from E to D. An equivalent formulation of Tukey reducibility was noticed by Schmidt in [19]. Given partial orderings D and E, a map $g : D \rightarrow E$ such that the image of each unbounded subset of D is an unbounded subset of E is called a *Tukey map* or an *unbounded map*. $E \geq_T D$ iff there is a Tukey map from D into E. If both $D \leq_T E$ and $E \leq_T D$, then we write $D \equiv_T E$ and say that D and E are Tukey equivalent. $\equiv_T$ is an equivalence relation, and $\leq_T$ on the equivalence classes forms a partial ordering. The equivalence classes can be called *Tukey types* or *Tukey degrees*.

In [26], Tukey introduced the Tukey ordering to develop the notion of Moore-Smith convergence in topology to the more general setting of directed partial orderings. The study of cofinal types and Tukey types of partial orderings often reveals useful information for the comparison of different partial orderings. For example, Tukey reducibility downward preserves calibre-like properties, such as c.c.c., property K, precalibre $\aleph_1$, σ-linked, and σ-centered (see [25]).

Satisfactory classification theories of Tukey degrees have been developed for several classes of ordered sets. The cofinal types of countable directed systems are 1 and ω (see [26]). Day found a classification of countable oriented systems (partially ordered sets) in [7] in terms of a three element basis. Assuming PFA, Todorcevic in [24] classified the Tukey degrees of directed partial orderings of

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cardinality $\aleph_1$ by showing that there are exactly five cofinal types, and in [25] classified the Tukey degrees of oriented systems (partially ordered sets) of size $\aleph_1$ in terms of a basis consisting of five forms of partial orderings. However, he also showed in [25] that there are at least $2^{\aleph_1}$ many Tukey incomparable separative σ-centered partial orderings of size $\mathfrak{c}$. This would preclude a satisfactory classification theory of all partial orderings of size continuum.

However, the structure of the Tukey types of particular classes of partial orderings of size continuum can yield useful information. This has been fully stressed first in the paper [9] by Fremlin who considered partially ordered sets occurring in analysis. After this, several papers appeared dealing with different classes of posets such as, for example, the paper [21] of Solecki and Todorcevic which makes a systematic study of the structure of the Tukey degrees of topological directed sets. The paper [17] of Milovich is the first paper after Isbell [11] to study Tukey degrees of ultrafilters on ω.

In this paper, we investigate the structure of the Tukey degrees of ultrafilters on ω ordered by reverse inclusion. For any ultrafilter $\mathcal{U}$ on ω, $(\mathcal{U}, \supseteq)$ is a directed partial ordering. We remark that for any two directed partial orderings D and E, $D \equiv_T E$ iff D and E are *cofinally similar*; that is, there is a partial ordering into which both D and E embed as cofinal subsets [26]. So for ultrafilters, Tukey equivalence is the same as cofinal similarity.

Another motivation for this study is that Tukey reducibility is a generalization of Rudin-Keisler reducibility.

Fact 1. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. If $\mathcal{U} \geq_{RK} \mathcal{V}$, then $\mathcal{U} \geq_T \mathcal{V}$.*

Proof. Take a function $h : \omega \to \omega$ satisfying $\mathcal{V} = h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$. Define $f : \mathcal{U} \to \mathcal{V}$ by $f(X) = h(X)$. Then f is a cofinal map. □

Thus arises the question: How different are Tukey and Rudin-Keisler reducibility? We shall study this question particularly for p-points.

We end the introduction with some notation and basic facts. All ultrafilters in this paper are have a base set which is countable, usually ω, but in Section 5 we also investigate ultrafilters on FIN, the family of finite subsets of ω.

Notation. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{U}_n$ $(n < \omega)$ be ultrafilters. We define the notation for the following ultrafilters.

(1) $\mathcal{U} \cdot \mathcal{V} = \{A \subseteq \omega \times \omega : \{i \in \omega : \{j \in \omega : (i,j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$.
(2) $\lim_{n\to\mathcal{U}} \mathcal{U}_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{j \in \omega : (n,j) \in A\} \in \mathcal{U}_n\} \in \mathcal{U}\}$.
(3) $\mathcal{U} \times \mathcal{V}$ is defined to be the ordinary cartesian product of $\mathcal{U}$ and $\mathcal{V}$ with the coordinate-wise ordering $\langle \supseteq, \supseteq \rangle$.
(4) $\Pi_{n<\omega} \mathcal{U}_n$ is the cartesian product of the $\mathcal{U}_n$ with its natural coordinate-wise product ordering. We will let $\mathcal{U}^\omega$ denote the cartesian product of ω many copies of $\mathcal{U}$.

Fact 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters.*

(1) *$\mathcal{U}\cdot\mathcal{V}\geq_T\mathcal{U}$ and $\mathcal{U}\cdot\mathcal{V}\geq_T\mathcal{V}$, and therefore $\mathcal{U}\cdot\mathcal{V}\geq_T\mathcal{U}\times\mathcal{V}$.*
(2) *If $\mathcal{U}_1\geq_T\mathcal{U}_0$ and $\mathcal{V}_1\geq_T\mathcal{V}_0$, then $\mathcal{U}_1\cdot\mathcal{V}_1\geq_T\mathcal{U}_0\cdot\mathcal{V}_0$.*
(3) *$\mathcal{U}\times\mathcal{V}\geq_T\mathcal{U}$.*
(4) *If $\mathcal{U}_1\geq_T\mathcal{U}_0$ and $\mathcal{V}_1\geq_T\mathcal{V}_0$, then $\mathcal{U}_1\times\mathcal{V}_1\geq_T\mathcal{U}_0\times\mathcal{V}_0$.*

2. Basic and basically generated ultrafilters

The following type of partial ordering was introduced by Solecki and Todorcevic in [21].

Definition 3 ([21])**.** Let D be a separable metric space and let $\leq$ be a partial ordering on D. We say that $(D,\leq)$ is *basic* if

(1) each pair of elements of D has the least upper bound with respect to $\leq$ and the binary operation of least upper bound from $D\times D$ to D is continuous;
(2) each bounded sequence has a converging subsequence;
(3) each converging sequence has a bounded subsequence.

Each ultrafilter is a separable metric space using the metric inherited from $\mathcal{P}(\omega)$ viewed as the Cantor space, and we define $\leq$ on an ultrafilter to be $\supseteq$. It is not hard to see that every bounded subset of an ultrafilter has a convergent subsequence. Thus, an ultrafilter is basic iff (3) holds.

The next Theorem shows that the basic ultrafilters are exactly the p-points.

Theorem 4. *An ideal $\mathcal{I}$ on $\mathcal{P}(\omega)$ containing all finite subsets of ω is basic relative to the Cantor topology iff $\mathcal{I}$ is a non-meager p-ideal. Hence, an ultrafilter is basic iff it is a p-point.*

Proof. Assume $\mathcal{I}$ is basic. Let $\langle n_k : k<\omega\rangle$ be an increasing sequence of integers. Note that each $[n_k,n_{k+1})\in\mathcal{I}$, since $\mathrm{Fin}\subseteq\mathcal{I}$. $[n_k,n_{k+1})\rightarrow\emptyset$; so by basicness, there is a subsequence whose union is in $\mathcal{I}$. Hence, $\mathcal{I}$ is nonmeager.

Let $\{A_n : n<\omega\}\subseteq\mathcal{I}$. We can assume that for each $n<\omega$, $A_n\subseteq A_{n+1}$. Let $A'_n=A_n\setminus n$. Then $A'_n\subseteq A_n$, so $A'_n\in\mathcal{I}$. $A'_n\rightarrow\emptyset$ in the Cantor topology, so since $\mathcal{I}$ is basic, there is a subsequence n_k such that $\bigcup_{k<\omega}A'_{n_k}\in\mathcal{I}$. Let $A=\bigcup_{k<\omega}A_{n_k}$. Then for each $n<\omega$, $A_n\subseteq^* A$, since for each n there is an $n_k>n$ such that $A_n\subseteq A_{n_k}\subseteq^* A'_{n_k}\subseteq A$. Thus, $\mathcal{I}$ is a p-ideal.

Now suppose $\mathcal{I}$ is a nonmeager p-ideal. Suppose $A_n,A\in\mathcal{I}$ and $A_n\rightarrow A$ in the Cantor topology. Take $B\in\mathcal{I}$ such that for each n, $A_n\subseteq^* B$. Let m_k be a strictly increasing sequence such that $m_0=0$ and

(1) $n\geq m_{k+1}$ implies $A_n\cap m_k=A\cap m_k$, and
(2) $n\leq m_k$ implies $A_n\setminus m_{k+1}\subseteq B$.

Either $C_0=\bigcup_{k<\omega}[m_{4k},m_{4k+2})\in\mathcal{I}$ or else $C_1=\bigcup_{k<\omega}[m_{4k+2},m_{4k+4})\in\mathcal{I}$, since $\mathcal{I}$ is nonmeager. Without loss of generality, assume $C_0\in\mathcal{I}$. Then $\bigcup_{k<\omega}A_{m_{4k+1}}\subseteq C_0\cup B\cup A$. Therefore, $\mathcal{I}$ is basic. □

Remark. From the proof, we can see that an ultrafilter is basic iff every sequence which converges to ω has a bounded subsequence.

The next definition gives a notion of ultrafilters which is weaker than p-point.

Definition 5. We say that an ultrafilter $\mathcal{U}$ on $\mathcal{P}(\omega)$ is *basically generated* if it has a filter basis $\mathcal{B} \subseteq \mathcal{U}$ (i.e. $\forall A \in \mathcal{U}\ \exists B \in \mathcal{B}\ B \subseteq A$) with the property that each sequence $\{A_n : n < \omega\} \subseteq \mathcal{B}$ converging to an element of $\mathcal{B}$ has a subsequence $\{A_{n_k} : k < \omega\}$ such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$.

Theorem 6. *Suppose that $\mathcal{U}$ and $\mathcal{U}_n$ are basically generated ultrafilters on $\mathcal{P}(\omega)$ by filter bases which are closed under finite intersection. Then $\mathcal{V} = \lim_{n\to\mathcal{U}} \mathcal{U}_n$ is basically generated by a filter basis which is closed under finite intersections. It follows that whenever $\mathcal{U}$ is basically generated by a filter basis closed under finite intersection, $\mathcal{U}^\alpha$ is basically generated by a filter basis closed under finite intersections, for each $\alpha < \omega_1$. (Here, we abuse notation and let $\mathcal{U}^2 = \mathcal{U} \cdot \mathcal{U}$, $\mathcal{U}^{\alpha+1} = \lim_{n\to\mathcal{U}} \mathcal{U}^\alpha$, and for limit ordinals $\alpha < \omega_1$, $\mathcal{U}^\alpha = \lim_{n\to\mathcal{U}} \mathcal{U}^{\beta_n}$ where β_n is an increasing sequence cofinal in α.)*

Proof. Let $\mathcal{B}, \mathcal{B}_n$ $(n < \omega)$ be filter bases of $\mathcal{U}$, $\mathcal{U}_n$ which are closed under finite intersection and which witness the fact that $\mathcal{U}$, $\mathcal{U}_n$ are basically generated. Let $p_0 : \omega \times \omega \to \omega$ be the projection map onto the first coordinate. Let $\mathcal{C} = \{A \in \mathcal{V} : p_0[A] \in \mathcal{B}$ and for each $n < \omega$, either $(A)_n = \emptyset$ or $(A)_n \in \mathcal{B}_n\}$. Then $\mathcal{C}$ is a filter basis for $\mathcal{V}$ which is closed under finite intersections. Consider a converging sequence $A_n \to B$ in $\mathcal{C}$. Note that $p_0[A_n] \to U$ for some $U \in \mathcal{U}$ containing $p_0[B]$. U might not be in $\mathcal{B}$, but $p_0[B]$ is in $\mathcal{B}$, since $B \in \mathcal{C}$. So for each $n < \omega$, letting $A'_n = A_n \cap (p_0[B] \times \omega)$, $A'_n \in \mathcal{C}$. Note that $A'_n \to B$, $p_0[A'_n] \to p_0[B]$, and all $p_0[A'_n] \in \mathcal{B}$, since $\mathcal{B}$ is closed under finite intersections. Taking a subsequence and reindexing, we have $\bigcap_{n<\omega} p_0[A'_n] \in \mathcal{U}$. Let $p = \bigcap_{n<\omega} p_0[A'_n]$. Note that $p \subseteq \bigcap_{n<\omega} p_0[A_n]$. Enumerate p as n_k. Then for each $k < \omega$ and each $m < \omega$, $(A'_m)_{n_k} = (A_m)_{n_k}$ since $n_k \in p \subseteq p_0[B]$. So, for each $k < \omega$, $(A_m)_{n_k} \to (B)_{n_k}$. Take a decreasing sequence $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_k \supseteq \ldots$ of infinite subsets of ω such that for each k, $\bigcap_{m \in M_k} (A_m)_{n_k} \in \mathcal{U}_{n_k}$. We may assume that $m_k = \min M_k$ is a strictly increasing sequence. Then $C = \bigcap_{k<\omega} A_{m_k} \in \mathcal{V}$. $\square$

Remark. If $\mathcal{U}$ is non-principal, then $\mathcal{U} \cdot \mathcal{U}$ is not a p-point. Thus, there are basically generated ultrafilters which are not p-points.

Theorem 7 (Isbell [11])**.** *There is an ultrafilter $\mathcal{U}_{\text{top}}$ on ω realizing the maximal cofinal type among all directed sets of cardinality continuum, i.e. $\mathcal{U}_{\text{top}} \equiv_T [\mathfrak{c}]^{<\omega}$.*

There are in fact $2^{\mathfrak{c}}$ many different ultrafilters of Tukey top degree, since any collection of independent sets can be used in a canonical way to construct an ultrafilter with top degree. Thus, already we see that for the case of the top Tukey degree, the Rudin-Keisler equivalence relation is strictly finer than the Tukey equivalence relation. Note also that $\mathcal{U}_{\text{top}}$ is not basically representable, or in other words,

Theorem 8. *If $\mathcal{U}$ is a basically generated ultrafilter on ω, then $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$.*

The next theorem gives a canonical form for cofinal maps on p-points. This theorem or similar ideas will be used in the majority of proofs in the rest of this paper.

Theorem 9. *Suppose $\mathcal{U}$ is a basic ultrafilter on ω and that $\mathcal{V}$ is an arbitrary ultrafilter on ω such that $\mathcal{V} \le_T \mathcal{U}$. Then there is a continuous monotone map $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ whose restriction to $\mathcal{U}$ is continuous and has cofinal range in $\mathcal{V}$.*

Proof. Let $g : \mathcal{V} \to \mathcal{U}$ be a Tukey map. Define $f : \mathcal{U} \to \mathcal{V}$ by $f(X) = \bigcap\{Y \in \mathcal{V} : g(Y) \supseteq X\}$. Since g is Tukey, f is well-defined. Note that f is monotone and has cofinal range in $\mathcal{V}$; thus f is a cofinal map.

We claim that there is an $\tilde{X} \in \mathcal{U}$ such that $f : \mathcal{U} \restriction \tilde{X} \to \mathcal{V}$ is continuous. For each $n < \omega$ take an $X_n \in \mathcal{U}$ such that $X_n \cap n = \emptyset$ and for each $s \subseteq n$, for each $k \le n$, if there is a $Y \in \mathcal{U}$ such that $s = Y \cap (n+1)$ and $k \notin f(Y)$, then $k \notin f(s \cup X_n)$. Let $Y \in \mathcal{U}$ be such that for each $n < \omega$, $Y \subseteq^* X_n$. Let $0 = n_0 < n_1 < \dots$ be such that for each $i < \omega$, for each $n \le n_i$, $Y \setminus n_{i+1} \subseteq X_n$. Without loss of generality, assume $X_0 \supseteq X_1 \supseteq \dots$. Let $Z = \bigcup_{i=0}^{\infty}[n_{2i+1}, n_{2i+2})$. Without loss of generality, assume that $Z \notin \mathcal{U}$. Let $\tilde{X} = Y \setminus Z$. Then $f \restriction \mathcal{U} \restriction \tilde{X}$ is continuous: Given $k < \omega$ and $W \in \mathcal{U} \restriction \tilde{X}$, letting i_k denote the least i for which $n_{2i_k+1} \ge k$ and letting $s = W \cap n_{2i_k+1}$, we have that $W \setminus n_{2i_k+1} \subseteq \tilde{X} \setminus n_{2i_k+1} = \tilde{X} \setminus n_{2i_k+2} \subseteq Y \setminus n_{2i_k+2} \subseteq X_{n_{2i_k+1}}$. Therefore, $k \in f(W)$ iff $k \in f(s \cup X_{n_{2i_k+1}})$ iff $k \in f(s \cup (\tilde{X} \setminus n_{2i_k+1}))$.

Now extend f on $\mathcal{U} \restriction \tilde{X}$ to all of $\mathcal{U}$ by defining $f'(X) = f(X \cap \tilde{X})$, for $X \in \mathcal{U}$. Then $f' : \mathcal{U} \to \mathcal{V}$ is monotone continuous, since for each $X \in \mathcal{U}$ and $k < \omega$, $k \in f'(X)$ iff $k \in f(X \cap \tilde{X})$ iff $k \in f(s \cup (\tilde{X} \setminus n_{2i_k+1}))$, where $s = X \cap \tilde{X} \cap n_{2i_k+1}$.

For an arbitrary $X \subseteq \omega$ set

$$f^*(X) = \bigcap\{f'(Z) : Z \supseteq X \text{ and } Z \text{ is cofinite}\}. \tag{1}$$

Let $Z \in \mathcal{U}$ be given. Letting $Z_n = (Z \cap n) \cup [n, \omega)$, $Z_n \to Z$, so $f'(Z_n) \to f'(Z)$, since f' is continuous on $\mathcal{U}$. Since each $f'(Z_n) \supseteq f'(Z)$, we have that $f^*(Z) = \bigcap_{n<\omega} f(Z_n) = f'(Z)$. So $f^* \restriction \mathcal{U} = f'$.

To see that f^* is continuous, let $Z \subseteq \omega$ and $k < \omega$, and let $Z_n = (Z \cap n) \cup [n, \omega)$. Then $k \in f^*(Z)$ iff for each $n < \omega$, $k \in f(Z_n)$ iff for each $n \ge n_{2i_k+1}$, $k \in f(Z_n)$ iff for each $n \ge n_{2i_k+1}$, $k \in f(Z_n \cap \tilde{X})$ iff $k \in f(s \cup \tilde{X} \setminus n_{2i_k+2})$, where $s = Z \cap n_{2i_k+1} \cap \tilde{X}$. By its definition, it is easy to see that f^* is monotone. $\square$

Remark. Note that this gives the canonical form of cofinal maps that is likely going to be the main object of study in this area from now on: Tukey reductions $\mathcal{U} \ge_T \mathcal{V}$ for $\mathcal{U}$ a p-point are all given using the formula (1) for f a monotone continuous map defined on the Fréchet filter.

Remark. Whereas the top Tukey degree has size $2^{\mathfrak{c}}$, the previous theorem implies that the Tukey degree of any p-point has size $\mathfrak{c}$.

Corollary 10. *Every $\le_T$-chain of p-points on ω has cardinality $\le \mathfrak{c}^+$.*

Proof. Note that Theorem 9 shows that every Tukey chain $\mathcal{F} \subseteq \{\text{p-points}\}$ is $\mathfrak{c}^+$-like, that is, $|\{\mathcal{V} \in \mathcal{F} : \mathcal{V} \le_T \mathcal{U}\}| \le \mathfrak{c}$ for all $\mathcal{U} \in \mathcal{F}$. □

Corollary 11. *Every family $\mathcal{F}$ of p-points on ω of cardinality $> \mathfrak{c}^+$ contains a subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ of equal size such that $\mathcal{U} \not\le_T \mathcal{V}$ whenever $\mathcal{U} \ne \mathcal{V}$ are in $\mathcal{F}_0$.*

Proof. Apply Hajnal's free-set lemma (see [12]). □

Remark. A similar trick was used by Rudin and Shelah in [20] in part of their proof that there are always $2^{\mathfrak{c}}$ many Rudin-Keisler incomparable ultrafilters.

Next, we use Theorem 9 to see that some strength of selective ultrafilters is preserved downward in the Tukey ordering.

Theorem 12. *Suppose $\mathcal{U} \ge_T \mathcal{V}$ for a pair of ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on ω. Suppose further that $\mathcal{U}$ is selective. Then $\mathcal{V}$ is basically generated.*

Proof. By Theorem 9, let $f : [\omega]^\omega \to \mathcal{P}(\omega)$ be a continuous monotone map such that $f[\mathcal{U}] \subseteq \mathcal{V}$ and $f[\mathcal{U}]$ generates $\mathcal{V}$. Shrinking to a cube $[M]^\omega$ for $M \in \mathcal{U}$ by applying the selective version of the Prömel-Voight canonical form of the Galvin-Prikry Theorem, we may assume that there is a Lipschitz map $\varphi : [\omega]^\omega \to \mathcal{P}(\omega)$ such that $\varphi(X) \subseteq X$ and a 1-1 homeomorphism $\psi : \text{range}(\varphi) \to \mathcal{P}(\omega)$ such that $f = \psi \circ \varphi$. With these reductions we can now claim that every converging sequence $X_n \to X$ of elements of $\mathcal{B} = f[\mathcal{U} \restriction M] \subseteq \mathcal{V}$ has a subsequence X_{n_k} such that $\bigcap_{k<\omega} X_{n_k} \in \mathcal{V}$. To see this, note that if $Y = \psi^{-1}(X)$ and $Y_n = \psi^{-1}(X_n)$, then $Y_n \to Y$. Let $K = \{A \in \mathcal{U} : \varphi(A) = Y\}$ and $K_n = \{A \in \mathcal{U} : \varphi(A) = Y_n\}$ $(n \in \omega)$. Then K and K_n are compact subsets of $\mathcal{U}$ such that $K_n \to K$. So in particular for an arbitrary choice $A_n \in K_n$, $(n \in \omega)$ we can find a subsequence A_{n_k} converging to a member B in K. So in particular A_{n_k} is a convergent sequence in $\mathcal{U}$. Since $\mathcal{U}$ is basic there is a further subsequence $A_{n_{k_i}}$ such that

$$A = \bigcap_{i<\omega} A_{n_{k_i}} \in \mathcal{U}.$$

It follows that $X_{n_{k_i}} = f(A_{n_{k_i}}) \supseteq f(A)$ for all $i < \omega$ and so in particular, $f(A) \in \mathcal{V}$ and $f(A) \subseteq \bigcap_{i<\omega} X_{n_{k_i}}$. □

It will be shown in Section 3 that for each selective ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$; hence $\mathcal{U} \equiv_T \mathcal{V}$ does not imply that $\mathcal{V}$ is selective.

Question 13. If $\mathcal{U}$ is a p-point and $\mathcal{U} \ge_T \mathcal{V}$, does it follow that $\mathcal{V}$ is basically generated?

Question 14. Can Theorem 9 be improved to show that if $\mathcal{U}$ is basically generated and $\mathcal{U} \geq_T \mathcal{V}$, then there is a continuous (or definable) monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ witnessing this?

More generally,

Question 15. If $\mathcal{V} \leq_T \mathcal{U} <_T [\mathfrak{c}]^{<\omega}$, then is there a continuous (or definable) monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ witnessing this?

One might first try to show that the existence of a continuous cofinal map propagates Tukey downwards, or in other words,

Question 16. Suppose that $\mathcal{U}$ is such that whenever $\mathcal{U} \geq_T \mathcal{V}$ then there is a continuous monotone cofinal map from $\mathcal{U}$ to $\mathcal{V}$. If $\mathcal{U} \geq_T \mathcal{W}$, then does it follow that for each $\mathcal{V} \leq_T \mathcal{W}$ there is a continuous monotone cofinal map from $\mathcal{W}$ into $\mathcal{V}$?

3. Comparing Tukey types of ultrafilters with $(\omega^\omega, \leq)$

In this section we investigate which ultrafilters are above $(\omega^\omega, \leq)$, where $h \leq g$ iff for each $n < \omega$, $h(n) \leq g(n)$.

Fact 17. *If $\mathcal{U}$ is a rapid ultrafilter, then $\mathcal{U} \geq_T \omega^\omega$.*

Proof. Define $f : \mathcal{U} \to \omega^\omega$ by letting $f(X)$ be the function which enumerates all but the least element of X in strictly increasing order. Then f is a cofinal map. □

Hence each selective ultrafilter and each q-point is Tukey above ω^ω.

Fact 18. *For each ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \geq_T \omega^\omega$.*

Proof. Define $f : \mathcal{U} \cdot \mathcal{U} \to \omega^\omega$ by $f(A) = \langle \min(A)_{n_k} \rangle$, where $\langle n_k : k < \omega \rangle$ enumerates those n for which $(A)_n \in \mathcal{U}$. Then f is a cofinal map. □

Theorem 19. *For any ultrafilters $\mathcal{U}, \mathcal{U}_n$ $(n < \omega)$, $\lim_{n \to \mathcal{U}} \mathcal{U}_n \leq_T \mathcal{U} \times \Pi_{n<\omega} \mathcal{U}_n$, where $\mathcal{U} \times \Pi_{n<\omega} \mathcal{U}_n$ is given its natural product ordering. In particular, $\mathcal{U} \cdot \mathcal{U} \leq_T \mathcal{U}^\omega$.*

Proof. Let $\mathcal{V} = \lim_{n \to \mathcal{U}} \mathcal{U}_n$. Recall the basis $\mathcal{B}$ for $\lim_{n \to \mathcal{U}} \mathcal{U}_n = \mathcal{V}$ from the proof of Theorem 6. It suffices to construct a Tukey map $g : \mathcal{B} \to \mathcal{U} \times \Pi_{n<\omega} \mathcal{U}_n$. Given $A \in \mathcal{B}$ let $g(A) = (p_0[A], (q_n(A) : n < \omega))$, where $q_n(A) = (A)_n$ if $n \in p_0[A]$ and $q_n(A) = \omega$ otherwise. To verify g is a Tukey map consider $(C, (D_n : n < \omega)) \in \mathcal{U} \times \Pi_{n<\omega} \mathcal{U}_n$ and let $\mathcal{X} = \{A \in \mathcal{B} : p_0[A] \supseteq C \text{ and } (\forall n < \omega) q_n(A) \supseteq D_n\}$. Let $B = \bigcap \mathcal{X}$. Then $p_0[B] \supseteq C$ and for each $n \in C$, $(B)_n \supseteq D_n$, so $B \in \mathcal{V}$. □

Theorem 20. *If $\mathcal{U}$ is a p-point, then $\mathcal{U}^\omega \equiv_T \mathcal{U} \times \omega^\omega$ and therefore $\mathcal{U}^\omega \equiv_T \mathcal{U} \cdot \mathcal{U} \cdot \mathcal{U}$.*

Proof. Given a sequence $(A_n)_{n<\omega} \in \mathcal{U}^\omega$, choose a $B \in \mathcal{U}$ and an $h : \omega \to \omega$ such that $B \setminus h(n) \subseteq A_n$ for each n, and set $g((A_n)_{n<\omega}) = (B, h)$. Then g is a Tukey map, so $\mathcal{U}^\omega \leq_T \mathcal{U} \times \omega^\omega$. On the other hand, $\omega^\omega \leq_T \mathcal{U} \cdot \mathcal{U} \leq_T \mathcal{U}^\omega$, by Fact 18 and Theorem 19. So $\mathcal{U} \times \omega^\omega \leq_T \mathcal{U} \times \mathcal{U}^\omega = \mathcal{U}^\omega$. □

Corollary 21. *If $\mathcal{V}$ is a p-point, $\mathcal{V} \geq_T \omega^\omega$, and $\mathcal{U}$ is any ultrafilter, then $\mathcal{U} \cdot \mathcal{V} \equiv_T \mathcal{U} \times \mathcal{V}$.*

Proof. By Theorem 19, $\mathcal{U} \cdot \mathcal{V} \leq_T \mathcal{U} \times \mathcal{V}^\omega$. Since $\mathcal{V}$ is a p-point, $\mathcal{V}^\omega \equiv_T \mathcal{V} \times \omega^\omega$, by Theorem 20. $\mathcal{V} \geq_T \omega^\omega$ implies that $\mathcal{V} \times \omega^\omega \equiv_T \mathcal{V}$. Therefore, $\mathcal{U} \cdot \mathcal{V} \leq_T \mathcal{U} \times \mathcal{V}^\omega \equiv_T \mathcal{U} \times \mathcal{V} \leq_T \mathcal{U} \cdot \mathcal{V}$. □

Theorem 22. *The following are equivalent for a p-point $\mathcal{U}$ on $\mathcal{P}(\omega)$.*
(1) *$\mathcal{U} \geq_T \omega^\omega$;*
(2) *$\mathcal{U} \equiv_T \mathcal{U}^\omega$;*
(3) *$\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$.*

Proof. Suppose $\mathcal{U} \geq_T \omega^\omega$. By Theorem 20, $\mathcal{U}^\omega \equiv_T \mathcal{U} \times \omega^\omega \leq_T \mathcal{U} \leq_T \mathcal{U}^\omega$. Suppose $\mathcal{U} \equiv_T \mathcal{U}^\omega$. Since always $\mathcal{U} \leq_T \mathcal{U} \cdot \mathcal{U}$, and $\mathcal{U} \cdot \mathcal{U} \leq_T \mathcal{U}^\omega$ by Theorem 19, we have that $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$. If $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$, then since $\mathcal{U} \cdot \mathcal{U} \geq_T \omega^\omega$, we have that $\mathcal{U} \geq_T \omega^\omega$. □

We remark that the p-point property was only used for (1) implies (2).

Corollary 23. *If $\mathcal{U}$ is a p-point of cofinality $< \mathfrak{d}$, then $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U}$.*

Remark. Such an ultrafilter $\mathcal{U}$ exists in any extension of a model of CH by a countable support iteration of length ω_2 of superperfect-set forcing since by a result of Shelah such an iteration preserves p-points.

Corollary 24. *If $\mathcal{U}$ is a rapid p-point then $\mathcal{U}^\omega \equiv_T \mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$.*

Remark. By Corollary 24, for each selective ultrafilter $\mathcal{U}$, the Tukey degree of $\mathcal{U}$ is strictly coarser than the Rudin-Keisler degree of $\mathcal{U}$, even though they both have cardinality $\mathfrak{c}$. That is, if $\mathcal{U}$ is selective, then $\mathcal{U} \cdot \mathcal{U}$ is not a p-point yet $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$. However, if $\mathcal{U} \equiv_{RK} \mathcal{V}$ then $\mathcal{V}$ is selective. We remark here that Todorcevic has shown that if $\mathcal{U}$ is selective, $\mathcal{V}$ is a p-point, and $\mathcal{U} \geq_T \mathcal{V}$, then $\mathcal{U} \geq_{RK} \mathcal{V}$, and hence, $\mathcal{V} \equiv_{RK} \mathcal{U}$. (See [18].) Hence, although the Tukey class of a selective ultrafilter includes non-p-points, any two selective ultrafilters in the same Tukey class are isomorphic.

Theorem 25. *Assuming $\mathfrak{p} = \mathfrak{c}$, there is a p-point $\mathcal{U}$ such that $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U}_{top}$.*

Proof. $\mathcal{U}$ is built to be generated by a $\supseteq^*$ chain $\langle A_\alpha : \alpha < \mathfrak{c} \rangle$ of infinite subsets of ω diagonalizing over all Souslin-measurable mappings of the form $f_\alpha : \omega^\omega \to [\omega]^\omega$ ($\alpha < \mathfrak{c}$). Let $\{X_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\omega)$. Given $\alpha < \mathfrak{c}$ and $\{A_\xi : \xi < \alpha\}$ we first find $A_\alpha \in [\omega]^\omega$ such that $A_\alpha \subseteq^* A_\xi$ for all $\xi < \alpha$ and also $A_\alpha \subseteq X_\alpha$ or $A_\alpha \subseteq X_\alpha^c$. We need to see that we can choose $A_{\alpha+1} \in [A_\alpha]^\omega$ such that $f_\alpha : \omega^\omega \to [\omega]^\omega$ fails to be a Tukey map for every nonprincipal ultrafilter $\mathcal{U}$ on ω such that $A_{\alpha+1} \in \mathcal{U}$. If there is $x \in \omega^\omega$ such that $A_\alpha \setminus f_\alpha(x)$ is infinite we take $A_{\alpha+1} = A_\alpha \setminus f_\alpha(x)$. Otherwise, $A_\alpha \subseteq^* f_\alpha(x)$ for all $x \in \omega^\omega$. Then we can find an $n \in \omega$ such that $P_n = \{x \in \omega^\omega : A_\alpha \setminus n \subseteq f_\alpha(x)\}$ is not bounded in ω^ω

relative to the ordering of eventual domination. In particular, there is $k \in \omega$ and $\{x_i : i < \omega\} \subseteq P_n$ such that $x_i(k) \geq i$ for all $i < \omega$. It follows that $\{x_i : i < \omega\}$ is unbounded in ω^ω but its image $\{f_\alpha(x_i) : i < \omega\}$ is bounded in any ultrafilter containing A_α.

Let $\mathcal{U}$ be the p-point generated by the tower $\{A_\alpha : \alpha < \mathfrak{c}\}$. We need to show that $\mathcal{U} \not\geq_T \omega^\omega$. Otherwise, applying Theorem 5.3 (i) of [21] there is a Souslin measurable map $f : \omega^\omega \to \mathcal{U}$. So there is an $\alpha < \mathfrak{c}$ such that $f_\alpha = f$. At this stage we have produced an infinite sequence $\{x_i : i < \omega\} \subseteq \omega^\omega$ that is unbounded in ω^ω but $\bigcap_{i<\omega} f(x_i) \in \mathcal{U}$. So f could not have been a Tukey map. □

Question 26. Is there an ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U} <_T \mathcal{U}_{top}$?

Remark. Using some assumptions like $\mathfrak{p} = \mathfrak{c}$, it seems possible to get Tukey chains of p-points of order-type $\mathfrak{c}^+$ which is, as we know, maximal possible. By Corollary 35 below, CH implies there are Tukey chains of p-points of length $\mathfrak{c}$. Dilip Raghavan has shown that, assuming CH, there is a Tukey chain of p-points isomorphic to the reals (see [18]).

Question 27. Is there an ultrafilter $\mathcal{U} <_T \mathcal{U}_{\text{top}}$ which is not Tukey reducible to any p-point?

Question 28. Is every basically generated ultrafilter Tukey reducible to a p-point?

Both of the preceding two questions are answered using the assumption $\mathcal{U} \not\geq_T \omega^\omega$ for any p-point $\mathcal{U}$ (which is true in the iterated superperfect extension). Namely, then $\mathcal{U} \cdot \mathcal{U} \not\leq_T \mathcal{V}$ for every ultrafilter $\mathcal{U}$ and every p-point $\mathcal{V}$.

Question 29. Is there a p-ideal I on ω which is not countably generated but $I \not\geq_T \omega^\omega$?

Remark. If $\mathfrak{b} \neq \mathfrak{d}$ there is such a p-ideal, so the question is whether we can get one with no extra set-theoretic assumptions.

Question 30. Does $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U} <_T \mathcal{U}_{\text{top}}$ imply $\mathcal{U}$ is basically generated?

4. Antichains, chains, and incomparable predecessors

We now investigate the structure of the Tukey degrees of p-points and selective ultrafilters in terms of which chains, antichains, and incomparable ultrafilters with a common upper bound embed into the Tukey degrees.

Theorem 31. (1) *Assuming cov$(\mathscr{M}) = \mathfrak{c}$, if κ is a cardinal such that $2^{<\kappa} = \mathfrak{c}$, then there are 2^κ pairwise Tukey incomparable selective ultrafilters.*

(2) *Assuming $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$, if κ is a cardinal such that $2^{<\kappa} = \mathfrak{c}$, then there are 2^κ pairwise Tukey incomparable p-points*

Proof. We start with the following lemma.

Lemma 32. *Suppose $\mathcal{U}_0, \mathcal{V}_0$ are filters but not ultrafilters. Suppose we are given a function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$. Then there are filters $\mathcal{U}_1 \supseteq \mathcal{U}_0$, $\mathcal{V}_1 \supseteq \mathcal{V}_0$ such that for any ultrafilters $\mathcal{U} \supseteq \mathcal{U}_1$, $\mathcal{V} \supseteq \mathcal{V}_1$, f is not not a cofinal map from $\mathcal{U}$ into $\mathcal{V}$.*

Proof. Suppose $\mathcal{U}_0, \mathcal{V}_0$ are filters but not ultrafilters, and $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$. If there is an $X \in \mathcal{U}_0^+$ such that $f(X)^c \in \mathcal{V}_0^+$, let $\mathcal{U}_1$ be the filter generated by $\mathcal{U}_0 \cup \{X\}$ and let $\mathcal{V}_1$ be the filter generated by $\mathcal{V}_0 \cup \{f(X)^c\}$. If not, let $\mathcal{U}_1 = \mathcal{U}_0$ and $\mathcal{V}_1 = \mathcal{V}_0$.

Let $\mathcal{U}$, $\mathcal{V}$ be ultrafilters extending $\mathcal{U}_1$, $\mathcal{V}_1$, respectively. Suppose towards a contradiction that $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a cofinal map witnessing $\mathcal{U} \geq_T \mathcal{V}$. Since $\mathcal{V}_0$ is not an ultrafilter, there is a $Y \in \mathcal{V}$ such that $Y^c \in \mathcal{V}_0^+$. There is an $X \in \mathcal{U}$ such that $f(X) \subseteq Y$, since f is a cofinal map. Thus, in the construction of $\mathcal{U}_1$ and $\mathcal{V}_1$, there is an $X' \in \mathcal{U}_0^+$ such that $f(X')^c \in \mathcal{V}_0^+$, and X' was put into $\mathcal{U}_1$ and $f(X')^c$ was put into $\mathcal{V}_1$. But then $f(X') \in \mathcal{V}$ and $f(X')^c \in \mathcal{V}$; contradiction. □

Now we prove (1). Let κ be the largest cardinal such that $2^{<\kappa} = \mathfrak{c}$. Then $2^\kappa > \mathfrak{c}$ and $\kappa \leq \mathfrak{c}$. Recall the following theorem of Ketonen [13]: $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ iff every filter base of size less than $\mathfrak{c}$ can be extended to a selective ultrafilter. Recall that $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ implies $\mathfrak{u} = \mathfrak{c}$.

Let P denote the collection of partitions of ω. Let $\pi : \mathfrak{c} \to P$ be a bijection. Let $\theta : \mathfrak{c} \to 2^{<\kappa}$ be such that for each $s \in 2^{<\kappa}$, $\theta^{-1}(s)$ is cofinal in $\mathfrak{c}$. Let $\sigma : \mathfrak{c} \to \mathcal{H}$, where $\mathcal{H}$ is the collection of continuous functions $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$. We do the construction in $\mathfrak{c}$ steps. Let $\mathcal{U}_{\langle\rangle}$ be the Fréchet filter on ω. At stage $\alpha < \mathfrak{c}$, let s denote $\theta(\alpha)$. If $\mathrm{lh}(s) = \gamma$ is a limit ordinal, let $\mathcal{U}_s = \bigcup_{\alpha<\gamma} \mathcal{U}_{s\restriction\alpha}$. If $\mathrm{lh}(s)$ is a successor ordinal, call it $\gamma + 1$, and for each $\zeta \leq \gamma$, $\mathcal{U}_{s\restriction\zeta}$ has been constructed but $\mathcal{U}_s$ has not yet been constructed, do the following things: (If we are not in this situation, do nothing and proceed to stage $\alpha + 1$.)

Let $u = s \restriction \zeta$. Let $T_\alpha = \{t \in 2^{<\kappa} : \mathcal{U}_t$ has already been constructed before stage $\alpha\} \bigcup \{u^\frown 0, u^\frown 1\}$. Note that T_α is a subtree of $2^{<\kappa}$. Let $L_\alpha = \{t \in 2^{\leq\kappa} : t$ is the union of a maximal chain in $T_\alpha\}$. (Note: some of the t in L_α may be elements of 2^κ; this is fine.) L_α lists the t for which either $t \in 2^\kappa$ and for all $\alpha < \kappa$, $\mathcal{U}_{t\restriction\alpha}$ has been constructed and $\mathcal{U}_t \supseteq \bigcup_{\alpha<\kappa} \mathcal{U}_{t\restriction\alpha}$ (here we continue indexing the filter corresponding to the branch t by $\mathcal{U}_t$ even though we will continue adding sets to it at the remaining stages); or $t \in 2^{<\kappa}$ and $\mathcal{U}_t$ has been constructed but $\mathcal{U}_{t^\frown i}$ has not been constructed for $i < 2$. Note that $u^\frown 0, u^\frown 1 \in L_\alpha$.

1. Choose $X \subseteq \omega$ such that $X, X^c \in \mathcal{U}_u^+$. Let $\mathcal{U}_{u^\frown 0}, \mathcal{U}_{u^\frown 1}$ be the filters generated by $\mathcal{U}_u \cup \{X\}, \mathcal{U}_u \cup \{X^c\}$, respectively.

2. Apply Lemma 32 for the first α functions from $\mathcal{H}$ to $\mathcal{U}_t, \mathcal{U}_v$ for all ordered pairs (t, v) with $t, v \in L_\alpha$, extending them to new filters (also denoted by $\mathcal{U}_t, \mathcal{U}_v$) so that each of the first α functions from $\mathcal{H}$ do not witness Tukey comparability of any ultrafilter extension of any pair $\mathcal{U}_t, \mathcal{U}_v$, for $(t, v) \in L_\alpha \times L_\alpha$, $t \neq v$. Since $|L_\alpha| < \mathfrak{c}$, the augmented bases do not generate ultrafilters.

3. Given $\langle I_n^\zeta : n < \omega\rangle$, $\zeta < \alpha$, the first α partitions listed by $\pi \restriction \alpha$, do the Ketonen argument to add at most α many subsets of ω to each $\mathcal{U}_t$, $t \in L_\alpha$, to

ensure selectivity of any ultrafilter containing $\mathcal{U}_t$ with respect to these α many partitions.

4. We fix at the beginning of the construction an enumeration $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$ of $[\omega]^\omega$. For each $t \in L_\alpha$ and each $\beta < \alpha$, if $B_\beta \in \mathcal{U}_t^+$, put B_β into $\mathcal{U}_t$. Otherwise, already $B_\beta^c \in \mathcal{U}_t$. This guarantees that after the $\mathfrak{c}$ steps of our construction, the filters $\mathcal{U}_v$, $v \in 2^\kappa$, will be ultrafilters.

Now, in $\mathfrak{c}$ steps, we have constructed for each $v \in 2^\kappa$ an ultrafilter $\mathcal{U}_v$ such that $\mathcal{U}_v$ is selective, and for $t \neq v$ in 2^κ, $\mathcal{U}_v \neq \mathcal{U}_t$, and $\mathcal{U}_v \not\leq_T \mathcal{U}_t$.

Hence, we have 2^κ Tukey incomparable selective ultrafilters.

For (2), the proof goes in exactly the same steps as for Theorem 31, only this time for 3., we just use Ketonen's argument from [13] to extend filter bases of size less than $\mathfrak{d}$ to p-points to ensure that the ultrafilters obtained in the end will be p-points. □

One way of making Tukey increasing chains of ultrafilters is by using κ-OK points. Kunen remarked in [14] that if $\mathcal{U}$ is κ-OK and $\kappa > \mathrm{cof}(\mathcal{U})$, then $\mathcal{U}$ is a p-point. It is easy to see the following.

Fact 33. *If $\mathcal{U}$ is κ-OK but not a p-point, then $\mathcal{U} \geq_T [\kappa]^{<\omega}$. Hence, if $\mathcal{U}$ is κ-OK but not a p-point, then cof($\mathcal{U}$) $= \kappa$ iff $\mathcal{U} \equiv_T [\kappa]^{<\omega}$.*

It follows that if there are κ-OK non p-points with cofinality κ for each uncountable $\kappa < \mathfrak{c}$, then there is a strictly increasing chain of ultrafilters of length α, where α is such that $\aleph_\alpha = \mathfrak{c}$.

We now give a general method for building Tukey increasing chains of p-points.

Theorem 34. *Assuming CH, for each p-point D there is a p-point E such that $E >_T D$.*

Proof. We use the notation from [2]. In [Theorem 6, [2]], Blass proved assuming MA that given a p-point D one can construct a p-point $E >_{RK} D$. Hence, $E \geq_T D$. His construction can be slightly modified to kill all possible cofinal maps so that we get a p-point E which is Tukey strictly above D.

Let D be a given p-point. Fix a bijective pairing $J : \omega \times \omega \to \omega$ with inverse (π_1, π_2), and identify ω with $\omega \times \omega$ via J. A subset $Y \subseteq \omega \times \omega$ is *small* iff the function $c_Y(i) := |\{y \in \omega : (i, y) \in Y\}|$ is bounded by some $n < \omega$ for all i in some $X \in D$. Otherwise Y is *large*. If E is an ultrafilter on $\omega \times \omega$, containing no small set, then $E >_{RK} D$.

Claim. Let $Y \subseteq \omega \times \omega$. Y is large iff there is a $W \in D$ such that $c_Y \restriction W$ is bounded below by a non-decreasing, unbounded step function on W.

Proof. First note that for any $Y \subseteq \omega \times \omega$, Y is large iff for each $n < \omega$, $\{i < \omega : c_Y(i) \leq n\} \notin D$ iff for each $n < \omega$, $\{i < \omega : c_Y(i) > n\} \in D$. Let $Y \subseteq \omega \times \omega$ be large. For each $n < \omega$, define $W_n = \{i < \omega : c_Y(i) > n\}$. Then each $W_n \in D$ and $W_n \supseteq W_{n+1}$. Since D is a p-point, there is a $W \in D$ such that for each $n < \omega$,

$W \subseteq^* W_n$. Let k_n be a strictly increasing sequence such that for each $n < \omega$, $W \setminus k_n \subseteq W_n$. Note that for each $i \in W \setminus k_n$, $c_Y(i) > n$. Therefore, for each $n < \omega$, for each $i \in W \cap (k_n, k_{n+1}]$, $c_Y(i) > n$. Hence, c_Y is bounded below on W by the function $h : W \to \omega$, where for each n, for each $i \in W \cap (k_n, k_{n+1}]$, $h(i) = n$.

For the reverse direction, if $Y \subseteq \omega \times \omega$, $W \in D$ and $c_Y \restriction W$ is bounded below by a non-decreasing unbounded step function, then for each $n < \omega$, $\{i \in W : c_Y(i) \leq n\}$ is finite, hence $\{i < \omega : c_Y(i) \leq n\} \notin D$. Therefore, Y is large. □

Do the construction of Blass (which builds a p-point $E \subseteq \omega \times \omega$ containing no small set) adding in the following additional step cofinally often in the construction of length $\mathfrak{c} = \omega_1$. At stage $\alpha < \mathfrak{c}$ in the construction, we have a filter base E_α. Enumerating all continuous monotone maps from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)$, let f denote the one under consideration at stage α. If $f"D$ does not generate an ultrafilter, there is nothing to do; so assume $f"D$ generates an ultrafilter. We shall show that there is a set Z such that $Z \cap Y$ and $Z^c \cap Y$ are large for each $Y \in E_\alpha$. Then if $Z \in f"D$, put Z^c into $E_{\alpha+1}$; and if $Z^c \in f"D$ then put Z into $E_{\alpha+1}$. Then f cannot be a cofinal map from any ultrafilter extending $E_{\alpha+1}$ into D.

Since we are assuming CH, the filter E_α is countably generated. By the induction hypothesis, every element of E_α is large. Let X_n $(n < \omega)$ be a base for E_α such that each $X_n \supseteq X_{n+1}$. Since each X_n is large, there is a $W_n \in D$ and a non-decreasing unbounded step function f_n defined on W_n such that for each $i \in W_n$, $c_{X_n}(i) \geq f_n(i)$. Without loss of generality, we can assume that each $W_n \supseteq W_{n+1}$. Since D is a p-point, let $W \in D$ satisfy for each $n < \omega$, $W \subseteq^* W_n$.

We shall build disjoint $Z_0, Z_1 \subseteq \omega \times \omega$ and a strictly increasing sequence $\langle k_n : n < \omega \rangle$ as follows. Let k_0 be least such that $f_0 \restriction ([k_0, \omega) \cap W_0) \geq 2$ and $W \setminus k_0 \subseteq W_0$. In general, choose $k_{m+1} > k_m$ satisfying

(1) for each $j \leq m$, $f_j \restriction ([k_m, \omega) \cap W_j) \geq 2(m+1)^2$;
(2) $W \setminus k_m \subseteq W_m$ (and hence for each $j < m$, $W \setminus k_m \subseteq W_j$)

Given $m < \omega$ and $i \in W \cap [k_m, k_{m+1})$, for each $j \leq m$, choose $x_{i,j,l}, y_{i,j,l}$, $l \leq m$, distinct in $\{z \in \omega : (i, z) \in X_j\} \setminus \{x_{i,q,l}, y_{i,q,l} : l \leq m,\ q < j\}$. (This is possible since for each $i \in W \cap [k_m, k_{m+1})$, for each $j \leq m$, $c_{X_j}(i) \geq f_j(i) \geq 2(m+1)^2$. For each $i \in W$, define m_i to be the integer m for which $i \in [k_m, k_{m+1})$. Define $Z_0 = \{(i, x_{i,j,l}) : i \in W,\ j \leq m_i,\ l \leq m_i\}$; $Z_1 = \{(i, y_{i,j,l}) : i \in W,\ j \leq m_i,\ l \leq m_i\}$. Note that Z_0, Z_1 are large, disjoint, and have large intersection with each X_n. Letting $Z = Z_0$, then both Z and Z^c have the desired properties.

Interweaving the above argument with Blass' construction, we obtain a p-point $E >_T D$. □

Remark. Dilip Raghavan has independently observed Theorem 34.

Corollary 35. *Assuming CH, there is a Tukey strictly increasing chain of p-points of order type* $\mathfrak{c}$.

Proof. In Theorem 7 of [2], Blass proved that MA implies that any RK increasing chain of p-points of length ω has an RK upper bound which is a p-point. The p-point E constructed in the above Theorem 34 is also RK strictly above D, so for any $\alpha < \omega_1$, we can construct ω-length chains of p-points $D_{\alpha+n}$, where each $D_{\alpha+n+1} >_T D_{\alpha+n}$ and $D_{\alpha+n+1} >_{RK} D_{\alpha+n}$ ($\alpha < \omega_1$) and then use [Theorem 7, [2]] to find a p-point RK above each $D_{\alpha+n}$, $n < \omega$, hence also Tukey above them. □

The following questions are to be answered assuming that p-points exist or some assumption that guarantees their existence.

Question 36. Is there a Tukey strictly increasing chain of p-points of length $\mathfrak{c}^+$?

The Tukey increasing chain of p-points constructed in the proof of Theorem 34 is also Rudin-Keisler increasing. This leads to the next question.

Question 37. Given any strictly Tukey increasing sequence of p-points of length ω, is there always a p-point Tukey above all of them?

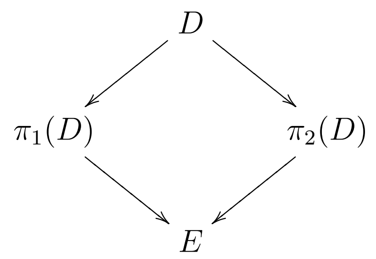
In particular,

Question 38. Given any p-point $\mathcal{V}$, is there a p-point $\mathcal{U}$ such that $\mathcal{U} >_T \mathcal{V}$, but $\mathcal{U}$ and $\mathcal{V}$ are RK-incomparable?

If the answer to Question 38 is no, then the answer to Question 37 is yes.

We now show that there are incomparable p-points with a common upper bound which is a p-point.

Theorem 39. *Assume Martin's Axiom. There is a p-point D with two Tukey-incomparable Tukey predecessors $\pi_1(D)$ and $\pi_2(D)$ which are also p-points, which in turn have a common Tukey lower bound E which is also a p-point. (In the following diagram, arrows represent strict Tukey reducibility.)*



Proof. In [Theorem 9, [2]], Blass proved that assuming Martin's Axiom, there is a p-point with two RK-incomparable predecessors. He used the following notions. A subset of $\omega \times \omega$ of the form $P \times Q$, where P and Q are subsets of ω of cardinality $n < \omega$, is called an *n-square*. A subset of $\omega \times \omega$ is called *large* if it includes an n-square for every n, and *small* otherwise. Blass' construction builds a p-point $D \subseteq \omega \times \omega$ consisting of large sets such that $\pi_1(D)$ and $\pi_2(D)$ are RK-incomparable. For $i = 1, 2$, $\pi_i(D) \leq_{RK} D$, hence $\pi_i(D)$ are also p-points and are $\leq_T D$. By cofinally weaving the following step into Blass' construction, we can build a p-point $D \subseteq \omega \times \omega$ of large sets such that $\pi_1(D)$ and $\pi_2(D)$ are Tukey-incomparable. As long as we only add large sets to the filter bases, Blass' construction can be simultaneously carried out.

Claim. Given $\mathcal{Y}$ a filter with base of size $< \mathfrak{c}$ and $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ monotone, there are large sets $W_i \subseteq \omega \times \omega$, $i = 1, 2$, such every set in the filter generated by $\mathcal{Y} \cup \{W_1, W_2\}$ is large, and for any ultrafilter $D \supseteq \{W_1, W_2\} \cup \mathcal{Y}$ consisting only of large sets, f is not a cofinal map from $\pi_i(D) \to \pi_j(D)$, for $i \neq j$.

Proof. By [Lemma 2, Section 6, [2]] (which uses MA), there is a large set X such that $X \subseteq^* Y$ for each $Y \in \mathcal{Y}$. We shall say that a set $Z \subseteq \omega \times \omega$ is *addable* to $\mathcal{Y}$ if $Z \cap Y$ is large for all $Y \in \mathcal{Y}$; equivalently, if $Z \cap X$ is large. For each $k < \omega$, let $L_k \subseteq X$ be a $(2k)$-square such that $\langle \pi_1(L_k) : k < \omega \rangle$, $\langle \pi_2(L_k) : k < \omega \rangle$ form block sequences; that is, for each $k < \omega$ and $i = 1, 2$, each element in $\pi_i(L_k)$ is less than each element in $\pi_i(L_{k+1})$. Let $I = \bigcup_{k<\omega} \pi_1(L_k)$ and $J = \bigcup_{k<\omega} \pi_2(L_k)$.

Suppose there is an infinite $I' \subseteq I$ such that letting $J' = (\omega \setminus f(I')) \cap J$ and $m_k = \min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\}$, the sequence $\langle m_k : k < \omega \rangle$ is unbounded. Then there is a strictly increasing subsequence $\langle m_{k_n} : n < \omega \rangle$. Hence, $I' \times J'$ is addable. Letting $W_1 = I' \times J'$ and D be any ultrafilter extending $\mathcal{Y} \cup \{W_1\}$, $I' \in \pi_1(D)$ and $f(I')$ is disjoint from J' which is in $\pi_2(D)$. Therefore, $f(I') \notin \pi_2(D)$.

Now suppose that for each infinite $I' \subseteq I$ letting $J' = (\omega \setminus f(I')) \cap J$, there is an $m < \omega$ such that $\min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\} \leq m$ for each $k < \omega$. Let $W_0 = \bigcup_{k<\omega} L_k$. Note that W_0 is addable. Let $\mathcal{Y}_0 = \mathcal{Y} \cup \{W_0\}$. Let D be any ultrafilter extending $\mathcal{Y}_0$ consisting only of large sets. Then $I \in \pi_1(D)$ and $J \in \pi_2(D)$. We claim that given $I' \subseteq I$ such that I' is in $\pi_1(D)$, there is a strictly increasing sequence $\langle k_n : n < \omega \rangle$ and an $m < \omega$ such that for each n, $|f(I') \cap \pi_2(L_{k_n})| \geq 2k_n - m$.

Let $I' \subseteq I$ be in $\pi_1(D)$ and let $J' = (\omega \setminus f(I')) \cap J$. Let $m < \omega$ be such that for each $k < \omega$, $\min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\} \leq m$. Let $I'' = \bigcup\{I' \cap \pi_1(L_k) : k < \omega$ and $|I' \cap \pi_1(L_k)| > m\}$. Then $I'' \in \pi_1(D)$; for its complement in I' cannot be $\pi_1(Z)$ for any large $Z \subseteq W_0$, and for any $I' \subseteq I$ in $\pi_1(D)$, there is a $Z \subseteq W_0$ in D for which $\pi_1(Z) = I'$. Let $\langle k_n : n < \omega \rangle$ enumerate in order the k for which $I'' \cap \pi_1(L_k) \neq \emptyset$. Then for each $n < \omega$, $|I'' \cap \pi_1(L_{k_n})| = |I' \cap \pi_1(L_{k_n})| > m$. So $|(\omega \setminus f(I')) \cap \pi_2(L_{k_n})| = |(\omega \setminus f(I')) \cap J \cap \pi_2(L_{k_n})| = |J' \cap \pi_2(L_{k_n})| \leq m$. Since $|\pi_2(L_{k_n})| = 2k_n$, we have that $|f(I') \cap \pi_2(L_{k_n})| \geq 2k_n - m$.

Divide each $\pi_2(L_k)$ into two disjoint sets of equal size, labeling one of them M_k. Note that $\omega \times \bigcup_{k<\omega} M_k$ is addable to $\mathcal{Y}_0$. Let $W_1 = W_0 \cap (\omega \times \bigcup_{k<\omega} M_k)$, $\mathcal{Y}_1 = \mathcal{Y}_0 \cup \{W_1\}$, and $J^* = \bigcup_{k<\omega} M_k$. Now for any ultrafilter D consisting only of large sets and extending $\mathcal{Y}_1$, $J^* \in \pi_2(D)$. Then f is not cofinal in $\pi_2(D)$, since f is monotone and for each $I' \subseteq I$ in $\pi_1(D)$, $f(I') \not\subseteq J^*$.

Thus, in each case we can find an addable W_1 such that for any ultrafilter D consisting of large sets and extending $\mathcal{Y} \cup \{W_1\}$, f is not a cofinal map from $\pi_1(D)$ into $\pi_2(D)$. Now use $\mathcal{Y} \cup \{W_1\}$ in place of $\mathcal{Y}$ and reverse the roles of 1 and 2 in the indexing to find a suitable W_2. □

Enumerate all continuous monotone maps from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)$ as f_α, $\alpha < \mathfrak{c}$. Let $\mathcal{Y}_\alpha \subseteq \omega \times \omega$ denote the filter constructed by stage $\alpha < \mathfrak{c}$. At stage $\alpha < \mathfrak{c}$, use the Claim for $\mathcal{Y}_\alpha$ and f_α while simultaneously mixing in Blass' construction to construct a p-point D extending $\bigcup_{\alpha<\mathfrak{c}} \mathcal{Y}_\alpha$ for which there are no Tukey maps between $\pi_1(D)$ and $\pi_2(D)$.

Since the p-point D in Theorem 39 is RK above $\pi_1(D)$ and $\pi_2(D)$, it follows from [Theorem 5, [2]] that there is a p-point which is RK (hence Tukey) below both $\pi_1(D)$ and $\pi_2(D)$. Thus, assuming MA, the diamond lattice embeds into the Tukey degrees of p-points. □

[Theorem 5, [2]] states that if countably many p-points have an RK upper bound which is a p-point, then they have an RK lower bound (which is necessarily a p-point).

Question 40. If countably many p-points have a Tukey upper bound which is a p-point, do they necessarily have a Tukey lower bound which is a p-point?

Question 41. Does every Tukey strictly decreasing sequence of p-points have a Tukey lower bound which is a p-point?

Remark. Laflamme showed in [15] that in the NCF model of [4], the RK ordering of p-points is upwards directed, and hence also downwards directed. Thus, in the NCF model, the Tukey degrees of p-points are both upwards and downwards directed. (We know by Theorem 6 that the class of basically generated ultrafilters with bases closed under finite intersections is upwards directed.) Recall that the cardinal inequality $\mathfrak{u} < \mathfrak{g}$ implies NCF (see [5]), so it is natural to ask the following.

Question 42. Does $\mathfrak{u} < \mathfrak{g}$ imply there is a minimal Tukey degree in the class of p-points?

5. Block-basic ultrafilters on FIN

In this section we study the Tukey ordering between idempotent ultrafilters $\mathcal{U}$ on the index set FIN and their canonical RK-predecessors $\mathcal{U}_{\min}$, $\mathcal{U}_{\max}$ and $\mathcal{U}_{\min,\max}$. Recall that an *idempotent ultrafilter* on the semigroup (FIN, $\cup$) is an ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U} \cup \mathcal{U} = \mathcal{U}$, where $\mathcal{U} \cup \mathcal{U}$ is defined to be $\{A \subseteq \text{FIN} : \{x \in \text{FIN} : \{y \in \text{FIN} : x \cup y \in A\} \in \mathcal{U}\} \in \mathcal{U}\}$. Recall also that the existence of idempotent ultrafilters on FIN was established by S. Glazer (see [6]). Note the following property of any idempotent ultrafilter $\mathcal{U}$ on FIN.

Fact 43. *If $\mathcal{U}$ is a nonprincipal idempotent ultrafilter on* FIN*, then $\mathcal{U}$ is neither p-point nor q-point.*

In [3], Blass showed that Glazer's proof easily adapts to show the following.

Theorem 44 (Theorem 2.1, [3]). *Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be a pair of nonprincipal ultrafilters on ω. Then there is an idempotent ultrafilter $\mathcal{U}$ on* FIN *such that $\mathcal{U}_{\min} = \mathcal{V}_0$ and $\mathcal{U}_{\max} = \mathcal{V}_1$.*

Corollary 45. *There exist idempotent ultrafilters on* FIN *realizing the maximal Tukey type* $\mathcal{U}_{\mathrm{top}}$.

Thus, one is naturally led to consider the conditions on idempotent ultrafilters $\mathcal{U}$ on FIN that would prevent $\mathcal{U}$ from having the maximal Tukey type.

Definition 46. An idempotent ultrafilter $\mathcal{U}$ on FIN is *block-generated* if it is generated by sets of the form $[X]$ where X is an infinite block-sequence.

It was shown in [3] that if $\mathcal{U}$ is a block-generated ultrafilter on FIN, then both $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are selective. (Block-generated ultrafilters are called *ordered-union ultrafilters* in [3].) So the existence of block-generated ultrafilters on FIN cannot be proved on the basis of the usual ZFC axioms of set theory, though using Hindman's Theorem one can easily establish the existence of such ultrafilters using CH or MA. We point out the following fact.

Fact 47. *Let* $\mathcal{U}$ *be a block-generated ultrafilter on* FIN. *Then*

(1) $\mathcal{U}_{\min,\max}$ *is neither a p-point nor a q-point.*
(2) *If* $\mathcal{U}_{\min}$ *is selective, then* $\mathcal{U}_{\min,\max}$ *is rapid.*

Proof. (1) Let $M_n = \{x_{\min,\max} : \min(x) = n\}$. Then $\{M_n : n < \omega\}$ is a partition of ω. If X is any block sequence, then $|[X]_{\min,\max} \cap M_n| = \omega$ for infinitely many n. So $\mathcal{U}_{\min,\max}$ is not a p-point.

Let $P_n = \{\iota(\{k, n\}) : k < n\}$, where ι is a fixed pairing function. Then for each $n \geq 1$, P_n is finite, and $\{P_n : n \geq 1\}$ is a partition of ω. If X is a block sequence, then $|[X]_{\min,\max} \cap P_n| > 1$ for infinitely many n. Hence, $\mathcal{U}_{\min,\max}$ is not a q-point.

(2) Given a strictly increasing function $g : \omega \to \omega$, without loss of generality assuming the coding function $\iota : [\omega]^2 \to \omega$ has the property that $\iota(\{m, n\}) \geq n$ for each $m < n$, let $k_l = 2^{l+1}$ for all $l < \omega$. Since $\mathcal{U}_{\min}$ is selective, there is an infinite block-sequence X such that $[X] \in \mathcal{U}$, $|X_{\min} \cap [0, g(k_2))| = 0$, and for each $l \geq 2$, $|X_{\min} \cap (g(k_l), g(k_{l+1}))| \leq 1$. Then $|[X]_{\min,\max} \cap g(n)| < n$ for each $n < \omega$. □

As noted above, no nontrivial idempotent ultrafilter on FIN is basic, so we are naturally led to the following relaxation of this notion.

Definition 48. A block-generated ultrafilter $\mathcal{U}$ is *block-basic* if whenever we are given a sequence (X_n) of infinite block sequences in FIN such that each $[X_n] \in \mathcal{U}$ and (X_n) converges to some infinite block sequence X such that $[X] \in \mathcal{U}$, then there is an infinite subsequence (X_{n_k}) such that $\bigcap_{k<\omega}[X_{n_k}] \in \mathcal{U}$.

Definition 49. Let $\mathrm{FIN}^{[n]}$ denote the collection of all block sequences of elements of FIN of length n. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *2-dimensional Ramsey Property* if for each finite coloring of $\mathrm{FIN}^{[2]}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $[X]^{[2]}$ is monochromatic. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *Ramsey Property* if for each $n < \omega$ and each finite coloring of $\mathrm{FIN}^{[n]}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and

$[X]^{[n]}$ is monochromatic. Let $\mathrm{FIN}^{[\infty]}$ denote the collection of all infinite block sequences of elements of FIN. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *∞-dimensional Ramsey Property* if for every analytic subset $\mathcal{A}$ of $\mathrm{FIN}^{[\infty]}$ there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $[X]^{[\infty]}$ is either included in or disjoint from $\mathcal{A}$. (For more information about ∞-dimensional Ramsey Theory, see [23].)

Theorem 50. *The following are equivalent for a block-generated ultrafilter $\mathcal{U}$ on* FIN*.*

(1) *$\mathcal{U}$ is block-basic.*
(2) *For every sequence (X_n) of infinite block sequences of* FIN *such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n.*
(3) *$\mathcal{U}$ has the 2-dimensional Ramsey property.*
(4) *$\mathcal{U}$ has the Ramsey property.*
(5) *$\mathcal{U}$ has the ∞-dimensional Ramsey property.*

Remark. (2) is called a *stable ordered-union ultrafilter* in [3].

Proof. The equivalence of (2), (3), (4) and (5) were established in Theorem 4.2 of [3].

(1) implies (2). Suppose $\mathcal{U}$ is block-basic. Let (X_n) be a sequence of block sequences of FIN such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n. Let m_n be a strictly increasing sequence such that $X_0 \geq X_1/m_1 \geq X_2/m_2 \geq \dots$. Let $Y_n = (\{l\} : l \leq m_n)^\frown(X_n/m_n)$. Then each $Y_n =^* X_n$ and $Y_n \rightarrow (\{l\} : l < \omega)$. By (1) there is a subsequence n_k such that $\bigcap_{k<\omega}[Y_{n_k}] \in \mathcal{U}$. Since $\mathcal{U}$ is block-generated, there is a Z such that $[Z] \in \mathcal{U}$ and $[Z] \subseteq \bigcap_{k<\omega}[Y_{n_k}]$. Then for each $n < \omega$, taking k such that $n_k > n$, we have that $X_n =^* Y_n \geq^* Y_{n_k} \geq Z$. Thus, (2) holds.

Now suppose that (2) holds. Since $\mathcal{U}$ is block-generated, (2) is equivalent to the statement (2)$'$: For every sequence (X_n) of infinite block sequences of FIN such that $[X_n] \in \mathcal{U}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n. Let (X_n) be a sequence of block sequences such that each $[X_n] \in \mathcal{U}$ and $(X_n) \rightarrow X$. By (2)$'$, let $Z \leq X_0$ be such that $[Z] \in \mathcal{U}$ and for each $n < \omega$, $Z \leq^* X_n$. Take m_k such that each $m_k = \min(z)$ for some $z \in Z$ and

(1) $n \geq m_{k+1} \rightarrow X_n \cap m_k = X \cap m_k$;
(2) $n \leq m_k \rightarrow X_n/m_{k+1} \geq Z$.

Let $Z_0 = \{z \in Z : \exists k(m_{4k} \leq \min(z) < m_{4k+2})\}$. If $[Z_0] \in \mathcal{U}$, then let $Y \leq Z_0, X$ such that $[Y] \in \mathcal{U}$. For each $k < \omega$, $X_{4k+3} \cap m_{4k+2} = X \cap m_{4k+2} \geq Y \cap m_{4k+2}$. For each $y \in Y$, $y \cap [m_{4k+2}, m_{4k+4}) = \emptyset$. $X_{4k+3}/m_{4k+4} \geq Z \geq Y$. Therefore, $\bigcap_{k<\omega}[X_{4k+3}] \supseteq [Y]$. If $[Z_0] \notin \mathcal{U}$, then since $\mathcal{U}$ is block-generated, there is a Z_1 such that $[Z_1] \in \mathcal{U}$ and $[Z_1] \subseteq [Z] \setminus [Z_0]$. Since $Z_1 \leq Z$ and $[Z_1] \cap [Z_0] = \emptyset$, for each $z \in Z$, if $\min(z) \in [m_{4k}, m_{4k+2})$ then $z \notin Z_1$. Therefore, $Z_1 \cap Z_0 = \emptyset$. Hence, for each $z \in Z_1$, $\min(z) \in [m_{4k+2}, m_{4k+4})$. Letting $Y \leq Z_1, X$ such that $[Y] \in \mathcal{U}$, $\bigcap_{k<\omega}[X_{4k+1}] \supseteq [Y]$. Hence, (1) holds. □

Corollary 51. *If $\mathcal{U}$ is a block-basic ultrafilter on* FIN*, then both $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are selective ultrafilters on ω.*

Applying Theorem 2.4 of [3], we get some sort of converse to this.

Corollary 52. *Assuming CH, for every pair $\mathcal{V}_0$ and $\mathcal{V}_1$ of non-RK-equivalent selective ultrafilters on ω, there is a block-basic ultrafilter $\mathcal{U}$ on* FIN *such that $\mathcal{U}_{\min} = \mathcal{V}_0$ and $\mathcal{U}_{\max} = \mathcal{V}_1$.*

Remark. Blass [3] proved that for block-generated ultrafilters $\mathcal{U}$ on FIN, we always have that $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are RK-incomparable. Thus, by a theorem of Todorcevic in [18], $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey-incomparable.

Our interest in block-basic ultrafilters on FIN is based on the following fact whose proof is analogous to that of Theorem 9.

Theorem 53. *Suppose $\mathcal{U}$ is a block-basic ultrafilter on* FIN *and that $\mathcal{U} \geq_T \mathcal{V}$ for some ultrafilter $\mathcal{V}$ on any countable index set I. Then there is a monotone continuous map $f : \mathcal{P}(\mathrm{FIN}) \to \mathcal{P}(I)$ such that $f''\mathcal{U}$ is a cofinal subset of $\mathcal{V}$.*

Similarly, one proves the following.

Theorem 54. *Suppose $\mathcal{U}$ is a block-basic ultrafilter on* FIN *and $\mathcal{V}$ is any ultrafilter on a countable index set I. If $\mathcal{U}_{\min,\max} \geq_T \mathcal{V}$, then there are an infinite block sequence $\tilde{X}$ such that $[\tilde{X}] \in \mathcal{U}$ and a monotone continuous function f from $\{[X]_{\min,\max} : X \leq \tilde{X}\}$ into $\mathcal{P}(I)$ whose restriction to $\{[X]_{\min,\max} : X \leq \tilde{X},\ [X] \in \mathcal{U}\}$ has cofinal range in $\mathcal{V}$.*

Proof. Let $\mathcal{B}$ be the collection of block sequences X such that $[X] \in \mathcal{U}$. Then $\{[X] : X \in \mathcal{B}\}$ is a base for $\mathcal{U}$. Let $\mathcal{C} = \{[X]_{\min,\max} : X \in \mathcal{B}\}$. Then $\mathcal{C}$ is a base for $\mathcal{U}_{\min,\max}$. For the sake of notation, let $\mathcal{W}$ denote $\mathcal{U}_{\min,\max}$. Let $g : \mathcal{V} \to \mathcal{W}$ be a Tukey map and let $f : \mathcal{W} \to \mathcal{V}$ by $f(W) = \bigcap\{V : g(V) \supseteq W\}$. Then f is a monotone cofinal map; so $f \restriction \mathcal{C}$ is a monotone cofinal map.

In a similar manner as in the proof of Theorem 9, we construct an $\tilde{X} \in \mathcal{B}$ such that the map f is continuous on $\{[W]_{\min,\max} : W \in \mathcal{B},\ W \leq \tilde{X}\}$. Let $\langle i_n : n < \omega\rangle$ be an enumeration of I. Let $X_0 = (\{0\}, \{1\}, \{2\}, \dots)$. Given X_n, take $X_{n+1} = (x_{n+1,0}, x_{n+1,1}, x_{n+1,2}, \dots) \leq X_n$ such that

(1) $\min(x_{n+1,0}) \geq n + 1$;
(2) For each block sequence $s \subseteq \mathcal{P}(n + 1)$, for each $k \leq n$, if there is a $Z \in \mathcal{B}$ such that $\min(Z) \geq n + 1$ and $i_k \notin f([s \cup Z]_{\min,\max})$, then $i_k \notin f([s \cup X_n]_{\min,\max})$.

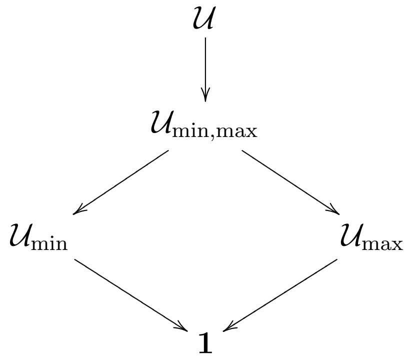
Since $\mathcal{U}$ is block-generated, there is a $Y \in \mathcal{B}$ such that for each $n < \omega$, $Y \leq^* X_n$. Let $l_0 = 0$ and for each $n < \omega$, let $l_{n+1} > l_n$ satisfy $l_{n+1} = \min(y)$ for some $y \in Y$ and $Y/l_{n+1} \leq X_{l_n}$.

Color $[Y]^{[2]}$ as follows: Let $h((y_0, y_1)) = 0$ if there is an $n < \omega$ such that $\max(y_0) < l_n$ and $l_{n+2} \le \min(y_1)$; 1 otherwise. Since $\mathcal{U}$ has the Ramsey property for pairs, there is a block-sequence $\tilde{X} \le Y$ such that h is constant on $[\tilde{X}]^{[2]}$. h cannot be constantly 1 on $[W]^{[2]}$ for any block sequence W. Thus, h is constantly 0 on $[\tilde{X}]^{[2]}$. Let (l_{n_j}) be a subsequence of (l_n) such that for each x in $\tilde{X}$, either $\max(x) < l_{n_{2j+1}}$ or $l_{n_{2j+2}} \le \min(x)$.

Suppose $W = (w_0, w_1, w_2, \dots) \le \tilde{X}$ and is in $\mathcal{B}$. Let $C = [W]_{\min,\max}$ and let $i \in I$. Let k be such that $i = i_k$. Take j large enough that $k < l_{n_{2j+1}}$ and there is an m such that $\max(w_m) < l_{n_{2j+1}}$ and $\min(w_{m+1}) \ge l_{n_{2j+2}}$. Note that $W/l_{n_{2j+1}} \le \tilde{X}/l_{n_{2j+1}} \subseteq Y/l_{n_{2j+1}} = Y/l_{n_{2j+2}} \le X_{l_{n_{2j+1}}}$. Thus, $i \notin f(C)$ iff $i \notin f([t \cup (W/l_{n_{2j+1}})]_{\min,\max})$, where $t = (w_0, \dots, w_m)$, iff $i \notin f([t \cup X_{l_{n_{2j+1}}}]_{\min,\max})$ iff $i \notin f([t \cup (\tilde{X}/l_{n_{2j+1}})]_{\min,\max})$. Thus, f is continuous on $\{[W]_{\min,\max} : w \in \mathcal{B},\ W \le \tilde{X}\}$.

In the natural way, f can be extended to a continuous monotone map from $\{[X]_{\min,\max} : X \le \tilde{X}\}$ into I. □

Theorem 55. *Assuming CH, there is a block-basic ultrafilter $\mathcal{U}$ on* FIN *such that $\mathcal{U}_{\min,\max} <_T \mathcal{U}$ and $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey incomparable. (In the following diagram, arrows represent strict Tukey reducibility.)*



Proof. First of all, note that for every block-generated ultrafilter $\mathcal{U}$ on FIN, we have that $\mathcal{U}_{\min,\max} = \mathcal{U}_{\min} \cdot \mathcal{U}_{\max} \equiv_T \mathcal{U}_{\min} \times \mathcal{U}_{\max}$. Recall that $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey incomparable, since they are non-isomorphic selective ultrafilters. Thus, it suffices to construct a block-basic ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\min,\max} <_T \mathcal{U}$. Assuming CH, one can construct a block-basic ultrafilter on FIN in the standard way (see [3]).

To ensure that $\mathcal{U} >_T \mathcal{U}_{\min,\max}$, interweave the following into the construction of $\mathcal{U}$. By Theorem 54, we can enumerate as $(f_\beta, \tilde{X}_\beta)$, $\beta < \mathfrak{c}$, all pairs $(f, \tilde{X})$ such that $\tilde{X} \in \mathrm{FIN}^{[\infty]}$ and $f : \{[Z]_{\min,\max} : Z \le \tilde{X}\} \to \mathcal{P}(\mathrm{FIN})$ is a monotone continuous function. At stage α in the construction, let Y be a block sequence such that for all $\beta < \alpha$, $Y \le^* X_\beta$, where $\{X_\beta : \beta < \alpha\}$ is a block-basis for the filter $\mathcal{U}_\alpha$ constructed by stage α. If there is no block sequence $Z \le Y, \tilde{X}$, then the domain of f is not contained in $\mathcal{U}_{\min,\max}$ for any $\mathcal{U}$ extending $\mathcal{U}_\alpha$. Now suppose there is a $Z \le Y, \tilde{X}$. Without loss of generality, assume $Y \le \tilde{X}$.

If there is a $W \le Y$ such that for each $W' \le W$, $f_\alpha([W']_{\min,\max}) \not\subseteq [W]$, then put $[W]$ into $\mathcal{U}_{\alpha+1}$. This ensures that f_α cannot be cofinal in any ultrafilter $\mathcal{U}$ extending $\mathcal{U}_{\alpha+1}$ generated by block sequences, since f_α is monotone.

Otherwise, for each $W \le Y$, there is a $W' \le W$ such that $f_\alpha([W']_{\min,\max}) \subseteq [W]$. $Y = (y_i)$. Fix $W = (w_i)$ to be the block sequence where each $w_i = y_{3i} \cup y_{3i+1} \cup y_{3i+2}$. Thus, $W \le Y$. Fix $W' \le W$ such that $f_\alpha([W']_{\min,\max}) \subseteq [W]$. $W' = (w'_j)$, where each $w'_j = \bigcup_{i \in I_j} w_i$, where I_j is some finite set. Let $m_j = \min(I_j)$ and $k_j = \max(I_j)$. Let $W'' = (w''_j)$, where each $w''_j = y_{3m_j} \cup y_{3k_j+2}$. Then $\min(w''_j) = \min(w'_j)$ and $\max(w''_j) = \max(w'_j)$ for all $j < \omega$; so $[W']_{\min,\max} = [W'']_{\min,\max}$. Note that $[W] \cap [W''] = \emptyset$, and $W'' \le Y$. Putting $[W'']$ into $\mathcal{U}_{\alpha+1}$, we will have that for any ultrafilter $\mathcal{U}$ extending $\mathcal{U}_\alpha$, $[W'']_{\min,\max} \in \mathcal{U}_{\min,\max}$. Hence, $f_\alpha([W'']_{\min,\max}) = f_\alpha([W']_{\min,\max}) \subseteq [W]$ which is disjoint from $[W'']$. Thus, the range of f_α will not be contained in $\mathcal{U}$. In this way, we can build $\mathcal{U}$ so that $\mathcal{U}_{\min,\max} \not\ge_T \mathcal{U}$, thus $\mathcal{U} >_T \mathcal{U}_{\min,\max}$. □

Question 56. If $\mathcal{U}$ is any block-basic ultrafilter, does it follow that $\mathcal{U} >_T \mathcal{U}_{\min,\max}$?

Remark. Note that the proof of the theorem shows that the generic filter for the forcing notion $(\mathrm{FIN}^{[\infty]}, \le^*)$ adjoins a block-basic ultrafilter $\mathcal{U}$ on FIN with the properties stated in Theorem 50. On the other hand, an argument analogous with the case of selective ultrafilters on ω (see the theorem of Todorcevic appearing in [8; 4.9]) shows that if there is a supercompact cardinal, then every block-basic ultrafilter $\mathcal{U}$ on FIN is generic over $L(\mathbb{R})$ for the forcing notion $(\mathrm{FIN}^{[\infty]}, \le^*)$. Thus, the conclusion of Theorem 4.9 in [8] is true for any block-basic ultrafilter $\mathcal{U}$ on FIN assuming the existence of a supercompact cardinal. This leads us also to the following related problem.

Problem 57. Assume the existence of a supercompact cardinal. Let $\mathcal{U}$ be an arbitrary block-basic ultrafilter on FIN. Show that the inner model $L(\mathbb{R})[\mathcal{U}]$ has exactly five Tukey types of ultrafilters on a countable index set.

This problem is based on the $\mathcal{U}$-version of Taylor's canonical Ramsey Theorem for FIN stating that for each map $f : \mathrm{FIN} \to \omega$, there is an $[X] \in \mathcal{U}$ such that $f \upharpoonright [X]$ is equivalent to one of the five mappings: constant, identity, min, max, (min,max) (see [1], [22]). If the answer to this problem is positive, then one can look at ultrafilters $\mathcal{U}$ on the index set FIN_k ($k = 1, 2, 3, \dots$) with analogous Ramsey-theoretic properties whose corresponding inner models $L(\mathbb{R})[\mathcal{U}]$ have different finite numbers of Tukey types. This will of course be based on Gower's Theorem for FIN_k and Lopez-Abad's canonical Ramsey Theorem for FIN_k (see [1], [16], [10]). For example, for a block-basic ultrafilter $\mathcal{U}$ on FIN_2, one could expect exactly 43 Tukey types of ultrafilters in $L(\mathbb{R})[\mathcal{U}]$.

The following is a subproblem of Problem 57.

Question 58. Is it true that for each block-basic $\mathcal{U}$, there are no Tukey types (a) strictly between $\mathcal{U}$ and $\mathcal{U}_{\min,\max}$, (b) strictly between $\mathcal{U}_{\min,\max}$ and $\mathcal{U}_{\min}$, and (c) strictly between $\mathcal{U}_{\min,\max}$ and $\mathcal{U}_{\max}$?

Question 59. Are there block-basic ultrafilters $\mathcal{U},\mathcal{V}$ on FIN which are Tukey equivalent but RK incomparable?

6. A characterization of ultrafilters which are not of Tukey top degree

In this section we investigate Isbell's question of whether ZFC implies that there is always an ultrafilter which does not have top Tukey degree.

We note that always $(\mathcal{U},\supseteq)\le_T(\mathcal{U},\supseteq^*)$; so if $(\mathcal{U},\supseteq^*)<_T[\mathfrak{c}]^\omega$, then $(\mathcal{U},\supseteq)<_T[\mathfrak{c}]^\omega$. Milovich showed in [17] that for any ultrafilter $\mathcal{U}$, there is an ultrafilter $\mathcal{W}$ such that $(\mathcal{W},\supseteq^*)\le_T(\mathcal{U},\supseteq)$. Thus, there is an ultrafilter $\mathcal{U}$ such that $(\mathcal{U},\supseteq)<_T[\mathfrak{c}]^{<\omega}$ if and only if there is an ultrafilter $\mathcal{W}$ such that $(\mathcal{W},\supseteq^*)<_T[\mathfrak{c}]^{<\omega}$.

CH implies the existence of p-points, so we can assume $\neg$CH. In this context, the following combinatorial principle holds.

Definition 60 (Todorcevic)**.** $\Diamond_{[\mathfrak{c}]^\omega}$ is the statement: There exist sets $S_A\subseteq A$, $A\in[\mathfrak{c}]^\omega$, such that for each $X\subseteq\mathfrak{c}$, $\{A\in[\mathfrak{c}]^\omega : X\cap A=S_A\}$ is stationary in $[\mathfrak{c}]^\omega$.

This can be modified to the following. $\Diamond^-_{[[\omega]^\omega]^\omega}$ is the statement: There exist ordered pairs $(\mathcal{U}_A,\mathcal{X}_A)$, where $A\in[[\omega]^\omega]^\omega$ and $\mathcal{X}_A\subseteq\mathcal{U}_A\subseteq A$, such that for each pair $(\mathcal{U},\mathcal{X})$ with $\mathcal{X}\subseteq\mathcal{U}$ and $\mathcal{X},\mathcal{U}\in[[\omega]^\omega]^{\mathfrak{c}}$, $\{A\in[[\omega]^\omega]^\omega : \mathcal{U}_A=\mathcal{U}\cap A$, $\mathcal{X}_A=\mathcal{X}\cap A\}$ is stationary in $[[\omega]^\omega]^\omega$.

Let $P_A=\{W\in[\omega]^\omega : \exists X\in\mathcal{U}_A(W\cap X=\emptyset)\}$, and let $Q_A=\{W\in[\omega]^\omega : \exists(B_n)_{n<\omega}\subseteq\mathcal{X}_A(\forall n<\omega,\ W\subseteq^* B_n)\}$. Let $D_A=P_A\cup Q_A$. Then for each $A\in[\omega]^\omega$, D_A is dense open in $[\omega]^\omega$: Let $Y\in[\omega]^\omega$. If $\mathcal{U}_A$ does not have the strong finite intersection property, then there are $U,V\in\mathcal{U}_A$ such that $|U\cap V|<\omega$. Then either $|Y\setminus U|=\omega$ or $|Y\setminus V|=\omega$. Thus, there is a $W\in[Y]^\omega$ such that for some $X\in\mathcal{U}_A$, $W\cap X=\emptyset$. Hence $W\in P_A$, and moreover, any $W'\in[W]^\omega$ is also in P_A. Suppose $\mathcal{U}_A$ has the strong finite intersection property. If $Y\notin\mathcal{U}_A^+$, then there is an $X\in\mathcal{U}_A$ such that $|Y\cap X|<\omega$. So $W=Y\setminus(Y\cap X)\in P_A$, and any $W'\in[W]^\omega$ is also in P_A. Otherwise, $Y\in\mathcal{U}_A^+$. Then there is a $W\in[Y]^\omega$ such that for each $B\in\mathcal{U}_A$, $W\subseteq^* B$, since $|\mathcal{U}_A|\le\omega$. Thus, $W\in Q_A$. Moreover, any $W'\in[W]^\omega$ is also in Q_A. Therefore, D_A is dense open in $[\omega]^\omega$.

Fact 61. *For any ultrafilter $\mathcal{U}$, $\{A\in[[\omega]^\omega]^\omega : \mathcal{U}\cap D_A\ne\emptyset\}$ is stationary.*

Proof. Let $\mathcal{U}$ be an ultrafilter and suppose that $\{A\in[[\omega]^\omega]^\omega : \mathcal{U}\cap D_A\ne\emptyset\}$ is not stationary. Then $\{A\in[[\omega]^\omega]^\omega : \mathcal{U}\cap D_A=\emptyset\}$ contains a club set, call it $\mathcal{C}$. Let $\mathcal{X}=\bigcup\{\mathcal{X}_A : A\in\mathcal{C}\}$. Let $X\in\mathcal{U}$. There are club many $A\in[\omega]^\omega$ with $X\in A$. Thus, there are club many A with $(A,\mathcal{U}\cap A)\prec([\omega]^\omega,\mathcal{U})$. By $\Diamond^-_{[[\omega]^\omega]^\omega}$, there is an $A\in[\omega]^\omega$ with $X\in A$ such that $\mathcal{U}\cap A=\mathcal{U}_A$ and $\mathcal{U}\cap A=\mathcal{X}_A$. Therefore, $\mathcal{U}\subseteq\mathcal{X}$.

We claim that for each $\mathcal{Y} \in [\mathcal{U}]^\omega$, $\mathcal{Y}$ has no pseudointersection in $\mathcal{U}$. Take an $A \in \mathcal{C}$ with $\mathcal{Y} \subseteq A$ such that $\mathcal{U} \cap D_A = \emptyset$, $\mathcal{U}_A = \mathcal{U} \cap A$, and $\mathcal{X}_A = \mathcal{U} \cap A$. Then $\mathcal{Y} \subseteq \mathcal{U} \cap A = \mathcal{X}_A$ and there is no pseudointersection of any infinite subset of $\mathcal{Y}$ in $\mathcal{U}$, since $Q_A \cap \mathcal{U} = \emptyset$. Contradiction, since $\mathcal{U}$ contains the Fréchet filter. □

Fact 62. *If $\mathcal{U} \cap D_A \neq \emptyset$ for club many $A \in [\omega]^\omega$, then $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$.*

Proof. Let $\mathcal{X} \in [\mathcal{U}]^{\mathfrak{c}}$. $\{A \in [[\omega]^\omega]^\omega : (A, \mathcal{U} \cap A, \mathcal{X} \cap A) \prec ([\omega]^\omega, \mathcal{U}, \mathcal{X})\}$ is club in $[[\omega]^\omega]^\omega$. $\{A \in [[\omega]^\omega]^\omega : A \cap \mathcal{U} = \mathcal{U}_A,\ A \cap \mathcal{X} = \mathcal{X}_A\}$ is stationary. If $\mathcal{U} \cap D_A \neq \emptyset$ for club many A, then there are stationary many A such that $\mathcal{U} \cap A = \mathcal{U}_A$, $\mathcal{X} \cap A = \mathcal{X}_A$, and either there is a $U \in \mathcal{U}_A$ and a $W \in \mathcal{U}$ such that $U \cap W = \emptyset$, which is impossible, or else there is a $W \in \mathcal{U}$ and $(B_n)_{n<\omega} \subseteq \mathcal{X}_A = A \cap \mathcal{X}$ such that for each $n < \omega$, $W \subseteq^* B_n$. Therefore, $\mathcal{U}$ is not of Tukey top degree. □

Fact 63. *If $\mathcal{U}$ is a p-point, then $\mathcal{U} \cap D_A \neq \emptyset$ for all $A \in [[\omega]^\omega]^\omega$.*

Proof. Let $A \in [[\omega]^\omega]^\omega$ be given. If $\mathcal{U}_A \not\subseteq \mathcal{U}$, then taking an $X \in \mathcal{U}_A \setminus \mathcal{U}$, $\omega \setminus X \in \mathcal{U} \cap P_A$. If $\mathcal{U}_A \subseteq \mathcal{U}$, then since $\mathcal{X}_A$ is countable, there is a $W \in \mathcal{U}$ which is a pseudointersection of $\mathcal{X}_A$. Hence, $W \in \mathcal{U} \cap Q_A$. □

Let $P'_A = \{W \in [\omega]^\omega : \forall X \in \mathcal{X}_A (W \subseteq^* X)\}$. Let $D'_A = P'_A \cup Q_A$. By the same proof as for D_A, we see that D'_A is dense open in $[\omega]^\omega$.

Fact 64. *If $\mathcal{U} \cap D'_A \neq \emptyset$ for club many A, then $\mathcal{U}$ is a p-point.*

Proof. Suppose that $\mathcal{C}$ is club in $[[\omega]^\omega]^\omega$ and for each $A \in \mathcal{C}$, $\mathcal{U} \cap D'_A \neq \emptyset$. Let $\mathcal{Y} \in [\mathcal{U}]^\omega$. Take A such that $\mathcal{Y} \subseteq A$, $(A, \mathcal{U} \cap A, \mathcal{U} \cap A) \prec ([\omega]^\omega, \mathcal{U}, \mathcal{U})$, $\mathcal{U}_A = \mathcal{U} \cap A$, $\mathcal{X}_A = \mathcal{X} \cap A$, and $\mathcal{U} \cap D'_A \neq \emptyset$. Then $\mathcal{Y} \subseteq \mathcal{U} \cap A = \mathcal{X}_A$. So since there is a $W \in \mathcal{U}$ such that for each $X \in \mathcal{X}_A$, $W \subseteq^* X$, there is a pseudo-intersection of $\mathcal{Y}$ in $\mathcal{U}$. □

Question 65. Can we use these dense sets, or similar ones, to obtain

(1) an ultrafilter which is not Tukey top?
(2) an ultrafilter which is not Tukey top but also is not basically generated?
(3) an ultrafilter which is basically generated but is not a Fubini limit of p-points?

7. Concluding remarks and problems

Recall that the properties of p-point and rapid are preserved under Rudin-Keisler reducibility.

Question 66. Which properties of ultrafilters are preserved under Tukey reducibility?

By Theorem 22, if a p-point $\mathcal{U} \geq_T \omega^\omega$, then $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$, which is not a p-point, so the property of being a p-point is not preserved by Tukey reducibility. However, we may ask the following.

Question 67. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{V}$, then is there a p-point $\mathcal{W}$ such that $\mathcal{W} \equiv_T \mathcal{V}$?

Question 68. Which lattices can be embedded into the Tukey degrees of p-points? In particular, are there two Tukey incomparable p-points which have no p-point as a common Tukey upper bound?

Question 69. Are there two Tukey non-comparable ultrafilters whose least upper bound is the top Tukey degree?

Question 70. Does every Tukey minimal degree contain a selective ultrafilter?

Question 71. What is the structure of the Rudin-Keisler degrees within the top Tukey degree?

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