

THE ROYAL
SWEDISH
ACADEMY OF
SCIENCES



**INSTITUT
MITTAG-LEFFLER**

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

**On Löwenheim-Skolem-Tarski numbers for
extensions of first order logic**

M. Magidor and J. Väänänen

REPORT No. 15, 2009/2010, fall

ISSN 1103-467X

ISRN IML-R- -15-09/10- -SE+fall

On Löwenheim-Skolem-Tarski numbers for extensions of first order logic*

Menachem Magidor
Institute of Mathematics
Hebrew University
Jerusalem, Israel

Jouko Väänänen[†]
Department of Mathematics
and Statistics
University of Helsinki
Helsinki, Finland

Abstract

We show that, assuming the consistency of a supercompact cardinal, the first (weakly) inaccessible cardinal can satisfy a strong form of a Löwenheim-Skolem-Tarski theorem for the equicardinality logic $L(I)$, a logic introduced in [4] strictly between first order logic and second order logic. On the other hand we show that in the light of present day inner model technology, nothing short of a supercompact cardinal suffices for this result. In particular, we show that the Löwenheim-Skolem-Tarski theorem for the equicardinality logic at κ implies the Singular Cardinals Hypothesis above κ as well as Projective Determinacy.

1 Introduction

The Löwenheim-Skolem Theorem is perhaps the most quoted result about first order logic. It shows the “local” character of first order formulas. The

*The authors finalized the paper during the special program “Mathematical Logic: Set theory and model theory” at the Mittag-Leffler Institute, Fall 2009. The authors would like to thank the Institute for its support.

[†]Research partially supported by grant 40734 of the Academy of Finland and by the EUROCORES LogICCC LINT programme.

truth of a first order sentence depends only on a small part of the set theoretical universe. For many purposes first order logic is ideal, but there are also interesting and useful extensions of first order logic.

Example 1 • *Second order logic L^2 extends first order logic with quantifiers of the form $\exists R\phi(R, x_0, \dots, x_{n-1})$, where the second order variable R ranges over n -ary relations on the universe for some fixed n .*

- *The logic $L(Q_1)$ extends first order logic with a new quantifier Q_1 binding one variable. The formula $Q_1x_0\phi(x_0, \dots, x_{n-1})$ has the meaning “there are uncountably many elements x_0 satisfying $\phi(x_0, \dots, x_{n-1})$ ”.*
- *The logic $L(Q_1^{MM})$ extends first order logic with a new quantifier Q_1^{MM} binding two variables. The formula $Q_1^{MM}x_0x_1\phi(x_0, \dots, x_{n-1})$ has the meaning “there is an uncountable set X such that any two elements x_0 and x_1 from X satisfy $\phi(x_0, \dots, x_{n-1})$ ”.*

Second order logic is in a sense the opposite of first order logic. It is powerful enough to capture exactly a large part of the set theoretical universe. The logics $L(Q_1)$ and $L(Q_{\aleph_1}^{MM})$ are more close to first order logic. The first is axiomatizable and so is the second, if we assume \diamond . In this paper we study the following two, in a sense intermediate, extensions of first order logic:

Example 2 • *Equicardinality logic $L(I)$ [4]. This logic extends first order logic by formulas of the form*

$$Ix_0y_0\phi(x_0, \dots, x_{n-1})\psi(y_0, \dots, y_{n-1})$$

with the meaning: “for given a_1, \dots, a_{n-1} and b_1, \dots, b_{n-1} the cardinality of the set of elements x_0 satisfying $\phi(x_0, a_1, \dots, a_{n-1})$ is the same as the cardinality of the set of elements y_0 satisfying $\psi(y_0, b_1, \dots, b_{n-1})$ ”.

- *Equicofinality logic $L(Q^{ec})$ [10]. This logic extends first order logic by formulas of the form*

$$Q^{ec}x_0x_1y_0y_1\phi(x_0, \dots, x_{n-1})\psi(y_0, \dots, y_{n-1})$$

with the meaning: “for given a_2, \dots, a_{n-1} and b_2, \dots, b_{n-1} , both the set of pairs of elements x_0 and x_1 satisfying $\phi(x_0, x_1, a_2, \dots, a_{n-1})$ and the set of pairs of elements y_0 and y_1 satisfying $\psi(y_0, y_1, b_2, \dots, b_{n-1})$ are linear orders, and moreover these linear orders have the same cofinality.”

The logics $L(I)$ and $L(Q^{ec})$ are in a clear sense between first order logic and second order logic. The results of this paper show that on the basis of ZFC alone there is mixed information as to whether $L(I)$ and $L(Q^{ec})$ are closer to first order logic or to second order logic.

Very little is known about the logic $L(Q^{ec})$. Shelah [10] conjectures that this logic is compact and axiomatizable. The hidden power of this logic is revealed in models with a wellordering. There the quantifier Q^{ec} can be used to pick elements of the well-ordering corresponding to regular cardinals. This puts severe limitations e.g. to the existence of small elementary submodels. In a sense, the stronger logic $L(I, Q^{ec})$ is better understood. At least we know that this logic is very far from compact and axiomatizable, because $L(I)$ is.

There is a quite general concept of a logic, that the above examples are special cases of. We define it as follows:¹

Definition 3 *Let τ be a fixed vocabulary. A logic L consists of*

1. *A set, also denoted by L , of “formulas” of L . If $\phi \in L$, then there is a natural number n_ϕ , called the length of the sequence of free variables,*
2. *A relation*

$$\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$$

between models of vocabulary τ , sequences $(a_0, \dots, a_{n_\phi-1})$ of elements of A and formulas $\phi \in L$. It is assumed that this relation satisfies the isomorphism axiom, that is, if $\pi : \mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$ and $\mathcal{B} \models \phi[\pi a_0, \dots, \pi a_{n_\phi-1}]$ are equivalent.

We call τ the vocabulary of the logic L .

Note, that no syntax is a priori assumed of a logic. The meaning of “ ϕ has a model”, and “the theory $T \subset L$ has a model” is obvious. We write $\mathcal{A} \equiv_L \mathcal{B}$ if $\mathcal{A} \models \phi$ and $\mathcal{B} \models \phi$ are equivalent for all $\phi \in L$ with $n_\phi = 0$. We write $\mathcal{A} \prec_L \mathcal{B}$ if $\mathcal{A} \models \phi[a_0, \dots, a_{n_\phi-1}]$ and $\mathcal{B} \models \phi[a_0, \dots, a_{n_\phi-1}]$ are equivalent for all $\phi \in L$ and all $a_0, \dots, a_{n_\phi-1} \in A$.

We now define two natural invariants for any logic L :

¹This is a little different than usual (e.g. [1, 6]) in that our logics have a fixed vocabulary.

Definition 4 *The Löwenheim-Skolem number $\text{LS}(L)$ of L is the smallest cardinal κ such that if a theory $T \subset L$ has a model, it has a model of cardinality $< \max(\kappa, |T|)$. The Löwenheim-Skolem-Tarski number $\text{LST}(L)$ of L is the smallest cardinal κ such that if \mathcal{A} is any τ -structure, then there is a substructure \mathcal{A}' of \mathcal{A} of cardinality $< \kappa$ such that $\mathcal{A}' \prec_L \mathcal{A}$.*

Note that $\text{LS}(L)$ always exists, because L is a set. In general there is no guarantee that $\text{LST}(L)$ exists, but if it exists, it is at least as big as $\text{LS}(L)$. We can think of the sizes of $\text{LS}(L)$ and $\text{LST}(L)$ as a “test” of how close the logic is to being first order. For first order logic these numbers are both \aleph_1 , and for $L(Q_1)$ and $L(Q_1^{\text{MM}})$ they are \aleph_2 . If κ is strongly inaccessible, then $\text{LST}(L_{\kappa\kappa}) = \kappa$.

The Löwenheim-Skolem numbers of $L(I)$ and $L(Q^{ec})$ are quite high in the hierarchy of cardinal numbers, certainly both cardinals are fixed points of the function $\alpha \mapsto \aleph_\alpha$. Whether the Löwenheim-Skolem number of $L(I)$ can be below the first weakly inaccessible, was asked in [15] and has been an open question ever since, but will be settled positively in this paper. On the other hand, in the inner model L^μ it is easy to see that $\text{LS}(L(I))$ is above the measurable cardinal.

For second order logic, $\text{LS}(L^2)$ is the supremum of Π_2 -definable ordinals ([14]), which means that it exceeds the first measurable, the first κ^+ -supercompact κ , and the first huge cardinal if they exist.

Theorem 5 ([7]) *1. Suppose κ is strong, then $\text{LS}(L^2) < \kappa$.*

2. $\text{LST}(L^2)$ exists if and only if supercompact cardinals exist, and then $\text{LST}(L^2)$ is the first of them.

Proof. For the first claim, suppose T is a theory in L^2 and T has a model \mathcal{A} . We may assume that the universe of \mathcal{A} is an ordinal δ . Let i be an embedding into M with critical point κ such that $T, \mathcal{A}, \mathcal{P}(\delta) \in M$. It is easy to prove by induction on formulas $\phi \in L^2$ that for all $\vec{a} \in A^n$ and $\vec{X} \in \mathcal{P}(A^{n_1}) \times \dots \times \mathcal{P}(A^{n_k})$ we have

$$\mathcal{A} \models \phi(\vec{a}, \vec{X}) \iff M \models \text{“}\mathcal{A} \models \phi(\vec{a}, \vec{X})\text{.”}$$

The point is that all subsets of A are in M . Thus $M \models \exists x(x \models T \text{ and } |x| < i(\kappa))$. Hence there is in V a model of T of cardinality $< \kappa$. For the second claim we refer to [7] but give the following argument for $\text{LST}(L^2) \leq \kappa$ for

supercompact κ since we will use it later: Suppose \mathcal{A} is a model of cardinality λ . Let i be an elementary embedding of V into a transitive M so that ${}^\lambda M \subseteq M$ and $i(\kappa) > \lambda$. Let \mathcal{B} be the pointwise image of \mathcal{A} under i . Since ${}^\lambda M \subseteq M$, $\mathcal{B} \in M$. It is easy to prove by induction on formulas $\phi \in L^2$ that for all $\vec{a} \in A^n$ and $\vec{X} \in \mathcal{P}(A^{n_1}) \times \dots \times \mathcal{P}(A^{n_k})$ we have

$$\begin{aligned} M \models \text{“}\mathcal{B} \models \phi(\vec{a}, \vec{X})\text{”} &\iff \mathcal{A} \models \phi(\vec{a}, \vec{X}) \\ &\iff M \models \text{“}i(\mathcal{A}) \models \phi(i(\vec{a}), i(\vec{X}))\text{”}. \end{aligned}$$

Thus $M \models \exists \mathcal{B}(\mathcal{B} \prec \mathcal{A} \text{ and } |B| < i(\kappa))$. Hence there is in V a model $\mathcal{C} \prec \mathcal{A}$ of T of cardinality $< \kappa$. \square

So second order logic meets the test of being very far from first order in terms of the size of its Löwenheim-Skolem numbers. We show that according to this test, $L(I)$ and $L(Q^{ec})$ can be close to second order logic but can also be, relatively speaking, close to first order logic.

The strongest large cardinal axiom from the point of view of Löwenheim-Skolem theorems is *Vopenka’s Principle*, which states that every proper class of structures of the same vocabulary has two members one of which is isomorphic to an elementary substructure of the other. In [12] an equivalent condition is given: Suppose A is a class. Let us call a cardinal κ *A-supercompact* if for all $\eta > \kappa$ there is $\alpha < \kappa$ and an elementary embedding

$$j : (V_\alpha, \in, A \cap V_\alpha) \rightarrow (V_\eta, \in, A \cap V_\eta)$$

with a critical point γ such that $j(\gamma) = \kappa$. It is proved in [12] that Vopenka’s Principle is equivalent to the statement that for every class A there is an *A-supercompact* cardinal. From this and the proof of Theorem 5 we get the following unpublished result of J. Stavi:

Theorem 6 *Vopenka’s Principle holds if and only if every logic has a Löwenheim-Skolem-Tarski number.*

For the intermediate logics $L(I)$ and $L(I, Q^{ec})$ the analogue of Theorem 5 (2) is the substantially less conclusive:

Theorem 7 ([13]) 1. *LST($L(I)$) exists only if inaccessible cardinals exist, and then LST($L(I)$) is at least as large as the first of them.*

2. *LST($L(I, Q^{ec})$) exists only if Mahlo cardinals exist, and then the cardinal LST($L(I, Q^{ec})$) is at least as large as the first of them.*

Proof. Let $\mathcal{A} = (R(\kappa^+), \epsilon)$, where $\kappa = \text{LST}(L(I))$. By the definition of $\text{LST}(L(I))$ there is a transitive set M of power $< \kappa$ and a monomorphism $i : (M, \epsilon) \rightarrow \mathcal{A}$ which preserves $L(I)$ -truth. Moreover, every M -cardinal is a real cardinal. Let λ be the largest cardinal in M . Clearly $i(\lambda) = \kappa > \lambda$. Let γ be the first ordinal moved by i . Trivially, γ is a limit cardinal. Suppose $f \in M$ is a cofinal δ -sequence in γ for some $\delta < \gamma$. Now $i(f)$ is a cofinal δ -sequence in $i(\gamma)$ whence $i(f)(\beta) > \gamma$ for some $\beta < \delta$. But $i(f)(\beta) = i(f(\beta)) = f(\beta) < \gamma$. Thus γ is weakly inaccessible in M , and therefore, $i(\gamma)$ is weakly inaccessible in V . The second claim is proved similarly. \square

The results of this paper explain why Theorem 7 is weaker than Theorem 5. The proof theoretic strength of the existence of either $\text{LST}(L(I))$ or $\text{LST}(L(I, Q^{ec}))$ exceeds substantially what follows from the mere size of these cardinals. Accordingly, and unlike $\text{LST}(L^2)$, the numbers $\text{LST}(L(I))$ and $\text{LST}(L(I, Q^{ec}))$ do not have to be very high in the scale of large cardinals. We will show in this paper that $\text{LST}(L(I))$ can be the first weakly inaccessible cardinal and $\text{LST}(L(I, Q^{ec}))$ can respectively be the first Mahlo cardinal. Also they can be of continuum size:

Theorem 8 ([13]) *Suppose κ is a supercompact cardinal and P is the notion of forcing C_κ . Let L be a provably \mathcal{C} -absolute logic which is provably a sublogic of L^2 . Then*

$$\Vdash_P \text{LST}(L(I, R)) \leq 2^\omega.$$

Proof. We give an outline of the proof for completeness. Suppose \mathcal{A} is a name for a finitary structure with universe λ in the P -forcing language. Let $i : V \rightarrow M$ be an elementary embedding of the universe such that $i(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $i''\kappa = \kappa$. Let \mathcal{B} be the point-wise image of \mathcal{A} under i . Using the fact that P preserve cardinals and cofinalities it is possible to show

$$M \models \text{“} \Vdash_{i(P)} i : \mathcal{B} \rightarrow_{L(I, R)} i(\mathcal{A}). \text{”} \tag{1}$$

It follows from this that

$$M \models \text{“} \Vdash_{i(P)} i(\mathcal{A}) \text{ has an } L\text{-elementary substructure of power } < i(\kappa) \text{”}.$$

Therefore

$$\Vdash_P \mathcal{A} \text{ has an } L\text{-elementary substructure of power } < \kappa.$$

\square

To see some of the strength of the Löwenheim-Skolem-Tarski Theorem for the equicardinality quantifier, let us recall the following observation from [13]: Let \mathcal{A} be the structure $(R(\kappa^+), \in)$. Let $\pi : (M, \in) \rightarrow (R(\kappa^+), \in)$ be an elementary embedding with M transitive and $|M| < \kappa$. If $\delta = M \cap On$, then $\pi \upharpoonright L_\delta : (L_\delta, \in) \rightarrow (L_{\kappa^+}, \in)$. Thus $0^\#$ exists. Obviously this argument can be considerably strengthened. We show in this paper that the existence of $LST(L(I))$ has enough combinatorial power to imply, when combined with current state of the inner model technology, Projective Determinacy.

2 The Failure of Squares

We have already alluded to the fact that the existence of $LST(L(I))$ has non-trivial consistency strength, for example, it implies $0^\#$. In this section we show that the existence of $LST(L(I))$ has a much stronger consistency strength, probably at the level of a supercompact cardinal.

We shall show that the existence of an $LST(K(I))$ cardinal implies that the combinatorial principle \square_λ fails for *every* $\lambda \geq \kappa$. For κ singular of cofinality ω we can do better than that and show that any reasonable version of \square_λ fails, in particular a consequence of any reasonable weakening of \square_λ (for $\text{cof}(\lambda) = \omega$) fails globally above $LST(L(I))$. The consequence we allude to is the existence of “good” scales.

We shall conclude this section by showing that assuming the consistency of a supercompact cardinal it is consistent that the first $LST(L(I))$ cardinal is the same as the first supercompact cardinal.

Definition 9 *The square principle \square_λ says: There is a sequence $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal} \rangle$, such that:*

1. C_α is closed unbounded subset of α .
2. The order type of C_α is always $\leq \lambda$.
3. If β is a limit point of C_α , then $C_\beta = C_\alpha \cap \beta$.

Theorem 10 *If κ is an $LST(L(I))$ number and $\lambda \geq \kappa$. Then \square_λ fails.*

Proof. Suppose $\langle C_\alpha : \alpha < \lambda^+ \text{ a limit ordinal} \rangle$, is a \square_λ sequence. Consider the structure

$$\mathcal{A} = \langle \lambda^+, \lambda, T, C \rangle,$$

where T is a function defined on the limit ordinals in λ^+ such that $T(\alpha)$ is the order-type of C_α , and C is a ternary relation such that $C(\alpha, \gamma, \eta)$ holds if and only if “ η is the γ -th member of C_α ”. Let \mathcal{B} be an $L(I)$ -elementary substructure of \mathcal{A} of cardinality $< \kappa$. It is easily verified that the order-type of the universe B of \mathcal{B} is a successor cardinal μ^+ , where μ is the order-type of $B \cap \lambda$. Let \mathcal{B}^* be the transitive collapse of \mathcal{B} . It is easily seen that \mathcal{B}^* has the form $\langle \mu^+, \mu, T^*, C^* \rangle$, where for some \square_μ -sequence $\langle C_\alpha^* : \alpha < \mu^+ \text{ a limit ordinal} \rangle$, T^* is a function defined on the limit ordinals in μ^+ such that $T^*(\alpha)$ is the order-type of C_α^* , and C^* is a ternary relation such that $C^*(\alpha, \gamma, \eta)$ holds if and only if “ η is the γ -th member of C_α^* ”. Let $\pi : B^* \rightarrow B$ be the inverse of the transitive collapse of \mathcal{B} . Let $\delta = \sup(B) = \sup(\pi''B^*)$. Note that $\delta < \lambda^+$ and $\text{cof}(\delta) = \mu^+$. C_δ is a closed unbounded subset of δ , $B = \{\pi(\alpha) : \alpha < \mu^+\} = \pi''B^*$ is cofinal in δ . So the set

$$A' = \{\eta < \delta : \eta \text{ limit point of } C_\delta \text{ and a limit point of } B\}$$

is closed unbounded in B_δ .

For $\eta \in A'$ let $\bar{\eta}$ be the minimal element of B^* such that $\pi(\bar{\eta}) \geq \eta$. Obviously, $\bar{\eta}$ is always defined because $\sup(A') = \sup \pi''B^* = \delta$. And if $\eta_1 < \eta_2$ in A' , then $\bar{\eta}_1 < \bar{\eta}_2$.

Claim: For $\eta \in A'$, η is a limit point of $C_{\pi(\bar{\eta})}$.

Proof. Otherwise, let ρ be $\sup(C_{\pi(\bar{\eta})} \cap \eta)$. So our assumption is $\rho < \eta$. As $\eta \in A'$, the range of π is cofinal in η , so there is ρ' such that

$$\rho < \pi(\rho') < \eta \leq \pi(\bar{\eta}).$$

By elementarity there is ρ'' such that

$$\pi(\rho') < \pi(\rho'') \in C_{\pi(\bar{\eta})}.$$

But clearly $\pi(\rho'') < \eta$, so we get a contradiction. \square (Claim.)

Claim: If $\eta_1 < \eta_2$ are in A' , then $\bar{\eta}_1$ is a limit point of $C_{\bar{\eta}_2}$.

Proof. Otherwise, let ρ be $\sup(C_{\pi(\bar{\eta}_2)}^* \cap \bar{\eta}_1)$. We assume $\rho < \bar{\eta}_1$, which means by definition of $\bar{\eta}_1$ that $\pi(\rho) < \eta_1$. By the previous claim η_2 is a limit point of $C_{\pi(\bar{\eta}_2)}$. So by the definition of the square principle

$$C_{\eta_2} = C_{\pi(\bar{\eta}_2)} \cap \eta_2.$$

Note that $\eta_2 \geq \pi(\bar{\eta}_1)$. By elementarity of π

$$\pi(\rho) = \sup(C_{\pi(\bar{\eta}_2)} \cap \pi(\bar{\eta}_1)) = \sup(C_{\eta_2} \cap \pi(\bar{\eta}_1)). \quad (2)$$

On the other hand η_1 and η_2 are in A' , hence they are limit points of C_δ , so $C_{\eta_2} = C_\delta \cap \eta_2$, so η_1 is a limit point of C_{η_2} . This contradicts (2). \square (Claim.)

It follows from the previous claim that if $\eta_1, \eta_2 \in A'$, then the order-type of $C_{\bar{\eta}_2}^*$ exceeds the order-type of $C_{\bar{\eta}_1}^*$. So $T^*(\bar{\eta}_2) > T^*(\bar{\eta}_1)$. The set A' being cofinal in δ , it has order-type at least μ^+ , so T^* is a monotone function from a set of ordinals of order-type $\geq \mu^+$ into μ , which is clearly a contradiction. \square (Theorem)

Weaker versions are the following list of weaker and weaker principles.

Definition 11 *The weak square principle $\square_{\kappa, \lambda}$ says: There is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa \text{ a limit ordinal} \rangle$, such that:*

1. \mathcal{C}_α is a set of closed unbounded subsets of α .
2. $|\mathcal{C}_\alpha| \leq \lambda$
3. The order type $\text{otp}(C)$ of each member C of \mathcal{C}_α is $\leq \kappa$.
4. If $C \in \mathcal{C}_\alpha$ and $\beta \in \lim(C)$, i.e. C is a limit point of C , then $C \cap \beta \in \mathcal{C}_\beta$.

The principle $\square_{\lambda, \lambda^+}$, the so called ‘‘silly square’’ is actually provable (see the proof of Lemma 17), so the weakest reasonable principle is $\square_{\lambda, \lambda}$. Our goal is now to show that if λ is singular of cofinality ω and above $\text{LST}(L(I))$, then $\square_{\lambda, \lambda}$ fails. This fact by itself indicates that the assumption of the existence of a $\text{LST}(L(I))$ cardinal has a large consistency strength. At the present it is not known how to get a model in which $\square_{\lambda, \lambda}$ fails even for a single singular λ without assuming a supercompact cardinal.

The way we shall prove the failure of $\square_{\lambda, \lambda}$ is by refuting an even weaker property: ‘‘The existence of a good sequence in λ^ω/FIN of length λ^+ .’’ The definitions and facts about ‘‘good sequences in λ^ω/FIN ’’ are due to Shelah and based on his pcf theory ([11]). Since we shall need a much simpler version of the notions and the basic lemmas, we include them for the sake of completeness.

We consider elements of On^ω ordered by eventual domination, i.e. for $f, g \in \text{On}^\omega$

$$f <^* g \text{ if } f(n) < g(n) \text{ for all but finitely many } n < \omega.$$

Definition 12 Suppose $\langle f_\alpha : \alpha < \mu \rangle$ is a $<^*$ -increasing sequence in On^ω .

- (i) A point $\delta \in \mu$ is called a good point for the sequence if there is a cofinal set $C \subseteq \delta$ and a function $\alpha \mapsto n_\alpha$ from C to ω such that if $\alpha < \beta$ in C and $k > \max(n_\alpha, n_\beta)$, then $f_\alpha(k) < f_\beta(k)$.
- (ii) The sequence is good, if there is a closed unbounded subset D of μ such that $\delta \in D$ implies that δ is a good point of the sequence.

Lemma 13 Suppose δ is a good point for the sequence $\langle f_\alpha : \alpha < \mu \rangle$ and D is any cofinal subset of δ . Then there is $E \subseteq D$ witnessing the goodness of δ .

Proof. Let C and $\alpha \mapsto n_\alpha$ witness the goodness of δ . W.l.o.g. $\text{otp}(C) = \text{otp}(D) = \text{cof}(\delta)$. Let $E \subseteq D$ be chosen so that for every $\gamma \in E$ there are $\gamma^-, \gamma^+ \in C$ in such a way that $\gamma^- < \gamma < \gamma^+$ and if $\gamma < \eta \in E$, then $\gamma^+ \leq \eta^-$. Let $m_\gamma \in \omega$ (for $\gamma \in E$) be such that if $i > m_\gamma$, then $f_{\gamma^-}(i) < f_\gamma(i) < f_{\gamma^+}(i)$. Let $n_\gamma^* = \max(n_{\gamma^-}, n_{\gamma^+}, m_\gamma)$. Now, if $i \geq \max(n_\gamma^*, n_\eta^*)$, then

$$f_\gamma(i) < f_{\gamma^+}(i) \leq f_{\eta^-}(i) < f_\eta(i).$$

□ (Lemma)

Theorem 14 (Shelah [11], see also [2] p.18) If $\text{cof}(\lambda) = \omega$ and $\square_{\lambda, \lambda}$ holds, then there is a good sequence in λ^ω of length λ^+ .

Proof. Fix a sequence of regular cardinals λ_n cofinal in λ . We shall actually get our sequence in $\prod_{n < \omega} \lambda_n \subseteq \lambda^\omega$. Note that every sequence of functions in $\prod_{n < \omega} \lambda_n$ of size λ has a $<^*$ -upper bound in $\prod_{n < \omega} \lambda_n$ (By taking $g(n) =$ the supremum of $f_n(n)$ for the first λ_{n-1} of our functions).

Fix a $\square_{\lambda, \lambda}$ -sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa \text{ a limit ordinal} \rangle$. Without loss of generality we can assume that $\text{otp}(C) < \lambda$ for each $C \in \mathcal{C}_\alpha$. (Indeed, if $\text{otp}(C) = \lambda$ when $C \in \mathcal{C}_\alpha$, then $\text{cof}(\alpha) = \omega$ and we can replace C by an ω -sequence cofinal in α . Note that this C is never used as an initial segment of $D \in \mathcal{C}_\beta$ for $\alpha < \beta$ because it would imply $\text{otp}(D) > \lambda$).

We define the $<^*$ -increasing sequence $\langle f_\alpha : \alpha < \lambda^+ \rangle$ in $\prod_{n < \omega} \lambda_n$ by induction. The successor stage is trivial: $f_{\alpha+1}(n) = f_\alpha(n)$. Suppose then α is limit. For each $C \in \mathcal{C}_\alpha$ we define a function in $\prod_n \lambda_n$ as follows:

$$g_C(i) = \begin{cases} \sup_{\beta \in C} g_\beta(i), & \text{if } \text{otp}(C) < \lambda_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}_\alpha \leq \lambda$, we can find $f_\alpha \in \prod_n \lambda_n$ such that $g_C <^* f_\alpha$ for every $C \in \mathcal{C}_\alpha$. Clearly $f_\beta <^* f_\alpha$ for every $\beta < \alpha$. We claim that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good sequence. Actually the claim is that every limit $\delta < \lambda^+$ is a good point of the sequence. If $\text{cof}(\delta) = \omega$, then we pick a cofinal sequence $\langle \delta_n : n < \omega \rangle$ in δ . Let $n(\delta_n)$ be such that for $i \geq n(\delta_n)$ we have

$$f_{\delta_{n-1}}(i) < f_{\delta_n}(i) < f_{\delta_{n+1}}(i).$$

Clearly the set $\{\delta_n : n < \omega\}$ and the map $\delta_n \mapsto n(\delta_n)$ witness the goodness of δ . If $\text{cof}(\delta) > \omega$, pick $C \in \mathcal{C}_\delta$ and let C^* be the set of limit points of C . Let n be such that $\text{otp}(C) < \lambda_n$ and also $g_C(i) \leq f_\alpha(i)$ for $i \geq n$. If $\beta < \beta' \in C^*$ and if $i \geq \max(n_\beta, n_{\beta'})$, we get $f_\beta(i) < g_{C \cap \beta'}(i) < f_{\beta'}(i)$ (because $\beta \in C \cap \beta'$). So the set C^* and the map $\beta \mapsto n_\beta$ witnesses the goodness of δ . \square (Theorem)

The result for the existence of $\text{LST}(L(I))$ number follows from

Theorem 15 *Suppose $\kappa = \text{LST}(LI)$ and $\lambda \geq \kappa$ with $\text{cof}(\lambda) = \omega$. Then there is no no good sequence in λ^ω of length λ^+ .*

Proof. Suppose $\text{cof}(\lambda) = \omega$. Suppose that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good sequence in λ^ω . Suppose D is a cub on λ^+ such that all points of D of cofinality $> \omega$ are good. Let

$$F = \{(\alpha, \beta, \gamma) \in \lambda^+ \times \omega \times \lambda : f_\alpha(\beta) = \gamma\},$$

and

$$\mathcal{A} = \langle \lambda^+, \lambda, <, F, D \rangle.$$

Since $\kappa = \text{LST}(LI)$ there is

$$\mathcal{B} = \langle B, B \cap \lambda, <, F \cap B^3, D \cap B \rangle \prec_{LI} \mathcal{A}$$

such that $|B| < \kappa$. Of course, $\omega \subset B$. Since

$$\forall x \neg Iyz(y < x)(z = z)$$

$$\forall x(x < \lambda \rightarrow \neg Iyz(y < x)(z < \lambda))$$

$$\forall y(\lambda < y \rightarrow Iuv(u < \lambda)(v < y))$$

are true in \mathcal{A} , they are true in \mathcal{B} and it follows that for some cardinal $\mu < \kappa$, $\text{otp}(B)$ is μ^+ and $\text{otp}(B \cap \lambda) = \mu$. Let $\delta = \text{sup}(B)$. Note that δ is a limit

point of D , and hence $\delta \in D$. Since B is cofinal in δ , $\text{cof}(\delta) = \mu^+$. By elementarity, each function f_α , $\alpha \in B$, maps ω into $B \cap \lambda$. We now argue that δ cannot be a good point of $\langle f_\alpha : \alpha < \lambda^+ \rangle$. Assume otherwise. Then there is a cofinal set $C \subseteq \delta$ and a function $\alpha \mapsto n_\alpha$ from C to ω such that if $\alpha < \beta$ in C and $k \geq \max(n_\alpha, n_\beta)$, then $f_\alpha(k) < f_\beta(k)$. By Lemma 13 we may assume $C \subseteq B$. Let C' be cofinal in C such that n_α is a fixed integer N for all $\alpha \in C'$. Now $\{f_\alpha(N) : \alpha \in C'\}$ is a subset of $B \cap \lambda$ which is of order-type μ^+ , a contradiction. \square (Theorem)

Corollary 16 *If $\kappa = \text{LST}(L(I))$, then $\square_{\lambda, \lambda}$ fails for every singular $\lambda \geq \kappa$ of cofinality ω . Hence, in particular, PD holds.*

The existence of $\text{LST}(L(I))$ also implies the Singular Cardinals Hypothesis above κ , i.e. if λ is singular $\geq \kappa$, then

$$\text{(SCH)} \quad \lambda^{\text{cof}(\lambda)} = \max(\lambda^+, 2^{\text{cof}(\lambda)}).$$

It follows from Silver's singular cardinals theorem that if λ violates the SCH and $\text{cof}(\lambda) > \omega$, then λ is a limit of cardinals that violate the SCH.

Lemma 17 ([11]) *If λ is a singular cardinal of cofinality ω and λ violates the SCH, then there is a good sequence in λ^ω of length λ^+ .*

Proof. By Shelah [11], if λ violates SCH and $\text{cof}(\lambda) = \omega$, then there is a sequence $\langle \lambda_n : n < \omega \rangle$ cofinal in λ such that $\prod_n \lambda_n / \text{FIN}$ has true cofinality λ^{++} , which implies that every set of functions in $\prod_n \lambda_n$ of cardinality λ^+ has a $<^*$ -upper bound in $\prod_n \lambda_n$. Now one can repeat the proof Theorem 14 by replacing in that proof the $\square_{\lambda, \lambda}$ -sequence by a $\square_{\lambda, \lambda^+}$ -sequence (the ‘‘Silly Square’’) and getting the good sequence in $\prod_n \lambda_n$. The silly square is always true, for if C_α is a cub subset of α of order type $\text{cof}(\alpha)$, we can let $\mathcal{C}_\alpha = \{C_\beta \cap \alpha : \beta < \lambda^+, \alpha \text{ limit point of } C_\beta\}$ and then $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$ witnesses $\square_{\lambda, \lambda^+}$. The proof works as before using the fact that for every $\alpha < \lambda^+$ $|\mathcal{C}_\alpha| \leq \lambda^+$ and that every set of functions in $\prod_n \lambda_n$ of cardinality λ^+ has a $<^*$ -upper bound. \square (Lemma)

Corollary 18 *If $\kappa = \text{LST}(L(I))$, then SCH holds above κ .*

Theorem 19 *If it is consistent to assume the existence of a supercompact cardinal, then it is consistent to assume that $\text{LST}(L(I))$ is the first supercompact cardinal.*

Proof. We refer to Magidor [8]. In this paper, assuming the existence of a supercompact cardinal, a model is constructed in which the first supercompact is the first strongly compact. It is achieved by forcing over a model in which κ is supercompact and arranging for SCH to fail for unboundedly many λ 's below κ , while preserving the supercompactness of κ . In the resulting model $\kappa \geq \text{LST}(L(I))$ cardinal (even $\kappa \geq \text{LST}(L^2)$). If $\mu < \kappa$ and $\mu \geq \text{LST}(L(I))$, then pick $\mu < \lambda < \kappa$ violating SCH to get a contradiction with Corollary 18. So $\kappa = \text{LST}(L(I))$ cardinal. \square (Theorem)

In the next section we shall show that $\text{LST}(L(I))$ can be much smaller than the first supercompact cardinal, namely it can be the first inaccessible, so we are in a true “identity crisis” situation.

3 The First Mahlo Cardinal

As we pointed out in Theorem 7, $\text{LST}(L(I, Q^{ec}))$ is, if it exists at all, at least as big as the first Mahlo cardinal. We now prove the consistency of $\text{LST}(L(I, Q^{ec}))$ being actually equal to the first Mahlo cardinal. As Corollary ?? shows, we have to start from a cardinal substantially larger than a Mahlo, even a strong cardinal is not enough. So we start from a supercompact cardinal.

Theorem 20 *It is consistent, relative to the consistency of a supercompact cardinal, that $\text{LST}(L(I, Q^{ec}))$ is the first Mahlo cardinal.*

Proof. Suppose κ is supercompact. We then make every $\rho < \kappa$ non-Mahlo. Suppose ρ is Mahlo. Let P_ρ be the set of closed bounded sets of singular cardinals $< \rho$ inversely ordered by end-extension, i.e. a weaker condition is an initial segment of a stronger condition. For every regular $\lambda < \rho$ the forcing notions P_ρ contains a λ -closed dense set $\{C : \max(C) > \lambda\}$. Therefore P_ρ cannot collapse cardinals $< \rho$ or change their cofinality. Moreover, P_ρ does not add new bounded subsets to ρ . On the other hand, $|P_\rho| = \rho$, so P_ρ preserves all cardinals and cofinalities. In particular P_ρ kills the Mahloness of ρ but preserves inaccessibility of ρ . Now we iterate this forcing. Suppose μ_α , $\alpha < \delta$, is an increasing sequence of Mahlo cardinals. Let $R_0 = P_{\mu_0}$. Suppose P_α has been defined. Let $P_\alpha \Vdash \tilde{Q}_\alpha = P_{\mu_\alpha}$ and $R_{\alpha+1} = R_\alpha \star \tilde{Q}_\alpha$. For limit α be let R_α be the direct limit of the previous stages, if α is inaccessible, and inverse limit otherwise. This will ensure that each R_α , α inaccessible,

will have the α -c.c. and will therefore preserve the Mahloness of each μ_β , $\beta \geq \alpha$. Let $R = R_\kappa$. Now $V^R \models \kappa = \text{LST}(L(I, Q^{ec}))$. To see why, suppose \mathcal{A} is a structure with universe λ , where $\lambda \geq \kappa$. Since κ is supercompact, there is $j : V \rightarrow M$, M transitive, such that ${}^\lambda M \subseteq M$ and $j(\kappa) > \lambda$. Note that $j(R) = R \star P_\kappa \star R_{>\kappa}$, where $R_{>\kappa}$ contains a κ -closed set. Since R has the κ -c.c. and $R_{>\kappa}$ is sufficiently closed, j can be extended to $j^* : V^R \rightarrow M^{j(R)}$. Now we can continue as above. \square

4 The First Inaccessible Cardinal

In this section we prove the main result of this paper:

Theorem 21 *If $ZFC +$ "There is a supercompact cardinal" is consistent, so is $ZFC +$ "There is an inaccessible cardinal" + " $\text{LST}(L(I))$ is the first inaccessible cardinal".*

The assumption of the consistency of a supercompact cardinal seems, on the basis of present technology, almost unavoidable. By Theorem 15 above, we know that the existence of $\text{LST}(L(I))$ implies the negation of $\square_{\lambda, \lambda}$ for every large enough λ . The only known way to get a model in which this holds is to start from a strongly compact cardinal. But the definition of strong compactness is not sufficient for getting reflection principles which seem to be necessary for getting the existence of $\text{LST}(L(I))$, so the assumption of a supercompact cardinal seems natural enough.

4.1 Outline of the Proof

We start with a supercompact cardinal κ . In our final model κ will be the first inaccessible cardinal, while preserving enough of the reflection properties of a supercompact cardinal, so that in the model κ will be $\text{LST}(L(I))$.

In the process of achieving this we force a closed unbounded set C of singular cardinals below κ . This will make κ non-Mahlo. We then collapse cardinals between consecutive elements of C so that none of them can be inaccessible. Thus κ has become the first inaccessible. But we have to be careful about the way in which we collapse cardinals in order to maintain enough reflection properties of κ , so that κ will be $\text{LST}(L(I))$. To argue that $\text{LST}(L(I)) = \kappa$ in the final model is similar to the argument of Theorem

4. Namely, suppose P is the forcing used to get our final model and \mathcal{A} is a name for a finitary structure in V^P with domain λ . Let $i : V \rightarrow M$ be an elementary embedding of the universe such that $i(\kappa) > \lambda$ and ${}^\lambda M \subseteq M$, where κ is the critical point of i (V is our ground model). P will be a forcing such that P as a forcing notion is a regular subforcing of $i(P)$. In $V^{i(P)}$ we can define an embedding $i^* : V^P \rightarrow M^{i(P)}$ extending i . By our assumption $i^* \upharpoonright \mathcal{A} \in V^P$, and $i^* \upharpoonright \mathcal{A}$ is an embedding of \mathcal{A} into $i^*(\mathcal{A})$. We would like i^* to preserve formulas of $L(I)$. Given that we are done because

$$M^{i(P)} \models \text{“}i^*(\mathcal{A}) \text{ has an } L(I)\text{-elementary substructure of cardinality } \lambda < i^*(\kappa)\text{”}.$$

By elementarity,

$$V^P \models \text{“}\mathcal{A} \text{ has an } L(I)\text{-elementary substructure of cardinality } < \kappa\text{”}.$$

To get i^* to preserve formulas of $L(I)$ we need that $i(P)/P$ collapses no cardinals $\leq \lambda$.

Suppose that when we collapsed cardinals between consecutive members of C we had some function $f : \kappa \rightarrow \kappa$ such that for a member δ of C no cardinal was collapsed between δ and $f(\delta)$. Let us also assume that $\lambda < i(f)(\kappa)$. (Note that κ is a limit point of $i^*(C)$.) So no cardinal between κ and $i(f)(\kappa)$ will be collapsed. In particular, all cardinals between κ and λ are preserved by $i(P)/P$.

Another issue is that κ is supposed to be a limit point of $i^*(C)$ hence in $M^{i(P)}$ it is supposed to be singular. In V^P it is supposed to be regular, indeed inaccessible. So we need $i(P)/P$ to make some regular cardinals singular. Since $i(P)$ “looks like P ”, we need P to make enough regular cardinals singular, so that $M^{i(P)} \models \text{“}\kappa \text{ is singular”}$.

The standard way of making a regular cardinal singular is by forcing with Prikry forcing on a measurable cardinal. Since we shall have to do it for many cardinals below κ , we shall have to iterate Prikry type forcings for measurable cardinals below κ .

So the forcing notion we shall use will be an iteration of several steps:

- (a) Iterated Prikry type forcing for every measurable $\lambda < \kappa$, where besides changing the cofinality of λ to ω we do some preparatory forcing for the additional steps, which will be relevant only to κ . We denote this forcing by Q_λ and the iteration up to κ by P_κ .

- (b) On κ we force a closed unbounded set C such that every limit point of C is singular. We denote this forcing by $\text{NM}(\kappa)$ (From “Non-Mahlo”).
- (c) We collapse cardinals between consecutive members of C making sure that if $\beta \in C$, then no cardinals are collapsed between β and $f(\beta)$ for an appropriate function $f : \kappa \rightarrow \kappa$. (We denote this forcing $\text{Col}(C)$).

The challenge will be to make sure that $\text{NM}(\kappa) * \text{Col}(C)$ embeds nicely into \mathbb{Q}_κ so that if $R = P_\kappa * \text{NM}(\kappa) * \text{Col}(C)$, then R embeds nicely into $i^*(R)$. This will be achieved by embedding $\text{NM}(\kappa) * \text{Col}(C)$ into \mathbb{Q}_κ , which is the κ -th stage in the iteration of $i^*(P_\kappa) = P_{i^*(\kappa)}$. We hope that these remarks make the following definition of the forcing notion somewhat less frightening.

4.2 The Forcing Construction

Our first step is to define the function $f : \kappa \rightarrow \kappa$ that will determine intervals where all the cardinals will be preserved. We assume that our ground model satisfies G.C.H. and that there is no inaccessible above κ . A classical lemma of Laver [5] proves the following:

Lemma 22 *Let κ be supercompact. Then there exists a function $h : \kappa \rightarrow V_\kappa$ such that for every x and every $\mu \geq \kappa$ there is a μ -supercompact embedding $j : V \rightarrow M$ (i.e. $M^\mu \subseteq M$, $j(\kappa) > \kappa$, $j(\alpha) = \alpha$ for $\alpha < \kappa$), such that $j(h)(\kappa) = x$.*

An easy corollary of Laver’s lemma is the following:

Lemma 23 *Let κ be supercompact such that there is no inaccessible cardinal above κ . Then there is a function $f : \kappa \rightarrow \kappa$ such that for all $\alpha < \kappa$, $\alpha < f(\alpha)$, there is no inaccessible cardinal λ with $\alpha < \lambda \leq f(\alpha)$, and for all $\mu \geq \kappa$ there is a μ -supercompact embedding $j : V \rightarrow M$ with $\mu < j(f)(\kappa)$.*

Proof Let h be the Laver-function from Lemma 22. Let $f(\alpha) = (h(\alpha))^+$ if $h(\alpha)$ is an ordinal $> \alpha$ such that there is no inaccessible cardinal λ with $\alpha < \lambda \leq h(\alpha)$. Define $f(\alpha) = \alpha^+$ otherwise. Apply Lemma 22 for μ and $x = \mu^+$. Note that in M there is no inaccessible cardinal λ with $\kappa < \lambda \leq j(h)(\kappa) = \mu^+$ so for $j(f)(\kappa)$ the first possibility of the definition of f holds and hence $j(f)(\kappa) = \mu^+ > \mu$. \square

So from now on we fix such a function f . Without loss of generality we can assume that the first inaccessible cardinal is closed under f . The cardinals that we shall be interested in will be cardinals $\lambda \leq \kappa$ such that λ is measurable and λ is closed under the function f . For such λ we define the forcing notion $\text{NM}(\lambda)$, which is intended to make λ non-Mahlo by forcing a closed unbounded set C of cardinals such that every limit point of C is singular. For technical simplicity it will be convenient to assume that if $\beta \in C$ and β' is the minimal member of C above β , then β' is inaccessible and $f(\beta) < \beta'$.

Definition 24 *Suppose λ is a measurable cardinal, closed under f . Then $\text{NM}(\lambda)$ is the set of all closed bounded subsets C of λ such that*

- (a) *Every member of C is a cardinal.*
- (b) *If β is a limit point of C , then β is singular.*
- (c) *If $\beta \in C$, and β' is the first point of C above β , then β' is inaccessible and $f(\beta) < \beta'$.*

The partial order \leq on $\text{NM}(\lambda)$ is defined by $D \leq C$ iff $D, C \in \text{NM}(\lambda)$ and D is an end-extension of C .

So the successor members of C are all regular, and limit points of C are closed under f . It is easy to see that if $C \in \text{NM}(\lambda)$, and C contains a point above $\mu < \lambda$, then $\{D : D \leq C\}$ is μ -closed. Hence it follows that forcing with $\text{NM}(\lambda)$ introduces no new μ -sequences of ordinals when $\mu < \lambda$. So λ remains regular, and since no new bounded subsets of λ are introduced, λ remains strongly inaccessible. Also, it is easy to see that if $G \subseteq \text{NM}(\lambda)$ is a generic filter, then $\bigcup G$ is a closed unbounded subset of λ . Every limit point of $\bigcup G$ is singular, so in the generic extension λ is a non-Mahlo inaccessible cardinal.

Since we are going to define many partial orders, we shall denote each of the relevant partial orders by \leq . Only in case of a possible confusion we shall add the subscript indicating the forcing notion (\leq_P for the partial order of P , etc). Also in some cases it will be convenient to define a preorder on the forcing notion (so we may write $p \leq q$ and $q \leq p$), so that we really mean the forcing notion is the equivalence classes of the relation “ $p \leq q$ and $q \leq p$ ”.

Given two regular cardinals $\mu < \rho$, $\text{Col}(\mu, < \rho)$ is the usual Levy collapse of all the cardinals δ with $\mu < \delta < \rho$ to μ . It is a μ -closed forcing notion and

if ρ is inaccessible or the successor of a regular cardinal, the forcing notion $\text{Col}(\mu, < \rho)$ satisfies the ρ -c.c.

Let C be a closed set of cardinals. For $\beta \in C - \{\text{sup}(C)\}$ let β' be the next point of C after β . We assume that if $\beta \in C$, then β' is inaccessible and $f(\beta) < \beta'$. The forcing notion

$$\text{Col}(C)$$

is defined to be the Easton product of $\text{Col}(f(\beta), < \beta')$ for $\beta \in C - \{\text{sup}(C)\}$. (Easton product means that it is the collection of all functions g defined on $C - \{\text{sup}(C)\}$ such that $g(\beta) \in \text{Col}(f(\beta), < \beta')$ and for regular δ the set $\{\beta < \delta : g(\delta) \neq \emptyset\}$ is bounded in δ .) Note that for our case, if $C \subseteq \lambda$ is the closed unbounded set introduced by $\text{NM}(\lambda)$, then the Easton condition simply means that if $g \in \text{Col}(C)$ then the cardinality of $\{\beta : g(\beta) \neq \emptyset\}$ is less than λ .

It is easy to see that if C is $\text{NM}(\lambda)$ -generic and the first member of C is below the first inaccessible, then if we force with $\text{Col}(C)$, then λ will be the first inaccessible.

4.2.1 Atomic Step

Let λ be a measurable cardinal $\leq \kappa$ closed under f . We shall describe a variation Q_λ of Prikry forcing for making λ singular of cofinality ω , while at the same time introducing a generic object over V to $\text{NM}(\lambda)$. Like Prikry forcing, Q_λ will introduce no new unbounded subsets of λ . Q_λ has an additional role which we shall explain below. Note that we have

$$V \subseteq V^{\text{NM}(\lambda)} \subseteq V^{Q_\lambda}.$$

(Note that in $V^{\text{NM}(\lambda)}$ the cardinal λ is still regular.)

The forcing Q_λ has an additional role which we shall explain below. Like Prikry-forcing, Q_λ will introduce no new bounded subsets of λ . The final stage of our iteration will be forcing with $\text{NM}(\kappa)$ followed by $\text{Col}(D)$, where D is the closed unbounded set introduced by $\text{NM}(\kappa)$. We shall need to embed the forcing $\text{NM}(\kappa) * \text{Col}^{V^{\text{NM}(\kappa)}}(D)$ into $Q_\kappa * \text{Col}^{V^{Q_\kappa}}(D)$. Unfortunately, $\text{Col}^{V^{Q_\kappa}}(D)$ in V^{Q_κ} is different from $\text{Col}^{V^{\text{NM}(\kappa)}}$ (Because, for instance, in $\text{NM}(\kappa)$ the cardinal κ is regular, but in V^{Q_κ} it is singular of cofinality ω , so the meaning of the Easton Product used in the definition of $\text{Col}(D)$ is very different.) But note that because $V^{\text{NM}(\kappa)}$ and V^{Q_κ} have the same bounded

subsets of κ , $\text{Col}^{V^{\text{NM}(\kappa)}}(D)$ is a sub-partial order of $\text{Col}^{V^{Q_\kappa}}(D)$. Also, if $G \subseteq \text{Col}^{V^{Q_\kappa}}(D)$ is a filter, $G \cap V$ is a filter in $\text{Col}^{V^{\text{NM}(\kappa)}}(D)$. The issue is the genericity of $G \cap V$ over $V^{\text{NM}(\kappa)}$. In general, $G \cap V$ does not have to be generic over $V^{\text{NM}(\kappa)}$. The additional role of Q_λ (for all $\lambda \leq \kappa$) will be to introduce a condition in $h \in \text{Col}^{V^{Q_\kappa}}(D)$ such that if $h \in G \subseteq \text{Col}^{V^{Q_\kappa}}(D)$ is a generic filter on V^{Q_κ} , then $G \cap V$ is generic over $V^{\text{NM}(\lambda)}$.

For the next definition we fix a normal ultrafilter U on λ .

Definition 25 Q_λ is the set of all objects p of the form

$$\langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle,$$

where

- (i) $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \lambda$, each α_i is closed under f .
- (ii) $A \subseteq \lambda$, $A \in U$.
- (iii) $c \in \text{NM}(\lambda)$, c contains some point below the first inaccessible, and $\alpha_i \in c$ for $i < n$.
- (iv) $h \in \text{Col}(c)$
- (v) C is a function defined on A such that for all $\beta \in A$ we have $C(\beta) \in \text{NM}(\lambda)$ with $\min(C(\beta)) = \beta$.
- (vi) H is a function defined on A such that for $\beta \in A$ we have $H(\beta) \in \text{Col}(C(\beta))$.
- (vii) For all $\beta \in A$ we have $\sup(c) < \beta$ and $\alpha_{n-1} < \beta$.
- (viii) For all $\beta, \beta' \in A$ such that $\beta < \beta'$ we have $\sup(C(\beta)) < \beta'$.

The intuitive meaning of the forcing is rather clear. The finite sequence $\alpha_0, \dots, \alpha_{n-1}$ is an initial segment of the ω -sequence that will be cofinal in λ . The set A is the set of possible candidates for extending the sequence $\alpha_0, \dots, \alpha_{n-1}$. As usual for Prikry type forcings, we require to have a large set of possible candidates to be members of the ω -sequence leading up to λ . The set c is an initial segment of the generic object in $\text{NM}(\lambda)$ that will be introduced by Q_λ , h is a partial information about an object that will eventually be a condition in $\text{Col}^{V^{Q_\lambda}}(D)$, where D will be the club introduced

by the $\text{NM}(\lambda)$ generic filter. For $\beta \in A$ the set $C(\beta)$ is a commitment, in case β will be added to the sequence, that the future generic object of $\text{NM}(\lambda)$ will contain $C(\beta)$. (Not as initial segment because it will have to continue c and possibly other $c(\gamma)$'s for γ 's that were included in the sequence before β .) Similarly for $H(\beta)$. These remarks motivate the definition of the partial order in Q_λ :

Definition 26 (*The partial order of Q_λ*). Suppose

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle$$

and

$$q = \langle \alpha_0^*, \dots, \alpha_{k-1}^*, A^*, c^*, h^*, C^*, H^* \rangle$$

are in Q_λ . We say that q extends p , in symbols $q \leq p$, if

- (i) $n \leq k$, $\alpha_i = \alpha_i^*$ for $i < n$, and $n \leq i < k$ implies $\alpha_i^* \in A$.
- (ii) $A^* \subseteq A$,
- (iii) c^* is an end extension of c (Hence it is an extension of c in the sense of $\text{NM}(\lambda)$.)
- (iv) If $n = k$, then $c^* = c$. If $n < k$, then $c^* \cap \alpha_n^* = c$ and for all $n \leq i < k-1$ the set $c^* \cap [\alpha_i^*, \alpha_{i+1}^*]$ is an end extension of $C(\alpha_i^*)$, and $c^* - \alpha_{k-1}^*$ is an end extension of $C(\alpha_{k-1})$. (Note that this condition together with clause (v) of Definition 25 implies that for $n \leq i < k$ we have $\alpha_i^* \in c^*$. *Why need to say this? Isn't it true by definition?*)
- (v) If $n = k$, then $h^* = h$. If $n < k$, then $h = h^* \upharpoonright c \cap \{\alpha_n^*\}$ (Namely, we do not add any additional information about the forcing condition we construct in $\text{Col}(D)$ below α_n^* .)
- (vi) For $n \leq i < k-1$ the set $h^* \upharpoonright c \cap [\alpha_i^*, \alpha_{i+1}^*]$ extends $H(\alpha_i^*)$ and $h^* \upharpoonright c \cap [\alpha_{k-1}^*, \kappa]$ extends $H(\alpha_{k-1}^*)$
- (vii) For all $\alpha \in A^*$ the condition $C^*(\alpha)$ is an extension (as a member of $\text{NM}(\lambda)$) of $C(\alpha)$ and $H^*(\beta)$ is an extension (as a member of $\text{Col}(C^*(\alpha))$) of $H(\alpha)$. Note that $H(\alpha)$, being a member of $\text{Col}(C(\alpha))$ can be considered also to be a member of $C^*(\alpha)$ since $C^*(\alpha)$ is an extensions of $C(\alpha)$.

If $n = k$ above, we say that q is a direct extension of p , in symbols $q \leq^* p$.

Notation: If $p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle$ is in Q_λ we call n the *length* of p and denote it by $n(p)$. Similarly

$$\begin{array}{lll}
\alpha(p) & = & \langle \alpha_0, \dots, \alpha_{n-1} \rangle \text{ the } \alpha\text{-part of } p \\
A(p) & = & A \text{ the } A\text{-part of } p \\
c(p) & = & c \text{ the } c\text{-part of } p \\
h(p) & = & h \text{ the } h\text{-part of } p \\
C(p) & = & C \text{ the } C\text{-part of } p \\
H(p) & = & H \text{ the } H\text{-part of } p
\end{array}$$

Note that if $\mu < \lambda$ then every decreasing sequence of direct extension of length $\leq \mu$ has a lower bound which is a direct extension of all the members of the sequence. To see this one uses the fact that the c parts and h parts of all the members of the sequence are the same, and if α is in the intersection of the A part of the member for the sequence and $\alpha > \mu$, then since $C(\alpha)$ is assumed to contain α , the sequence of the c -parts has a lower bound because $\text{NM}(\lambda)$ is μ^+ -closed under a condition containing a point above μ . A similar argument takes care of the H part.

The following lemma is a typical one for Prikry type forcing. Its proof is a straight-forward generalization of similar lemmas proved before (e.g. Magidor [9]):

Lemma 27 *Let Φ be a statement in the forcing language for Q_λ and $p \in Q_\lambda$. Then there exists a direct extension q of p such that q decides Φ .*

As usual, it follows from the lemma that Q_λ introduces no new bounded subsets of λ .

Lemma 28 *Let $G \subseteq Q_\lambda$ be generic over V . Let $D_G = \{c(p) : p \in G\}$. Then $D_G \subseteq \text{NM}(\lambda)$ generates a generic filter of $\text{NM}(\lambda)$ over V .*

Proof: It is immediate that for $p, p' \in G$ either $c(p) \leq c(p')$ or $c(p') \leq c(p)$, so D_G generates a filter. We just have to prove its genericity. Let $E \subseteq \text{NM}(\lambda)$, $E \in V$, be dense open in $\text{NM}(\lambda)$. We have to show that $E \cap D_G \neq \emptyset$. Let

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle \in Q_\lambda.$$

We show that an extension of p forces that $E \cap D_G \neq \emptyset$. For every $\alpha \in A$ the set $c \cup C(\alpha)$ is a condition in $\text{NM}(\lambda)$, so it has an extension in E . Denote this extension by $C^*(\alpha)$. Let

$$p^* = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C^*, H \rangle.$$

This is an extension of p (in fact a direct extension). Any extension of p^* of length $> n$ forces $E \cap D_G \neq \emptyset$. \square (Lemma)

We abuse notation by denoting also $\cup D_G$ by D_G . It is a club in λ in which limit points are all singulars. Its minimal point is below the first inaccessible cardinal. (Recall clause (iii) of Definition 25.) As usual, $\bigcup\{\alpha(p) : p \in G\}$ is an ω -sequence cofinal in λ , so λ has cofinality ω in $V[G]$.

Now let us consider the h -parts of conditions in G . Let $H_G = \bigcup\{h(p) : p \in G\}$

Lemma 29 $V[G] \models H_G \in \text{Col}(D_G)$.

Proof: H_G is clearly a partial function defined on $\beta \in D_G$. Note that if p, q are in G , each of them of length $> n$, then if α_n is the n -th coordinate (in both of them) of their α -part, then $h(p) \upharpoonright \alpha_n = h(q) \upharpoonright \alpha_n$. It means that

$$H_G \upharpoonright \alpha_n = h(p) \text{ for any } p \text{ of length } > n \text{ belonging to } G. \quad (3)$$

Now $h(p) \in \text{Col}(c(p))$, so for every $\beta' \in \text{dom}(H_G)$ we have $H_G(\beta) \in \text{Col}(f(\beta), < \beta')$, where β' is the first member of D_G above β . By (3) also $H_G \upharpoonright \alpha_n$ has Easton support, but since $\lambda = \sup_{n < \omega} \alpha_n$ is singular, it follows that the support constraint in $\text{Col}^{V[G]}(D)$ is satisfied by H_G and hence $H_G \in \text{Col}(D_G)$. \square (Lemma 29)

The next lemma explains the role of the h part of the conditions in Q_λ .

Lemma 30 *Let $G \subseteq Q_\lambda$ be generic over V and D_G, H_G as above. Let $G^* \subseteq \text{Col}^{V[G]}(D_G)$ be generic over $V[G]$ such that $H_G \in G^*$. Let $G^{**} = \text{Col}^{V[D_G]}(D_G) \cap G^*$. Then G^{**} is a generic filter in $\text{Col}^{V[D_G]}(D_G)$ over $V[D_G]$. (Note that $\text{Col}^{V[D_G]}(D_G) \subseteq \text{Col}^{V[G]}(D_G)$.)*

Proof: G^{**} is obviously a filter. We have to verify genericity. So let \mathring{E} be an $\text{NM}(\lambda)$ -term, which is forced to be a dense open subset of $\text{Col}^{V[D_G]}(D_G)$ and let E_G be its realization in $V[D_G]$. Let $p \in Q_\lambda$. We shall extend p to

a condition q such that $q \Vdash G^* \cap \dot{E} \neq \emptyset$. Then, since $E_G \in V[D_G]$, $q \in G$ implies

$$G^{**} \cap E_G = G^* \cap V[D_G] \cap E_G \neq \emptyset.$$

Assume

$$p = \langle \alpha_0, \dots, \alpha_{n-1}, A, c, h, C, H \rangle.$$

Without loss of generality we can assume that $\alpha \in A$ implies α is inaccessible. Fix $\alpha \in A$ and consider $c \cup C(\alpha)$ which is a condition in $\text{NM}(\lambda)$ such that α belongs to it. Assume for a while that $c \cup C(\alpha) \in D_G$. Note that in $V[D_G]$, $\text{Col}(D_G)$ can be represented as the cartesian product $\text{Col}(c \cup \{\alpha\}) \times \text{Col}(D_G - \alpha)$. Since α is inaccessible, $\text{Col}(c \cup \{\alpha\})$ has cardinality α . $\text{Col}(D_G - \alpha)$ is α^+ -closed. (Remember that we start collapsing cardinals above α at $f(\alpha) > \alpha$.) So by standard arguments the following set is dense open subset of $\text{Col}(D_G - \alpha)$ in $V[D_G]$:

$$\begin{aligned} E^* &= \{g : g \in \text{Col}(D_G - \alpha), \\ &\quad \{g' : g' \in \text{Col}(c \cup \{\alpha\}), (g, g') \in E\} \\ &\quad \text{is dense open in } \text{Col}(c \cup \{\alpha\})\} \end{aligned}$$

So there is an extension c_α^* of $c \cup C(\alpha)$ in $\text{NM}(\lambda)$ such that some \dot{g}_α satisfies

$$c_\alpha^* \Vdash \dot{g}_\alpha \in \text{Col}(D_G - \alpha) \wedge \dot{g}_\alpha \in E^* \wedge \dot{g}_\alpha \leq H(\alpha).$$

(Note that since $c_\alpha^* \leq c \cup C(\alpha)$ and $H(\alpha) \in \text{Col}(C(\alpha))$, $c_\alpha^* \Vdash H(\alpha) \in \text{Col}(D_G - \alpha)$.) Without loss of generality we can assume \dot{g} is essentially a bounded subset of λ and since forcing with $\text{NM}(\lambda)$ does not add bounded subsets of λ , we can assume that we have a particular g_α which we can assume to be in $\text{Col}(c_\alpha^*)$ such that

$$c_\alpha^* \Vdash \check{g}_\alpha \in \text{Col}(D_G - \alpha) \wedge \check{g}_\alpha \in E^* \wedge \check{g}_\alpha \leq H(\alpha).$$

The condition c_α^* must be of the form $c \cup c_\alpha^{**}$ where c_α^{**} is an end extension of $C(\alpha)$. So we define the extension of p to be

$$q = \langle \alpha_0, \dots, \alpha_{n-1}, A^*, c, h, C^*, H^* \rangle,$$

where $C^*(\alpha) = c_\alpha^{**}$ and $H^*(\alpha) = g_\alpha$. The set A^* is a subset of A in U such that each $\alpha \in A$ satisfies the conditions (v), (vi) and (viii) of Definition 25 with respect to the new C and H parts.

Now assume $q \in G$, let q' be an extension of q of length $> n$. Let α be the n -th element of the α -part of q' . Then $\alpha \in A^*$. The c part of q' extends $c \cup C^*(\alpha) = c \cup c_\alpha^{**}$. The h part extends $h \cup H^*(\alpha) = h \cup g_\alpha$. Now assume $q' \in G$. It means that D_G is an end extension of $c \cup c_\alpha^{**}$. In $V[G]$ we can also represent $\text{Col}(D_G)$ as $\text{Col}(c \cup \{\alpha\}) \times \text{Col}(D_G - \alpha)$. Note that H_G extends $h \cup g_\alpha$ and $H_G \in G^*$, so $g_\alpha \in G^* \cap V[D_G] = G^{**}$. Hence in $V[D_G]$ the set

$$E' = \{g' : g' \in \text{Col}(c \cup \{\alpha\}), (g', g_\alpha) \in E_G\}$$

is dense open.

But $\text{Col}(c \cup \{\alpha\})$ is the same in $V[D_G]$ and in $V[G]$. (Because it is a set of bounded subsets of λ and all bounded subsets of λ in $V[D_G]$ and in $V[G]$ are also in V .) The filter $G^* \cap \text{Col}(c \cup \{\alpha\})$ is generic in $\text{Col}(c \cup \{\alpha\})$ over $V[G]$, so $G^* \cap E' \neq \emptyset$. Let $g' \in G^* \cap E'$, $(g', g_\alpha) \in E_G$, $(g', g_\alpha) \in G^*$, $(g', g_\alpha) \in V[D_G]$. So $(g', g_\alpha) \in E_G \cap G^* \cap V[D_G] = E_G \cap G^{**} \neq \emptyset$. \square (Lemma 30).

We also want to show that Q_λ collapses no cardinals. By Lemma 27 forcing with Q_λ introduces no new bounded subsets of λ , so no new cardinals $\leq \kappa$ then λ is collapsed **Unclear meaning**.

We assume G.C.H., Q_λ has cardinality $2^\lambda = \lambda^+$ so no cardinals above λ^+ are collapsed. So the only cardinal we have to consider is λ^+ . Forcing with Q_λ makes λ singular of cofinality ω , so if λ^+ is not a cardinal it becomes singular of cofinality $< \lambda$. In order to handle this case we need arguments similar to Magidor [9]. First we need the following definition:

Definition 31 Let $q \leq p$ be as in Definition 26. Let

$$r = \langle \alpha_0^*, \dots, \alpha_{k-1}^*, A^{**}, c^*, h^*, C \upharpoonright A^{**}, H \upharpoonright A^{**} \rangle,$$

where A^{**} is the set $\{\beta \in A : \alpha_{k-1}^* < \beta\}$.

We call r the *Interpolant* of p, q ($\text{Int}(p, q)$). The condition r is the maximal extension of p such that q is a direct extension of it. Note that knowing p , the condition r is determined by $\alpha_n^*, \dots, \alpha_{k-1}^*, c^*$ and h^* . Since the cardinality of possible $\alpha_n^*, \dots, \alpha_{k-1}^*, c^*, h^*$ is λ , we get that the set $\{\text{Int}(q, p) : q \leq p\}$ is of size λ .

The following lemma is exactly like Theorem 2.6 in Magidor [9].

Lemma 32 Let D be a dense open subset of Q_λ and let $p \in Q_\lambda$ have length n . Then there exists a direct extension q of p ($q \leq^* p$) such that if $t \leq q$ and $t \in D$, then $\text{Int}(t, q) \in D$.

Lemma 33 *In V^{Q_λ} the cardinal λ^+ is still a cardinal.*

Proof: Suppose p forces that τ is a function $\mu \rightarrow \lambda^+$, $\mu < \lambda$, and τ is cofinal in λ^+ . For $\alpha < \mu$, let D_α consist of such q that either q is incompatible with p , or $q \leq p$ and q forces a value for $\tau(\alpha)$. Use Lemma 32 μ times to get a \leq^* -decreasing sequence $\langle q_\alpha : \alpha \leq \mu \rangle$ where $q_0 = q$, and $q_{\alpha+1}$ satisfies Lemma 32 with respect to D_α . For every $\alpha < \mu$ the possible values for $\tau(\alpha)$ when we force below q_μ are determined by a member of $\{Int(t, q_\mu) : t \leq q_\mu\}$, so it has cardinality $\leq \mu$. Hence the range of τ is included in a set of V of cardinality $\mu \cdot \lambda < \lambda^+$. So the range of τ is bounded in λ^+ , a contradiction. \square (Lemma 33)

The forcing $Col(D_G)$ of course collapses cardinals but the following lemma will be useful:

Lemma 34 *Let G, G^*, G^{**} be as in Lemma 30. Then:*

- (a) *The only V -cardinals collapsed in $V[G][G^*]$ are the cardinals in the interval $(f(\beta), \beta')$, where $\beta \in D_G$ and β' is the next member of D_G above β .*
- (b) *The only V -cardinals collapsed in $V[D_G][G^{**}]$ are the cardinals in the interval $(f(\beta), \beta')$, where $\beta \in D_G$ and β' is the next member of D_G above β .*
- (c) *$V[G][G^*]$ and $V[D_G][G^{**}]$ have the same cardinals.*
- (d) *$V[G][G^*]$ and $V[D_G][G^*]$ have the same bounded subsets of λ .*

Proof: (a) is standard, after we have Lemma 33. The only possible problem is again λ^+ , but if it collapsed, it becomes singular of cofinality $< \lambda$. The forcing $Col(D_G)$ is such that every μ -sequence of ordinals is introduced by $Col(D_G \upharpoonright \rho)$ for some $\rho < \lambda$, so it is of cardinality $< \lambda$. So λ^+ is not collapsed.

(b) follows from (a) for cardinals $< \lambda$. G^* is generic over $V[D_G]$ with respect to a forcing notion of size λ , so no cardinal above λ is collapsed.

(c) follows immediately from (a) and (b).

(d) follows from the fact that a bounded subset of λ introduced by $Col(D_G)$ is introduced by $Col(D_G \cap \beta)$ for some $\beta < \lambda$. This is true for both $V[D_G]$ and $V[G]$. $Col(D_G \cap \beta)$ is the same in $V[D_G]$ and $V[G]$, and $G^{**} \cap Col(D_G \cap \beta) = G^* \cap Col(D_G \cap \beta)$. So (d) follows. \square (Lemma 34)

4.2.2 Iteration

We would like to iterate the forcing Q_λ for all measurable $\lambda < \kappa$. The scheme of iteration we shall use is the scheme introduced by Gitik [3]. Our terminology follows (with minor changes) the terminology of [3].

Definition 35 *Suppose λ is a regular cardinal. A forcing notion $\langle P, \leq \rangle$ is said to be λ -Prikrý if there is a partial order $\leq^* \subseteq \leq$ on P such that*

- (a) *Every \leq^* -decreasing sequence of length less than λ has a \leq^* -lower bound.*
- (b) *For every statement Φ in the forcing language for P and for every $p \in P$ there is $q \in P$ such that $q \leq^* p$ and q decides Φ .*

We call \leq^ the direct extension relation.*

Note that we do *not* assume that any two strict extensions of p are necessarily compatible. The remarks above show that if λ is measurable and U is a normal ultrafilter on λ , then Q_λ is a λ -Prikrý forcing notion.

When we refer to Prikrý forcing notions in the sequel we assume that they are given with \leq^* , that is, they are of the form $\langle P, \leq, \leq^* \rangle$. We shall also assume that each forcing notion P is given with its maximal element $\mathbf{1}_P$.

Definition 36 *An iteration*

$$\langle \langle P_\alpha : \alpha \leq \mu \rangle, \langle Q_\alpha : \alpha < \mu \rangle \rangle$$

is called a Gitik iteration of Prikrý forcings if the following holds: Each P_α is a forcing notion of sequences of length α such that

- (i) *If $p = \langle \tau_\beta : \beta < \alpha \rangle \in P_\alpha$ and $\gamma < \alpha$, then $p \upharpoonright \gamma = \langle \tau_\beta : \beta < \gamma \rangle \in P_\gamma$.*
- (ii) *If $p = \langle \tau_\beta : \beta < \alpha \rangle \in P_\alpha$ and $\gamma < \alpha$ then $p \upharpoonright \gamma \Vdash_{P_\gamma} \tau_\gamma \in Q_\gamma$, where Q_γ is a P_γ -name forced over P_γ to denote a γ -Prikrý forcing with the partial orders $\leq_\gamma, \leq_\gamma^*$.*
- (iii) *The sequence has Easton support, namely for every regular $\gamma \leq \alpha$ the set $\{\beta < \gamma : \tau_\beta \neq \mathbf{1}_{Q_\beta}\}$ has cardinality $< \gamma$.*
- (iv) *Q_γ is the trivial forcing notion unless both γ is Mahlo and $\Vdash_{P_\beta} |Q_\beta| < \gamma$ for every $\beta < \gamma$.*

The partial order on P_α is defined as follows: Suppose $q = \langle \tau_\beta^* : \beta < \alpha \rangle$ and $p = \langle \tau_\beta : \beta < \alpha \rangle$. Then $q \leq p$ if there is a finite set $B \subseteq \{\beta < \alpha : \tau_\beta \neq \mathbf{1}_{Q_\beta}\}$ such that:

- (a) If $\beta \notin B$ such that $\tau_\beta \neq \mathbf{1}_{Q_\beta}$, then $q \upharpoonright \beta \Vdash \tau_\beta^* \leq_\beta^* \tau_\beta$.
- (b) If $\beta \in B$ or $\tau_\beta = \mathbf{1}_{Q_\beta}$, then $q \upharpoonright \beta \Vdash \tau_\beta^* \leq_\beta \tau_\beta$.

(Namely, we can take a non-direct extension of any point β in which $\tau_\beta = \mathbf{1}_{Q_\beta}$ and only at finitely many points β in which $\tau_\beta \neq \mathbf{1}_{Q_\beta}$.) The direct extension for P_α is defined as $q \leq^* p$ if in the above definition we can take $B = \emptyset$.

Let us fix now a Gitik iteration $\langle \langle P_\alpha : \alpha \leq \mu \rangle, \langle Q_\alpha : \alpha < \mu \rangle \rangle$ of Prikry forcings. Now Lemma 1.3 of Gitik [3] is essentially:

Lemma 37 *Let α be Mahlo such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$. Then P_α has cardinality $\leq \alpha$ and it satisfies the $\alpha - c.c.$.*

Lemma 1.4 of Gitik [3] is essentially:

Lemma 38 *Let Φ be a statement for the forcing language for P_μ and $p \in P_\mu$. Then there is $q \leq^* p$ such that $q \Vdash \Phi$ or $q \Vdash \neg\Phi$.*

It follows that if α is the first such that Q_α is not the trivial forcing notion, then P_μ is α -Prikry. Also, if α is a Mahlo cardinal such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$, then in V^{P_α} we can consider

$$\langle \langle P_\beta/P_\alpha : \alpha \leq \beta \leq \mu \rangle, \langle Q_\beta : \alpha \leq \beta < \mu \rangle \rangle$$

to be a Gitik iteration of Prikry forcing notions, so in particular we get:

Lemma 39 *If $\alpha < \mu$ is a Mahlo cardinal such that $\Vdash_{P_\gamma} |Q_\gamma| < \alpha$ for all $\gamma < \alpha$, then*

- (i) P_μ/P_α is an α -Prikry forcing notion.
- (ii) Every bounded subset of α in V^{P_μ} belongs already to V^{P_α} . (So, in particular, no α satisfying the above requirement is collapsed.)

The next lemma is a variation of Lemma 1.5 in Gitik [3] and it deals with the preservation of measurable cardinals by the Gitik iterations:

Lemma 40 *Let $\alpha \leq \mu$ be measurable such that $2^\alpha = \alpha^+$ and $\Vdash_{P_\beta} |Q_\beta| < \alpha$ for all $\beta < \alpha$. Let $A = \{\beta < \alpha : \Vdash_{P_\beta} \text{“}Q_\beta \text{ is the trivial forcing notion”}\}$. Let U be a normal ultrafilter on α such that $A \in U$. Then in V^{P_α} the filter U can be extended to a normal ultrafilter. In particular, α remains measurable.*

Proof: Let j be the ultrapower embedding $j : V^\kappa/U \rightarrow M$. Note that $j(\langle\langle P_\beta : \beta \leq \alpha \rangle, \langle Q_\beta : \beta < \alpha \rangle\rangle)$ is in M a Gitik iteration of Prikry forcings of length $j(\alpha)$. Let us denote the new iteration $\langle\langle P_\beta^* : \beta \leq j(\alpha) \rangle, \langle Q_\beta^* : \beta < j(\alpha) \rangle\rangle$. Since $\Vdash_{P_\beta} |Q_\beta| < \alpha$ for all $\beta < \alpha$, we get that for all $\beta < \alpha$ $|P_\beta| < \alpha$, $Q_\beta^* = Q_\beta$, $P_\beta^* = P_\beta$ and also $P_\alpha^* = P_\alpha$. Our assumption that $A \in U$ translates into

$$\Vdash_{P_\alpha} \text{“}Q_\alpha \text{ is the trivial forcing notion.”}$$

So

$$M^{P_\alpha} \Vdash \text{“}P_{j(\alpha)}/P_\alpha \text{ is an } \alpha^+ \text{-Prikry forcing notion.”}$$

The forcing P_α satisfies the α -c.c. and has cardinality α with $2^\alpha = \alpha^+$, so we can enumerate in V in a sequence of length α^+ all P_α -terms forced to denote subsets of α . Let this list be $\langle \dot{A}_\delta : \delta < \alpha^+ \rangle$. Note that since M is closed under α -sequences, initial segments of the sequence $\langle j(\dot{A}_\delta) : \delta < \alpha^+ \rangle$ are in M . Now we argue in V^{P_α} . By induction define a \leq^* -decreasing sequence $\langle p_\delta : \delta < \alpha^+ \rangle$ in $P_{j(\alpha)}/P_\alpha$ such that for each $\delta < \alpha^+$ the condition $p_{\delta+1}$ decides the statement ‘ $\alpha \in j(\dot{A}_\delta)$ ’. ($j(\dot{A}_\delta)$ is a $P_{j(\alpha)}^* = j(P_\alpha)$ -term, but in M^{P_α} we can consider it to be a $P_{j(\alpha)}^*/P_\alpha$ -term. By $P_{j(\alpha)}^*/P_\alpha$ we can find such a condition $p_{\delta+1} \leq^*$ -extending p_δ . Every initial segment of the sequence $\langle p_\delta : \delta < \alpha^+ \rangle$ is in M , so at limit stages δ we can take p_δ to be a \leq^* -lower bound for $\langle p_\eta : \eta < \delta \rangle$.) Now in V^{P_α} define the ultrafilter U^* extending U by

$$A_\delta \in U^* \iff p_{\delta+1} \Vdash \kappa \in j(\dot{A}_\delta).$$

It is easy to check that U^* is well-defined and that it is a normal ultrafilter in V^{P_α} extending U . \square (Lemma 40)

4.3 The Final Model

We are now ready to define our final model in which the first inaccessible will be a LST($L(I)$). Let us fix a supercompact cardinal κ . Recall that we assume G.C.H. to hold in V . We start the construction by defining a Gitik iteration $\langle\langle P_\alpha : \alpha \leq \kappa \rangle, \langle Q_\alpha : \alpha < \kappa \rangle\rangle$ of Prikry forcings. of length κ . The

iteration will be defined if we inductively define Q_α . By induction it will be clear that for $\alpha < \gamma$, γ Mahlo, we have $\Vdash |Q_\alpha| < \gamma$. So we define:

- (i) If α is not measurable in V then Q_α is the trivial forcing notion.
- (ii) If α is measurable in V , then we pick a normal ultrafilter U on α such that $A = \{\beta < \alpha : \beta \text{ non-measurable}\} \in U$.

Then α and U satisfy the assumptions of Lemma 37. So in V^{P_α} the cardinal α is still measurable with a normal ultrafilter extending U . Define Q_α to be the Q_α as defined in Section 4.2.1. It is an α -Prikrý forcing and its cardinality is 2^α which is less than the next Mahlo cardinal. So the iteration is defined. Since P_κ is of cardinality κ and satisfies the κ -c.c., the cardinal κ is still Mahlo in V^{P_κ} . (In fact, by Lemma 37 it is still measurable.) So now we force over V^{P_α} with $\text{NM}(\kappa)$ to get a closed unbounded subset D of κ such that each of the limit points of D is singular, and then we force with $\text{Col}(D)$. So our final forcing notion is

$$P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D).$$

Forcing with $\text{Col}(D)$ makes sure that there are no inaccessible cardinals $< \kappa$. Forcing with $\text{NM}(\kappa)$ keeps κ inaccessible, similarly for $\text{Col}(D)$, so in $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ the cardinal κ is the first inaccessible. Our goal will be achieved when we show:

Theorem 41 *In $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ the cardinal κ is $\text{LST}(L(I))$.*

Proof: Denote $V^* = V^{P_\kappa}$. In V^* the cardinal κ is still measurable, so by picking a normal ultrafilter U on κ we can define the forcing Q_κ and force with it over V^* . Let G be the generic filter in Q_κ . By Lemma 28 we can define from G an $\text{NM}(\kappa)$ filter D_G which is going to be generic over V^* . We can force further with $\text{Col}(D_G)$. Let G^* be the generic filter, where we can assume that $H_G \in G^*$. By Lemma 30 $G^{**} = G^* \cap V^*[D_G]$ is generic over $V^*[D_G]$ with respect to $\text{Col}^{V^*[D_G]}(D_G)$. So $V^*[D_G][G^{**}]$ is a $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$ generic extension of V . By Lemma 34 $V^*[D_G][G^{**}]$ has the same bounded sequences of elements of κ and the same cardinals as $V^*[G][G^*]$.

It is now easily seen that given any generic $D \times H$ over V^* with respect to $\text{NM}(\kappa) \star \text{Col}(D)$, we can (by doing further forcing) assume that $D = D_G$ and $H = G^{**}$, where $G \subseteq Q_\kappa$ is generic over V^* . Assume otherwise. Let

$(c, h) \in \text{NM}(\kappa) \star \text{Col}(D)$ force the negation of our claim. Without loss of generality $h \in \text{Col}(c)$. Pick a condition p in Q_λ where the c -part is c and its h -part is h . Assume G is Q_κ -generic (over V^*) such that $p \in G$. Note that $c \in D_G$ and H_G extends h . Pick a generic G^* in $\text{Col}(D_G)$ such that $H_G \in G^*$. Obviously $h \in G^* \cap V^*[D_G] = G^{**}$. So if we consider the generic pair $D_G \star G^{**}$, (c, h) belongs to it, but this is a contradiction. So we proved (Using also Lemma 34):

Lemma 42 $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ has a forcing extension which is of the form $V_2 = V^{P_\kappa \star Q_\kappa \star \text{Col}(D_G)}$, where V_1 and V_2 have the same cardinals and the same bounded sequences of elements of κ .

Of course, the extension we describe in Lemma 42 does not preserve cofinalities because in $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ the cardinal κ is regular while in $V_2 = V^{P_\kappa \star Q_\kappa \star \text{Col}(D_G)}$ it has cofinality ω .

We resume the proof of Theorem 41. So we are given in $V_1 = V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ a structure $\mathcal{A} = \langle \lambda, R_1, R_2, \dots \rangle$ (without loss of generality we may assume that the universe of the structure is an ordinal λ). We have to get a substructure \mathcal{A}' of \mathcal{A} such that $\mathcal{A}' \prec_{L(I)} \mathcal{A}$ and $|A'| < \kappa$. Without loss of generality we may assume that $\kappa^+ \leq \lambda$.

In V the cardinal κ is supercompact, so we fix in V an embedding $j : V \rightarrow M$ such that κ is the critical point of j , $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and by our definition of the function f we can assume that $j(f)(\kappa) > \lambda$. The structure \mathcal{A} is the realization of a term \dot{A} in the forcing language for $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$. We can assume $\dot{A} \in M$ because $|P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)| = \kappa$.

Consider in M the forcing notion $j(P_\kappa) \star j(\text{NM}(\kappa)) \star j(\text{Col}(D))$. The forcing notion $j(P_\kappa)$ is (in M) an iteration of length $j(\kappa)$. Its first κ steps are exactly like in V (They are defined in V_κ which is the same as in M). The κ -th step of the iteration is Q_κ , so $j(P_\kappa) = P_\kappa \star Q_\kappa \star T$, where T is the iteration in M between κ and $j(\kappa)$.

Lemma 43 One can force over V_1 to get a generic filter in the forcing $j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))$ such that in the resulting model there is an embedding

$$j^* : V_1 \rightarrow M^{j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))}$$

extending j such that V_1 and $M^{j(P_\kappa) \star j(Q_\kappa) \star j(\text{Col}(D))}$ have the same cardinals $\leq \lambda$.

Proof: By Lemma 42, we can force over V_1 , not collapsing any cardinals, to get a model V_2 of the form $V^{P_\kappa \star Q_\kappa \star \text{Col}(D)}$, where the generic filter for $\text{Col}(D)$ over $V^{P_\kappa \star \text{NM}(\kappa)}$ is of the form $G^{**} = G^* \cap V^{P_\kappa \star \text{NM}(\kappa)}$, and where furthermore G^* is the generic filter in $\text{Col}(D)$ over V^{Q_κ} . The generic filter for $P_\kappa \star Q_\kappa$ provides the generic filter for the first $\kappa + 1$ steps of the iteration of $j(P_\kappa)$ over M . (Note that Q_κ is the same in the sense of M^{P_κ} and V^{P_κ}). Follow this forcing by forcing with T . So we get a generic filter for $P_\kappa \star Q_\kappa \star T = j(P_\kappa)$. Recall that we assumed that κ is the last inaccessible in V so there are no inaccessibles (and hence no measurable) between κ and λ . Since $M^\lambda \subseteq M$, the same is true in M . So the iteration of $j(P_\kappa)$ between κ and λ^+ is the trivial iteration, so T is a Gitik iteration of μ -Prikrý forcings for $\lambda^+ \leq \mu$. It means that

$$\mathcal{P}(\lambda) \cap V^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap M^{P_\kappa \star Q_\kappa} = \mathcal{P}(\lambda) \cap V^{j(P_\kappa)}.$$

Hence no cardinals $\leq \lambda$ are collapsed in $M^{j(P_\kappa)}$. We now have to force with $j(\text{NM}(\kappa))$ which is $\text{NM}(j(\kappa))$ in the sense of $M^{j(P_\kappa)}$. The club D introduced by $\text{NM}(\kappa)$ is in $V^{P_\kappa \star Q_\kappa}$ so it belongs to $M^{P_\kappa \star Q_\kappa} \subseteq M^{j(P_\kappa)}$. In $M^{P_\kappa \star Q_\kappa}$ the cardinal κ is singular of cofinality ω . So $D \cup \{\kappa\}$ is a condition in $\text{NM}(j(\kappa))$ (It is a closed subset of $j(\kappa)$, every limit point, including κ , is singular. The other conditions are easily verified.) So we can pick a generic filter D^* in $\text{NM}(j(\kappa))$ such that $D \cup \{\kappa\}$ is an initial segment of it. Forcing with $\text{NM}(j(\kappa))$ over $M^{j(P_\kappa)}$ does not add any bounded subsets of $j(\kappa)$, so again no cardinals $\leq \lambda$ are collapsed.

Now we have to pick a generic filter for $\text{Col}(D^*)$. The set $D \cup \{\kappa\}$ is an initial segment of D^* so

$$\text{Col}(D^*) = \text{Col}(D) \star \text{Col}(D^* - \kappa).$$

(The collapses are not in the sense of $M^{j(P_\kappa)}$ but since $M^{j(P_\kappa)}$ agrees with $V^{P_\kappa \star Q_\kappa}$ on P_κ , $\text{Col}(D)$ is the same in the sense of $V^{P_\kappa \star Q_\kappa}$ and $M^{j(P_\kappa)}$). The filter G^* is generic in $\text{Col}(D)$, so we pick a generic filter for $\text{Col}(D^*)$ such that its restriction to $\text{Col}(D \cup \{\kappa\})$ is exactly G^* . Now we reach a crucial point for which we introduced the function f .

We defined $\text{Col}(D)$ such that if $\beta \in D$, then $\text{Col}(D)$ does not collapse any cardinals between β and $f(\beta)$. Since $\kappa \in D^*$, the forcing $\text{Col}(D^*)$ does not collapse any cardinals between κ and $j(f)(\kappa)$. But $j(f)(\kappa) > \lambda$, so $\text{Col}(D^*)$ collapses below λ only cardinals collapsed by $\text{Col}(D^* \cap \kappa) = \text{Col}(D)$. So it means that below λ the models $M^{j(P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D))}$ and $V^{P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)}$ have the same cardinals.

Denote by H the generic filter over V in $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D)$. We have defined a generic filter in $j(P_\kappa \star \text{NM}(\kappa) \star \text{Col}(D))$. Let us denote this by H^* . We claim that the particular way we have defined H^* guarantees that H^* satisfies a condition which is known as “the master condition” i.e.

Claim: If $p \in H$, then $j(p) \in H^*$.

Proof: The condition $p \in H$ is of the form (q, s, t) where $q \in P_\kappa$, s is a term denoting an element of $\text{NM}(\kappa)$ in V^{P_κ} and t is a term denoting a member of $\text{Col}(D)$ in $V^{P_\kappa \star \text{NM}(\kappa)}$. The generic filter we picked for $j(P_\kappa)$ extends the generic filter for P_κ , so $q \in H$ implies that $q = j(q) \in H^*$.

The generic filter we picked for $\text{NM}(j(\kappa))$ is an end extension of the generic filter picked for $\text{NM}(\kappa)$, so since s denotes a bounded subset of κ introduced by P_κ , $j(s) = s$ and $s \in H$ implies $j(s) \in H^*$. The generic filter for $\text{Col}(D)$ is $G^{**} = G^* \cap V^{P_\kappa \star \text{NM}(\kappa)}$. The way we picked the generic filter for $\text{Col}(D^*)$ was such that $G^{**} \subseteq H^*$. Note that t denotes a subset of $V_\kappa^{P_\kappa}$ of cardinality $< \kappa$ in V^{P_κ} , so $j(t) = t$. But $t \in G^*$ so $t \in G^{**}$, so $j(t) = t \in H^*$. (We abuse notation by denoting a term and its realization by the same symbol). So we have actually showed that for $p \in H$ we have $j(p) = (j(q), j(s), j(t)) \in H^*$. \square (Claim)

Once we have the master condition we can as usual define the extension j^* of j by defining j^* in the realization $[t]_H$ of an $P_\kappa \star \text{NM}(\kappa) \star \text{Col}(\kappa)$ -term t as

$$j^*([t]_H) = [j(t)]_{H^*}.$$

It is a standard argument that given the assumptions of the claim, j^* is well-defined and it is elementary.

When we described the construction of H^* we argued that the cardinals $\leq \lambda$ in $V[H]$ are the same as in $M[H^*]$. So the lemma is verified. \square (Lemma 43)

So we have two universes $V[H]$ and $M[H^*]$ which agree on cardinals $\leq \lambda$. Moreover, we have $j \subseteq j^*$ which is elementary

$$j^* : V[H] \rightarrow M[H^*].$$

The structure \mathcal{A} is in $V[H]$ and because $M^\lambda \subseteq M$ we have $\mathcal{A} \in M[H^*]$ and $j \upharpoonright \mathcal{A} = j \upharpoonright \lambda \in M \subseteq M[H^*]$.

Suppose $\Phi(x_1, \dots, x_n)$ is a formula in the logic $L(I)$ and suppose $a_1, \dots, a_n \in A$. Now:

$$M[H^*] \models “\mathcal{A} \models \Phi(a_1, \dots, a_n)” \text{ iff } V[H] \models “\mathcal{A} \models \Phi(a_1, \dots, a_n)” \quad (4)$$

This is because $M[H^*]$ agrees with $V[H]$ on the cardinals $\leq \lambda$ which are all the cardinals relevant for evaluating the truth of $\Phi(a_1, \dots, a_n)$. On the other hand, from j^* being elementary we get:

$$V[H] \models \text{“}\mathcal{A} \models \Phi(a_1, \dots, a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), \dots, j^*(a_n))\text{”}.$$

Hence

$$M[H^*] \models \text{“}\mathcal{A} \models \Phi(a_1, \dots, a_n)\text{” iff } M[H^*] \models \text{“}j^*(\mathcal{A}) \models \Phi(j^*(a_1), \dots, j^*(a_n))\text{”}.$$

So

$$M[H^*] \models j^* \upharpoonright A = j \upharpoonright A \text{ is an } L(I)\text{-elementary embedding of } \mathcal{A} \text{ into } j^*(\mathcal{A}).$$

Since $M[H^*] \models |A| \leq \lambda < j^*(\kappa)$, we get

$$M[H^*] \models \text{“There is an } L(I)\text{-elementary substructure of } j^*(\mathcal{A}) \text{ of cardinality } < j^*(\kappa)\text{”}.$$

By j^* being elementary,

$$V[H] \models \text{“There is an } L(I)\text{-elementary substructure of } \mathcal{A} \text{ of cardinality } < \kappa\text{”}.$$

□ (Theorem 41)

This ends the proof of Theorem 21. □

We have shown that, assuming the consistency of a supercompact cardinal, it is consistent to assume that $\text{LST}(L(I))$ exists and moreover, we can consistently assume that it is either the first supercompact cardinal, or something much smaller, namely the first (weakly) inaccessible cardinal. A fortiori, then $\text{LST}(L(I))$ can be consistently equal to $\text{LST}(L^2)$ or also consistently different from $\text{LST}(L^2)$. Moreover, we have shown that even the existence of $\text{LST}(L(I))$ implies the consistency of large cardinals. In many respects the existence of $\text{LST}(L(I))$ seems, in the light of present day knowledge, like Martin’s Maximum, and the cardinal $\text{LST}(L(I))$ behaves – be it small or large - as \aleph_2 in the presence of Martin’s Maximum. But $\text{LST}(L(I))$ makes no claims about the size of the continuum: If it is consistent that there are supercompact cardinals, then it is consistent on the one hand that $\text{LST}(L(I))$ exists and $2^\omega = \aleph_1$ and on the other hand that $\text{LST}(L(I)) = 2^\omega$ ([13]).

References

- [1] Jon Barwise and Solomon Feferman, editors. *Model-theoretic logics*. Perspectives in Mathematical Logic. Springer-Verlag, New York, 1985.
- [2] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales and stationary reflection. *J. Math. Log.*, 1(1):35–98, 2001.
- [3] Moti Gitik. Changing cofinalities and the nonstationary ideal. *Israel J. Math.*, 56(3):280–314, 1986.
- [4] Klaus Hartig. Uber einen Quantifikator mit zwei Wirkungsbereichen. In *Colloq. Found. Math., Math. Machines and Appl. (Tihany, 1962)*, pages 31–36. Akad. Kiado, Budapest, 1965.
- [5] Richard Laver. Making the supercompactness of κ indestructible under κ -directed closed forcing. *Israel J. Math.*, 29(4):385–388, 1978.
- [6] Per Lindstrom. On extensions of elementary logic. *Theoria*, 35:1–11, 1969.
- [7] Menachem Magidor. On the role of supercompact and extendible cardinals in logic. *Israel J. Math.*, 10:147–157, 1971.
- [8] Menachem Magidor. How large is the first strongly compact cardinal? or A study on identity crises. *Ann. Math. Logic*, 10(1):33–57, 1976.
- [9] Menachem Magidor. On the singular cardinals problem. I. *Israel J. Math.*, 28(1-2):1–31, 1977.
- [10] Saharon Shelah. Generalized quantifiers and compact logic. *Trans. Amer. Math. Soc.*, 204:342–364, 1975.
- [11] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. , Oxford Science Publications.
- [12] Robert M. Solovay, William N. Reinhardt, and Akihiro Kanamori. Strong axioms of infinity and elementary embeddings. *Ann. Math. Logic*, 13(1):73–116, 1978.

- [13] Jonathan Stavi and Jouko Väänänen. Reflection principles for the continuum. In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 59–84. Amer. Math. Soc., Providence, RI, 2002.
- [14] Jouko Väänänen. Abstract logic and set theory. I. Definability. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math.*, pages 391–421. North-Holland, Amsterdam, 1979.
- [15] Jouko Väänänen. Boolean-valued models and generalized quantifiers. *Ann. Math. Logic*, 18(3):193–225, 1980.