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Asymptotic probabilities of extension properties

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ASYMPTOTIC PROBABILITIES OF EXTENSION PROPERTIES

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ABSTRACT. We consider a set $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ of *finite* structures such that all members of $\mathbf{K}_n$ have the same universe, the cardinality of which approaches ∞ as $n \to \infty$. Each structure in $\mathbf{K}$ may have a nontrivial underlying pregeometry and on each $\mathbf{K}_n$ we consider a probability measure, either the uniform measure, or what we call the *dimension conditional measure*. The main questions are: What conditions imply that for every extension axiom φ, compatible with the defining conditions of $\mathbf{K}$, the probability that φ is true in a member of $\mathbf{K}_n$ approaches 1 as $n \to \infty$? And what conditions imply that this is not the case, possibly in the strong sense that the mentioned probability approaches 0 for some φ?

If each $\mathbf{K}_n$ is the set of structures with universe $\{1, \ldots, n\}$, in a fixed relational language, in which certain "forbidden" structures cannot be weakly embedded and $\mathbf{K}$ has the free amalgamation property, then there is a condition (concerning the set of forbidden structures) which, if we consider the uniform measure, gives a dichotomy; i.e. the condition holds if and only if the answer to the first question is 'yes'. In general, we do not obtain a dichotomy, but we do obtain a condition guaranteeing that the answer is 'yes' for the first question, as well as condition guaranteeing that the answer is 'no'; and we give examples showing that in the gap between these conditions the answer may be either 'yes' or 'no'. This analysis is made for both the uniform measure and for the dimension conditional measure. The later measure has closer relation to random generation of structures and is more "generous" with respect to satisfiability of extension axioms. Some zero-one laws are derived from the results. The main application is the case of l-colourable structures, for some fixed integer $l \geq 2$.
Keywords: Finite structure, asymptotic probability, extension axiom, zero-one law, colouring.

Contents

1. INTRODUCTION

Extension axioms have been used as a technical tool for proving zero-one laws [10, 13, 14, 15, 17], but they also have other implications which will be explained below. Extension axioms, by their definition, express possibilities of extending a structure that are compatible (or "consistent") with the definition of a given class of structures under consideration. So given a structure $\mathcal{M}$ from this class, the set of extension axioms which are satisfied in $\mathcal{M}$ tells which possibilities of extending substructures of $\mathcal{M}$, in ways

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compatible with the context, are actually realized in the particular structure $\mathcal{M}$. Thus, extension axioms have a combinatorial interest of their own.

If we consider the class of all finite L-structures, where L is a language with finite relational vocabulary, then it follows from the proof of the zero-one law (as presented in [10, 13, 15]) that, for every extension axiom, almost all sufficiently large finite L-structures satisfy it. Hence the interesting case to study is the case when there are some restrictions on the structures under consideration. For example, we could restrict ourselves to the class of finite structures in which some particular structure cannot be (weakly) embedded; for instance, the class of triangle-free graphs. Specific classes of this kind have been studied extensively. An overview with emphasis on graphs and partial orders is found in [21]; see also [17, 20]. An overview with focus on zero-one laws is found in [22]; it takes up, among other things, the number theoretic approach to zero-one laws which was first developed by K. Compton, and which is the subject a book by S. Burris [3]. However, none of the previously published research focuses specificly on searching for "dividing lines" for asymptotic probabilities of extension properties in a general model theoretic setting. That is the purpose of this article, as well as deriving consequences such as zero-one laws.

The general framework of this article is the following. For some language L, $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ is a set of *finite* L-structures such that all members of $\mathbf{K}_n$ have the same universe; by convention an initial segment of $\{1, 2, 3, \ldots\}$. In addition each $\mathcal{M} \in \mathbf{K}$ may have a nontrivial closure operator which makes it into a pregeometry; in this case, the closure operator is uniformly definable on all members of $\mathbf{K}$ in the sense described in Definition 7.1. An important special case is when the closure (and pregeometry) is *trivial*, by which we mean that every subset of any structure from $\mathbf{K}$ is closed.

Suppose that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M} \in \mathbf{K}$ and that A and B are closed subsets of M. For a structure $\mathcal{N}$, the *$\mathcal{B}/\mathcal{A}$-extension axiom* holds for $\mathcal{N}$ if for every embedding τ of $\mathcal{A}$ into $\mathcal{N}$ there is an embedding π of $\mathcal{B}$ into $\mathcal{N}$ which extends τ. If the dimension of B is at most $k + 1$, then we call it a *k-extension axiom* of $\mathbf{K}$. If the closure is trivial then dimension is the same as cardinality.

If L no constant symbols we allow A to be empty, and in this case the $\mathcal{B}/\mathcal{A}$-extension axiom expresses that there exists a copy of $\mathcal{B}$ in the ambient structure. Hence, if $\mathcal{M}$ satisfies all k-extension axioms, then every $\mathcal{B} \in \mathbf{K}$ of dimension at most $k + 1$ can be embedded into $\mathcal{M}$; in this case one may say that $\mathcal{M}$ is '$(k + 1)$-universal for $\mathbf{K}$'. By involving pebble games [16, 19] it follows that if L is relational and $\mathcal{M} \in \mathbf{K}$ satisfies all k-extension axioms of $\mathbf{K}$, then $\mathcal{M}$ has the following 'homogeneity property, up to k-variable expressibility': Whenever $\bar{a}, \bar{a}'$ are tuples of elements and there is an isomorphism from the closure of $\bar{a}$ to the closure of $\bar{a}'$ which sends a_i to a'_i, then $\bar{a}$ and $\bar{a}'$ satisfy exactly the same formulas in which at most k distinct variables occur.

If the class $\mathbf{K}^*$ of all structures which can be embedded into some member of $\mathbf{K}$ has (up to taking isomorphic copies) the joint embedding property and the amalgamation property, then a structure $\mathcal{M}$ exists which satisfies all k-extension axioms of $\mathbf{K}$ for every $k \in \mathbb{N}$; because we can let $\mathcal{M}$ be the so-called Fraïssé limit of $\mathbf{K}^*$. However, if $\mathbf{K}$ contains arbitrarily large (finite) structures, then the Fraïssé limit of $\mathbf{K}^*$ is infinite. The question whether, for every $k \in \mathbb{N}$, there exists a *finite* $\mathcal{M} \in \mathbf{K}$ which satisfies every k-extension axiom of $\mathbf{K}$ may be hard. For instance, the problem [5] whether there is a finite triangle-free graph which satisfies every 4-extension axiom is still open. By using the fact that the proportion of triangle-free graphs with vertices $1, \ldots, n$ which are bipartite approaches 1 as n approaches infinity [12, 17], it is straightforward to derive that the proportion of all triangle-free graphs which vertices $1, \ldots, n$ which satisfy all 3-extension axioms approaches 0 as n approaches infinity. The main results (Theorems 3.15, 3.17, 7.28 and 7.29) are concerned with the question of when, for some k and large enough n, it is usual (or unusual), in senses to be made precise, that structures in $\mathbf{K}_n$ satisfy all k-extension axioms.

For the moment, assume that, for each n, μ_n is a probability measure on $\mathbf{K}_n$. Let $Th_\mu(\mathbf{K})$ be the set of sentences φ such that the μ_n-probability that φ is true in a member of $\mathbf{K}_n$ approaches 1 as n approaches infinity. Also assume that $\mathbf{K}^*$, as defined above, satisfies the joint embedding and amalgamation properties and let $Th_{\mathbb{F}}(\mathbf{K})$ be the complete theory of the Fraïssé limit of $\mathbf{K}^*$. If, moreover, the closure is trivial on all members of $\mathbf{K}$, it is straightforward to see that $Th_\mu(\mathbf{K}) = Th_{\mathbb{F}}(\mathbf{K})$ if and only if $Th_{\mathbb{F}}(\mathbf{K})$ contains all extension axioms of $\mathbf{K}$. (We can get rid of the assumption that the closure is trivial if we assume that it is "well-behaved", as in Section 7; and then we argue like in Section 10.2.)

The rest of the introduction is devoted to explaining, roughly, the results of this article. We try to appeal to the readers intuition rather than giving the full definitions of notions involved; but sometimes references to these definitions are given.

We start, in Sections 3 – 5 , by considering $\mathbf{K}$ such that all $\mathcal{M} \in \mathbf{K}$ have trivial closure, so dimension is the same as cardinality. Also, until Section 6 we consider only the uniform measure. The first result, Theorem 3.4, gives a dichotomy for the special case when, for a fixed language L, with finite relational vocabulary, and set $\mathbf{F}$ of "forbidden" L-structures, $\mathbf{K}_n$ defined to be the set of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that no $\mathcal{F} \in \mathbf{F}$ can be *weakly embedded* into $\mathcal{M}$ (see Section 2.1). If every $\mathcal{F} \in \mathbf{F}$ is "simple" in a sense which is made precise in Theorem 3.4, then for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as n approaches infinity; and $\mathbf{K}$ has a zero-one law. On the other hand, if there is at least one "non-simple" $\mathcal{F} \in \mathbf{F}$, then for some $0 \leq c < 1$ and $2|F|$-extension axiom φ, the proportion of $\mathcal{M} \in \mathbf{K}_n$ in which φ is true never exceeds c; if the language has no unary relation symbols, then this proportion approaches 0 as n approaches infinity. It may nevertheless be the case that $\mathbf{K}$ has a zero-one law, as in the example of triangle-free graphs [17].

Theorem 3.4, just described, is proved by using the more general Theorems 3.15 and 3.17. In Theorems 3.15 and 3.17 we have no assumptions about how $\mathbf{K}$ is defined. We will call a structure $\mathcal{A}$ *permitted* if it can be embedded into some structure in $\mathbf{K}$. For the sake of simplifying this introductory description of the results, let's assume that every permitted structure is isomorphic to some structure in $\mathbf{K}$; in other words, we assume that $\mathbf{K}$ is, up to taking isomorphic copies, closed under substructures (the 'hereditary property'). The key concept will be that of *substitutions of permitted structures* in a permitted (super)structure $\mathcal{M}$, that is, the act of replacing, in $\mathcal{M}$, the interpretations (of relation symbols) on the universe of $\mathcal{A} \subseteq \mathcal{M}$ by the interpretations in another permitted structure $\mathcal{A}'$ with the same universe as $\mathcal{A}$. If whenever $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{M}$ are permitted, $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ and $\mathcal{A}'$ have the same universe, the result of "replacing $\mathcal{A}$ by $\mathcal{A}'$ in $\mathcal{M}$", denoted $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$, is a permitted structure, then, for every extension axiom of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ in which it is true approaches 1 as n approaches infinity. This statement is a consequence of Theorem 3.15 which, essentially, is a reformulation, with the terminology used here, of known results – although this may not be obvious at first sight.

If, however, there exist permitted $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{M}$ such that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ is not permitted – we say "forbidden" –, but the reverse substitution, that is, the replacement of $\mathcal{A}'$ by $\mathcal{A}$, *never* produces a forbidden structure from a permitted one, then one of the following holds: (a) $\mathbf{K}$ fails to satisfy the free amalgamation property, or (b) there is an extension axiom φ of $\mathbf{K}$ and $0 \leq c < 1$ such that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ never exceeds c; and if there are no unary relation symbols, then this proportion approaches 0. This statement is a consequence of Theorem 3.17 and its corollary. From these results we also get information, in case (a), about an instance of free amalgamation which fails, and in case (b), about the extension axiom φ. Theorem 3.17 is proved by a counting argument.

One proves, under the assumption that $\mathbf{K}$ has the free amalgamation property, that for a properly chosen extension axiom φ it is the case that for every $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ, there are sufficiently many $\mathcal{N} \in \mathbf{K}_n$ which do *not* satisfy φ.

There is a third possibility, other than those considered in the previous two paragraphs. It is possible that there are permitted $\mathcal{A}$ and $\mathcal{A}'$ with the same universe such that the substitution of $\mathcal{A}'$ for $\mathcal{A}$ in some permitted (super)structure $\mathcal{M}$ may produce a forbidden (not permitted) structure, but whenever this happens then the reverse substitution of $\mathcal{A}$ for $\mathcal{A}'$ in some permitted $\mathcal{N}$, say, may also produce a forbidden structure. In this case it is possible that for every extension axiom φ of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ in which φ is true approaches 1 as n approaches infinity. But it is also possible that for some extension axiom φ of $\mathbf{K}$, the proportion of structures in $\mathbf{K}$ in which φ is true approaches 0 as n approaches infinity. Section 4 gives examples showing this. The same section also gives examples for which Theorem 3.17, and possibly its corollary, apply. These examples show how the rather technical Theorem 3.17 and its corollary can be used. Some examples in Section 4 also serve the purpose of illustrating differences between the uniform probability measure and conditional probability measures, which are introduced in Section 6; these examples will be re-examined in Section 6. Section 5 is devoted to the proof of Theorem 3.17.

In Section 6 conditional probability measures (on $\mathbf{K}_n$) are introduced, motivated and illustrated with examples (that we have already met in Section 4). One reason for introducing these are that the conditions which, according to Theorem 3.15, guarantee that for every extension axiom of $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, the proportion of structures in $\mathbf{K}_n$ which satisfy it approaches 1 as n approaches infinity, are rather restrictive. The conditional measures that we consider – or the *dimension conditional measures*, to be precise – are more permissive with respect to satisfiability of extension axioms. This is made precise by Lemma 7.26 and Example 4.3, for instance. Another motivation for considering conditional measures is that they are more closely related to random generation of finite structures. While the uniform measure is conceptually simple it may, for some $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, be unclear what type of random generation procedure will, for any $\mathcal{M} \in \mathbf{K}_n$, generate $\mathcal{M}$ with probability exactly $1/|\mathbf{K}_n|$. Often, as in the case of l-coloured, or l-colourable, structures (graphs, for example), the most obvious generation procedure – first randomly assign colours, then randomly assign relationships (e.g. edges) so that the colouring is respected – corresponds to conditional measures, in the sense of this article. A third reason for considering conditional measures is simply that they may, in some situations, offer a simpler analysis of asymptotic problems than does the uniform measure, while they are still natural in the sense of being related to random generation of finite structures. Finally we note that a conditional measure may sometimes coincide with a uniform measure, on the same set of structures. Theorem 9.2 and Proposition 9.3 give an example of when this happens.

In Section 7 we start working in a context where the structures that we consider have underlying (possibly nontrivial) pregeometries, and 'dimension' takes over the role of 'cardinality'. By a pregeometry on a structure $\mathcal{M}$ we mean a closure operator $\mathrm{cl}_{\mathcal{M}}$ which operates on subsets of the universe of $\mathcal{M}$ and satisfies certain conditions [1, 15]; moreover we require that $\mathrm{cl}_{\mathcal{M}}$ is uniformly definable in all structures considered (Definition 7.1 and Assumption 7.10). The context considered previously is a special case of the framework of Section 7. The main results of this section, Theorems 7.28, 7.29 and 7.31, apply to the dimension conditional measure, which is a conditional measure that "considers" closed sets of dimension 0 first, then closed subsets of dimension 1, then of dimension 2, and so on. These theorems are related to Theorems 3.15 and 3.17. Theorems 7.28 and 7.29 represent the "positive" side of things, like Theorem 3.15, showing that if certain conditions are satisfied, then for every extension axiom of $\mathbf{K}$ the probability (with the dimension conditional measure) that it holds in a member of $\mathbf{K}_n$ approaches 1 as n approaches infinity; and from this a zero-one law is derived. The conditions in question require, as in Section 3, that whenever $\mathcal{M}$ is permitted, then certain "substitutions", or "replacements", of interpretations can be made in $\mathcal{M}$ without producing a forbidden (not permitted) structure. Also, there is a requirement that the underlying pregeometry, and possibly some other structure which is never changed, is *polynomially k-saturated*. This roughly means that for every $k \in \mathbb{N}$ and all sufficiently large n and every $\mathcal{M} \in \mathbf{K}_n$, the reduct of $\mathcal{M}$ to the sublanguage which defines the pregeometry satisfies every k-extension axiom (with respect to the set of such reducts); and moreover, the truth of a k-extension axiom has many different witnesses compared to the size of the universe.

The last result of Section 7, Theorem 7.31, is a "cousin" of Corollary 3.18 and tells that if there are permitted $\mathcal{A}$ and $\mathcal{A}'$ such that $\mathbf{K}$ *accepts* (Definition 7.20) the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ but *not* the reverse subsitution $[\mathcal{A}' \triangleright \mathcal{A}]$, then $\mathbf{K}$ fails to have the *independent amalgamation property*, or for some extension axioms φ and ψ, the probability, with the dimension conditional measure, that $\varphi \wedge \psi$ holds in a member of $\mathbf{K}_n$ approaches 0 as n approaches infinity. The proofs of Theorems 7.28, 7.29 and 7.31 appear in Section 10.

Sections 8 and 9 study *l-colourable* structures in some fixed relational language. Examples 7.22 and 7.23 show that *l-coloured* structures can be treated within the context developed in Section 7. Theorem 7.29 implies that l-coloured structures have a zero-one law. Since l-colourable structures can be viewed as reducts of l-coloured structures we will also consider a "reduct version" of the dimension conditional measure. With this probability measure it is *not* true that all extension axioms (of l-coloured structures) hold almost surely; but we can show that all extension axioms of a certain kind, called the *l-colour compatible extension axioms*, hold almost surely in sufficiently large structures; and this is enough for subsequently deriving a zero-one law for l-colourable structures.

In Section 9 we study extension axioms and zero-one laws for l-coloured structures and for l-colourable structures *if* one considers the *uniform* probability measure instead of the reduct version of the dimension conditional measure, which was used in Section 8. Theorem 9.2 gives a condition, concerning the distribution of colours, which implies that, with the uniform probability measure, every (l-colour compatible) extension axiom will almost surely hold in sufficiently large l-coloured (or l-colourable) structures, and this implies a zero-one law. The same theorem also states a connection between on the one hand l-coloured structures and their extension axioms, and on the other hand, l-colourable structures and l-colour compatible extension axioms. Proposition 9.3 gives a condition, about the number of colours and the arities of the relation symbols involved, which, if it holds, implies that Theorem 9.2 can be applied.

All results of the article hold also if one restricts attention to structures in which every relation symbol (of arity at least 2) is interpreted as an irreflexive and symmetric relation. All proofs work out in the same way under this assumption (as well as without it).

The results in Sections 8 and 9 may be useful in contexts which do not directly speak about colourings. Suppose that for some $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ and probability measure μ_n on $\mathbf{K}_n$ there is $l \in \mathbb{N}$ such that, for n large enough, $\mathcal{M} \in \mathbf{K}_n$ is almost surely l-colourable. If we know that every l-colourable structure (with universe an initial segment of $\{1, 2, \ldots\}$) belongs to $\mathbf{K}$ and that the set of L-structures which are l-colourable has a zero-one law for the measures μ_n, then also $\mathbf{K}$ has a zero-one law for the same measures. This approach was used in [17] when proving that if $\mathbf{K}_n$ is the set of $(l+1)$-clique-free graphs (or $\mathcal{K}_{l+1}$-free graphs) with universe $\{1, \ldots, n\}$, then $\mathbf{K}$ has a zero-one law for the uniform probability measure. The authors of [17] first proved that almost all $(l+1)$-clique-free graphs are l-colourable, with a relatively even distribution of colours, and then that the l-colourable graphs have a zero-one law.

Section 10 gives the proofs of the main theorems of Section 7. The definitions appearing in sections 5 and 10 are only used within those sections.

The notions of 'polynomial k-saturation' and 'acceptance of substitutions' in Section 7 are versions, adapted to the context of this article, of the notions 'polynomial k-saturation' and 'k-independence hypothesis' in [9]. This is suffiently clear for polynomial k-saturation, but it is perhaps harder to see the relationship between admittance of (k-)substitutions and the k-independence hypothesis. However, in both cases the essential difference between Section 7 of this article and [9] is that in [9] complete types of an infinite structure are considered, while here we consider types with only quantifier-free formulas of tuples enumerating the universe of a closed substructure of some permitted structure. But in this article we avoid speaking about such types since it is equally convenient to speak about (sub)structures and formulas describing them up to isomorphism. Lemmas 10.5 – 10.9, as well as their proofs, are adaptations to the context of this article of Lemmas 2.16 – 2.22 in [9]. The results of this article have their beginnings in considerations from two directions. On the one hand, trying to understand asymptotic satisfiability of extension axioms – conditions implying that they almost surely hold, and conditions implying that some almost surely fail – and on the other hand, trying to understand if some zero-one laws for finite structures were hidden in the probabilistic arguments used in [9].

2. Preliminaries

2.1. **Languages, structures and embeddings.** For basic notions not explained here the reader is refered to [15, 10]. By a ***language*** L we mean the set of (first-order) formulas that can be built up from a ***vocabulary*** (also called ***signature***) which is a set of relation, constant and/or function symbols. We consider the identity symbol '$=$' as a logical symbol which we may always use, together with connectives and (first-order) quantifiers, to build formulas; so '$=$' is never mentioned when we describe the symbols of a vocabulary. If the vocabulary has no constant or function symbols, then we call it ***relational***.

Structures will be denoted by "caligraphic", or "script'", letters: $\mathcal{A}$, $\mathcal{B}$, $\ldots$, $\mathcal{M}$, $\mathcal{N}$, $\ldots$. Their universes will be denoted by the corresponding non-caligraphic letter A, B, $\ldots$, M, N, $\ldots$, or with bars around the letter; for instance, $|\mathcal{M}|$ as well as M denote the universe of $\mathcal{M}$. The cardinality of a set X is denoted by $|X|$; and the cardinality of the (universe of) the structure $\mathcal{M}$ is denoted by $\|\mathcal{M}\|$, or by $|M|$. Boldface letters always denote classes, usually sets, of structures. Sequences, or tuples, of elements are denoted by $\bar{a}, \bar{b}, \ldots$; and $|\bar{a}|$ denotes the length of the sequence $\bar{a}$. By '$\bar{a} \in M$' we mean that $\bar{a}$ is a sequence such that all of its elements belong to the set M. Sometimes we write $\bar{a} \in M^n$ to show that $\bar{a}$ has length n. By $\mathrm{rng}(\bar{a})$, the ***range*** (or ***image***) of $\bar{a}$, we denote the set of all elements that occur in $\bar{a}$. If L has no constant symbols, then we allow an L-structure to have an empty universe.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are L-structures, where L is, as usual, a language. A function $f : M \to N$ is called a ***weak embedding*** of $\mathcal{M}$ (in)to $\mathcal{N}$ if f is *injective* and:

(1) For every constant symbol c, $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$.
(2) For every function symbol g, of arity r, say, and every $\bar{a} \in M^r$, $f(g^{\mathcal{M}}(\bar{a})) = g^{\mathcal{N}}(f(\bar{a}))$.
(3) for every relation symbol, R, of arity r, say, if $\bar{a} \in R^{\mathcal{M}}$ then $f(\bar{a}) \in R^{\mathcal{N}}$, where $f(\bar{a}) = (f(a_1), \ldots, f(a_r))$ if $\bar{a} = (a_1, \ldots, a_r)$.

We say that f is an ***embedding*** if f is *injective* and (1), (2) and the following hold:

(3') for every relation symbol, R, of arity r, say, $\bar{a} \in R^{\mathcal{M}}$ if and only if $f(\bar{a}) \in R^{\mathcal{N}}$.

We say that $\mathcal{M}$ is ***(weakly) embeddable*** into $\mathcal{N}$ if there exists a (weak) embedding from $\mathcal{M}$ to $\mathcal{N}$. Thus, a bijective embedding is the same as an isomorphism. We say that $\mathcal{M}$ is a ***weak substructure*** of $\mathcal{N}$, denoted $\mathcal{M} \subseteq_w \mathcal{N}$, if $M \subseteq N$ and the identity mapping $id : M \to N$ is a weak embedding. We call $\mathcal{M}$ a ***substructure*** of $\mathcal{M}$, denoted $\mathcal{M} \subseteq \mathcal{N}$, if $M \subseteq N$ and the identity mapping $id : M \to N$ is an embedding. $\mathcal{A}$ is a *proper* (weak) substructure of $\mathcal{M}$ if $\mathcal{A}$ is a (weak) substructure of $\mathcal{M}$ and $\mathcal{A} \neq \mathcal{M}$. The symbol '$\subset$' means 'proper subset' or 'proper substructure'.

If $\mathcal{M}$ is a structure and $A \subseteq M$, then $\mathcal{M}\restriction A$ denotes the substructure of $\mathcal{M}$ which is generated by A (the smallest substructure $\mathcal{N}$ of $\mathcal{M}$ such that $A \subseteq N$); so if the vocabulary is relational, then $|\mathcal{M}\restriction A| = A$. If L_0 is a language such that $L_0 \subseteq L$ and $\mathcal{M}$ is an L-structure, then $\mathcal{M}\restriction L_0$ denotes the reduct of $\mathcal{M}$ to L_0.

Since we will several times speak about graphs, we note that, with graph theoretic terminology, if $\mathcal{M}$ and $\mathcal{N}$ are graphs, then $\mathcal{M}$ is a ***subgraph*** of $\mathcal{N}$ if and only if $\mathcal{M}$ is a weak substructure of $\mathcal{N}$; and $\mathcal{M}$ is an ***induced subgraph*** of $\mathcal{N}$ if and only if $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Suppose that R is a relation symbol from the vocabulary of the language of $\mathcal{M}$. Then a tuple $\bar{a}$ of elements from M is called an ***R-relationship*** of M if $\bar{a} \in R^{\mathcal{M}}$ (or equivalently, if $\mathcal{M} \models R(\bar{a})$). If the symbol '$R$' is clear from the context, or if it does not matter which R we refer to, then we may just call an R-relationship a ***relationship***. Sometimes we consider only structures $\mathcal{M}$ in which every relation symbol R is interpreted as an irreflexive and symmetric relation. In this case an *R-relationship* of $\mathcal{M}$ is a set $\mathrm{rng}(\bar{a})$ such that $\bar{a} \in R^{\mathcal{M}}$. So for graphs in general, a relationship is the same as a directed edge; and if we consider only undirected graphs, a relationship is the same as an (undirected) edge.

Remark 2.1. All results in this article hold also if we assume that the interpretations of relation symbols are always irreflexive and symmetric. The proofs in this case are either the same as, or obvious modifications of, the given proofs.

2.2. **Amalgamation.** Let $\mathbf{K}$ be a class of finitely generated L-structures, where L has a countable vocabulary, and let $\widehat{\mathbf{K}}$ be the class consisting of all L-structures $\mathcal{M}$ such that $\mathcal{M}$ is isomorphic to a member of $\mathbf{K}$; so $\widehat{\mathbf{K}}$ is "closed under isomorphism". See [15] (Chapter 7), for example, for definitions of the following notions: ***hereditary property***, or being ***closed under substructures*** as we sometimes say here, ***amalgamation property*** and ***joint embedding property***. We say that $\mathbf{K}$ has any of these properties if $\widehat{\mathbf{K}}$ has it. If the vocabulary of L has only relation symbols then the amalgamation property implies the joint embedding property; but in general the later property is not implied by the first.

If $\widehat{\mathbf{K}}$ has all three properties, then the so-called *Fraïssé limit* $\mathcal{M}_{\mathbf{K}}$ of $\widehat{\mathbf{K}}$ exists [15]. $\mathcal{M}_{\mathbf{K}}$ has the following properties: $\mathcal{M}_{\mathbf{K}}$ is countable, every finitely generated $\mathcal{A} \subseteq \mathcal{M}_{\mathbf{K}}$ belongs to $\widehat{\mathbf{K}}$; every $\mathcal{A} \in \mathbf{K}$ can be embedded into $\mathcal{M}_{\mathbf{K}}$, and if $\mathcal{A} \subseteq \mathcal{M}_{\mathbf{K}}$ is finitely generated and $\mathcal{A} \subseteq \mathcal{B} \in \mathbf{K}$, then there is an embedding $f : \mathcal{B} \to \mathcal{M}_{\mathbf{K}}$ such that $f\restriction A$ is the identity function [15]. The Fraïssé limit $\mathcal{M}_{\mathbf{K}}$ of $\widehat{\mathbf{K}}$, if it exists, is also called the Fraïssé limit of $\mathbf{K}$.

We will consider the following (stronger) variant of the amalgamation property: We say that $\widehat{\mathbf{K}}$ (and $\mathbf{K}$) has the ***free amalgamation property*** if whenever $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \widehat{\mathbf{K}}$, $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{C}$ and $B \cap C = A$, then there is $\mathcal{D} \in \widehat{\mathbf{K}}$ such that $\mathcal{B} \subseteq \mathcal{D}$, $\mathcal{C} \subseteq \mathcal{D}$.

2.3. **Pregeometries.** The notion of ***(combinatorial) pregeometry***, also called ***matroid***, will play a role in sections 7 and 10. See [15] (Chapter 4.6), or [1] (Chapter II.3), for a definition. We use the following terminology when (A, cl) is a pregeometry, with ***closure operator*** cl which maps every $X \subseteq A$ to some closed $Y \subseteq A$. For $X, Y, Z \subseteq A$,

X ***is independent from*** Y ***over*** Z if $X \cap \mathrm{cl}(\emptyset) = \emptyset$ and for every $a \in X$, $a \in \mathrm{cl}(Y \cup Z) \iff a \in \mathrm{cl}(Z)$. In the special case that $Z = \emptyset$ we say that X ***is independent from*** Y. Because of the 'exchange property' of pregeometries, independence is symmetric with respect to X and Y. We say that $a \in A$ is independent from $Y \subseteq A$ over $Z \subseteq A$ if $\{a\}$ is independent from Y over Z. A set $X \subseteq A$ is called ***independent*** if for every $a \in X$, a is independent from $X - \{a\}$ (over $\emptyset$). The ***dimension*** of $X \subseteq A$ is the supremum of the cardinalities of independent subsets of X. A set $X \subseteq A$ is called ***closed*** if $\mathrm{cl}(X) = X$. If $cl(X) = X$ for every $X \subseteq A$ then we call (A, cl) the ***trivial pregeometry*** on A.

2.4. **Zero-one laws.** Suppose that, for $n \in \mathbb{N}$, $\mathbf{K}_n$ is a set of L-structures and that μ_n is a probability measure on $\mathbf{K}_n$. We say that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ has a ***zero-one law*** if for every L-sentence φ, $\lim_{n\to\infty} \mu_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big)$ exists and is 0 or 1. When saying "φ is almost surely true (or false)" we mean that the limit is 1 (or 0). If μ_n is the uniform probability measure we may instead say that "almost all $\mathcal{M} \in \mathbf{K}$ satisfy φ" if the limit is 1. By the ***almost sure theory of*** $\mathbf{K}$ (with respect to the measures μ_n), we mean the set of sentences φ such that the probability that φ is true in $\mathbf{K}_n$ approaches 1 as $n \to \infty$.

3. Permitted structures and substitutions

From this section and until Section 7 we work within the following framework:

Assumptions and terminology 3.1. Fix a first-order language L with finite relational vocabulary. Let m_n, $n \in \mathbb{N}$, be a sequence of positive integers. For every $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a set of L-structures with universe $\{1, \ldots, m_n\}$; and let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. A structure $\mathcal{M}$ is called ***represented (with respect to*** $\mathbf{K}$***)*** if it is isomorphic to a structure in $\mathbf{K}$. A structure $\mathcal{M}$ is called ***permitted (with respect to*** $\mathbf{K}$) if it is embeddable into a structure in $\mathbf{K}$. A structure which is not permitted is called ***forbidden***. Since we fix $\mathbf{K}$ for rest of the section we sometimes omit the phrase "with respect to $\mathbf{K}$".

Observe that if $\mathbf{K}$ has the hereditary property, then a structure is permitted if and only if it is represented. In this section and the next, all examples of $\mathbf{K}$ which are considered in some detail have the hereditary property. However, since the results do not depend on this we do not assume it. (One example of $\mathbf{K}$ which is *not* closed under substructures is given by letting $\mathbf{K}_n$ be the set of triangle-free graphs with universe $\{1, \ldots, n\}$ and diameter 2.)

Definition 3.2. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are permitted structures and that $\mathcal{A}$ is a proper substructure of $\mathcal{B}$.

(i) The $\mathcal{B}/\mathcal{A}$***-extension axiom*** (or the $\mathcal{B}$***-extension axiom over*** $\mathcal{A}$) holds, by definition, in $\mathcal{M}$ if the following is true:

> For every embedding τ of $\mathcal{A}$ into $\mathcal{M}$ there exists an embedding π of $\mathcal{B}$ into $\mathcal{M}$ which extends τ (i.e. $\pi(a) = \tau(a)$ whenever $a \in A$).

The $\mathcal{B}/\mathcal{A}$-extension axiom can be expressed by a first-order sentence of the form

$$\forall x_1, \ldots, x_n \exists y_1, \ldots, y_m \big(\varphi(x_1, \ldots, x_n) \longrightarrow \psi(x_1, \ldots, x_n, y_1, \ldots, y_m)\big),$$

where φ and ψ are quantifier-free. If the language has no constant symbols, then we allow the possibility that the universe of $\mathcal{A}$ is empty, in which case the $\mathcal{B}/\mathcal{A}$-extension axiom is called the $\mathcal{B}/\emptyset$-extension axiom. It is then expressed by an existential formula

$$\exists y_1, \ldots, y_m \psi(y_1, \ldots, y_m).$$

(ii) If $|\mathcal{B}| \leq k+1$, then the $\mathcal{B}/\mathcal{A}$-extension axiom is called a k***-extension axiom of*** $\mathbf{K}$; or, if we do not care about k, just an ***extension axiom of*** $\mathbf{K}$. If $\mathbf{K}$ is clear from the context we may omit saying "of $\mathbf{K}$".

Remark 3.3. *If there are probability measures μ_n on $\mathbf{K}_n$, for $n \in \mathbb{N}$, such that for every extension axiom φ of $\mathbf{K}$, the μ_n-probability that $\mathcal{M} \in \mathbf{K}_n$ satisfies φ approaches 1 as n approaches ∞, then $\mathbf{K}$ has a zero-one law for the measures μ_n.* The usual proof of this statement does not depend on the measures μ_n. It is proved in [13, 10, 15, 22] (for example) by collecting into a theory $T_{\mathbf{K}}$ all extension axioms, together with sentences expressing the possbile isomorphism types of substructures of members of $\mathbf{K}$. The general idea of the argument is as follows. By the assumptions in the above statement and compactness, $T_{\mathbf{K}}$ is consistent. By a back-and-forth argument one then proves that $T_{\mathbf{K}}$ is countably categorical and therefore complete. The completeness of $T_{\mathbf{K}}$ (and compactness) implies that $\mathbf{K}$ has a zero-one law.

If we define $\mathbf{K}$ by forbidding certain weak substructures, and the thus obtained $\mathbf{K}$ has the free amalgamation property, then we have the following "dichotomy".

Theorem 3.4. *Let $\mathbf{F}$ be a set of L-structures and, for every $n \in \mathbb{N}$, let $\mathbf{K}_n$ consist of exactly those L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that* no *$\mathcal{F} \in \mathbf{F}$ is* weakly *embeddable into $\mathcal{M}$ (so in particular, every member of $\mathbf{F}$ is forbidden). Assume that $\mathbf{K}_n \neq \emptyset$ for all sufficiently large n and that $\mathbf{K}$ has the free amalgamation property. Consider the following condition:*

(∗) *There are $\mathcal{F} \in \mathbf{F}$, a relation symbol R of arity r, say, and $\bar{a} \in F^r$ such that* $\mathrm{rng}(\bar{a})$ *is a* proper *subset of F, $\bar{a} \in R^{\mathcal{F}}$, and if $\mathcal{P}$ is constructed by removing the R-relationship $\bar{a}$, but making no other changes in $\mathcal{F}$, then $\mathcal{P}$ is permitted.*

If (∗) is false, then, for every $k \in \mathbb{N}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms of $\mathbf{K}$ approaches 1 as $n \to \infty$. If (∗) is true, then letting $\mathcal{F} \in \mathbf{F}$, R and $\bar{a}$ be any witnesses of property (∗) and letting α be the number of permitted structures with universe $\{1, \ldots, |\mathrm{rng}(\bar{a})|\}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all $(2|F| - |\mathrm{rng}(\bar{a})| - 1)$-extension axioms of $\mathbf{K}$ never exceeds $1 - 1/(1 + \alpha)$. Moreover, if L has no unary relation symbols, then the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all $(2|F| - |\mathrm{rng}(\bar{a})| - 1)$-extension axioms approaches 0 as $n \to \infty$.

Theorem 3.4 is a consequence of theorems 3.15 and 3.17. Since one may see it as an application of these theorems, we give the proof of Theorem 3.4 as Example 4.1 in Section 4. The argument in Example 4.1 gives some information about what happens if $\mathbf{K}$ does not have the free amalgamation property.

Remark 3.5. One may ask if the assumption that there are no unary relation symbols is necessary for the last statement of Theorem 3.4. The author does not have an example showing that this statement fails without the assumption that there are no unary relation symbols, *if* we assume, as in Theorem 3.4, that $\mathbf{K}$ has the free amalgamation property. But Example 4.2 shows that when it is assumed that there are no unary relation symbols in Theorem 3.17, then this assumption is necessary.

Two examples follow, one for which (∗) in Theorem 3.4 does not hold, and one for which (∗) holds.

Example 3.6. Suppose that L has only one binary relation symbol R and that $\mathbf{F} = \{\mathcal{A}, \mathcal{B}\}$, where $A = \{1\}$, $R^{\mathcal{A}} = \{(1,1)\}$, $B = \{1,2\}$ and $R^{\mathcal{B}} = \{(1,2),(2,1)\}$. If $\mathbf{K}_n$ and $\mathbf{K}$ are defined as in Theorem 3.4, then an L-structure is permitted if and only if it is an irreflexive and antisymmetric directed graph. Moreover, the property (∗) fails for $\mathbf{F}$.

Example 3.7. (**$\mathcal{K}_l$-free graphs**) It is not difficult to define $\mathbf{F}$ for which the property (∗) holds, but let us mention an example which has been studied in some detail [17]. Let L have only one binary relation symbol R and consider only structures in which R is interpreted as an irreflexive and symmetric relation, that is, an undirected graph without loops. Let $l \geq 3$ and let $\mathcal{K}_l$ be the complete (undirected) graph with vertices

$1,\dots,l$. If $\mathbf{F} = \{\mathcal{K}_l\}$ then condition $(*)$ holds, since the removal of one edge from $\mathcal{K}_l$ creates a permitted graph. It is easy to see that $\mathbf{K}$ has the free amalgamation property. By Theorem 3.4, the proportion of $\mathcal{M} \in \mathbf{K}_l$ which satisfy all $(2l-3)$–extension axioms of $\mathbf{K}$ approaches 0 as $n \to \infty$. For $l = 3$ at least, this conclusion is not new. Because the proportion of $\mathcal{K}_3$-free graphs (*triangle-free* graphs) which are bipartite approaches 1 as $n \to \infty$ [12, 17]; and a graph is bipartite if and only if it has no cycle of odd length; moreover, it is easy to see that a 5-cycle exists in every $\mathcal{K}_3$-free graph which satisfies all 3-extension axioms.

Remark 3.8. Even if, for some extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ does not approach 1, $\mathbf{K}$ may nevertheless have a zero-one law with respect to the uniform probability measure. For example, it has been shown [17] that, for every $l \geq 3$, if $\mathbf{F} = \{\mathcal{K}_l\}$ where $\mathcal{K}_l$ is the complete graph with l vertices, and $\mathbf{K}_n$ and $\mathbf{K}$ are defined as in Example 3.7, then $\mathbf{K}$ has a zero-one law for the uniform probability measure.

Definition 3.9. Let $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ be structures and suppose that $\mathcal{A}$ is a proper substructure of $\mathcal{B}$.
(i) We say that the ***$\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least*** m (or that the ***$\mathcal{B}$-multiplicity over $\mathcal{A}$ in $\mathcal{M}$ is at least*** m) if the following holds:

> Whenever σ is an embedding of $\mathcal{A}$ into $\mathcal{M}$, then there are embeddings σ_i of $\mathcal{B}$ into $\mathcal{M}$, for $i = 1,\dots,m$, such that each σ_i extends σ and if $i \neq j$ then $\sigma_i(B) \cap \sigma_j(B) = \sigma(A)$.

The ***$\mathcal{B}/\mathcal{A}$-multiplicity is*** m if it is at least m but not at least $m+1$.
(ii) We say that $\mathcal{M}$ ***has (at least)*** n ***copies of*** $\mathcal{A}$ if there are (at least) n different substructures $\mathcal{A}'_1,\dots,\mathcal{A}'_n$ of $\mathcal{M}$ such that each $\mathcal{A}'_i$ is isomorphic to $\mathcal{A}$.

Remark 3.10. Observe the following relationships between extension axioms and multiplicity, where we assume that $\mathcal{A} \subset \mathcal{B}$.
(i) $\mathcal{M}$ satisfies the $\mathcal{B}/\mathcal{A}$-extension axiom if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 1.
(ii) Suppose that there are a structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{B} \to \mathcal{C}$ and $\sigma_2 : \mathcal{B} \to \mathcal{C}$ such that $\sigma_1 \restriction A = \sigma_2 \restriction A$ and $\sigma_1(B) \cap \sigma_2(B) = \sigma_1(A)$. If $\mathcal{M}$ satisfies the $\mathcal{C}/\mathcal{A}$-extension axiom then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 2.

Definition 3.11. Suppose that the vocabulary of L does not contain any constant symbol. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ be L-structures such that $\mathcal{A} \subseteq \mathcal{M}$ and $|\mathcal{A}| = |\mathcal{B}|$. We define $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ to be the structure obtained by "replacing $\mathcal{A}$ by $\mathcal{B}$ inside $\mathcal{M}$", or more precisely, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is defined to be the structure with the same universe as $\mathcal{M}$ which satisfies the following conditions: For every n and every relation symbol R of arity n,

(1) if $(a_1,\dots,a_n) \in A^n$, then $(a_1,\dots,a_n) \in R^{\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]} \iff (a_1,\dots,a_n) \in R^{\mathcal{B}}$, and
(2) if $(a_1,\dots,a_n) \in M^n - A^n$, then $(a_1,\dots,a_n) \in R^{\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]} \iff (a_1,\dots,a_n) \in R^{\mathcal{M}}$.

The notation $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ may be read as $\mathcal{M}$ ***with $\mathcal{A}$ replaced by*** $\mathcal{B}$, or $\mathcal{M}$ ***with $\mathcal{B}$ substituted for*** $\mathcal{A}$.

Definition 3.12. Let $\mathcal{A}$ and $\mathcal{B}$ be permitted structures (with respect to $\mathbf{K}$) with the same universe.
(i) We say that $\mathbf{K}$ ***admits the substitution*** $[\mathcal{A} \triangleright \mathcal{B}]$ if for every represented $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is a represented structure.
(ii) We say that $\mathbf{K}$ ***weakly admits the substitution*** $[\mathcal{A} \triangleright \mathcal{B}]$ if for every represented $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is a permitted structure.
(iii) If $|\mathcal{A}| = |\mathcal{B}|$ and $\|\mathcal{A}\| \leq k$, then we call $[\mathcal{A} \triangleright \mathcal{B}]$ a ***k-substitution***.
(iv) If $\mathbf{K}$ (weakly) admits every k-substitution $[\mathcal{A} \triangleright \mathcal{B}]$, where $\mathcal{A}$ and $\mathcal{B}$ are permitted structures (with the same universe), then we say that $\mathbf{K}$ ***(weakly) admits k-substitutions***.

When speaking of a substitution $[\mathcal{A} \triangleright \mathcal{B}]$ we always assume that $\mathcal{A}$ and $\mathcal{B}$ have the same universe. Note that if every permitted structure is represented, which is the case if $\mathbf{K}$ has the hereditary property, then $\mathbf{K}$ admits a substitution $[\mathcal{A} \triangleright \mathcal{B}]$ if and only if $\mathbf{K}$ weakly admits $[\mathcal{A} \triangleright \mathcal{B}]$.

Lemma 3.13. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are permitted structures with the same universe such that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is weakly admitted. Then for every permitted $\mathcal{M}$ such that $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted.*

Proof. Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ satisfy the premisses of the lemma and that $[\mathcal{A} \triangleright \mathcal{M}]$ is weakly admitted. Since $\mathcal{M}$ is permitted there is a represented structure $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$ (recall that the class of represented structures is closed under isomorphism). Since the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is weakly admitted, $\mathcal{N}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted, so there is a represented $\mathcal{N}'$ such that $\mathcal{N}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}'$. By assumption $\mathcal{A} \subseteq \mathcal{M} \subseteq \mathcal{N}$, so $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}[\mathcal{A} \triangleright \mathcal{B}] \subseteq \mathcal{N}'$, which means that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted (because $\mathcal{N}'$ is represented). □

Remark 3.14. Suppose that ρ is the supremum of the arities of relation symbols in the vocabulary of L.
(i) It is straighforward to see that if $\mathbf{K}$ admits ρ-substitutions, then $\mathbf{K}$ admits k-substitutions for every $k \in \mathbb{N}$; because every k-substitution can be achieved by performing, in sequence, finitely many ρ-substitutions.
(ii) By using Lemma 3.13, it follows, similarly as in (i), that if $\mathbf{K}$ weakly admits ρ-substitutions, then $\mathbf{K}$ weakly admits k-substitutions for every k.

Remember that a structure $\mathcal{M}$ satisfies the $\mathcal{B}/\mathcal{A}$-extension axiom if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least 1. The next theorem is essentially a rephrasing, with the terminology of this article, of a partial result which is used to prove that every nontrivial parametric class of L-structures has a labeled zero-one law ([10] Theorem 4.2.3, [18]). A class $\mathbf{C}$ of finite L-structures is *nontrivial and parametric*, in the sense of [10, 18], if and only if $\mathbf{C}$ is the class of represented structures with respect to some $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ which admits k-substitutions for every k, and $\mathbf{K}_n$ is a nonempty set of L-structures with universe $\{1, \ldots, n\}$. The result refered to in [10, 18] is a generalization of the well-known zero-one law for 'random structures' [13, 14], which in the present context amounts to considering the uniform probability measure on the set $\mathbf{K}_n$ of *all* L-structures with universe $\{1, \ldots, n\}$ (assuming that the arity of at least one relation symbol, besides '=', is greater than 1).

Theorem 3.15. [10, 13, 14, 18] *Let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ where each $\mathbf{K}_n$ is a nonempty set of L-structures with universe $\{1, \ldots, m_n\}$ and $\lim_{n \to \infty} m_n \to \infty$. Suppose that $\mathbf{K}$ admits k-subsitutions and let p be any positive integer. Whenever $\mathcal{A} \subset \mathcal{B}$ are permitted and $|\mathcal{B}| \leq k$, then the proportion of structures $\mathcal{M} \in \mathbf{K}_n$ such that the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least p approaches 1 as n approaches ∞.*

Since Theorem 3.15 is not quite the same as similar results refered to [10, 13, 14, 18], we give a sketch of its proof.

Proof sketch. For simplicity, consider the case when $\|\mathcal{B}\| = \|\mathcal{A}\| + 1 \leq k$ and $p = 2$. For every positive $d \in \mathbb{N}$, let α_d denote the number of different permitted structures with universe $\{1, \ldots, d\}$. Let $\mathcal{M} \in \mathbf{K}_n$. Since $\mathbf{K}$ admits k-substitutions it follows that for every $d \leq k$, every permitted structure $\mathcal{P}$ with universe $\{1, \ldots, d\}$, all distinct $i_1, \ldots, i_d \in \{1, \ldots, m_n\}$, and $\mathcal{M} \in \mathbf{K}_n$, the probability that $j \mapsto i_j$ is an embedding of $\mathcal{P}$ into $\mathcal{M}$ is $1/\alpha_d$, with the uniform probability measure. Suppose that $\mathcal{A}'$ is a copy of $\mathcal{A}$ with universe $A' = \{i_1, \ldots, i_d\} \subset M = \{1, \ldots, m_n\}$, so $d < k$. For every $j \in \{1, \ldots, \lfloor m_n/2 \rfloor\} - A'$, the probability that $\mathcal{M} \restriction \{i_1, \ldots, i_d, j\}$ is a copy of $\mathcal{B}$ is at least $1/\alpha_{d+1}$. Therefore the probability that there is no $j \in \{1, \ldots, \lfloor m_n/2 \rfloor\} - A'$ such that this holds is at most $\big(1-1/\alpha_{d+1}\big)^{\lfloor m_n/2\rfloor - d}$. There are at most $\binom{m_n}{d}$ copies of $\mathcal{A}$ in $\mathcal{M}$ and therefore the probability that some copy $\mathcal{A}' \subseteq \mathcal{M}$ of $\mathcal{A}$ cannot be extended to a copy of $\mathcal{B}$ by adding an element from $\{1, \ldots, \lfloor m_n/2\rfloor\} - A'$ is at most $\binom{m_n}{d}\big(1-1/\alpha_{d+1}\big)^{\lfloor m_n/2\rfloor - d}$ which approaches 0 as n approaches ∞ (because we assume that $\lim_{n\to\infty} m_n = \infty$). In the same way, the probability that some copy $\mathcal{A}' \subseteq \mathcal{M}$ of $\mathcal{A}$ cannot be extended to a copy of $\mathcal{B}$ by adding an element from $\{\lfloor m_n/2\rfloor + 1, \ldots, m_n\} - A'$ approaches 0 as $n \to \infty$. It follows that the probability that the $\mathcal{A}/\mathcal{B}$-multiplicity of $\mathcal{M} \in \mathbf{K}_n$ is less than 2 approaches 0 as $n \to \infty$. □

With Theorem 3.15 at hand it remains to study what happens, asymptotically, with extension axioms and multiplicities when there are permitted $\mathcal{A}$ and $\mathcal{B}$ (with the same universe) such that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is not admitted with respect to $\mathbf{K}$. The assumption that, for some permitted $\mathcal{A}$ and $\mathcal{B}$, the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is not admitted is not enough, even if we assume that $\mathbf{K}$ has the hereditary property and free amalgamation property, to produce an extension axiom φ of $\mathbf{K}$ such that the proportion of structures in $\mathbf{K}_n$ satisfying φ does *not* approach 1 as $n \to \infty$. In this context it may, or may not, be the case that for every extension axiom, the proportion of structures in $\mathbf{K}_n$ in which it is true approaches 1. Examples 4.6 and 4.7 show this.

But if $\mathbf{K}$ has the hereditary property and free amalgamation property and there are permitted $\mathcal{A}$ and $\mathcal{B}$ with the same universe such that $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, and permitted $\mathcal{M}$ such that $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden, then (by Corollary 3.18) the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2|M| - 1)$-extension axioms never exceeds some $c < 1$; and if there are no unary relation symbols, then this proportion approaches 0 as $n \to \infty$. If we do not assume that $\mathbf{K}$ has the hereditary and free amalgamation properties, then we can still obtain a related result (Theorem 3.17) if we add another assumption on $\mathcal{A}$ and $\mathcal{B}$. In the case that $\mathbf{K}$ has the hereditary property and free amalgamation property, Lemma 3.16, below, implies that we can find permitted $\mathcal{A}$ and $\mathcal{B}$ which satisfy this added assumption.

Recall that if $\mathbf{K}$ has the hereditary property and the amalgamation property, then the notions 'permitted structure' and 'represented structure' coincide, and therefore the notions 'admit' (some subsitution) and 'weakly admit' (the same substitution) coincide.

Lemma 3.16. *Suppose that* $\mathbf{K}$ *has the hereditary property and the free amalgamation property. Assume that* $\mathcal{A}$ *and* $\mathcal{B}$ *are permitted structures with the same universe and that the substitution* $[\mathcal{A} \triangleright \mathcal{B}]$ *is admitted, but* $[\mathcal{B} \triangleright \mathcal{A}]$ *is not admitted. Then there are permitted* $\mathcal{A}'$ *and* $\mathcal{B}'$ *such that*

(1) $A' = B' \subseteq A$,
(2) *the substitution* $[\mathcal{A}' \triangleright \mathcal{B}']$ *is admitted but* $[\mathcal{B}' \triangleright \mathcal{A}']$ *is not admitted, and*
(3) *for every proper subset* $U \subset A'$, $\mathcal{A}' \restriction U = \mathcal{B}' \restriction U$.

Moreover, if $\mathcal{M}$ *is permitted and* $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ *is forbidden, then there is permitted* $\mathcal{M}'$ *with* $M' = M$ *such that* $\mathcal{M}'[\mathcal{B}' \triangleright \mathcal{A}']$ *is forbidden.*

Proof. With '$\subset$' we mean 'proper subset' or 'proper substructure'. It suffices to prove that if $\mathcal{A}' = \mathcal{A}$ and $\mathcal{B}' = \mathcal{B}$ do not satisfy (1) – (3), then there is $U \subset A$ such that if $\mathcal{A}' = \mathcal{A} \restriction U$ and $\mathcal{B}' = \mathcal{B} \restriction U$, then $[\mathcal{A}' \triangleright \mathcal{B}']$ is admitted but $[\mathcal{B}' \triangleright \mathcal{A}']$ is not admitted. (Because if $A' = B'$ is a singleton set, then (3) trivially holds.) The last statement of the lemma will follow from the proof that there exist $\mathcal{A}'$ and $\mathcal{B}'$ satisfying (1) – (3).

First we prove the following:

Claim. If $U \subset A$, $\mathcal{U} = \mathcal{A} \restriction U$ and $\mathcal{V} = \mathcal{B} \restriction U$, then the substitution $[\mathcal{U} \triangleright \mathcal{V}]$ is admitted.

Proof of Claim. Let $\mathcal{M}$ be any permitted structure and suppose that $\mathcal{U} \subseteq \mathcal{M}$. We need to show that $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}]$ is permitted. By the free amalgamation property there is a permitted $\mathcal{C}$ such that $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{M} \subseteq \mathcal{C}$ and $A \cap M = U$. Since $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, $\mathcal{C}[\mathcal{A} \triangleright \mathcal{B}]$ is permitted. From $\mathcal{M} \subseteq \mathcal{C}$ and $A \cap M = U$ we get $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}] \subseteq \mathcal{C}[\mathcal{A} \triangleright \mathcal{B}]$, so $\mathcal{M}[\mathcal{U} \triangleright \mathcal{V}]$ is permitted (and hence represented). □

Suppose that for some $U \subset A$, $\mathcal{A}{\restriction}U \neq \mathcal{B}{\restriction}U$. (Otherwise $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$ satisfy (1) – (3).) Let $U_1, \ldots, U_l$ be an enumeration of all proper subsets $U_i \subset A = B$ such that $\mathcal{A}{\restriction}U_i \neq \mathcal{B}{\restriction}U_i$. By assumption there is a permitted $\mathcal{M}$ such that $\mathcal{B} \subset \mathcal{M}$ and $\mathcal{N} = \mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden. For $i = 1, \ldots, l$, let $\mathcal{U}_i = \mathcal{A}{\restriction}U_i$ and, by induction, define $\mathcal{N}_0 = \mathcal{M}$, $\mathcal{V}_1 = \mathcal{M}{\restriction}U_1$, $\mathcal{N}_{i+1} = \mathcal{N}_i[\mathcal{V}_{i+1} \triangleright \mathcal{U}_{i+1}]$ and $\mathcal{V}_{i+1} = \mathcal{N}_i{\restriction}U_{i+1}$. Let $\mathcal{A}' = \mathcal{N}_l{\restriction}A$. Then $\mathcal{N} = \mathcal{M}[\mathcal{B} \triangleright \mathcal{A}] = \mathcal{N}_l[\mathcal{A}' \triangleright \mathcal{A}]$.

If every one of the substitutions $[\mathcal{V}_1 \triangleright \mathcal{U}_1], \ldots, [\mathcal{V}_l \triangleright \mathcal{U}_l]$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ is admitted, then $\mathcal{N}$ is permitted, which contradicts the assumption about $\mathcal{N}$. First suppose that for some i, the substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is not admitted. By the claim, $[\mathcal{U}_i \triangleright \mathcal{V}_i]$ is admitted, so we are done (remember the first paragraph of the proof).

Now suppose that for every i, the substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is admitted, and consequently $[\mathcal{A}' \triangleright \mathcal{A}]$ is not admitted. By the definition of $\mathcal{A}'$, we have $\mathcal{A}'{\restriction}U = \mathcal{A}{\restriction}U$ for every $U \subset A = A'$. Hence, we are done if we can show that the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ is admitted. By the definition of $\mathcal{U}_i$, $\mathcal{V}_i$, $i = 1, \ldots, l$ and $\mathcal{A}'$, the result of the substitution $[\mathcal{B} \triangleright \mathcal{A}']$, in any permitted structure, can be achieved by the performing the substitutions

$$[\mathcal{V}_1 \triangleright \mathcal{U}_1], \ldots, [\mathcal{V}_l \triangleright \mathcal{U}_l]$$

sequentially in the order from left to right. By assumption, every substitution $[\mathcal{V}_i \triangleright \mathcal{U}_i]$ is admitted, and hence $[\mathcal{B} \triangleright \mathcal{A}']$ is admitted. Since the result of the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ can be achieved by first performing the substitution $[\mathcal{A} \triangleright \mathcal{B}]$, which is admitted by assumption, and then $[\mathcal{B} \triangleright \mathcal{A}']$, it follows that $[\mathcal{A} \triangleright \mathcal{A}']$ is admitted.

Now we verify the last statement of the lemma. When starting with $\mathcal{M}$ and then performing the substitutions $[\mathcal{V}_1 \triangleright \mathcal{U}_1]$, …, $[\mathcal{V}_l \triangleright \mathcal{U}_l]$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ in this order, then, since $\mathcal{N}$ is forbidden, there is a *first* structure during this process which is is forbidden. For $\mathcal{M}'$ we take the last structure during the process such that it and every structure before it is permitted. □

Theorem 3.17. *Assume that $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted structures such that $\mathcal{S}_\mathcal{P} \subseteq \mathcal{P}$, $|\mathcal{S}_\mathcal{P}| = |\mathcal{S}_\mathcal{F}|$, $\|\mathcal{S}_\mathcal{P}\| = k$, $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden, but the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted. Morover, assume that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P}{\restriction}|\mathcal{U}| = \mathcal{S}_\mathcal{F}{\restriction}|\mathcal{U}|$. Let α be the number of different permitted structures with universe $\{1, \ldots, k\}$ (so $\alpha \geq 2$).*
(i) For every n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(a) *$\mathcal{M}$ contains a copy of $\mathcal{S}_\mathcal{F}$, and*
(b) *the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2*

never exceeds $1 - 1/(1 + \alpha)$.
(ii) Suppose that there exist a permitted structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{P} \to \mathcal{C}$ and $\sigma_2 : \mathcal{P} \to \mathcal{C}$ such that $\sigma_1(|\mathcal{P}|) \cap \sigma_2(|\mathcal{P}|) = \sigma_1(|\mathcal{S}_\mathcal{P}|)$ and $\sigma_1{\restriction}|\mathcal{S}_\mathcal{P}| = \sigma_2{\restriction}|\mathcal{S}_\mathcal{P}|$. Then, for every n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ that satisfy all $(2\,\|\mathcal{P}\| - k - 1)$-extension axioms never exceeds $1 - 1/(1 + \alpha)$.
(iii) Suppose that L has no unary relation symbols. The proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(c) *$\mathcal{M}$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom, where $\mathcal{U} \subseteq \mathcal{S}_\mathcal{F}$ and $\|\mathcal{U}\| = 1$, and*
(d) *the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2*

approaches 0 as n approaches ∞.

Corollary 3.18. *Suppose that $\mathbf{K}$ has the hereditary property and the free amalgamation property. Also assume that there are permitted structures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ such that $A = B$ and the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted, but $\mathcal{M}[\mathcal{B} \triangleright \mathcal{A}]$ is forbidden.*

Then the proportion of structures in $\mathbf{K}_n$ *which satisfy all* $(2|M|-1)$*-extension axioms never exceeds* $1-1/(1+\alpha)$*, where* α *is the number of permitted structures with universe* A*. If the language has no unary relation symbols then this proportion approaches 0 as* $n \to \infty$*.*

Proof of Corollary 3.18. Assume that $\mathbf{K}$ has the hereditary property and free amalgamation property, and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ satisfy the assumptions of the corollary. From Lemma 3.16 it follows that there are permitted structures $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ which satisfy the assumptions of Theorem 3.17 and $|\mathcal{S}_\mathcal{P}| \subseteq |\mathcal{A}|$ and $|\mathcal{P}| = |\mathcal{M}|$. Since $\mathbf{K}$ has the free amalgamation property, part (ii) of Theorem 3.17 implies that the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_\mathcal{P}\| - 1)$-extension axioms never exceeds $1-1/(1+\alpha')$, where α' is the number of permitted structures with universe $|\mathcal{S}_\mathcal{P}|$. Note that if α is the number of permitted structures with universe A, then, since $\|\mathcal{S}_\mathcal{P}\| \leq |A|$, we have $1-1/(1+\alpha') \leq 1-1(1+\alpha)$.

Every structure in $\mathbf{K}$ which satisfies all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_\mathcal{P}\| - 1)$-extension axioms satisfies both (c) and (d) in part (iii) of Theorem 3.17. So if the language has no unary relation symbols the proportion of structures in $\mathbf{K}_n$ which satisfy all $(2\,\|\mathcal{P}\| - \|\mathcal{S}_\mathcal{P}\| - 1)$-extension axioms must approach 0 as $n \to \infty$. Since $2|M|-1 = 2|P|-1 \geq 2\,\|\mathcal{P}\| - \|\mathcal{S}_\mathcal{P}\| - 1$ we are done. □

4. Examples

In all examples, $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$ has the hereditary property.

Example 4.1. (**Forbidden weak substructures and proof of Theorem 3.4.**) Let L have a finite relational vocabulary, and let $\mathbf{F}$ be a set of finite L-structures. For $n \in \mathbb{N}$, let $\mathbf{K}_n$ be the set of all L-structures $\mathcal{M}$ with universe $\{1,\ldots,n\}$ such that *no* $\mathcal{F} \in \mathbf{F}$ can be weakly embedded into $\mathcal{M}$. Then a structure $\mathcal{A}$ is forbidden if and only if some $\mathcal{F} \in \mathbf{F}$ can be weakly embedded into $\mathcal{A}$. It follows that there exists (at least) one ***minimal forbidden*** structure $\mathcal{F}_{min} \in \mathbf{F}$ in the sense that every *proper* weak substructure of $\mathcal{F}_{min}$ is permitted.

If $\mathcal{F}_{min}$ does not have any relationship at all, that is, if $\mathcal{F}_{min}$ is just a finite set of cardinality m, say, then $\mathbf{K}_n = \emptyset$ for every $n \geq m$. Since we are only interested in the case when $\mathbf{K}_n \neq \emptyset$ for arbitrarily large $n \in \mathbb{N}$, we now assume that every minimal forbidden structure has at least one relationship. From this it follows that $\mathbf{K}_n \neq \emptyset$ for every $n \in \mathbb{N}$, because the assumption ensures that the set $\{1,\ldots,n\}$ without any structure belongs to $\mathbf{K}_n$.

Let $\mathcal{F}_{min}$ be any minimal forbidden structure. By assumption, for some relation symbol R, $R^{\mathcal{F}_{min}}$ is nonempty, so we can remove a relationship $\bar{a}$ from $R^{\mathcal{F}_{min}}$ and call the resulting structure $\mathcal{P}$. Note that $\mathcal{P}$ is permitted (since $\mathcal{F}_{min}$ is minimal forbidden), and that $\mathcal{F}_{min}$ and $\mathcal{P}$ have the same universe which includes $\mathrm{rng}(\bar{a})$. Let $\mathcal{S}_\mathcal{F} = \mathcal{F}_{min}\!\upharpoonright\!\mathrm{rng}(\bar{a})$ and $\mathcal{S}_\mathcal{P} = \mathcal{P}\!\upharpoonright\!\mathrm{rng}(\bar{a})$. Then $\mathcal{S}_\mathcal{P}$ is permitted, because it is a substructure of $\mathcal{P}$, and $\mathcal{P}$ is permitted. If $\mathrm{rng}(\bar{a}) = |\mathcal{F}_{min}|$ then $\mathcal{S}_\mathcal{F} = \mathcal{F}_{min}$ which is forbidden. If this holds for *every* choice of minimal forbidden $\mathcal{F}_{min}$ and $\mathcal{S}_\mathcal{F}$ as defined above, then $(*)$ in Theorem 3.4 does not hold, and it is straightforward to verify that $\mathbf{K}$ admits k-substitutions for every k. In this case, Theorems 3.15 imply that, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$; and hence $\mathbf{K}$ has a zero-one law for the uniform measure, by Remark 3.3.

Now suppose that there is a minimal forbidden $\mathcal{F}_{min}$ and R such that for some $\bar{a} \in R^{\mathcal{F}_{min}}$, $\mathrm{rng}(\bar{a})$ is a proper subset of $|\mathcal{F}_{min}|$. Then $(*)$ in Theorem 3.4 holds and $\mathcal{S}_\mathcal{F} = \mathcal{F}_{min}\!\upharpoonright\!\mathrm{rng}(\bar{a})$ is a *proper* substructure of $\mathcal{F}_{min}$, and since the latter is minimal forbidden, $\mathcal{S}_\mathcal{F}$ is permitted. Hence $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted, but $\mathcal{F}_{min} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden. (The notions 'admitted' and 'weakly admitted' coincide here because the notions 'permitted' and 'represented' coincide in this example.) But since the removal of a relationship from a permitted structure will never (in the present context) produce a forbidden structure, the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted. Moreover, by the definition of $\mathcal{S}_\mathcal{F}$ and $\mathcal{S}_\mathcal{P}$, they agree on all proper subsets of their common universe. Thus, Theorem 3.17 is applicable. By part (i) of Theorem 3.17, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

(a) $\mathcal{M}$ contains a copy of $\mathcal{S}_\mathcal{F}$, and
(b) the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2

never exceeds $1 - 1(1+\alpha)$, where α is the number of permitted structures with universe $\{1, \ldots, |\text{rng}(\bar{a})|\}$. If $\mathbf{K}$ has the free amalgamation property (which is assumed in Theorem 3.4), then part (ii) of Theorem 3.17 is applicable, and it follows that the proportion of structures in $\mathbf{K}$ which satisfy all $(2|P| - |\text{rng}(\bar{a})| - 1)$-extension axioms never exceeds $1 - 1/(1+\alpha)$. And if the language has no unary relation symbols and $\mathbf{K}$ has the free amalgamation property, then this proportion approaches 0 as $n \to \infty$, by part (iii) of Theorem 3.17. Note that $\|\mathcal{F}_{min}\| = |P|$, so Theorem 3.4 is proved.

Example 4.2. This example shows that when, in Theorem 3.17, it is assumed that the language has no unary relation symbols, then this assumption is necessary. (The author does not have a corresponding example if one adds the assumption that $\mathbf{K}$ has the free amalgamation property, as in Theorem 3.4 and 3.18.)

Let P_1 and P_2 be unary relation symbols and let L be a language the vocabulary of which is finite, relational and contains P_1 and P_2. For $n \in \mathbb{N}$, let $\mathbf{K}_n$ consist of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that

$$\begin{aligned}
&\text{at most one element in } M \text{ satisfies } P_1(x),\\
&\text{at most one element in } M \text{ satisfies } P_2(x), \text{ and}\\
&\mathcal{M} \models \neg\exists x, y\big(P_1(x) \wedge P_2(y)\big).
\end{aligned}$$

$\mathbf{K}_n$ can also be described in the following way, by forbidden weak substructures. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have universe $\{1,2\}$ and the following interpretations: $(P_1)^\mathcal{A} = \{1\}$, $(P_2)^\mathcal{A} = \{2\}$, $(P_1)^\mathcal{B} = \{1,2\}$, $(P_2)^\mathcal{B} = \emptyset$, $(P_1)^\mathcal{C} = \emptyset$, $(P_2)^\mathcal{C} = \{1,2\}$ and $R^\mathcal{A} = R^\mathcal{B} = R^\mathcal{C} = \emptyset$ for every other relation symbol R in the vocabulary.

Then $\mathbf{K}_n$ can also be described as the set of all L-structures $\mathcal{M}$ such that no $\mathcal{F} \in \mathbf{F} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ is weakly embeddable in $\mathcal{M}$. Note that $\mathbf{F}$ satisfies the condition labelled $(*)$ in Theorem 3.4, so if α is the number of permitted structures with universe $\{1\}$, then the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all 3-extension axioms never exceeds $1 - 1/(1+\alpha)$. We have $\alpha \geq 3$, and if the only unary relation symbols of L are P_1 and P_2 then $\alpha = 3$.

Next, we show that the the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all 0-extension axioms (i.e. $\mathcal{N}/\emptyset$-extension axioms with N a singleton set) never exceeds 1/2. For every n, $\mathbf{K}_n$ can be partitioned into three parts: one part, $\mathbf{X}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_1(x)$; another part, $\mathbf{Y}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_2(x)$; and a third part, $\mathbf{Z}_n$, consisting of all $\mathcal{M} \in \mathbf{K}_n$ which do not satisfy any of $\exists x P_1(x)$ or $\exists x P_2(x)$. The definition of $\mathbf{K}_n$ implies that, for each n, $|\mathbf{X}_n| = |\mathbf{Y}_n| = n|\mathbf{Z}_n|$. Let $\mathcal{A}' = \mathcal{A}\restriction\{1\}$. Then $\mathcal{A}'$ is permitted and $\mathcal{A}' \models P_1(1)$. Moreover, for every n, the $\mathcal{A}'/\emptyset$-extension axiom holds exactly for those $\mathcal{M} \in \mathbf{K}_n$ which belong to $\mathbf{X}_n$, and we have

$$\frac{|\mathbf{X}_n|}{|\mathbf{K}_n|} = \frac{|\mathbf{X}_n|}{|\mathbf{X}_n| + |\mathbf{Y}_n| + |\mathbf{Z}_n|} = \frac{n|\mathbf{Z}_n|}{n|\mathbf{Z}_n| + n|\mathbf{Z}_n| + |\mathbf{Z}_n|} = \frac{1}{2 + 1/n} \;\to\; \frac{1}{2} \quad \text{as } n \to \infty.$$

Example 4.3. (**Graph with a restricted unary predicate**) Let the vocabulary of L consist of a unary relation symbol Q and a binary relation symbol R. Let $\mathbf{K}_n$ be the set of L structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that $R^\mathcal{M}$ is irreflexive and symmetric (i.e. an undirected graph) and

$$\mathcal{M} \models \forall x, y\big(R(x,y) \to (\neg Q(x) \wedge \neg Q(y))\big).$$

We use notation which suggests how Theorem 3.17 will be used. Define $\mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{F}$, $\mathcal{P}$ and $\mathcal{F}$ as follows: let $|\mathcal{S}_\mathcal{P}| = |\mathcal{S}_\mathcal{F}| = \{a\}$; $Q^{\mathcal{S}_\mathcal{P}} = R^{\mathcal{S}_\mathcal{P}} = \emptyset$; $Q^{\mathcal{S}_\mathcal{F}} = \{a\}$, $R^{\mathcal{S}_\mathcal{F}} = \emptyset$; $|\mathcal{P}| = \{a, b\}$, $Q^\mathcal{P} = \emptyset$, $R^\mathcal{P} = \{(a,b),(b,a)\}$; and $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$. Then $\mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{F}$ and $\mathcal{P}$ are permitted, but $\mathcal{F}$ is forbidden, since $\mathcal{F} \models Q(a) \wedge R(a,b)$. Hence, the substitution $[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is not admitted, but the reverse substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted, because we can always remove a Q-relationship without producing a forbidden structure.

By Theorem 3.17 (i), the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_\mathcal{F}$ and whose $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity is at least two is not larger than $1 - 1/(1+2) = 2/3$. In this example we can do much better, asymptotically speaking, and show that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_\mathcal{F}$, or equivalently, which satisfy $\exists x Q(x)$, approaches 0 as $n \to \infty$. We can argue as follows to see this. First let

$$\mathbf{X}_n = \big\{\mathcal{M} \in \mathbf{K}_n : \ Q(x) \text{ is satisfied by at least two elements in } |\mathcal{M}|\big\},$$
$$\mathbf{Y}_n = \big\{\mathcal{M} \in \mathbf{K}_n : \ Q(x) \text{ is satisfied by a unique element in } |\mathcal{M}|\big\}.$$

Since

$$\frac{|\mathbf{Y}_n|}{|\mathbf{K}_n|} = \frac{n 2^{\binom{n-1}{2}}}{2^{\binom{n}{2}}} = \frac{n}{2^{n-1}} \to 0,$$

as $n \to \infty$, it is sufficient to show that $|\mathbf{X}_n| \leq |\mathbf{Y}_n|$. For $\mathcal{M} \in \mathbf{X}_n$, let $a \in |\mathcal{M}| = \{1, \ldots, n\}$ be minimal such that $\mathcal{M} \models Q(a)$, and let $\mathcal{M}'$ be defined as follows: $|\mathcal{M}'| = \{1, \ldots, n\}$, $Q^{\mathcal{M}'} = \{a\}$ and let $R^{\mathcal{M}'}$ be the symmetric closure of

$$R^\mathcal{M} \cup \big\{(b,c) : b \in Q^\mathcal{M} - \{a\}, \quad c \in \{1, \ldots, n\} - Q^\mathcal{M}\big\}.$$

Note that $\mathcal{M}' \in \mathbf{Y}_n$. It is now easy to verify that the map $\mathcal{M} \mapsto \mathcal{M}'$ from $\mathbf{X}_n$ to $\mathbf{Y}_n$ is injective; thus $|\mathbf{X}_n| \leq |\mathbf{Y}_n|$.

Since $\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \neg\exists Q(x)\}$ is the set of all (undirected) graphs with vertices $1, \ldots, n$ it follows that, with the uniform probability measure, the almost sure theory of $\mathbf{K}$ is identical to the almost sure theory of all undirected graphs, and consequently $\mathbf{K}$ has a zero-one law for the uniform probability measure. Since the complete theory of the Fraïssé-limit of $\mathbf{K}$ contains the sentence $\exists x Q(x)$ it is different from the almost sure theory of $\mathbf{K}$, for the uniform measure. As we will see later, for the 'dimension conditional probability measure' (where dimension equals cardinality in this example), the almost sure theory of $\mathbf{K}$ is identical to the complete theory of the Fraïssé-limit of $\mathbf{K}$.

Example 4.4. (**Partially coloured binary relation.**) Let the vocabulary of L consist of one binary relation symbol R and two unary relation symbols P_1, P_2. $\mathbf{K}_n$ consists of all L-structures $\mathcal{M}$ with universe $\{1, \ldots, n\}$ such that

$$\mathcal{M} \models \forall x \neg\big(P_1(x) \wedge P_2(x)\big), \quad \text{and}$$
$$\mathcal{M} \models \forall x, y\big(R(x,y) \ \to \ \big[\neg\big(P_1(x) \wedge P_1(y)\big) \ \wedge \ \neg\big(P_2(x) \wedge P_2(y)\big)\big]\big).$$

We can think of P_i as representing the colour 'i'. Before using Theorem 3.17 to get some information about $\mathbf{K}_n$ we consider the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy $\exists x P_i(x)$. Let

$$\mathbf{X}_n = \big\{\mathcal{M} \in \mathbf{K}_n : (P_1)^\mathcal{M} = \emptyset\big\}.$$

For every $\mathcal{M} \in \mathbf{X}_n$, let

$$\mathbf{Y}_n(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}_n : R^\mathcal{N} = R^\mathcal{M}, \ (P_1)^\mathcal{N} \neq \emptyset\big\},$$

and note that $|\mathbf{Y}_n(\mathcal{M})| \geq n$, by considering the case $|(P_1)^\mathcal{N}| = 1$. Moreover, if $\mathcal{M}, \mathcal{M}' \in \mathbf{X}_n$ are different, then $\mathbf{Y}_n(\mathcal{M})$ is disjoint from $\mathbf{Y}_n(\mathcal{M}')$. Hence, there are at least n as many $\mathcal{M} \in \mathbf{K}_n$ satisfying $\exists x P_1(x)$ as there are $\mathcal{N} \in \mathbf{K}_n$ not satisfying $\exists x P_1(x)$.

Therefore the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that $\mathcal{M} \models \exists x P_1(x)$ approaches 1 as $n \to \infty$. The same argument works for P_2.

For an L-structure $\mathcal{M}$ and $a \in |\mathcal{M}|$, let us say that a is *blank* or *uncoloured* (in $\mathcal{M}$) if $\mathcal{M} \models \neg P_1(a) \wedge \neg P_2(a)$. Let $\mathcal{S}_\mathcal{P}$ have universe $\{a\}$ where a is blank in $\mathcal{S}_\mathcal{P}$ and $R^{\mathcal{S}_\mathcal{P}} = \emptyset$. Let $\mathcal{S}_\mathcal{F}$ also have universe $\{a\}$ where a has colour 1 in $\mathcal{S}_\mathcal{F}$ (i.e. $\mathcal{S}_\mathcal{F} \models P_1(a)$) and $R^{\mathcal{S}_\mathcal{F}} = \emptyset$. Then $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted and it is easily seen that the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted, because making a point blank never violates the conditions for being permitted (with respect to $\mathbf{K}$). But if one point in an R-relationship is coloured by i, then colouring the other point in the same R-relationship by the same colour i produces a forbidden structure; so the substitution $[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is not admitted. Now we apply Theorem 3.17. Let $|\mathcal{P}| = \{a, b\}$, $(P_1)^\mathcal{P} = \{b\}$ and $R^\mathcal{P} = \{(a, b)\}$. Since we now that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain a copy of $\mathcal{S}_\mathcal{F}$ (i.e satisfy $\exists x P_1(x)$) approaches 1 as $n \to \infty$, it follows that for arbitrarily small $\varepsilon > 0$ and all sufficiently large n, the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2 never exceeds $\big(1 - 1/(1 + 3)\big) + \varepsilon = 3/4 + \varepsilon$. Observe that the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2 if and only if $\mathcal{M}$ satifies the extension axiom

$$\varphi = \forall x \exists y, z\big([\neg P_1(x) \wedge \neg P_2(x)] \ \to \ [R(x, y) \wedge R(x, z) \wedge P_1(y) \wedge P_1(z)]\big).$$

Next we show that the for every *positive* $k \in \mathbb{N}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which have at least $k + 1$ elements satisfying P_1 does not exceed $1/2^{k-1}$; this will point out a difference compared to Example 4.5 below. For positive integers k, let Let $A \subseteq \{1, \ldots, n\}$ be nonempty and let

$$\mathbf{Z}_{n,k} = \big\{\mathcal{M} \in \mathbf{K}_n : k < \big|(P_1)^\mathcal{M}\big|\big\},$$
$$\mathbf{Z}'_{n,k} = \big\{\mathcal{M} \in \mathbf{K}_n : k = \big|(P_1)^\mathcal{M}\big|\big\}.$$

Observe that for every $\mathcal{M} \in \mathbf{Z}_{n,k}$ there are $\mathcal{M}_1 = f_1(\mathcal{M})$ and $\mathcal{M}_2 = f_2(\mathcal{M})$ such that

$(P_1)^{\mathcal{M}_1}$ is the set of (exactly) the k smallest members of $(P_1)^\mathcal{M}$,

$(P_2)^{\mathcal{M}_1} = (P_2)^\mathcal{M}$,

$R^{\mathcal{M}_1} = R^\mathcal{M} \cup \big\{(a, b) : a \text{ is the smallest member of } (P_1)^{\mathcal{M}_1},\ b \in (P_1)^\mathcal{M} - (P_1)^{\mathcal{M}_1}\big\}$,

$(P_1)^{\mathcal{M}_2}$ is the set of (exactly) the k smallest members of $(P_1)^\mathcal{M}$,

$(P_2)^{\mathcal{M}_2} = (P_2)^\mathcal{M}$, and

$R^{\mathcal{M}_2} = R^\mathcal{M} \cup \big\{(a, b) : a \in (P_1)^{\mathcal{M}_2},\ b \in (P_1)^\mathcal{M} - (P_1)^{\mathcal{M}_2}\big\}$.

It is straightforward to check that both maps $f_i : \mathbf{Z}_{n,k} \to \mathbf{Z}'_{n,k}$ are injective. Moreover, if $k \geq 2$, then clearly $f_1(\mathcal{M}) \neq f_2(\mathcal{M})$ for every $\mathcal{M} \in \mathbf{Z}_{n,k}$. It follows that if $k \geq 2$, then $2|\mathbf{Z}_{n,k}| \leq |\mathbf{Z}'_{n,k}|$. Thus, for every $k \geq 2$ we have

$$|\mathbf{Z}_{n,1}| \geq |\mathbf{Z}'_{n,2}| \geq 2|\mathbf{Z}_{n,2}| \geq 2|\mathbf{Z}'_{n,3}| \geq 2 \cdot 2|\mathbf{Z}_{n,3}| \geq \ldots \geq 2^{k-1}|\mathbf{Z}_{n,k}|,$$

and hence

$$|\mathbf{Z}_{n,k}| \leq \frac{|\mathbf{Z}_{n,1}|}{2^{k-1}} \quad \text{and therefore} \quad \frac{|\mathbf{Z}_{n,k}|}{|\mathbf{K}_n|} \leq \frac{|\mathbf{Z}_{n,k}|}{|\mathbf{Z}_{n,1}|} \leq \frac{1}{2^{k-1}}.$$

In other words, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which have at least $k + 1$ elements which satisfy P_1 is at most $1/2^{k-1}$.

Consequently, for any $f : \mathbb{N} \to \mathbb{R}$ which tends to ∞ as $n \to \infty$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which have at least $f(n)$ elements with colour 1 approaches 0 as $n \to \infty$. Since the same argument works for the colour 2 and we can take $f(n) = \sqrt{n}$ (for example), it follows that for arbitrarily small $\varepsilon > 0$ there is n_ε such that for all $n > n_\varepsilon$ the proportion of $\mathcal{M} \in \mathbf{K}_n$ such that

$$\frac{\text{number of nonblank elements in } \mathcal{M}}{\text{number of blank elements in } \mathcal{M}} > \varepsilon$$

is less than ε. Finally we point out that, for the extension axiom φ defined above, the proportion of $\mathcal{M} \in \mathbf{K}_n$ in which φ is true actually approaches 0, as $n \to \infty$. The reason for this is that, as we have shown, the proportion of $\mathcal{M} \in \mathbf{K}_n$ in which the number of blank elements is at least $n - 2\sqrt{n}$ approaches 1 as $n \to \infty$. Since $\lim_{n\to\infty}(n - 2\sqrt{n}) = \infty$, one can argue similarly as in the proof of part (iii) of Theorem 3.17 (although in this example, we have unary relation symbols) and conclude that the proportion of $\mathcal{M} \in \mathcal{K}_n$ such that $\mathcal{M} \models \varphi$ approaches 0 as n tends to ∞.

Example 4.5. (**Coloured binary relation.**) Let $\mathbf{K}_n$ be defined as in Example 4.4 *except* that we add the condition that there are *no* blank elements, that is, every $\mathcal{M} \in \mathbf{K}_n$ satisfies $\forall x\big(P_1(x) \vee P_2(x)\big)$. By Proposition 9.3, there is a constant $0 < \mu < 1$ such that, for $i = 1, 2$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ with at least μn elements with colour i approaches 1 as $n \to \infty$. Hence, Theorem 9.2 implies that for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ approaches 1 as $n \to \infty$. Since $\mathbf{K}$ has the hereditary property and the free amalgamation property, Lemma 3.16 and Theorem 3.17 (part (ii)) implies that there does *not* exist permitted $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{F}}$ such that the substitution $[\mathcal{S}_{\mathcal{F}} \triangleright \mathcal{S}_{\mathcal{P}}]$ is admitted and $[\mathcal{S}_{\mathcal{P}} \triangleright \mathcal{S}_{\mathcal{F}}]$ is not admitted. However, since changing one colour to another in a permitted structure may produce a forbidden structure, there are permitted $\mathcal{A}$ and $\mathcal{A}'$ (with singleton universes) such that *none* of the substitutions $[\mathcal{A} \triangleright \mathcal{A}']$ and $[\mathcal{A}' \triangleright \mathcal{A}]$ is admitted.

The last two examples (as well as Example 4.5) show that if $\mathbf{K}$ neither satisfies the conditions of Theorem 3.15, nor the conditions of Theorem 3.17 (or Corollary 3.18), then it may, or may not, be the case that for every extension axiom φ of $\mathbf{K}$ the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. In contrast to examples 4.2 – 4.5, the last two examples of this section do not have any unary relations.

Example 4.6. (**Complete bipartite graph.**) For all $r, s \in \mathbb{N}$, let $\mathcal{K}_{r,s}$ denote the undirected graph with vertices $a_1, \ldots, a_r, b_1, \ldots, b_s$ and an edge connecting a_i and b_j for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, and no other edges. $\mathcal{K}_{0,s}$ and $\mathcal{K}_{r,0}$ are independent sets (no edges at all) with s and r vertices, respectively.

For every $n \in \mathbb{N}$, let $\mathbf{K}_n$ be the set of all graphs with vertices $1, \ldots, n$ which are isomorphic to $\mathcal{K}_{r,s}$ for some r, s. Clearly, by adding an edge to any represented $\mathcal{M}$ with at least 3 nodes, we create a forbidden graph. Also, by removing an edge from any $\mathcal{K}_{r,s}$ such that $r + s \geq 3$ and $\min(r, s) \geq 1$, we create a forbidden graph.

It is easy to see that if $s, r \geq k + 1$, then $\mathcal{K}_{r,s}$ satisfies all k-extension axioms of $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. Also, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are isomorphic to some $\mathcal{K}_{r,s}$ with $r, s \geq k + 1$ approaches 1 as $n \to \infty$. It follows that, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. It is straightforward to verify that the class of represented structures is closed under taking substructures (so 'permitted' is the same as 'represented') and has the free amalgamation property. By Corollary 3.18, there does not exist any permitted $\mathcal{A}$ and $\mathcal{B}$ with $A = B$ such that $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted and $[\mathcal{B} \triangleright \mathcal{A}]$ is not admitted.

Example 4.7. (N**-free bipartite graph.**) Let $\mathcal{N}$ be the graph with vertices $1, 2, 3, 4$ where $\{i, i + 1\}$ is an edge for $i = 1, 2, 3$ and there are no other edges. Let $\mathbf{K}_n$ consist of all bipartite graphs with vertices $1, \ldots, n$ in which $\mathcal{N}$ cannot be embedded.

From the assumption that $\mathcal{N}$ is not embeddable into any represented $\mathcal{M}$ it follows easily that every represented $\mathcal{M}$ is isomorphic to a disjoint union of graphs of the form $\mathcal{K}_{r,s}$ (with no other edges than those occuring within some copy of $\mathcal{K}_{r,s}$). Note that if we remove an edge from a represented structure, then we may create a forbidden structure; and if we add an edge to a represented structure, then we may also create a forbidden structure. Clearly $\mathbf{K}$ has the hereditary property, and it is straightforward to verify that $\mathbf{K}$ has the free amalgamation property (which would no longer be true if we would omit the requirement about bipartiteness [7]). By the use of Lemma 3.16 and the fact that the language has only one relation symbol one easily verifies that there does not exist permitted $\mathcal{A}$ and $\mathcal{B}$ such that the substitution $[\mathcal{A} \triangleright \mathcal{B}]$ is admitted but $[\mathcal{B} \triangleright \mathcal{A}]$ is not admitted.

We now show that if $\mathcal{A}$ is the graph consisting of only one vertex a and $\mathcal{B}$ has edge set $\{a, b\}$ and a is adjacent to b, then the probability that $\mathcal{M} \in \mathbf{K}_n$ satisfies the $\mathcal{B}/\mathcal{A}$–extension axiom approaches 0 as $n \to \infty$. Let $\mathbf{X}_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ which do *not* contain any connected component which is a singleton, and let $\mathbf{X} = \bigcup_{n>1} \mathbf{X}_n$. Note that the class of represented structures with respect to $\mathbf{X}$ are closed under taking disjoint unions and extracting connected components; thus, the class of represented structures with respect to $\mathbf{C}$ is *adequate* in the sense of [4], which we will use. Let $\mathbf{C}_n$ be the set of connected represented structures with respect to $\mathbf{C}$ which have universe $\{1, \ldots, n\}$; so every member of $\mathbf{C}_n$ is isomorphic to $\mathbf{K}_{r,s}$ for some $r, s \geq 1$ with $r + s = n$. We have $|\mathbf{C}_n| = 2^{n-1} - 1$; because every $\mathcal{M} \in \mathbf{C}_n$ is determined by a partition of $\{1, \ldots, n\}$ into two nonempty parts. Since $|\mathbf{C}_n| = 2^{n-1} - 1 = O(n^{n/2})$, it follows from Theorem 7 in [4] that

$$\frac{n|\mathbf{X}_{n-1}|}{|\mathbf{X}_n|} \to \infty \quad \text{as} \quad n \to \infty.$$

Let $\mathbf{Y}_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ that contain at least one connected component which is a singleton, and let $\mathbf{Y}'_n$ be the set of $\mathcal{M} \in \mathbf{K}_n$ that contain exactly one connected component which is a singleton. Observe that

$$\mathbf{X}_n = \mathbf{K}_n - \mathbf{Y}_n \text{ and } |\mathbf{Y}'_n| = n|\mathbf{X}_{n-1}|.$$

It follows that

$$\frac{|\mathbf{X}_n|}{|\mathbf{K}_n|} \leq \frac{|\mathbf{X}_n|}{|\mathbf{Y}'_n|} = \frac{|\mathbf{X}_n|}{n|\mathbf{X}_{n-1}|} \to 0 \quad \text{as} \quad n \to \infty.$$

In other words, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which contain at least one connected component which is a singleton approaches 1 as $n \to \infty$. But for every such $\mathcal{M}$, the $\mathcal{B}/\mathcal{A}$–extension axiom fails. Nevertheless, $\mathbf{K}$ has a zero-one law for the uniform probability measure, which follows from Theorem 7 in [4] and the above observed fact that $|\mathbf{C}_n| = 2^{n-1} - 1 = O(n^{n/2})$.

Finally we mention that $\mathbf{K}_n$ could alternatively have been defined as the set of all graphs with universe $\{1, \ldots, n\}$ in which neither $\mathcal{N}$ nor a "triangle" (i.e. 3-clique) can be embedded. Using this definition one can show, by induction, that no represented structure contains an odd cycle; so every represented structure is bipartite.

5. Proof of Theorem 3.17

Let L have a finite relational vocabulary and let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$, where every $\mathbf{K}_n$ is a set of L-structures with universe $\{1, \ldots, m_n\}$ and $\lim_{n \to \infty} m_n = \infty$. Suppose that $\mathcal{P}$, $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ are permitted structures such that $\mathcal{S}_\mathcal{P} \subseteq \mathcal{P}$, $|\mathcal{S}_\mathcal{P}| = |\mathcal{S}_\mathcal{F}|$, $\|\mathcal{S}_\mathcal{P}\| = k$, $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is forbidden, but the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted. Morover, assume that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P} \restriction |\mathcal{U}| = \mathcal{S}_\mathcal{F} \restriction |\mathcal{U}|$. Let α be the number of different permitted structures with universe $\{1, \ldots, k\}$ (so $\alpha \geq 2$).

We use the following terminology:

Definition 5.1. (i) A pair of structures $(\mathcal{A}, \mathcal{B})$ is called a ***coexisting pair*** if $\mathcal{A}$ and $\mathcal{B}$ have the same universe.
(ii) We say that two coexisting pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are ***isomorphic*** if there is a

bijection $\sigma : |\mathcal{A}| \to |\mathcal{A}'|$ which is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ as well as from $\mathcal{B}$ to $\mathcal{B}'$.
(iii) If $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are isomorphic coexisting pairs then we may say that $(\mathcal{A}', \mathcal{B}')$ is a ***copy*** of $(\mathcal{A}, \mathcal{B})$.

Lemma 5.2. *Suppose that $\mathcal{S}_\mathcal{P}$ is a proper substructure of $\mathcal{P}$ and that $\mathcal{M}$ is represented. If $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ is a copy of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ and $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$, then the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ is 0.*

Proof. Without loss of generality (by just renaming elements) we may assume that $\mathcal{S}_\mathcal{P} = \mathcal{S}_\mathcal{P}^\mathcal{M} \subseteq \mathcal{M}$ and that $\mathcal{S}_\mathcal{F} = \mathcal{S}_\mathcal{F}^\mathcal{M}$. We show that the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is 0. Suppose for a contradiction that it is at least 1. Without loss of generality, we may assume that $\mathcal{P} = \mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is a substructure of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$, so in particular, the common universe of $\mathcal{F}$ and $\mathcal{P} = \mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is a subset of the universe of $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ and of $\mathcal{M}$. For each relation symbol R, of arity r say, we consider the interpretation of R in $\mathcal{M}{\restriction}|\mathcal{F}|$. If $\bar{a} \in |\mathcal{S}_\mathcal{F}|^r$, then

$$\bar{a} \in R^{\mathcal{M}\restriction F} \iff \bar{a} \in R^{\mathcal{S}_\mathcal{F}} \iff \bar{a} \in R^\mathcal{F} \quad (\text{since } \mathcal{S}_\mathcal{F} \subset \mathcal{F}).$$

If $\bar{a} \in |\mathcal{F}|^r - |\mathcal{S}_\mathcal{F}|^r$, then we use the definition of substitutions (Definition 3.11) and get

$$\bar{a} \in R^{\mathcal{M}\restriction F} \iff \bar{a} \in R^\mathcal{M} \iff \bar{a} \in R^{\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]}$$
$$\iff \bar{a} \in R^{\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]\restriction F} \iff \bar{a} \in R^{\mathcal{F}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]} \iff \bar{a} \in R^\mathcal{F}.$$

So whenever $\bar{a} \in |\mathcal{F}|^r$ we have $\bar{a} \in R^\mathcal{M}$ if and only if $\bar{a} \in R^\mathcal{F}$. Since the argument holds for every relation symbol R it follows that the forbidden structure $\mathcal{F}$ is a substructure of $\mathcal{M}$, which contradicts that $\mathcal{M}$ is represented. □

Definition 5.3. Let the expression 'mult$(\mathcal{A}/\mathcal{B}; \mathcal{M}) \geq n$' mean 'the $\mathcal{A}/\mathcal{B}$-multiplicity of $\mathcal{M}$ is at least n'.

Lemma 5.4. *Suppose that $\mathcal{M}, \mathcal{N} \in \mathbf{K}_n$ are different and that* mult$(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{M}) \geq 2$ *and* mult$(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{N}) \geq 2$. *Let $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ be copies of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ such that $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N} \subseteq \mathcal{N}$. If $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}] = \mathcal{N}[\mathcal{S}_\mathcal{F}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{N}]$ then $\mathcal{S}_\mathcal{F}^\mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N}$ have the same universe U and $\mathcal{M}$ and $\mathcal{N}$ are different only on U (that is, for every relation symbol R, if $\bar{a}$ belongs to exactly one of the relations $R^\mathcal{M}$ and $R^\mathcal{N}$, then $\bar{a} \in U$.)*

Proof. Let $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ be copies of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ such that $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{N} \subseteq \mathcal{N}$. Then there are maps $\sigma_\mathcal{M} : |\mathcal{S}_\mathcal{F}^\mathcal{M}| \to |\mathcal{S}_\mathcal{F}|$ and $\sigma_\mathcal{N} : |\mathcal{S}_\mathcal{F}^\mathcal{N}| \to |\mathcal{S}_\mathcal{N}|$ such that:

- $\sigma_\mathcal{M}$ is an isomorphism from $\mathcal{S}_\mathcal{F}^\mathcal{M}$ to $\mathcal{S}_\mathcal{F}$ and from $\mathcal{S}_\mathcal{P}^\mathcal{M}$ to $\mathcal{S}_\mathcal{P}$, and
- $\sigma_\mathcal{N}$ is an isomorphism from $\mathcal{S}_\mathcal{F}^\mathcal{N}$ to $\mathcal{S}_\mathcal{F}$ and from $\mathcal{S}_\mathcal{P}^\mathcal{N}$ to $\mathcal{S}_\mathcal{P}$.

Let $\{a_1, \ldots, a_k\}$ be the universe of $\mathcal{S}_\mathcal{F}^\mathcal{M}$ (and of $\mathcal{S}_\mathcal{P}^\mathcal{M}$) and let $\{b_1, \ldots, b_k\}$ be the universe of $\mathcal{S}_\mathcal{F}^\mathcal{N}$ (and of $\mathcal{S}_\mathcal{P}^\mathcal{N}$).

Suppose, for a contradiction, that

(I) $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}] = \mathcal{H} = \mathcal{N}[\mathcal{S}_\mathcal{F}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{N}]$ and that
(II) $\{a_1, \ldots, a_k\} \neq \{b_1, \ldots, b_k\}$.

Then

$$\mathcal{M} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{M}] \quad \text{and} \quad \mathcal{N} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}]. \tag{1}$$

Recall the assumption that for every proper substructure $\mathcal{U} \subset \mathcal{S}_\mathcal{P}$, $\mathcal{S}_\mathcal{P}{\restriction}|\mathcal{U}| = \mathcal{S}_\mathcal{F}{\restriction}|\mathcal{U}|$. Since $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ and $(\mathcal{S}_\mathcal{P}^\mathcal{N}, \mathcal{S}_\mathcal{F}^\mathcal{N})$ are copies of $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$, it follows that $\mathcal{S}_\mathcal{P}^\mathcal{M}$ and $\mathcal{S}_\mathcal{F}^\mathcal{M}$ agree on all proper subsets of their common universe; and the same with $\mathcal{M}$ replaced by $\mathcal{N}$. From (1) it follows that

$$\text{if } U \subseteq \{1, \ldots, m_n\} \text{ and } |U| < k, \text{ then } \mathcal{M}{\restriction}U = \mathcal{H}{\restriction}U = \mathcal{N}{\restriction}U. \tag{2}$$

Since $\mathcal{H}\restriction\{b_1,\dots,b_k\} = \mathcal{S}_\mathcal{P}^\mathcal{N}$ and $\mathcal{M}$ is obtained from $\mathcal{H}$ by the substitution $\mathcal{M} = \mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{M}]$, which *only* affects the interpretations of relation symbols on $\{a_1,\dots,a_k\}$, assumption (II) together with (2) implies that

$$\mathcal{M}\restriction\{b_1,\dots,b_k\} = \mathcal{S}_\mathcal{P}^\mathcal{N}.$$

Since the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2, there are $\mathcal{P}_i \subseteq \mathcal{M}$ and isomorphisms $\sigma_i : \mathcal{P}_i \to \mathcal{P}$ such that $\mathcal{S}_\mathcal{P}^\mathcal{N} \subset \mathcal{P}_i$, $\sigma_i\restriction|\mathcal{S}_\mathcal{P}^\mathcal{N}| = \sigma_\mathcal{N}$, for $i = 1, 2$, and $|\mathcal{P}_1| \cap |\mathcal{P}_2| = \{b_1,\dots,b_k\} = |\mathcal{S}_\mathcal{P}^\mathcal{N}|$. By assumption (I), $\mathcal{H}$ is obtained from $\mathcal{M}$ by the substitution $\mathcal{H} = \mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ which only affects the interpretations of relation symbols on $\{a_1,\dots,a_k\}$. This together with (II), (2) and the choice of $\mathcal{P}_1$ and $\mathcal{P}_2$ so that $|\mathcal{P}_1| \cap |\mathcal{P}_2| = \{b_1,\dots,b_k\}$ implies that for $i = 1$ or $i = 2$, $\mathcal{H}\restriction|\mathcal{P}_i| = \mathcal{P}_i$. Choose i so that

$$\mathcal{H}\restriction|\mathcal{P}_i| = \mathcal{P}_i. \tag{3}$$

Since $\mathcal{S}_\mathcal{P}^\mathcal{N} \subset \mathcal{P}_i$ and $\sigma_i : \mathcal{P}_i \to \mathcal{P}$ is an isomorphism such that $\sigma_i\restriction|\mathcal{S}_\mathcal{P}^\mathcal{N}| = \sigma_\mathcal{N}$, the substitution $[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}]$ changes $\mathcal{P}_i$ to a structure which is isomorphic with $\mathcal{F}$, that is, $\mathcal{P}_i[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}] \cong \mathcal{F}$, via the isomorphism σ_i. By applying (1) and (3) we get

$$\mathcal{N}\restriction|\mathcal{P}_i| = (\mathcal{H}[\mathcal{S}_\mathcal{P}^\mathcal{N} \triangleright \mathcal{S}_\mathcal{F}^\mathcal{N}])\restriction|\mathcal{P}_i| \cong \mathcal{F}.$$

Hence the substructure of $\mathcal{N}$ with universe $|\mathcal{P}_i|$ is isomorphic to the forbidden structure $\mathcal{F}$, Therefore $\mathcal{N}$ is not represented, which contradicts that $\mathcal{N} \in \mathbf{K}_n$.

So if (I) holds then (II) is false and hence all the structures $\mathcal{S}_\mathcal{F}^\mathcal{M}$, $\mathcal{S}_\mathcal{P}^\mathcal{M}$, $\mathcal{S}_\mathcal{F}^\mathcal{N}$ and $\mathcal{S}_\mathcal{P}^\mathcal{N}$ have the same universe, say U. Consequently, from the assumption (I), if R is a relation symbol of arity r, say, and $\bar{a} \in \{1,\dots,m_n\}^r$ belongs to exactly one of $R^\mathcal{M}$ and $R^\mathcal{N}$, then $\bar{a} \in U$. □

Definition 5.5. (i) For every L-structure $\mathcal{M}$, let $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})$ denote the set of all structures of the form $\mathcal{M}[\mathcal{S}_\mathcal{F}^\mathcal{M} \triangleright \mathcal{S}_\mathcal{P}^\mathcal{M}]$ where $(\mathcal{S}_\mathcal{P}^\mathcal{M}, \mathcal{S}_\mathcal{F}^\mathcal{M})$ is a copy of the coexisting pair $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ and $\mathcal{S}_\mathcal{F}^\mathcal{M} \subseteq \mathcal{M}$. (If $\mathcal{M}$ contains no copy of $\mathcal{S}_\mathcal{F}$ then $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) = \emptyset$)
(ii) For every n, let $\mathbf{\Omega}_n$ denote the set of all $\mathcal{M} \in \mathbf{K}_n$ such that $\text{mult}(\mathcal{P}/\mathcal{S}_\mathcal{P}; \mathcal{M}) \geq 2$.
(iii) Let α be the number of different permitted L-structures with universe $\{1,\dots,k\}$.

Lemma 5.6. *If $\mathcal{M}_1,\dots,\mathcal{M}_{\alpha+1} \in \mathbf{\Omega}_n$ and $\mathcal{M}_i \neq \mathcal{M}_j$ whenever $i \neq j$, then*

$$\bigcap_{1\leq i\leq \alpha+1} \mathbf{\Sigma}(\mathcal{M}_i; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) = \emptyset.$$

In other words, for every structure $\mathcal{N}$, it can belong to $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})$ for at most α distinct $\mathcal{M} \in \mathbf{\Omega}_n$.

Proof. Suppose for a contradiction that $\mathcal{M}_1,\dots,\mathcal{M}_{\alpha+1} \in \Omega_n$ are distinct and that $\mathcal{N} \in \mathbf{\Sigma}(\mathcal{M}_i; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})$ for every $i \in \{1,\dots,\alpha+1\}$. Then there are copies $(\mathcal{S}_\mathcal{P}^{\mathcal{M}_i}, \mathcal{S}_\mathcal{F}^{\mathcal{M}_i})$ of $(\mathcal{S}_\mathcal{P}, \mathcal{S}_\mathcal{F})$ such that $\mathcal{S}_\mathcal{F}^{\mathcal{M}_i} \subseteq \mathcal{M}_i$ and $\mathcal{N} = \mathcal{M}_i[\mathcal{S}_\mathcal{F}^{\mathcal{M}_i} \triangleright \mathcal{S}_\mathcal{P}^{\mathcal{M}_i}]$ for every $i \in \{1,\dots,\alpha+1\}$. By Lemma 5.4, all $\mathcal{S}_\mathcal{F}^{\mathcal{M}_i}$, $i \in \{1,\dots,\alpha+1\}$, have the same universe, which we denote by U, and for every pair $i, j \in \{1,\dots,\alpha+1\}$ of distinct numbers, $\mathcal{M}_i$ and $\mathcal{M}_j$ are different only on U. The assumption that $\mathcal{M}_i \neq \mathcal{M}_j$ if $i \neq j$ now implies that for all distinct $i, j \in \{1,\dots,\alpha+1\}$, $\mathcal{M}_i\restriction U \neq \mathcal{M}_j\restriction U$. Since $|U| = k$, this contradicts the choice of α, being the number of all different permitted L-structures with universe $\{1,\dots,k\}$. □

Now we have the tools for proving part (i) of Theorem 3.17, and then the other parts of the theorem. Let $n \in \mathbb{N}$ and let $\mathbf{\Omega}_n^*$ be the set of all $\mathcal{M} \in \mathbf{K}_n$ such that

(a) $\mathcal{M}$ contains a copy of $\mathcal{S}_\mathcal{F}$, and
(b) the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2.

By (b) and the definition of $\mathbf{\Omega}_n$, $\mathbf{\Omega}_n^* \subseteq \mathbf{\Omega}_n$. Since every $\mathcal{M} \in \mathbf{\Omega}_n^*$ contains a copy of $\mathcal{S}_\mathcal{F}$, it follows that for every $\mathcal{M} \in \mathbf{\Omega}_n^*$, $\mathbf{\Sigma}_n(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) \neq \emptyset$. Since the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is admitted, $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) \subseteq \mathbf{K}_n$ for every $\mathcal{M} \in \mathbf{K}_n$. By Lemma 5.2, for every $\mathcal{M} \in \mathbf{\Omega}_n^*$, $\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) \subseteq \mathbf{K}_n - \mathbf{\Omega}_n^*$. Lemma 5.6 now implies that

$$\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| \geq \Big| \bigcup_{\mathcal{M} \in \mathbf{\Omega}_n^*} \mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}) \Big| \geq \frac{\big|\mathbf{\Omega}_n^*\big|}{\alpha}$$

$$\text{and hence} \quad \alpha\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| \geq \big|\mathbf{\Omega}_n^*\big|.$$

From this we get

$$\frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\big|\mathbf{K}_n\big|} = \frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\big|\mathbf{\Omega}_n^*\big| + \big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|} \geq \frac{\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|}{\alpha\big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big| + \big|\mathbf{K}_n - \mathbf{\Omega}_n^*\big|} = \frac{1}{\alpha + 1}.$$

Thus, the proportion of $\mathcal{M} \in \mathbf{K}_n$ *not* satisfying both (a) and (b) is at least $1/(1+\alpha)$. This concludes the proof of part (i) of Theorem 3.17.

Part (ii) of Theorem 3.17 is a straightforward consequence of part (i). For if there exist a permitted structure $\mathcal{C}$ and embeddings $\sigma_1 : \mathcal{P} \to \mathcal{C}$ and $\sigma_2 : \mathcal{P} \to \mathcal{C}$ such that $\sigma_1(|\mathcal{P}|) \cap \sigma_2(|\mathcal{P}|) = \sigma_1(|\mathcal{S}_\mathcal{P}|)$, $\sigma_1 \restriction |\mathcal{S}_\mathcal{P}| = \sigma_2 \restriction |\mathcal{S}_\mathcal{P}|$ and $\mathcal{M} \in \mathbf{K}_n$ satisfies all $(2\,\|\mathcal{P}\| - k - 1)$-extension axioms, then conditions (a) and (b) in part (i) of Theorem 3.17 are satisfied.

Now we prove part (iii) of Theorem 3.17. Here we have added the assumption that L has no unary relation symbols, so there is a unique (up to isomorphism) permitted structure with a singleton universe. (In fact this is sufficient for what we want to prove.) Let $\mathcal{U} \subset \mathcal{S}_\mathcal{F}$ be such that $\|\mathcal{U}\| = 1$. Note that since $\mathcal{S}_\mathcal{F} \not\cong \mathcal{S}_\mathcal{P}$ (and $|\mathcal{S}_\mathcal{F}| = |\mathcal{S}_\mathcal{F}|$) we have $\|\mathcal{S}_\mathcal{F}\| > 1$. Suppose that $\mathcal{M} \in \mathbf{K}_n$ is such that

(c) $\mathcal{M}$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom, and
(d) the $\mathcal{P}/\mathcal{S}_\mathcal{P}$-multiplicity of $\mathcal{M}$ is at least 2.

Since $\|\mathcal{M}\| = m_n$, there are m_n distinct copies of $\mathcal{U}$ in $\mathcal{M}$. Each one of these copies of $\mathcal{U}$ is, by (c), included in a copy of $\mathcal{S}_\mathcal{F}$, so we get at least m_n/k distinct copies of $\mathcal{S}_\mathcal{F}$ in $\mathcal{M}$. By (d), $\big|\mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})\big| \geq m_n/k$. By Lemma 5.2, *no* $\mathcal{N} \in \mathbf{\Sigma}(\mathcal{M}; \mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P})$ satisfies (d).

Hence, if $\mathbf{E}_n$ is the set of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy both (c) and (d), then, by Lemma 5.6,

$$\big|\mathbf{K}_n - \mathbf{E}_n\big| \geq \frac{m_n\big|\mathbf{E}_n\big|}{k\alpha} \quad \text{and hence} \quad \frac{\big|\mathbf{E}_n\big|}{\big|\mathbf{K}_n\big|} \leq \frac{|\mathbf{E}_n|}{|\mathbf{K}_n - \mathbf{E}_n|} \leq \frac{k\alpha}{m_n}.$$

As $\lim_{n\to\infty} m_n = \infty$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy both (c) and (d) approaches 0 as n approaches ∞. This concludes the proof of Theorem 3.17.

6. Conditional probability measures

In Sections 3 – 5 we saw that a condition that ensures that, for every extension axiom, it is true in almost all suffiently large structures, is that every substitution involving (only) permitted structures is admitted. And if this condition does not hold it may happen that some extension is false in almost all sufficiently large structures. In this section we start to develop a theory of conditional probability measures on finite sets of structures. When using this measure we can include more examples of sets of finite structures for which any extension axiom is almost surely true in all sufficiently large structures under consideration. Such examples include Examples 4.3, 4.4 and 4.5, and more generally, coloured structures and partially coloured structures (as in examples 7.22 – 7.24). But there are other examples, such as $\mathcal{K}_l$-free graphs ($l \geq 3$) which are not included; that is, also with the conditional measures considered here there is an extension axiom which almost surely fails for sufficiently large $\mathcal{K}_l$-free graphs.

Although the uniform probability measure is conceptually simple, it does not necessarily correspond to the probability measure associated with a method for randomly generating a structure of some specified kind. The conditional measures to be considered are more closely related to probability measures associated with random generation of structures of a given kind. This is the first point that will be stressed below, after the next two definitions.

Definition 6.1. Let $\mathbf{C}_0$ and $\mathbf{C}_1$ be finite sets of structures and let $\mathbb{P}_0$ be a probability measure on $\mathbf{C}_0$. Suppose that

(1) for every $\mathcal{A} \in \mathbf{C}_0$ there is at least one $\mathcal{B} \in \mathbf{C}_1$ such that $\mathcal{A} \subseteq_w \mathcal{B}$, and
(2) for every $\mathcal{B} \in \mathbf{C}_1$ there is a *unique* $\mathcal{A} \in \mathbf{C}_0$ such that $\mathcal{A} \subseteq_w \mathcal{B}$. We denote such $\mathcal{A}$ by $\mathcal{B}{\restriction}0$.

Then we define the ***uniformly*** $\mathbb{P}_0$***-conditional probability measure*** $\mathbb{P}_1$ on $\mathbf{C}_1$ as follows:

For every $\mathcal{B} \in \mathbf{C}_1$, the probability of $\mathcal{B}$ in $\mathbf{C}_1$ is

$$\mathbb{P}_1(\mathcal{B}) = \frac{1}{\big|\{\mathcal{B}' \in \mathbf{C}_1 : \mathcal{B}'{\restriction}0 = \mathcal{B}{\restriction}0\}\big|} \cdot \mathbb{P}_0(\mathcal{B}{\restriction}0),$$

and for $\mathbf{X} = \{\mathcal{B}_1, \ldots, \mathcal{B}_n\} \subseteq \mathbf{C}_1$ (where $\mathbf{X}$ is enumerated without repetition)

$$\mathbb{P}_1(\mathbf{X}) = \sum_{i=1}^{n} \mathbb{P}_1(\mathcal{B}_i).$$

Definition 6.2. More generally, assume that $\mathbf{C}_0, \ldots, \mathbf{C}_r$ are finite sets of structures such that, for every $i = 0, \ldots, r-1$, (1) and (2) in Definition 6.1 hold if $\mathbf{C}_0$ and $\mathbf{C}_1$ are replaced by $\mathbf{C}_i$ and $\mathbf{C}_{i+1}$, respectively. Let $\mathbb{P}_0$ denote the uniform probability measure on $\mathbf{C}_0$ (i.e. all elements of $\mathbf{C}_0$ have the same probability $1/|\mathbf{C}_0|$). By induction, define $\mathbb{P}_{i+1}$ to be the uniformly $\mathbb{P}_i$-conditional probability measure, for $i = 0, \ldots, r-1$. We call the probability measure $\mathbb{P}_r$ on $\mathbf{C}_r$, thus obtained, the ***uniformly*** $(\mathbf{C}_0, \ldots, \mathbf{C}_{r-1})$***-conditional probability measure***.

Example 6.3. Let us first illustrate the definitions by considering Example 4.3, where $\mathbf{K}_n$ is the set of graphs with vertices $1, \ldots, n$ (with edge relation represented by R) and a unary relation symbol P subject to the condition: $R(a,b) \iff \neg P(a)$ and $\neg P(b)$. We have proved (see Example 4.3) that with the uniform probability mesure, the probability of $\exists x P(x)$ holding in $\mathcal{M} \in \mathbf{K}_n$ approaches 0 as $n \to \infty$. Next we show that with a naturaly chosen conditional measure, the probability that $\exists x P(x)$ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 1 as $n \to \infty$.

Observe that whenever L_0 is a sublanguage of a languge L and $\mathcal{M}$ is an L-structure then $\mathcal{M}{\restriction}L_0 \subseteq_w \mathcal{M}$. Let L denote the language considered in Example 4.3, with one binary relation symbol R and one unary relation symbol P, and let L_0 be the sublanguage of L whose vocabulary contains only P. For every n, let $\mathbf{K}_n{\restriction}L_0 = \{\mathcal{M}{\restriction}L_0 : \mathcal{M} \in \mathbf{K}_n\}$. Note that for every n, if $\mathbf{C}_0 = \mathbf{K}_n{\restriction}L_0$ and $\mathbf{C}_1 = \mathbf{K}_n$, then conditions (1) and (2) in Definition 6.1 hold. Hence, for every n, the uniformly $(\mathbf{K}_n{\restriction}L_0)$-conditional probability measure on $\mathbf{K}_n$ is well-defined. Now, the claim that the probability, with this measure, that (the extension axiom) $\exists x P(x)$ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 1 as $n \to \infty$, is a consequence of Theorem 7.28. But for this simple example it suffices to observe that the probability of $\mathcal{M} \in \mathbf{K}_n$, with the uniformly $(\mathbf{K}_n{\restriction}L_0)$-conditional measure, is the probability of obtaining $\mathcal{M}$ by the following generating procedure: First go through every $i \in \{1, \ldots, n\}$ and with probability $1/2$ let it satisfy $P(x)$; then take the set $\{i_1, \ldots, i_m\}$ of all vertices which do not satisfy $P(x)$, and for each unordered pair $\{i,j\}$ of elements from $\{i_1, \ldots, i_m\}$ assign an edge to it with probability $1/2$. So the probability that no $i \in \{1, \ldots, n\}$ satisfies $P(x)$ is $1/2^n$, which approaches 0 as $n \to \infty$.

Example 6.4. Let us now consider Example 4.4 (partially coloured binary relation), where the vocabulary of L is $\{R, P_1, P_2\}$, R is binary and P_i, $i = 1, 2$, are unary, and thought of as "colours". $\mathbf{K}_n$ consists of all structures with universe $\{1, \ldots, n\}$ such that the universe is partially coloured with respect to the relation R, that is, every element has at most one colour (1 or 2), and it may be uncoloured (or "blank"), and whenever $R(a, b)$ holds, then a and b cannot be coloured with the same colour. We saw in Example 4.4 that with the uniform measure, for $i = 1, 2$, the probability that $\mathcal{M} \in \mathbf{K}_n$ has at least one element with colour i approaches 1 as $n \to \infty$. But the probability that, for any $i = 1, 2$ and any positive $k \in \mathbb{N}$, $\mathcal{M} \in \mathbf{K}_n$ has at least $k + 1$ elements with colour i is at most $1/2^k$. Thus, with regard to the uniform probabilty measure, a typical member of $\mathbf{K}_n$ has at least one element with colour i, for $i = 1, 2$. But for $k > 0$ at most $1/2^{k-1}$ of $\mathcal{M} \in \mathbf{K}_n$ have at least $k + 1$ elements with the same colour.

How can we design a procedure that generates – by possibly making some random assignments on the way – $\mathcal{M} \in \mathbf{K}_n$ in such a way that the probability of ending up with an $\mathcal{M} \in \mathbf{K}_n$ with exactly k elements with colour 1 is the same as the proportion of $\mathcal{M} \in \mathbf{K}_n$ which have exactly k elements with colour 1? The author does not know, and the point is that, in general, it may not be easy to conceive of a generating procedure, of structures from a given set, such that the probability measure associated with the generating procedure is identical to the uniform probability measure on the given set of structures.

Recall, from Example 4.4, that there is an extension axiom φ such that the probability, with the uniform mesure, that φ holds in $\mathcal{M} \in \mathbf{K}_n$ approaches 0 as $n \to \infty$. But if we apply the following generating procedure of $\mathcal{M} \in \mathbf{K}_n$, then, for *every* extension axiom φ, the probability of ending up with an $\mathcal{M} \in \mathbf{K}_n$ which satisfies φ approaches 1 as $n \to \infty$. For every $i \in \{1, \ldots, n\}$, with probability 1/3 let it have colour 1, colour 2 or be blank; then go through all pairs (i, j) such that i and j are not coloured with the same colour and let $(i, j) \in R^{\mathcal{M}}$ with probability 1/2. The probability of obtaining, in this way, a structure $\mathcal{M} \in \mathbf{K}_n$ is the same as the probability of $\mathcal{M}$ with the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure on $\mathbf{K}_n$, where L_0 is the sublanguage of L whose vocabualary is $\{P_1, P_2\}$ and $\mathbf{K}_n \restriction L_0 = \{\mathcal{M} \restriction L_0 : \mathcal{M} \in \mathbf{K}_n\}$. By letting the underlying geometry of every structure in $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ be trivial (see Remark 7.2) and applying Theorem 7.28 it follows that, for every extension axiom φ of $\mathbf{K}$, the probability, with the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure, that φ holds in $\mathbf{K}_n$ approaches 1 as $n \to \infty$; and by Theorem 7.29, $\mathbf{K}$ has a zero-one law. We have in particular shown that the asymptotic probability, with the uniform probability mesure, of a first order definable property in $\mathbf{K}$ may be different from the asymptotic probability of the same property when the $(\mathbf{K}_n \restriction L_0)$-conditional measure is used.

Before taking underlying pregeometries into account, we collect a technical lemma which will be used later.

Lemma 6.5. *Suppose that $\mathbf{C}_0, \ldots, \mathbf{C}_k$ are finite sets of structures such that, for every $i = 0, \ldots, k - 1$, (1) and (2) in Definition 6.1 hold if $\mathbf{C}_0$ and $\mathbf{C}_1$ are replaced by $\mathbf{C}_i$ and $\mathbf{C}_{i+1}$, respectively. For $r = 1, \ldots, k$, let $\mathbb{P}_r$ denote the uniformly $(\mathbf{C}_0, \ldots, \mathbf{C}_{r-1})$-conditional probability measure on $\mathbf{C}_r$. If $1 \leq r \leq s \leq k$ and $\mathcal{A} \subseteq \mathbf{C}_r$, then*

$$\mathbb{P}_r(\mathcal{A}) = \mathbb{P}_{r+1}\big(\{\mathcal{B} \in \mathbf{C}_{r+1} : \mathcal{A} \subseteq_w \mathcal{B}\}\big)$$
$$\textit{and } \mathbb{P}_r(\mathcal{A}) = P_s\big(\{\mathcal{B} \in \mathbf{C}_s : \mathcal{A} \subseteq_w \mathcal{B}\}\big).$$

Proof. The second identity follows from the first by induction, and the first identity is a straightforward consequence of Definitions 6.2 and 6.1. □

7. Underlying pregeometries

Definition 7.1. (i) We call an L-structure $\mathcal{A}$ a ***pregeometry*** if

(1) there is a closure operation $\mathrm{cl}_{\mathcal{A}}$ on A such that $(A, \mathrm{cl}_{\mathcal{A}})$ is a pregeometry,
(2) for all $n \in \mathbb{N}$ there is a formula $\theta_n(x_1, \ldots, x_{n+1}) \in L$ such that for all $a_1, \ldots, a_{n+1} \in A$, $a_{n+1} \in \mathrm{cl}_{\mathcal{A}}(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \models \theta_n(a_1, \ldots, a_{n+1})$, and
(3) if $X \subseteq A$ is closed with respect to $\mathrm{cl}_{\mathcal{A}}$ (i.e. $\mathrm{cl}_{\mathcal{A}}(X) = X$), then X is closed under interpretations of (eventual) function symbols and constant symbols; so X is the universe of a substructure of $\mathcal{A}$.

(ii) Let $\mathbf{K}$ be a class of L-structures. We call $\mathbf{K}$ a ***pregeometry*** if every $\mathcal{A} \in \mathbf{K}$ is a pregeometry and for every $n \in \mathbb{N}$ there is a formula $\theta_n(x_1, \ldots, x_{n+1}) \in L$ such that for every $\mathcal{A} \in \mathbf{K}$ and all $a_1, \ldots, a_{n+1} \in A$, $a_{n+1} \in \mathrm{cl}_{\mathcal{A}}(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \models \theta_n(a_1, \ldots, a_{n+1})$.

Remark 7.2. For every structure $\mathcal{A}$, if $\mathrm{cl}_{\mathcal{A}}(X) = X$ for every $X \subseteq A$, then $\mathcal{A}$ is a pregeometry in the sense of Definition 7.1 (i). This pregeometry is often called ***trivial*** or ***degenerate***. It may happen that for a structure $\mathcal{A}$ there is more than one way to define a pregeometry on A. As noted, we always have a trivial pregeometry on A. But if, for example, $\mathcal{A}$ is a vector space over some finite field (formalized as a first-order structure in a suitable way), then we can also let $\mathrm{cl}_{\mathcal{A}}(X)$ be the linear span of X, and then $\mathrm{cl}_{\mathcal{A}}$ becomes a pregeometry on A. When saying that a structure $\mathcal{A}$ is a pregeometry we assume that some particular pregeometry on A (in the sense of Definition 7.1 (i)) is fixed, and if we say that a class of L-structures $\mathbf{K}$ is a pregeometry we assume that, for every $\mathcal{A} \in \mathbf{K}$, some pregeometry $\mathrm{cl}_{\mathcal{A}}$ is fixed on A and that the condition in Definition 7.1 (ii) holds.

Assumption 7.3. For the rest of this section we assume that $\mathbf{K}$ is a class of L-structures which is a pregeometry, and that the formulas $\theta_n(x_1, \ldots, x_{n+1})$ define the pregeometry in the sense of Definition 7.1 (ii). (Later, in Assumption 7.10, we will add some more assumptions.)

Definition 7.4. (i) As in Sections 3 – 6, we say that structure $\mathcal{A}$ is ***represented (with respect to $\mathbf{K}$)*** if $\mathcal{A}$ is isomorphic to some structure in $\mathbf{K}$. We say that $\mathcal{A}$ is ***permitted (with respect to $\mathbf{K}$)*** if it can be embedded into some structure in $\mathbf{K}$; or equivalently, if it is a substructure of some represented structure. And a structure which is not permitted (with respect to $\mathbf{K}$) is ***forbidden (with respect to $\mathbf{K}$)***. Note that every represented structure is a pregeometry on which the closure operator is defined by $\theta_n(x_1, \ldots, x_{n+1})$, $n \in \mathbb{N}$. This is what we mean when speaking about a pregeometry and closure on a represented structure.
(ii) If $\mathcal{M}$ is a pregeometry, then the notation $\mathcal{A} \subseteq_{cl} \mathcal{M}$ means that $\mathcal{A}$ is a substructure of $\mathcal{M}$ and $\mathrm{cl}_{\mathcal{M}}(A) = A$. In words, we express '$\mathcal{A} \subseteq_{cl} \mathcal{M}$' by saying that $\mathcal{A}$ ***is a closed substructure of*** $\mathcal{M}$.

Definition 7.5. The notion of $\mathcal{B}/\mathcal{A}$-multiplicity is defined as before, except that we require that $\mathcal{A}$ and $\mathcal{B}$ are closed in some superstructure. More precisely: Suppose that there is a represented $\mathcal{N}$ such that $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{N}$ and both A and B are closed in $\mathcal{N}$. We say that the ***$\mathcal{B}/\mathcal{A}$-multiplicity of a (represented) structure $\mathcal{M}$ is at least*** m if the following holds:

> whenever $\mathcal{A}' \subseteq_{cl} \mathcal{M}$ and $\sigma : \mathcal{A}' \to \mathcal{A}$ is an isomorphism, then there are $\mathcal{B}'_i \subseteq_{cl} \mathcal{M}$ and isomorphisms $\sigma_i : \mathcal{B}'_i \to \mathcal{B}$, for $i = 1, \ldots, m$, such that $\mathcal{A}' \subseteq \mathcal{B}'_i$, $\sigma_i \restriction A' = \sigma$ and $B'_i \cap B'_j = A'$ whenever $i \neq j$.

The ***$\mathcal{B}/\mathcal{A}$-multiplicity is*** m if it is at least m but not at least $m+1$.

Remark 7.6. Observe that we can express, in first-order logic, that sets are closed (or not) in a uniform way. For if $\gamma_n(x_1, \ldots, x_n)$ denotes the formula

$$\neg\exists x_{n+1}\Big(\theta_n(x_1, \ldots, x_n, x_{n+1}) \ \wedge \ \bigwedge_{i=1}^{n} x_i \neq x_{n+1}\Big),$$

then for every $\mathcal{M} \in \mathbf{K}$ and all $a_1, \ldots, a_n \in M$, $\mathcal{M} \models \gamma_n(a_1, \ldots, a_n)$ if and only if $\{a_1, \ldots, a_n\}$ is closed in $\mathcal{M}$. It follows that whenever $\mathcal{N}$ is represented and $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ are closed substructures of $\mathcal{M}$, then, for every $m \in \mathbb{N}$, there is a sentence φ_m such that for every represented $\mathcal{M}$, $\mathcal{M} \models \varphi_m$ if and only if the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least m.

Definition 7.7. For represented $\mathcal{M}$ and closed substructures $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{M}$, the ***$\mathcal{B}/\mathcal{A}$-extension axiom*** is the statement expressing that the $\mathcal{B}/\mathcal{A}$-multiplicity is at least 1. As noted in Remark 7.6, this statement is expressible with a first-order sentence.

Note that if the closure operator of (structures in) $\mathbf{K}$ is trivial, then the definitions of extension axioms and multiplicity coincide with those given earlier; so the earlier setting is a special case of the current setting.

Definition 7.8. Let $\mathbf{K}$ be a class of L-structures and let $(\mathcal{M}_n : n \in \mathbb{N})$ be a sequence of structures from $\mathbf{K}$.
(i) We say that the sequence $(\mathcal{M}_n : n \in \mathbb{N})$ is ***polynomially k-saturated*** if there are a sequence of numbers $(\lambda_n : n \in \mathbb{N})$ with $\lim_{n\to\infty} \lambda_n = \infty$ and a polynomial $P(x)$ such that for every $n \in \mathbb{N}$:

(1) $\lambda_n \leq |M_n| \leq P(\lambda_n)$, and
(2) whenever $\mathcal{N}$ is represented and $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{N}$ are closed (in $\mathcal{N}$) and $\dim_{\mathcal{N}}(A)+1 = \dim_{\mathcal{N}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}_n$ is at least λ_n.

(ii) We say that $\mathbf{K}$ is ***polynomially k-saturated*** if there are $\mathcal{M}_n \in \mathbf{K}$, for $n \in \mathbb{N}$, such that the sequence $(\mathcal{M}_n : n \in \mathbb{N})$ is ***polynomially k-saturated***.

Example 7.9. While it is possible to construct many different $\mathbf{K}$ which are k-saturated (by application of Theorem 7.28) the kind of pregeometries that are present in examples that the author can construct are rather limited. So let us look at examples of $\mathbf{K}$ which are polynomially k-saturated for every $k \in \mathbb{N}$ and which do not have any more structure than what is necessary for defining the pregeometry. The cases known are on the one hand the trivial pregeometry and on the other hand (possibly projective or affine variants of) linear spaces over a fixed, but arbitrary, finite field.

If L has empty vocabulary and $\mathcal{E}_n$ is the unique L-structure with universe $\{1, \ldots, n\}$ (with trivial closure operator), then it is straightforward to check that $(\mathcal{E}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$.

Now suppose that $\mathcal{G}_n$ is a vector space with dimension n with universe $\{1, \ldots, p^n\}$ over a finite field F of order p. Let $\mathrm{cl}_{\mathcal{G}_n}$ be linear span. To view $\mathcal{G}_n$ as a first order structure we can let scalar multiplication be represented by unary function symbols (one for every element in F), vector addition by a binary function symbol, and let there be a constant symbol for the zero vector. Then $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a pregeometry in the sense of Definition 7.1. The proof of Lemma 3.5 in [9] shows that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated, for every $k \in \mathbb{N}$. In [9] it is explained how one can "transform" $\mathcal{G}_n$ into first-order structure, which represents a projective space $\mathcal{P}_n$ or affine space $\mathcal{A}_n$ over F of dimension n. By the argument leading to Proposition 3.4 in [9] it follows that $(\mathcal{P}_n : n \in \mathbb{N})$ and $(\mathcal{A}_n : n \in \mathbb{N})$ are polynomially k-saturated, for every $k \in \mathbb{N}$.

There are other "linear geometries" (see [6]) which involve quadratic forms. These may be candidates for other polynomially k-saturated sequences of pregeometries; but for reasons explained in Problem 3.8 in [9], the author has not been able to prove or disprove it.

From now on we work within the following context, in addition to the assumptions already made (see Assumption 7.3).

Assumption 7.10. From now on we assume the following:

(1) $L_0 \subseteq L$ are first-order languages with vocabularies V_0 and V, respectively, and $V - V_0$ is finite and relational.
(2) $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is a set of L_0-structures which is a pregeometry, in the sense of Definition 7.1. Moreover, assume that the formulas $\theta_n(x_1, \ldots, x_{n+1}) \in L_0$, for $n \in \mathbb{N}$, define the pregeometry in the sense of Definition 7.1.
(3) For $n \in \mathbb{N}$, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ is a set of expansions to L of $\mathcal{G}_n$, and $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$. For each $\mathcal{A} \in \mathbf{K}$, $\mathrm{cl}_{\mathcal{A}}$ is, by definition, the same as $\mathrm{cl}_{\mathcal{A} \restriction L_0}$, where the latter is the same as $\mathrm{cl}_{\mathcal{G}_n}$ for some n, because $\mathcal{A} \restriction L_0 = \mathcal{G}_n$ for some n.
(4) Whenever $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then $\mathcal{A}$ is represented.

Remark 7.11. (i) Note that point (4) in Assumption 7.10 says that the class of represented structures is closed under closed substructures.
(ii) If the closure is trivial, then (4) is equivalent to the hereditary property (for $\mathbf{K}$).
(iii) Analogoues of the main theorems of this section can be stated and proved without assumption (4), but then, to get such results, the notion of 'acceptance of substitutions' (Definition 7.20) must be modified, and becomes more complicated. The author opted, in this case, for simplicity rather than some more generality.

Definition 7.12. Let $\mathcal{A} \in \mathbf{K}$ and let d be a natural number.
(i) The ***d-dimensional reduct*** of $\mathcal{A}$, denoted $\mathcal{A} \restriction d$, is the weak substructure of $\mathcal{A}$ which is defined as follows:

(a) $\mathcal{A} \restriction d$ has the same universe as $\mathcal{A}$.
(b) Every symbol in the vocabulary of L_0 is interpreted in the same way in $\mathcal{A} \restriction d$ as in $\mathcal{A}$.
(c) For every relation symbol R which belongs to the vocabulary of L but *not* to the vocabulary of L_0, and for every tuple $\bar{a}$ from the universe of $\mathcal{A}$,
$$\bar{a} \in R^{\mathcal{A} \restriction d} \iff \dim_{\mathcal{A}}(\bar{a}) \leq d \text{ and } \bar{a} \in R^{\mathcal{A}}.$$

(ii) $\mathbf{K} \restriction d = \{\mathcal{A} \restriction d : \mathcal{A} \in \mathbf{K}\}$.
(iii) $\mathbf{K}_n \restriction d = \{\mathcal{A} \restriction d : \mathcal{A} \in \mathbf{K}_n\}$.

Remark 7.13. (i) Observe that if there is no relation symbol whose arity is greater than d, then for every $\mathcal{A} \in \mathbf{K}$, $\mathcal{A} \restriction d = \mathcal{A}$; hence $\mathbf{K} \restriction d = \mathbf{K}$ and $\mathbf{K}_n \restriction d = \mathbf{K}_n$ for every n.
(ii) By Definition 7.12, for every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, the sequence $\mathbf{K}_n \restriction 0, \mathbf{K}_n \restriction 1, \ldots, \mathbf{K}_n \restriction r$ satisfies the conditions for $\mathbf{C}_0, \ldots, \mathbf{C}_r$ in Definition 6.2. Hence, for every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, the uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction r - 1)$-conditional measure is well-defined on $\mathbf{K}_n \restriction r$.

Remark 7.14. Note that for a an L-structure $\mathcal{M} \in \mathbf{K}$ we have different kinds of "reducts", and the same symbol '$\restriction$' is used in all contexts, but the symbol following '$\restriction$' is a key, besides the context, to what is meant. For a sublanguage $L' \subseteq L$, $\mathcal{M} \restriction L'$ is the reduct of $\mathcal{M}$ to L' in the usual "language wise" sense. For a subset $X \subseteq M$, $\mathcal{M} \restriction X$ denotes the substructure of $\mathcal{M}$ which is generated by X. And for a natural number d, $\mathcal{M} \restriction d$ denotes the d-dimensional reduct of $\mathcal{M}$, which is a *weak* substructure of $\mathcal{M}$, but not necessarily a substructure.

Definition 7.15. (i) Let ρ be equal to the largest arity of a relation symbol in the vocabulary of L. Note that if $r \geq \rho$ then for every $\mathcal{A} \in \mathbf{K}$, $\mathcal{A} \restriction r = \mathcal{A}$; hence $\mathbf{K} \restriction r = \mathbf{K}$

and $\mathbf{K}_n \restriction r = \mathbf{K}_n$ for every n.
(ii) For every $n \in \mathbb{N}$, let $\mathbb{P}_{n,0}$ denote the uniform probability measure on $\mathbf{K}_n \restriction 0$. For every $n \in \mathbb{N}$ and every positive $r \in \mathbb{N}$, let $\mathbb{P}_{n,r}$ denote the uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction r-1)$-conditional measure on $\mathbf{K}_n \restriction r$.
(iii) The uniformly $(\mathbf{K}_n \restriction 0, \ldots, \mathbf{K}_n \restriction \rho - 1)$-conditional measure $\mathbb{P}_{n,\rho}$ on $\mathbf{K}_n = \mathbf{K}_n \restriction \rho$ is also denoted by δ_n and called the ***dimension conditional measure*** on $\mathbf{K}_n$.

Example 7.16. Suppose that L and $\mathbf{K}_n$ are defined like in any of Examples 4.3 – 4.5, let L_0 be the language with empty vocabulary, and let the underlying pregeometry be trivial. If L_0 is defined as in the corresponding example, then the dimension conditional measure on $\mathbf{K}_n$ is, by definition, the same as the uniformly $(\mathbf{K}_n \restriction 0, \mathbf{K}_n \restriction 1)$-conditional measure on $\mathbf{K}_n$, which in turn is identical to the uniformly $(\mathbf{K}_n \restriction L_0)$-conditional measure, considered in the mentioned examples; this follows straightforwardly from the definitions. Examples with nontrivial underlying pregeometry will appear later.

Definition 7.17. We say that the pregeometry $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is ***uniformly bounded*** if there is a function $u : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every $X \subseteq |\mathcal{G}_n|$, $\big|\mathrm{cl}_{\mathcal{G}_n}(X)\big| \leq u\big(\dim_{\mathcal{G}_n}(X)\big)$.

Remark 7.18. The trivial pregeometries and the pregemetries obtained from vector spaces over finite fields are uniformly bounded. However, the assumption on uniform boundedness is only used in Theorem 7.29. More examples of uniformly bounded pregeometries are obtained by applying the amalgamation construction first developed by E. Hrushovski; the variants which produce countably categorical limit structures [11]. However, the cases of such constructions known to the author do *not* produce pregeometries which are polynomially k-saturated for all k; this can be seen by considering the arguments in Section 2 of [8]. The author does not know an example of a pregeometry (in the sense of this paper) $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ which is not uniformly bounded and each $\mathcal{G}_n$ is *finite*, as we always assume here.

Terminology 7.19. When saying that two represented structures $\mathcal{A}$ and $\mathcal{A}'$ ***agree on*** L_0 ***and on closed proper substructures*** we mean that $\mathcal{A} \restriction L_0 = \mathcal{A}' \restriction L_0$ (so in particular, $\mathrm{cl}_{\mathcal{A}} = \mathrm{cl}_{\mathcal{A}'}$) and whenever $\mathcal{U} \subseteq_{cl} \mathcal{A}$ and $\dim_{\mathcal{A}}(U) < \dim_{\mathcal{A}}(A)$, then $\mathcal{A} \restriction U = \mathcal{A}' \restriction U$.

The next definition generalizes the notion of 'admitting substitutions' from Section 3 to the context of this section.

Definition 7.20. Let $\mathcal{A}$ and $\mathcal{A}'$ be represented structures. Note that, in part (i) and (ii) of the next definition, the property defined can only hold if $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 and on closed proper substructures; so that is the situation which is of interest.
(i) We say that $\mathbf{K}$ ***accepts the substitution*** $[\mathcal{A} \triangleright \mathcal{A}']$ ***over*** L_0 if whenever $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then there is a represented $\mathcal{N}$ such that $\mathcal{N} \restriction L_0 = \mathcal{M} \restriction L_0$, $\mathcal{N} \restriction |\mathcal{A}'| = \mathcal{A}'$ and if $\mathcal{U} \subseteq_{cl} \mathcal{N}$, $\dim_{\mathcal{N}}(U) \leq \dim_{\mathcal{N}}(A')$ and $\mathcal{U} \neq \mathcal{A}'$, then $\mathcal{N} \restriction U = \mathcal{M} \restriction U$.
(ii) We say that $\mathbf{K}$ ***accepts*** k***-substitutions over*** L_0 if whenever $\mathcal{A}$ and $\mathcal{A}'$ are represented structures which agree on L_0 and on closed proper substructures, and $\dim_{\mathcal{A}}(A) = \dim_{\mathcal{A}'}(A') \leq k$, then $\mathbf{K}$ accepts the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0.

Remark 7.21. (i) It is easy to see the following: If there is, up to isomorphism, a unique represented structure with dimension 0, then $\mathbf{K}$ accepts 0-substitutions over L_0.
(ii) Let ρ be the supremum of the arities of all relation symbols that belong to the vocabulary of L but *not* to the vocabulary of L_0. It is straightforward to verify that if $\mathbf{K}$ accepts ρ-substitutions over L_0, then, for every $k \in \mathbb{N}$, $\mathbf{K}$ accepts k-substitutions over L_0.

Example 7.22. (**Coloured structures of the *first kind*.**) For the sake of having a uniform terminology in this example, and the next, let us have the following convention.

For $F = \{1\}$ let L_F be the language with empty vocabulary V_F and let $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$, where $\mathcal{G}_n$ is the unique L_F-structure with universe $\{1, \ldots, n\}$. In this case call $\mathbf{G}^F$ the *vector space pregeometry over* $\{1\}$.

For any finite field F, the *vector space pregeometry over* F refers to the pregeometry $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ defined in Examples 7.9; so $\mathcal{G}_n$ is a vector space over F of dimension n, and L_F and V_F is the language and vocabulary, respectively, of $\mathcal{G}_n$.

Let $\mathbf{G}^F = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be the vector space pregeometry over F, where F is a finite field or $\{1\}$. Then let $l \geq 2$ and assume that $L_{col} \supset L_F$, "the colour language" is the language with vocabulary $V_{col} = V_F \cup \{P_1, \ldots, P_l\}$ where all P_i are unary relation symbols, called *colours*. Also assume that $L_{rel} \supset L_F$, "the language of relations", has a vocabulary V_{rel} such that $V_{rel} - V_F$ contains only finitely many relation symbols, of any arity. Let L the language with vocabulary $V = V_{col} \cup V_{rel}$. For every positive $n \in \mathbb{N}$ define $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ to be set of expansions $\mathcal{M}$ of $\mathcal{G}_n$ to L that satisfy the following three *colouring conditions (of the first kind)*:

(1) $\mathcal{M} \models \forall x\big(P_1(x) \vee \ldots \vee P_l(x)\big)$.
(2) For all distinct $i, j \in \{1, \ldots, l\}$, and all $a, b \in M - \mathrm{cl}_{\mathcal{M}}(\emptyset)$ such that $a \in \mathrm{cl}_{\mathcal{M}}(b)$, $\mathcal{M} \models \neg\big(P_i(a) \wedge P_j(b)\big)$. (In other words: any two linearly dependent non-zero elements must have the same colour.)
(3) If $R \in V_{rel}$ has arity $m \geq 2$, $\mathcal{M} \models R(a_1, \ldots, a_m)$, $i, j \in \{1, \ldots, m\}$ and $i \neq j$, then there are $b, c \in \mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ such that for every $k \in \{1, \ldots, l\}$, $\mathcal{M} \models \neg\big(P_k(b) \wedge P_k(c)\big)$; that is, at least two elements in $\mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ have different colours.

It is now straightforward to verify that, for every F considered, $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$ accepts k-substitutions over L_F, for every $k \in \mathbb{N}$. And as mentioned in Example 7.9, $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$. It is also uniformly bounded. Thus, with this setup of $(\mathcal{G}_n : n \in \mathbb{N})$ and $\mathbf{K}$ the premises of Theorems 7.28 and 7.29 (below) are satisfied. This example and the next will be studied more in Sections 8 and 9.

Example 7.23. (**Coloured structures of the *second kind*.**) The colourings of the first kind are the convention within hypergraph theory [2] (where they are just called 'colourings'), but here we would also like to consider another sort of colourings, those of the "second kind". Here, $\mathbf{G}^F$, L_F, L_{col}, L_{rel} and L are defined as in Example 7.22. Let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$, where $\mathbf{K}_n$ consists of those L-expansions $\mathcal{M}$ of $\mathcal{G}_n$ which satisfy (1) and (2) from the previous example and

(3') If $R \in V_{rel}$ has arity $m \geq 2$, $\mathcal{M} \models R(a_1, \ldots, a_m)$, $b, c \in \mathrm{cl}_{\mathcal{M}}(a_1, \ldots, a_m)$ and b is independent from c (i.e. $b \notin \mathrm{cl}_{\mathcal{M}}(c)$), then for every $k \in \{1, \ldots, l\}$, $\mathcal{M} \models \neg\big(P_k(b) \wedge P_k(c)\big)$; that is, *every* pair of mutually independent elements b and c in the closure of $a_1, \ldots, a_m$ have different colours.

Again, it is straightforward to verify that, for every F considered, $\mathbf{K}$ accepts k-substitutions over L_F, for every $k \in \mathbb{N}$.

Example 7.24. (**Other variations**) In the previous two examples, it is also possible to consider projective or affine spaces over a finite field, instead of a vector space. And, by droping condition (1), one can consider partial colorings of the first, or the second, kind. For all these variations, $\mathbf{K}$ accepts k-substitutions for every $k \in \mathbb{N}$.

Example 7.25. In this example, elements as well as pairs of elements can be coloured and some restrictions are imposed. Suppose that, for every n, $\mathcal{G}_n$ is a projective space over the 2-element field and let L_0 be the language of $\mathcal{G}_n$. Let $L \supset L_0$ contain, besides the symbols of L_0, three unary relation symbols P_1, P_2, P_3, three binary relation symbols R_1, R_2, R_3 and one ternary relation symbol S. We can think of the P_i as colours of elements, and the R_i as colours of pairs. For every n, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ consists of all expansions $\mathcal{M}$ of $\mathcal{G}_n$ to L which satisfy the following conditions:

(a) For every 2-dimensional subspace $X \subseteq M$, if no pair $(a, b) \in X^2$ is coloured, then at least one point in X is coloured.
(b) For every two dimensional subspace $X \subseteq M$, if some pair $(a, b) \in X^2$ is coloured, then there are *not* two different points in X with the same colour (but two different points may be uncoloured).
(c) If $\mathcal{M} \models S(a, b, c)$, then $\{a, b, c\}$ is independent and if $(d_1, d_2), (e_1, e_2) \in \mathrm{cl}_{\mathcal{M}}(a, b, c)$, then (d_1, d_2) and (e_1, e_2) do not have the same colour (but both may be uncoloured).

We show that $\mathbf{K}$ accepts 3-substitutions over L_0. Since no relation symbol has arity greater than 3 it follows (see Remark 7.21) that $\mathbf{K}$ accepts k-substitutions over L_0 for every $k \in \mathbb{N}$.

Let $\mathcal{A}, \mathcal{A}$ be represented and assume that $\mathcal{A}\restriction L_0 = \mathcal{A}' L_0$ and that $\mathcal{A}$ and $\mathcal{A}'$ agree on all closed proper substructures. We must show that if $\mathcal{M}$ is represented and $\mathcal{A} \subseteq_{cl} \mathcal{M}$, then there exists a represented $\mathcal{N}$ such that $\mathcal{N}\restriction L_0 = \mathcal{M}\restriction L_0$, $\mathcal{N}\restriction A' = \mathcal{A}'$ and whenever $\mathcal{U} \subseteq_{cl} \mathcal{N}$, $\dim_{\mathcal{N}}(U) \leq \dim_{\mathcal{N}}(A')$, and $U \neq A'$, then $\mathcal{N}\restriction U = \mathcal{M}\restriction U$.

First suppose that $\dim_{\mathcal{M}}(A) = 1$. Let $\mathcal{M}' = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$, according to Definition 3.11 (Since $\mathcal{A}\restriction L_0 = \mathcal{A}'\restriction L_0$, the substitution involves only interpretations of relation symbols). Then go through all $\mathcal{B} \subseteq_{cl} \mathcal{M}'$ of dimension 2; whenever we meet such $\mathcal{B}$ which is forbidden we can change some binary relationships (R_i, $i = 1, 2, 3$), but not change any unary relationships (P_i, $i = 1, 2, 3$), and thus get a permitted substructure. When this has been done for all 2-dimensional closed substructures, call the result $\mathcal{M}''$; so all 2-dimensional substructures of $\mathcal{M}''$ are permitted. Then we can just remove all S-relationships from $\mathcal{M}''$ so that in the resulting structure $\mathcal{N}$ the interpretation of S is empty. It now follows from the construction of $\mathcal{N}$ and (a) – (c) that $\mathcal{N}$ is represented. And whenever $U \subseteq N$ is 1-dimensional and different from A', then $\mathcal{N}\restriction U = \mathcal{M}\restriction U$.

Now suppose that $\dim_{\mathcal{M}}(A) = 2$. Let $\mathcal{M}' = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$. Then $\mathcal{M}'$ and $\mathcal{M}$ agree on all closed 1- or 2-dimensional subsets which are different from A'. By removing all S-relationships from $\mathcal{M}'$ we get $\mathcal{N}$ which is represented and such that $\mathcal{N}$ and $\mathcal{M}$ agree on all closed 1- or 2-dimensional subsets which are different from A'. Moreover, $\mathcal{N}\restriction A' = \mathcal{A}'$.

Finally, suppose that $\dim_{\mathcal{A}}(A) = 3$. Both $\mathcal{A}$ and $\mathcal{A}'$ satisfy (a) – (c) (because they are permitted) and $\mathcal{A}$ and $\mathcal{A}'$ agree, by assumption, on substructures of dimension 2. Hence $\mathcal{N} = \mathcal{M}[\mathcal{A} \triangleright \mathcal{A}']$ and $\mathcal{M}$ agree on subsets of dimension 2 and on closed subsets of dimension 3 which are different from A'. Since $\mathcal{A}'$ is represented, and hence satisfies (a) – (c), $\mathcal{N}$ is represented.

The next lemma tells that the notion of 'accepting k-substitutions over L_0' is indeed a generalization of the notion of 'admitting k-substitutions'.

Lemma 7.26. *Let L_0 be the language with empty vocabulary and let $\mathcal{G}_n$ be the unique L_0-structure with universe $\{1, \ldots, m_n\}$ (with the trivial geometry), where we assume that the sequence $m_n \in \mathbb{N}$ tends to infinity. Let L be any language with finite relational vocabulary. Suppose that, for every n, $\mathbf{K}_n$ is a set of L-structures with universe $\{1, \ldots, m_n\}$; in other words, $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$ is a set of expansions of $\mathcal{G}_n$ to L; and let $\mathbf{K} = \bigcup_{n \in \mathbb{N}} \mathbf{K}_n$. For every $k \in \mathbb{N}$, if $\mathbf{K}$ admits k-substitutions (in the sense of Definition 3.12), then $\mathbf{K}$ accepts k-substitutions over L_0.*

Proof. One just checks that, under the assumptions, $\mathbf{K}$ does indeed accept k-substitutions over L_0, according to Definition 7.20. □

Definition 7.27. For every $n \in \mathbb{N}$ and every L-sentence φ,

$$\text{let } \delta_n(\varphi) \text{ be an abbreviation for } \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big).$$

Recall assumptions 7.10.

Theorem 7.28. *Let $k > 0$. Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ uniformly bounded, polynomially k-saturated and that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}(\mathcal{G}_n)$ accepts k-substitutions over L_0. Then:*

(i) *For every $(k-1)$-extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$.*
(ii) *$\mathbf{K}$ polynomially k-saturated.*

Theorem 7.29. *Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is uniformly bounded and polynomially k-saturated for every $k \in \mathbb{N}$. Also assume that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}(\mathcal{G}_n)$ accepts k-substitutions over L_0 for every $k \in \mathbb{N}$. Then, for every L-sentence φ, either $\lim_{n\to\infty} \delta_n(\varphi) = 0$ or $\lim_{n\to\infty} \delta_n(\varphi) = 1$.*

For the last theorem of this section we need a definition.

Definition 7.30. We say that $\mathbf{K}$ has the ***independent amalgamation property*** if the following holds: Whenever $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ are represented, $\mathcal{A} \subseteq_{cl} \mathcal{B}_i$, for $i = 1, 2$, and $B_1 \cap B_2 = A$, then there is a represented $\mathcal{C}$ such that $\mathcal{B}_i \subseteq_{cl} \mathcal{C}$ for $i = 1, 2$.

Theorem 7.31. *Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is uniformly bounded and polynomially k-saturated for every $k \in \mathbb{N}$. Assume that, up to isomorphism, there is a unique represented structure (with respect to $\mathbf{K}$) with dimension 0 (a particular case of this is when the closure of $\emptyset$ is $\emptyset$). Let $k \in \mathbb{N}$ be minimal such that $\mathbf{K}$ does not accept k-substitutions over L_0 and suppose that $\mathcal{A}$ and $\mathcal{A}'$ are represented structures (with respect to $\mathbf{K}$) such that $\mathcal{A}$ and $\mathcal{A}'$ have dimension k, agree on L_0 and on closed proper substructures, $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, but does* not *accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ over L_0. Then at least one of the following holds:*

(i) *$\mathbf{K}$ does not have the independent amalgamation property.*
(ii) *There are $\beta < 1$ and extension axioms φ and ψ such that for all sufficiently large n, $\delta_n(\varphi \wedge \psi) < \beta$. If $k > 1$, then $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$.*

The proof shows that if the assumptions of the theorem hold and one particular instance of the independent amalgamation property is satisfied, then case (ii) holds; more information about this instance of independent amalgamation and φ and ψ is given by the proof.

The proofs of Theorems 7.28 – 7.31 are given in Section 10.

8. Reducts and l-colourable structures

In this section, for each $n \in \mathbb{N}$, $\mathbf{K}_n$ is defined as in Example 7.22 for $F = \{1\}$, or as in Example 7.23 for $F = \{1\}$. Note that '$F = \{1\}$' means that the universe of every $\mathcal{M} \in \mathbf{K}_n$ is $\{1, \ldots, n\}$ and that the pregeometry is trivial (i.e. $\mathrm{cl}_{\mathcal{M}}(X) = X$ for every $\mathcal{M} \in \mathbf{K}_n$ and every $X \subseteq M$). As usual, let $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$. The notation L_{col} (the language of the l colours), L_{rel} (the language of relations) and L mean the same as in the mentioned examples. *But we add the assumption that all relation symbols of the vocabulary of L_{rel} have arity at least 2.* (Colouring unary relations is not so interesting.)

The expression 'l-coloured' may be used with respect to colourings of the *first kind*, or of the *second kind*. If we consider colourings of the second kind we add the assumption that l is at least as great as the arity of every relation symbol in the vocabulary of L_{rel}; for otherwise the interpretations of some relation symbol(s) will be empty for all l-coloured structures, and then there is no point in having this (or these) relation symbol(s). Theorem 8.1 holds for both kinds of colourings. It is open whether the theorem holds also in the case when F is a finite field, which gives a nontrivial pregeometry, and $\mathbf{K}_n$ is defined as in the mentioned examples. Observe that if there are no relation symbols of arity greater than 2, then the two kinds of colourings coincide. A structure which is isomorphic with one in $\mathbf{K}$ is called ***l-coloured***. Note that being l-coloured is equivalent to being represented with respect to $\mathbf{K}$.

For each n, let

$$\mathbf{C}_n = \{\mathcal{M}\restriction L_{rel} : \mathcal{M} \in \mathbf{K}_n\} \quad \text{and} \quad \mathbf{C} = \bigcup_{n \in \mathbb{N}} \mathbf{C}_n.$$

A structure which is isomorphic to one in $\mathbf{C}$, will be called ***l-colourable***. Being l-colourable is equivalent to being represented with respect to $\mathbf{C}$. It is clear that an L_{rel}-structure $\mathcal{M}$ is l-colourable if and only if there is $f : M \to \{1, \dots, l\}$ such that the expansion $\mathcal{M}'$ of $\mathcal{M}$ to L, defined by $\mathcal{M}' \models P_i(a)$ if and only if $f(a) = i$, belongs to $\mathbf{K}_n$. Therefore we can, when convenient, use functions instead of the relation symbols $P_1, \dots, P_l$ to represent colourings.

In this section, $\delta_n^{\mathbf{K}}$ denotes the dimension conditional measure on $\mathbf{K}_n$ (see Definition 7.15). For each n, we consider the measure, $\delta_n^{\mathbf{C}}$ on $\mathbf{C}_n$ which is inherited from $\mathbf{K}_n$ in the following sense:

$$\text{For every } \mathbf{X} \subseteq \mathbf{C}_n, \ \delta_n^{\mathbf{C}}(\mathbf{X}) = \delta_n^{\mathbf{K}}\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M}\restriction L_{rel} \in \mathbf{X}\}\big).$$

For every sentence L_{rel}-sentence φ, let $\delta_n^{\mathbf{C}}(\varphi) = \delta_n^{\mathbf{C}}\big(\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \varphi\}\big)$.

Theorem 8.1. *For every sentence $\varphi \in L_{rel}$, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 0$ or $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$. This holds for both kinds of colourings considered.*

8.1. **Proof of Theorem 8.1.** The general pattern of the proof is a familiar one. We collect into a theory $T_{\mathbf{C}}$ a certain type of extension axioms (to be called 'l-colour compatible extension axioms') together with sentences which describe all possible isomorphism types of structures in $\mathbf{C}$. Then we show that for every $\psi \in T_{\mathcal{C}}$, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\psi) = 1$, which implies (via compactness) that $T_{\mathbf{C}}$ is consistent. After this we show that $T_{\mathbf{C}}$ is complete by showing that it is countably categorical. The zero-one law is now a straightforward consequence of the previously proven facts, together with compactness.

Remark 8.2. We can *not* expect that for every extension axiom φ of $\mathbf{C}$, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$. For example, suppose that the vocabulary of L_{rel} contains only one relation symbol which is binary, and that $l = 2$. Then there is *no* 2-colourable L_{rel}-structure which satisfies all 3-extension axioms of $\mathbf{C}$. For if $\mathcal{M}$ would be such a structure, then it is easy to see that $\mathcal{M}$ would contain a 5-cycle (does not matter if it is directed or not) which contradicts that $\mathcal{M}$ is 2-colourable.

In order to define the type of extension axioms that are useful in this context, we need to find a way of expressing, with an L_{rel}-formula, that two elements in an L-structure have the same colour. In fact, it suffices to find an L_{rel}-formula $\xi(y, z)$ such that with $\delta_n^{\mathbf{K}}$-probability approaching 1 as $n \to \infty$: if $\mathcal{M} \in \mathbf{K}_n$ and $a, b \in M$, then $\mathcal{M} \models \xi(a, b)$ if and only if a and b have the same colour in $\mathcal{M}$. The following lemma is a first step in that direction:

Lemma 8.3. *There is an l-colourable structure $\mathcal{S}$ and distinct $a, b \in S$ such that the following hold:*

(a) *Whenever $\gamma : S \to \{1, \dots, l\}$ is a colouring of $\mathcal{S}$, then $\gamma(a) = \gamma(b)$; in other words, whenever $\mathcal{S}$ is l-coloured then a and b get the same colour.*
(b) *For every $i \in \{1, \dots, l\}$, there is a colouring $\gamma_i : S \to \{1, \dots, l\}$ of $\mathcal{S}$ such that $\gamma(a) = \gamma(b) = i$.*

Proof. The argument divides into two cases depending on whether we consider colourings of the first kind, or of the second kind.

First assume that we work with colourings of the second kind. Let $S = \{0, 1, \dots, l\}$. Let R be any symbol from the vocabulary of L_{rel}, so the arity of R, call it ρ, is at least

2. By assumption, since we work with colourings of the second kind now, $l \geq \rho \geq 2$. Let $R^{\mathcal{S}}$ consist exactly of all tuples $(s_1, \dots, s_\rho)$ of *distinct* elements from S such that

$$\{s_1, \dots, s_\rho\} \subseteq S - \{0\} \quad \text{or} \quad \{s_1, \dots, s_\rho\} \subseteq S - \{1\}.$$

For all other relation symbols Q of the vocabulary of L_{rel}, let $Q^{\mathcal{S}} = \emptyset$. Note that there is no relationship in $\mathcal{S}$ which contains both 0 and 1. Therefore any assignment of the same colour $i \in \{1, \dots, l\}$ to 0 and 1 can be extended to an l-colouring of $\mathcal{S}$. Also note that every l-colouring of $\mathcal{S}$ must give all elements in $S - \{0\}$ different colours; and it must give all elements in $S - \{1\}$ different colours. Since $|S| = l - 1$ there is no other choice but giving 0 and 1 the same colour. Hence the lemma holds for $\mathcal{S}$ with $a = 0$ and $b = 1$.

Now assume that we work with colourings of the first kind. By assumption all relation symbols in the vocabulary of L_{rel} have arity at least 2. Let ρ be the minimum of the arities of relation symbols in the vocabulary of L_{rel}, so $\rho \geq 2$, and let R be a relation symbol in the vocabulary of L_{rel} which has arity ρ. Let $S = \{0, 1, \dots, (\rho - 1)l\}$ and let $R^{\mathcal{S}}$ consist exactly of all tuples $(s_1, \dots, s_\rho)$ of *distinct* elements from S such that

$$\{s_1, \dots, s_\rho\} \subseteq S - \{0\} \quad \text{or} \quad \{s_1, \dots, s_\rho\} \subseteq S - \{1\}.$$

For all other relation symbols Q of the vocabulary of L_{rel}, let $Q^{\mathcal{S}} = \emptyset$. Note that there is no relationship in $\mathcal{S}$ which contains both 0 and 1.

Suppose that both 0 and 1 are assigned the colour 1. We show that there is a colouring $\gamma : S \to \{1, \dots, l\}$ of $\mathcal{S}$ such that $\gamma(0) = \gamma(1) = 1$. This will prove (b), because any permutation of the colours of an l-colouring gives a new l-colouring. Assign the colour 1 to exactly $\rho - 2$ elements $s_1, \dots, s_{\rho-2} \in S - \{0, 1\}$. So exactly $\rho - 1$ elements of $S = \{0, 1, \dots, (\rho - 1)l\}$ have been assigned the colour 1; and these elements are $0, 1, s_1, \dots, s_{\rho-2}$. Hence

$$\big|\{S - \{0, 1, s_1, \dots, s_{\rho-2}\}\big| = (\rho - 1)l - 1 - (\rho - 2) = (\rho - 1)(l - 1),$$

so $S - \{0, 1, s_1, \dots, s_{\rho-2}\}$ can be partitioned into $l - 1$ parts each of which contains exactly $\rho - 1$ elements. Consequently, we can, for each colour $i \in \{2, \dots, l\}$, assign the colour i to exactly $\rho - 1$ elements in $S - \{0, 1, s_1, \dots, s_{\rho-2}\}$. Since no colour has been assigned to more that $\rho - 1$ elements, the result is an l-colouring of $\mathcal{S}$.

It remains to prove (a). So assume that $\gamma : S \to \{1, \dots, l\}$ is a colouring of $\mathcal{S}$. Note that $\big|S - \{0\}\big| = \big|S - \{1\}\big| = (\rho - 1)l$. By the definition of $\mathcal{S}$, every ρ-tuple of distinct elements $(s_1, \dots, s_\rho) \in (S - \{0\})^\rho$ is an R-relationship. Hence, for every colour $i \in \{1, \dots, l\}$, we must have $\big|\gamma^{-1}(i) \cap (S - \{0\})\big| = \rho - 1$. Suppose that $\gamma(1) = 1$. (If $\gamma(1) \in \{2, \dots, l\}$ the argument is analogous.) Assume, for a contradiction, that $\gamma(0) = i \neq 1$. Above we concluded that $\big|\gamma^{-1}(i) \cap (S - \{0\})\big| = \rho - 1$. Since $\gamma(0) = i$ we get $\big|\gamma^{-1}(i)\big| = \rho$, and as $\gamma(1) \neq i$, we get $\gamma^{-1}(i) \subseteq S - \{1\}$. Hence, there are distinct $s_1, \dots, s_\rho \in \gamma^{-1}(i) \subseteq S - \{1\}$. By the definition of $\mathcal{S}$, $(s_1, \dots, s_\rho) \in R^{\mathcal{S}}$. Since γ assigns all elements $s_1, \dots, s_\rho$ the colour i, this contradicts that γ is a colouring of $\mathcal{S}$. This shows that the lemma holds also for colourings of the first kind if we take $a = 0$ and $b = 1$. □

Notation 8.4. Let $\mathcal{S}$ be an l-colourable structure and $a, b \in S$ distinct elements such that Lemma 8.3 is satisfied. Note that we must have $|S| \geq 3$. Without loss of generality we assume that $|\mathcal{S}| = S = \{1, \dots, s\}$ for some $s \geq 3$ and that $a = s - 1$ and $b = s$. Hence every assignment of the same colour to $s - 1$ and s can be extended to an l-colouring of $\mathcal{S}$, and every l-colouring of $\mathcal{S}$ gives $s - 1$ and s the same colour.

Let $\chi_{\mathcal{S}}(x_1, \dots, x_s)$ be a quantifier-free L_{rel}-formula which expresses the L_{rel}-isomorhism type of $\mathcal{S}$; in other words, for every L_{rel}-structure $\mathcal{M}$ and all $a_1, \dots, a_s \in M$, $\mathcal{M} \models \chi_{\mathcal{S}}(a_1, \dots, a_{s+1})$ if and only if the map $a_i \mapsto i$ is an isomorphism from $\mathcal{M}{\restriction}\{a_1, \dots, a_s\}$ to $\mathcal{S}$.

Let $\xi(y,z)$ be the the formula

$$y = z \ \vee \ \exists u_1, \ldots, u_{s-2} \chi_{\mathcal{S}}(u_1, \ldots, u_{s-2}, y, z).$$

For $n, k \in \mathbb{N}$ let $\mathbf{X}_{n,k} \subseteq \mathbf{K}_n$ be the set of all $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms with respect to $\mathbf{K}$.

Lemma 8.5. *For every* $k \in \mathbb{N}$, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) = 1$.

Proof. As mentioned in Examples 7.9 and 7.22, for every $k \in \mathbb{N}$, the trivial pregeometry is polynomially k-saturated and $\mathbf{K}$ accepts k-substitutions over the language with empty vocabulary. By Theorem 7.28 (i), for every extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\varphi) = 1$. The lemma follows since there are only finitely many k-extension axioms. □

Lemma 8.6. *Suppose that* $k \geq \|\mathcal{S}\|$ *and* $\mathcal{M} \in \mathbf{X}_{n,k}$. *For all* $a, b \in M$, $\mathcal{M} \models \xi(a,b)$ *if and only if* a *and* b *have the same colour in* $\mathcal{M}$ *(i.e. for some* $i \in \{1, \ldots, l\}$, $\mathcal{M} \models P_i(a) \wedge P_i(b)$*).*

Proof. Let $k \geq \|\mathcal{S}\|$ and $\mathcal{M} \in \mathbf{X}_{n,k}$. Suppose that $\mathcal{M} \models \xi(a,b)$. If $a = b$ then a and b have the same colour, so suppose that $a \neq b$. Then there are $m_1, \ldots, m_{s-2} \in M$ such that

$$\mathcal{M} \models \chi_{\mathcal{S}}(m_1, \ldots, m_{s-2}, a, b).$$

It follows that the L_{rel}-reduct of $\mathcal{M}\restriction\{m_1, \ldots, m_{s-2}, a, b\}$ is isomorphic with $\mathcal{S}$ via the L_{rel}-isomorphism $m_i \mapsto i$, for $i = 1, \ldots, s-2$, $a \mapsto s-1$ and $b \mapsto s$. Then we get an l-colouring of $\mathcal{S}$ by letting i get the same colour as m_i, for $i = 1, \ldots, s-2$, letting $s-1$ get the same colour as a, and letting s get the same colour as b. From Lemma 8.3 it follows that $s-1$ and s must have the same colour in $\mathcal{S}$; hence a and b must have the same colour in $\mathcal{M}$.

Now suppose that $a, b \in M$ are distinct elements which have the same colour in $\mathcal{M}$, that is, for some colour $i \in \{1, \ldots, l\}$, $\mathcal{M} \models P_i(a) \wedge P_i(b)$. By Lemma 8.3 and Notation 8.4, there is an l-coloured structure $\mathcal{S}_i$ such that $\mathcal{S}_i\restriction L_{rel} = \mathcal{S}$ and $\mathcal{S}_i \models P_i(s-1) \wedge P_i(s)$. Let $\mathcal{S}_i' = \mathcal{S}_i\restriction\{s-1, s\}$. Since a and b have the same colour in $\mathcal{M}$, there is no binary relationship of $\mathcal{M}$ which includes both a and b. Hence $\mathcal{M}\restriction\{a,b\}$ has no other relationships than the colour of a and of b which is i in both cases. By the properties of $\mathcal{S}$ (given by Lemma 8.3 and Notation 8.4), $\mathcal{S}_i'$ has no other relationships than the colour of $s-1$ and of s which is i in both cases. Hence, any bijection between $\{s-1, s\}$ and $\{a, b\}$ is an isomorphism between $\mathcal{S}_i$ and $\mathcal{M}\restriction\{a,b\}$. Since $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \|\mathcal{S}\|$, it follows that $\mathcal{M}$ satisfies the $\mathcal{S}_i/\mathcal{S}_i'$-extension axiom. This implies that there are $m_1, \ldots, m_{s-2} \in M$ such that the map $m_i \mapsto i$, for $i = 1, \ldots, s-2$, $a \mapsto s-1$ and $b \mapsto s$, is an isomorphism from $\mathcal{M}\restriction\{m_1, \ldots, m_{s-2}, a, b\}$ to $\mathcal{S}_i$. Since $\mathcal{S}_i\restriction L_{rel} = \mathcal{S}$ we get $\mathcal{M} \models \chi_{\mathcal{S}}(m_1, \ldots, m_{s-2}, a, b)$, so $\mathcal{M} \models \xi(a,b)$. □

Next, we define 'l-colour compatible extension axioms'. Suppose that $\mathcal{B}$ is l-colourable (and finite) and let $\mathcal{A} \subset \mathcal{B}$. Let $\widehat{\mathcal{B}}$ be an l-coloured structure such that $\widehat{\mathcal{B}}\restriction L_{rel} = \mathcal{B}$; in other words, $\widehat{\mathcal{B}}$ is an l-colouring of $\mathcal{B}$. Without loss of generality we assume that $A = \{1, \ldots, \alpha\}$ and $B = \{1, \ldots, \beta\}$, so $\alpha < \beta$. Let $\chi_{\mathcal{A}}(x_1, \ldots, x_\alpha)$ and $\chi_{\mathcal{B}}(x_1, \ldots, x_\beta)$ be quantifier-free L_{rel}-formulas which express the isomorphism types of $\mathcal{A}$ and $\mathcal{B}$, respectively; so for any L_{rel}-structure $\mathcal{M}$, $\mathcal{M} \models \chi_{\mathcal{A}}(m_1, \ldots, m_\alpha)$ if and only if the map $m_i \mapsto i$ is an isomorphism from $\mathcal{M}\restriction\{m_1, \ldots, m_\alpha\}$ to $\mathcal{A}$; and similarly for $\chi_{\mathcal{B}}$.

If $\|\mathcal{A}\| \geq 1$ then let $\theta_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\alpha)$ be the conjunction of

- all $\xi(x_i, x_j)$ such that $1 \leq i, j \leq \alpha$ and i and j have *the same* colour in $\widehat{\mathcal{B}}$ and
- all $\neg\xi(x_i, x_j)$ such that $1 \leq i, j \leq \alpha$ and i and j have *different* colours in $\widehat{\mathcal{B}}$.

If $\|\mathcal{A}\| = 0$ then we do not need to define $\theta_{\widehat{\mathcal{B}}}$. Since $\mathcal{A}$ is a proper substructure of $\mathcal{B}$ we have $\|\mathcal{B}\| \geq 1$. Let $\theta^*_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\beta)$ be the conjunction of

- all $\xi(x_i, x_j)$ such that $1 \leq i, j \leq \beta$ and i and j have *the same* colour in $\widehat{\mathcal{B}}$ and
- all $\neg\xi(x_i, x_j)$ such that $1 \leq i, j \leq \beta$ and i and j have *different* colours in $\widehat{\mathcal{B}}$.

If $\|\mathcal{A}\| \geq 1$ then we call the following sentence an ***instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom***:

$$\forall x_1, \ldots, x_\alpha \exists x_{\alpha+1}, \ldots, x_\beta\big(\\ \big[\chi_\mathcal{A}(x_1, \ldots, x_\alpha) \ \wedge \ \theta_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\alpha)\big] \rightarrow \big[\chi_\mathcal{B}(x_1, \ldots, x_\beta) \ \wedge \ \theta^*_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\beta)\big] \ \big).$$

If $\|\mathcal{A}\| = 0$, then we call

$$\exists x_1, \ldots, x_\beta\big(\chi_\mathcal{B}(x_1, \ldots, x_\beta) \ \wedge \ \theta^*_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\beta)\big)$$

an ***instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom***. Since there are only finitely many l-colourings of $\mathcal{B}$ there are only finitely many instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. The ***l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom*** is, by definition, the conjunction of all instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. If $|\mathcal{B}| \leq k+1$ then the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom is also called an ***l-colour compatible k-extension axiom***.

Lemma 8.7. *Suppose that $\mathcal{B}$ is l-colourable (and finite) and let $\mathcal{A} \subset \mathcal{B}$. Let φ denote the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. If $k \geq \max(\|\mathcal{S}\|, \|\mathcal{B}\|)$ and $\mathcal{M} \in \mathbf{X}_{n,k}$, then $\mathcal{M} \models \varphi$.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$, φ, k and $\mathcal{M}$ satisfy the premisses of the lemma, so in particular $\mathcal{M} \in \mathbf{X}_{n,k} \subseteq \mathbf{K}_n$. We consider only the case when $\|\mathcal{A}\| \geq 1$, since the case when $\|\mathcal{A}\| = 0$ is analogous. Without loss of generality we assume that $A = \{1, \ldots, \alpha\}$ and $B = \{1, \ldots, \beta\}$ where $\alpha < \beta$. It suffices to prove that every instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom is true in $\mathcal{M}$. So let $\widehat{\mathcal{B}}$ be an l-coloured structure such that $\widehat{\mathcal{B}}{\restriction}L_{rel} = \mathcal{B}$. Then the instance of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom which is obtained from $\widehat{\mathcal{B}}$ has the form

$$\forall x_1, \ldots, x_\alpha \exists x_{\alpha+1}, \ldots, x_\beta\big(\\ \big[\chi_\mathcal{A}(x_1, \ldots, x_\alpha) \ \wedge \ \theta_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\alpha)\big] \rightarrow \big[\chi_\mathcal{B}(x_1, \ldots, x_\beta) \ \wedge \ \theta^*_{\widehat{\mathcal{B}}}(x_1, \ldots, x_\beta)\big] \ \big).$$

where $\chi_\mathcal{A}$, $\chi_\mathcal{B}$, $\theta_{\widehat{\mathcal{B}}}$ and $\theta^*_{\widehat{\mathcal{B}}}$ are as described above (before the lemma). Let $\widehat{\mathcal{A}} = \widehat{\mathcal{B}}{\restriction}\{1, \ldots, \alpha\}$. It follows that $\widehat{\mathcal{A}}{\restriction}L_{rel} = \mathcal{A}$. Suppose that $m_1, \ldots, m_\alpha \in M$ and

$$\mathcal{M} \models \chi_\mathcal{A}(m_1, \ldots, m_\alpha) \ \wedge \ \theta_{\widehat{\mathcal{B}}}(m_1, \ldots, m_\alpha).$$

From $\mathcal{M} \models \chi_\mathcal{A}(m_1, \ldots, m_\alpha)$ it follows the map $m_i \mapsto i$ is an isomorphism from the L_{rel}-reduct of $\mathcal{M}{\restriction}\{m_1, \ldots, m_\alpha\}$ to $\mathcal{A}$. From $\mathcal{M} \models \theta_{\widehat{\mathcal{B}}}(m_1, \ldots, m_\alpha)$, the definition of $\theta_{\widehat{\mathcal{B}}}$ and Lemma 8.6 it follows that the colouring of $\mathcal{M}{\restriction}\{m_1, \ldots, m_\alpha\}$ is a permutation of the colouring of $\widehat{\mathcal{A}}$. More precisely, there is a permutation π of $\{1, \ldots, l\}$ such that, for every $i \in \{1, \ldots, \alpha\}$ and every colour $j \in \{1, \ldots, l\}$, $\widehat{\mathcal{A}} \models P_j(i) \iff \mathcal{M} \models P_{\pi(j)}(m_i)$. Let $\widehat{\mathcal{B}}_\pi$ be an L-structure such that $\widehat{\mathcal{B}}_\pi{\restriction}L_{rel} = \widehat{\mathcal{B}}{\restriction}L_{rel}$ and for every $i \in |\widehat{\mathcal{B}}|$ and every colour $j \in \{1, \ldots, l\}$, $\widehat{\mathcal{B}} \models P_j(i) \iff \widehat{\mathcal{B}}_\pi \models P_{\pi(j)}(i)$. Then let $\widehat{\mathcal{A}}_\pi = \widehat{\mathcal{B}}_\pi{\restriction}\{1, \ldots, \alpha\}$. By assumption, $\mathcal{M} \in \mathbf{X}_{n,k}$ and $k \geq \max(\|\mathcal{S}\|, \|\mathcal{D}\|)$, so $\mathcal{M}$ satisfies the $\widehat{\mathcal{B}}_\pi/\widehat{\mathcal{A}}_\pi$-extension axiom. This implies that there are $m_{\alpha+1}, \ldots, m_\beta \in M$ such that the map $m_i \mapsto i$, for $i = 1, \ldots, \beta$ is an isomorphism from $\mathcal{M}{\restriction}\{m_1, \ldots, m_\beta\}$ to $\widehat{\mathcal{B}}_\pi$. From this and the choice of $\widehat{\mathcal{B}}_\pi$ it follows that

$$\mathcal{M} \models \chi_\mathcal{B}(m_1, \ldots, m_\beta) \ \wedge \ \theta^*_{\widehat{\mathcal{B}}}(m_1, \ldots, m_\beta),$$

which completes the proof. □

Corollary 8.8. *For every l-colour compatible extension axiom φ, $\lim_{n\to\infty} \delta_n^{\mathbf{C}}(\varphi) = 1$.*

Proof. Let φ be an l-colour compatible extension axiom. Since $\varphi \in L_{rel}$ we have

$$\{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \models \varphi\} = \{\mathcal{N}{\restriction}L_{rel} : \mathcal{N} \in \mathbf{K}_n \text{ and } \mathcal{N} \models \psi\},$$

so by the definition of $\delta_n^{\mathcal{C}}$ we get $\delta_n^{\mathbf{C}}(\varphi) = \delta_n^{\mathbf{K}}(\varphi)$, for every n. Therefore it suffices to show that $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\varphi) = 1$. For some l-colourable L_{rel}-structures $\mathcal{C} \subset \mathcal{D}$, φ is the l-colour compatible $\mathcal{D}/\mathcal{C}$-extension axiom. Take $k \geq \max(\|\mathcal{D}\|, \|\mathcal{S}\|)$. By Lemma 8.5, $\lim_{n\to\infty} \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k}) = 1$. By Lemma 8.7, for every n and every $\mathcal{M} \in \mathbf{X}_{n,k}$, $\mathcal{M}$ satisfies φ, so $\delta_n^{\mathbf{C}}(\varphi) \geq \delta_n^{\mathbf{K}}(\mathbf{X}_{n,k})$. □

For every integer $n > 0$ let $\mathcal{M}_{(n,1)}, \ldots, \mathcal{M}_{(n,m_n)}$ be an enumeration of all isomorphism types of l-colourable structures of cardinality at most n. Let $\chi_i^n(x_1, \ldots, x_n)$ describe the isomorphism type of $\mathcal{M}_{(n,i)}$ in such a way that we require that all variables $x_1, \ldots, x_n$ actually occur in χ_i^n. It means that if $\|\mathcal{M}_{(n,i)}\| < n$, then $\chi_i^n(x_1, \ldots, x_n)$ must express that some variables refer to the same element, by saying '$x_k = x_l$' for some $k \neq l$. For every $n \in \mathbb{N}$ let ψ_n denote the sentence

$$\forall x_1, \ldots, x_n \bigvee_{i=1}^{m_n} \bigvee_{\pi} \chi_i^n(x_{\pi(1)}, \ldots, x_{\pi(n)}),$$

where the second disjunction ranges over all permutations π of $\{1, \ldots, n\}$. Then let $T_{iso} = \{\psi_n : n \in \mathbb{N},\ n > 0\}$ and note that every ψ_n is true in every l-colourable structure. Let T_{ext} consist of all l-colour compatible extension axioms and let $T_{\mathbf{C}} = T_{iso} \cup T_{ext}$. By Corollary 8.8 and compactness, $T_{\mathbf{C}}$ is consistent. Since $T_{ext} \subset T_{\mathbf{C}}$, every model of $T_{\mathbf{C}}$ is infinite. In order to prove Theorem 8.1 it is enough to prove that $T_{\mathbf{C}}$ is complete.

Lemma 8.9. *$T_{\mathbf{C}}$ is countably categorical and therefore complete.*

Proof. Suppose that the L_{rel}-structures $\mathcal{M}$ and $\mathcal{M}'$ are countable models of $T_{\mathbf{C}}$. We prove that $\mathcal{M} \cong \mathcal{M}'$ by a back and forth argument, in which partial isomorphisms between $\mathcal{M}$ and $\mathcal{M}'$ are extended step by step. It suffices to prove the following:

Claim. *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are finite substructures of $\mathcal{M}$ and $\mathcal{M}'$, respectively, and that f is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ such that for all $a, b \in A$, $\mathcal{M} \models \xi(a,b) \iff \mathcal{M}' \models \xi(f(a), f(b))$. For every $m \in M - A$ (or $m' \in M' - A'$), there are finite substructures $\mathcal{B} \subseteq \mathcal{M}$, $\mathcal{B}' \subseteq \mathcal{M}'$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}'$ such that $m \in B$ (or $m' \in B'$), g extends f and for all $a, b \in B$, $\mathcal{M} \models \xi(a,b) \iff \mathcal{M}' \models \xi(g(a), g(b))$.*

So suppose that $\mathcal{A}$ and $\mathcal{A}'$ are finite substructures of $\mathcal{M}$ and $\mathcal{M}'$, respectively, and that f is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ such that for all $a, b \in A$, $\mathcal{M} \models \xi(a,b) \iff \mathcal{M}' \models \xi(f(a), f(b))$. Let $m \in M - A$. Define $\mathcal{B} = \mathcal{M}{\restriction}A \cup \{m\}$. Since $\mathcal{M} \models T_{\mathbf{C}} \supset T_{iso}$, $\mathcal{B}$ is a finite l-colourable L_{rel}-structure, so there is an l-coloured structure $\widehat{\mathcal{B}}$ such that $\widehat{\mathcal{B}}{\restriction}L_{rel} = \mathcal{B}$. From $\widehat{\mathcal{B}}$ we get an instance φ of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom. Since $\mathcal{M}' \models T_{\mathbf{C}} \supset T_{ext}$, there are $\mathcal{B}' \subset \mathcal{M}'$ and an isomorphism $g : \mathcal{B} \to \mathcal{B}'$ such that g extends f and for all $a, b \in B$, $\mathcal{M} \models \xi(a,b) \iff \mathcal{M}' \models \xi(g(a), g(b))$. The last property is guaranteed by the definition of instances of the l-colour compatible $\mathcal{B}/\mathcal{A}$-extension axiom and Lemma 8.6. Note that the argument works for the 'base case' when $A = A' = \emptyset$ and f is the empty map. If $m' \in M' - A'$, then we argue symmetrically. □

9. The uniform measure and l-colourable structures

In this section the meaning of L_{col}, L_{rel}, L, $\mathbf{K}_n$, $\mathbf{K}$, $\mathbf{C}_n$ and $\mathbf{C}$ are the same as in the previous section. So $l \geq 2$ is fixed, the pregeometry is always the trivial one, $\mathbf{K}_n$ is the set of l-coloured L_{rel}-structures (so an l-colouring specified) with universe $\{1, \ldots, n\}$, while $\mathbf{C}_n$ is the set of l-colourable L_{rel}-structures (where no l-colouring is specified) with universe $\{1, \ldots, n\}$. As in the previous section, we may describe l-colourings either with the colour symbols $P_1, \ldots, P_l$, or by using a function from the universe of a structure into $\{1, \ldots, l\}$. The above may refer to colourings of the first kind as well as colourings of the second kind (see Examples 7.22 and 7.23). Theorem 9.2 holds for both kinds of colourings, but Proposition 9.3 is proved only for the second kind of colourings; the author conjectures that it holds for the first kind of colourings as well.

Definition 9.1. Let $m \in \mathbb{R}$ and suppose that $\mathcal{M}$ is an L_{rel}-structure and that $\gamma : M \to \{1, \ldots, l\}$ is an l-colouring of $\mathcal{M}$. We say that γ is an ***m-rich l-colouring***, or that $\mathcal{M}$ is ***m-richly l-coloured by γ***, if for every $i \in \{1, \ldots, l\}$, $\big|\{a \in M : \gamma(a) = i\}\big| \geq m$. Analogously, we say that $\mathcal{M} \in \mathbf{K}$ (or $\mathcal{M}{\restriction}L_{col}$) is ***$m$-richly l-coloured*** if for every $i \in \{1, \ldots, l\}$, $\big|\{a \in M : \mathcal{M} \models P_i(a)\}\big| \geq m$. We say that an L_{rel}-structure ***has an m-rich l-colouring*** if there is $\mathcal{N} \in \mathbf{K}$ such that $\mathcal{M} = \mathcal{N}{\restriction}L_{rel}$ and $\mathcal{N}$ is m-richly l-coloured; or equivalently, if there is an m-rich l-colouring $\gamma : M \to \{1, \ldots, l\}$.

The notion of an *l-colour compatible extension axiom* was defined in Section 8, before Lemma 8.8.

Theorem 9.2. *Let $f : \mathbb{N} \to \mathbb{R}$ be such that for all $k \in \mathbb{N}$ and every $0 < \alpha < 1$, $\lim_{n\to\infty} n^k \cdot \alpha^{f(n)} = 0$. (For example, if μ is positive then $f(n) = \mu n$ and $f(n) = n^\mu$ satisfy this condition.) The following holds for colourings of the first and of the second kind.*
(i) Suppose that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly l-coloured approaches 1 as $n \to \infty$. Then, for every extension axiom φ of $\mathbf{K}$, the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy φ approaches 1 as $n \to \infty$. Consequently, $\mathbf{K}$ has a 0-1 law for the uniform probability measure.
(ii) Suppose that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich l-colouring approaches 1 as $n \to \infty$. Then, for every l-colour compatible extension axiom φ of $\mathbf{C}$, the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy φ approaches 1 as $n \to \infty$. Consequently, $\mathbf{C}$ has a 0-1 law for the uniform probability measure.
(iii) If $k \geq l$ and the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms of $\mathbf{K}$ approaches 1 as $n \to \infty$, then the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy all l-colour compatible k-extension axioms of $\mathbf{C}$ approaches 1 as $n \to \infty$.

Proposition 9.3. *Suppose that ρ, the supremum of the arities of relation symbols in L_{rel}, is equal to l, the number of colours. Then there is $0 < \mu < 1$ such that the following holds for colourings of the* ***second kind****:*
(i) the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are μn-richly l-coloured approaches 1 as $n \to \infty$, and
(ii) the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have a μn-rich l-colouring approaches 1 as $n \to \infty$.
If $\rho = l = 2$, then (a) and (b) also hold for colourings of the first kind.

9.1. **Proof of Theorem 9.2.** In this section, $l \geq 2$ is fixed so we may occasionally say 'colouring' instead of 'l-colouring'. The proof is the same for colorings of the first and of the second kind, so we will only speak of colourings, without specifying the kind. Suppose that for all $k \in \mathbb{N}$ and every $0 < \alpha < 1$, $\lim_{n\to\infty} n^k \cdot \alpha^{f(n)} = 0$. We first prove (i) and then derive (ii) from (i).

As said in Remark 3.3, the zero-one law for $\mathbf{K}$, with the uniform measure, follows if we can show that for every extension axiom of $\mathbf{K}$, the proportion of structures in $\mathbf{K}_n$ which satisfy it approaches 1 as $n \to \infty$. Suppose that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly coloured approaches 1 as $n \to \infty$.

Let φ be an extension axiom of $\mathbf{K}$. It suffices to consider the case when φ has only one existential quantifier, so let φ have the form

$$\forall x_1, \ldots, x_k \exists x_{k+1} \big(\psi(x_1, \ldots, x_k) \to \psi'(x_1, \ldots, x_k, x_{k+1})\big),$$

where ψ and ψ' are quantifier-free.

For every L_{col}-structure $\mathcal{A}$ (or equivalently, for every $\mathcal{A} \in \mathbf{K} \restriction 1$), let

$$\mathbf{E}_L(\mathcal{A}) = \{\mathcal{M} \in \mathbf{K} : \mathcal{M}\restriction L_{col} = \mathcal{A}\}.$$

Since we assume that the proportion of $\mathcal{M} \in \mathbf{K}_n$ which are $f(n)$-richly coloured approaches 1 as $n \to \infty$, it is sufficient to prove that

(a) for every $\varepsilon > 0$ there is n_ε such that for every $n > n_\varepsilon$, if $\mathcal{A} \in \mathbf{K}_n\restriction 1$ is a $f(n)$-rich colouring, then the proportion of $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ which satisfy φ is at least $1 - \varepsilon$.

The proof of (a) is a slight variant of the well known proof that, with the uniform measure, the probability that an extension axiom is true in a randomly picked structure (without any restrictions on its relations, and with at least one relation with arity > 1) with universe $\{1, \ldots, n\}$ approaches 1 as n tends to infinity (see [14, 13, 10, 15]).

Suppose that $\mathcal{A} \in \mathbf{K}_n\restriction 1$ is an $f(n)$-rich colouring. Let α be the number of nonequivalent quantifier-free L-formulas with free variables (exactly) $x_1, \ldots, x_{k+1}$. We show that, with the uniform measure, the probability that $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ does *not* satisfy φ approaches 0 as $n \to \infty$; moreover, the convergence is uniform in the sense that it depends only on $n = \|\mathcal{A}\|$, and from this (a) follows.

Note that the only restriction on the interpretations of relation symbols from L_{rel} in structures in $\mathbf{E}_L(\mathcal{A})$ is that the interpretations respect the colouring of $\mathcal{A}$ (in the appropriate sense, depending on whether we consider colourings of the first or second kind). Suppose that $\bar{a} = (a_1, \ldots, a_k) \in |\mathcal{A}|^k$, $\mathcal{M} \in \mathbf{E}_L(\mathcal{A})$ and $\mathcal{M} \models \psi(\bar{a})$. Let $a_{k+1} \in |\mathcal{A}|$ be any of the at least $f(n)$ elements which have the colour, say i, which is specified for x_{k+1} by $\psi'(x_1, \ldots, x_{k+1})$. Then the probability, with the uniform measure, that, for such a_{k+1}, $\mathcal{M} \models \psi'(a_1, \ldots, a_k, a_{k+1})$ is at least $1/\alpha$. So the probability that this is not true is at most $1 - 1/\alpha$; and the probability that $\mathcal{M} \not\models \psi'(a_1, \ldots, a_k, a)$ for every one of the at least $f(n)$ elements a with colour i is at most $(1 - 1/\alpha)^{f(n)}$. There are n^k choices of $\bar{a} \in |\mathcal{A}|^k$ for which $\exists x_{k+1}\psi'(\bar{a}, x_{k+1})$ could fail to be true in $\mathcal{M}$, so the probability that $\mathcal{M} \not\models \varphi$ is at most $n^k \cdot (1 - 1/\alpha)^{f(n)} \to 0$ as $n \to \infty$; by the assumption about $f(n)$. Since we get the same expression '$n^k \cdot (1 - 1/\alpha)^{f(n)}$' for every $f(n)$-rich colouring $\mathcal{A} \in \mathbf{K}_n\restriction 1$ we have proved (a), and hence (i).

The proof of parts (ii) and (iii) och the theorem will use the following lemma which is valid for colourings of both kinds:

Lemma 9.4. *There is $m \in \mathbb{N}$ so that the following holds: Whenever both $\mathcal{N}, \mathcal{N}' \in \mathbf{K}$ satisfy all m-extension axioms of $\mathbf{K}$ and $\mathcal{N}\restriction L_{rel} = \mathcal{N}'\restriction L_{rel}$, then the colouring of $\mathcal{N}'$ is a permutation of the colouring of $\mathcal{N}$; in other words, two elements have the same colour in $\mathcal{N}$ if and only if they have the same colour in $\mathcal{N}'$.*

Proof. Let $\mathcal{S}$ and $\xi(y, z)$ be the structure and formula which where defined in Notation 8.4 and let $m = \|\mathcal{S}\|$. Suppose that both $\mathcal{N}, \mathcal{N}' \in \mathbf{K}$ satisfy all m-extension axioms of $\mathbf{K}$ and $\mathcal{N}\restriction L_{rel} = \mathcal{N}'\restriction L_{rel}$. Recall that ξ is an L_{rel}-formula. Let $a, b \in |\mathcal{N}| = |\mathcal{N}'|$. By Lemma 8.6, which holds for colourings of both kinds: a and b have the same colour in $\mathcal{N} \iff \mathcal{N} \models \xi(a, b) \iff \mathcal{N}\restriction L_{rel} \models \xi(a, b) \iff \mathcal{N}'\restriction L_{rel} \models \xi(a, b) \iff \mathcal{N}' \models \xi(a, b) \iff$ a and b have the same colour in $\mathcal{N}'$. □

Now we are ready to prove (ii) and (iii) of the theorem. Choose m so that the conclusion of Lemma 9.4 holds. Fix an arbitrary $k \geq m$ and let

$$\mathbf{X}_n^{\mathbf{K}} = \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ satisfies all } k\text{-extension axioms of } \mathbf{K}\},$$
$$\mathbf{X}_n^{\mathbf{C}} = \{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \text{ satisfies all colour compatible } k\text{-extension axioms of } \mathbf{C}\},$$
$$\mathbf{Y}_n^{\mathbf{K}} = \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } f(n)\text{-richly coloured}\},$$
$$\mathbf{Y}_n^{\mathbf{C}} = \{\mathcal{M} \in \mathbf{C}_n : \mathcal{M} \text{ has an } f(n)\text{-rich colouring}\}.$$

Observe that with the notation used in the proof of part (i) we have

$$\mathbf{Y}_n^{\mathbf{K}} = \bigcup \{\mathbf{E}_L(\mathcal{A}) : \mathcal{A} \in \mathbf{K}_n \restriction 1\},$$

and (a) implies that

$$\text{(b)} \qquad \lim_{n\to\infty} \frac{|\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}|}{|\mathbf{Y}_n^{\mathbf{K}}|} = 1.$$

Note that for every colouring of $\mathcal{M} \in \mathbf{C}$ the colours can be permuted in $l!$ ways. Therefore,

$$\text{(c)} \qquad |\mathbf{K}_n| \geq l!|\mathbf{C}_n|.$$

From Lemma 9.4 and the assumption that $k \geq l$ it follows that

$$\text{(d)} \qquad |\mathbf{X}_n^{\mathbf{K}}| = l!|\mathbf{X}_n^{\mathbf{C}}| \text{ and } |\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}| = l!|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}|.$$

By (c) and (d) we have

$$\text{(e)} \qquad \frac{|\mathbf{X}_n^{\mathbf{K}}|}{|\mathbf{K}_n|} \leq \frac{l!|\mathbf{X}_n^{\mathbf{C}}|}{l!|\mathbf{C}_n|} = \frac{|\mathbf{X}_n^{\mathbf{C}}|}{|\mathbf{C}_n|} \leq 1.$$

Recall that the definitions of $\mathbf{X}_n^{\mathbf{K}}$ and $\mathbf{X}_n^{\mathbf{C}}$ depend on k and the assumption that $k \geq m$. If the proportion of $\mathcal{M} \in \mathbf{K}_n$ which satisfy all k-extension axioms of $\mathbf{K}$ approaches 1 as $n \to \infty$, then $|\mathbf{X}_n^{\mathbf{K}}|/|\mathbf{K}_n| \to 1$, which by (e) implies that $|\mathbf{X}_n^{\mathbf{C}}|/|\mathbf{C}_n| \to 1$ as $n \to \infty$. Thus (iii) is proved.

It remains to prove part (ii). So assume that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$. In other words,

$$\text{(f)} \qquad \lim_{n\to\infty} \frac{|\mathbf{Y}_n^{\mathbf{C}}|}{|\mathbf{C}_n|} = 1.$$

By (c) and (d),

$$\text{(g)} \qquad \frac{|\mathbf{X}_n^{\mathbf{K}} \cap \mathbf{Y}_n^{\mathbf{K}}|}{|\mathbf{Y}_n^{\mathbf{K}}|} \leq \frac{l!|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}|}{l!|\mathbf{Y}_n^{\mathbf{C}}|} = \frac{|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}|}{|\mathbf{C}_n|} \cdot \frac{|\mathbf{C}_n|}{|\mathbf{Y}_n^{\mathbf{C}}|} \leq 1.$$

Now (b), (f) and (g) imply that

$$\text{(h)} \qquad \lim_{n\to\infty} \frac{|\mathbf{X}_n^{\mathbf{C}} \cap \mathbf{Y}_n^{\mathbf{C}}|}{|\mathbf{C}_n|} = 1.$$

We have derived (h), for arbitrary $k \geq m$, under the assumption that the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1, as $n \to \infty$. Since every l-colour compatible extension axiom is an l-colour compatible k-extension axiom for all sufficiently large k, we have proved: If the proportion of $\mathcal{M} \in \mathbf{C}_n$ which have an $f(n)$-rich colouring approaches 1 as $n \to \infty$, then for every l-colour compatible extension axiom φ, the proportion of $\mathcal{M} \in \mathbf{C}_n$ which satisfy φ approaches 1 as $n \to \infty$.

Now we can define $T_{\mathbf{C}}$ exactly as in Section 8 just before Lemma 8.9. Then $T_{\mathbf{C}}$ is a consistent and complete theory and for every finite $\Delta \subset T_{\mathbf{C}}$, the proportion of $\mathcal{M} \in \mathbf{C}_n$

such that $\mathcal{M} \models \Delta$ approaches 1 as $n \to \infty$. By the completeness of $T_{\mathbf{C}}$ and compactness, $\mathbf{C}$ has a zero-one law for the uniform probability measure.

9.2. **Proof of Proposition 9.3.** Suppose that ρ, the supremum of the arities of relation symbols in L_{rel}, is equal to $l \geq 2$, where l is the number of colours. Note that if $l = 2$, then there is no difference between colourings of the first kind and the second kind; so in this case, what holds for the second kind also holds for the first kind. Essentially the same proof works for (i) and for (ii), so we prove (i) and only indicate where some words should be changed to get a proof of (ii).

Let $f : \mathbb{N} \to \mathbb{R}$ be such that $f(n) \geq 1$ for all sufficiently large n. Let $c : \{1, \ldots, n\} \to \{1, \ldots, l\}$ be a function, intended to be viewed as a potential colouring, and let n_i be the number of elements which have colour i (i.e. m such that $c(m) = i$). Suppose that R is a k-ary relation symbol (so $k \leq l = \rho$). Then the number of interpretations of R on $\{1, \ldots, n\}$ which respect this colouring c *in the sense of second kind* (of colourings) is

$$2^{\sum n_{i_1} \cdot \ldots \cdot n_{i_k}},$$

where the sum ranges over all k-tuples $(i_1, \ldots, i_k) \in \{1, \ldots, l\}^k$ of *different* colours. Note that if $k < l$, then, for this sum, we have

$$\sum n_{i_1} \cdot \ldots \cdot n_{i_k} \leq l^k \cdot n^k \leq l^{l-1} \cdot n^{l-1}.$$

Moreover, if $k = l$ and at least one colour is assigned to less than $f(n)$ elements, then

$$\sum n_{i_1} \cdot \ldots \cdot n_{i_k} = \sum n_{i_1} \cdot \ldots \cdot n_{i_l} \leq l^l \cdot f(n) \cdot n^{l-1}.$$

Therefore, for some positive constants α, β and γ, the number of structures in $\mathbf{K}_n$ which are coloured in such a way that at least one colour is assigned to less than $f(n)$ elements is at most

$$s_n = l^n \cdot 2^{\alpha n^{l-1} + \beta f(n) n^{l-1}} = 2^{\gamma n + \alpha n^{l-1} + \beta f(n) n^{l-1}}.$$

(In the case of proving (ii) we consider $\mathbf{C}_n$ instead of $\mathbf{K}_n$, but we get the same upper bound s_n for the number of structures in $\mathbf{C}_n$ which have a colouring, of the second kind, such that at least one colour is assigned to less than $f(n)$ elements.)

Let $0 < d < \lfloor 1/l \rfloor$. Recall the assumption that $\rho = l$, that is, at least one relation symbol has arity l. This implies that the number of structures in $\mathbf{K}_n$ in which every colour is assigned to at least dn elements is at least

$$t_n = 2^{(dn)^l}.$$

(In the case of proving (ii), we get the same lower bound t_n for the number of structures in $\mathbf{C}_n$ which can be coloured in such a way that every colour is assigned to at least dn elements.)

Recall the assumption that $f(n) \geq 1$ for all sufficiently large n. For some positive constant δ and all sufficiently large n, we have

$$s_n \leq 2^{\delta f(n) n^{l-1}},$$

and hence

$$\begin{aligned} \frac{s_n}{t_n} &\leq \frac{2^{\delta f(n) n^{l-1}}}{2^{d^l n^l}} = 2^{\delta f(n) n^{l-1} - d^l n^l} \\ &= 2^{n^{l-1}\left(\delta f(n) - d^l n\right)} = 2^{\delta n^{l-1}\left(f(n) - \frac{d^l}{\delta} n\right)} \\ &= 2^{-\delta n^{l-1}\left(\frac{d^l}{\delta} n - f(n)\right)}. \end{aligned}$$

We see that if there exists $\varepsilon > 0$ such that for all sufficiently large n, $\frac{d^l}{\delta}n - f(n) > \varepsilon$, then $s_n/t_n \to 0$ as $n \to \infty$. This condition holds if (for example), $f(n) = d^l n/2\delta$, and in this case

$$\frac{s_n}{|\mathbf{K}_n|} \leq \frac{s_n}{t_n} \to 0 \quad \text{as } n \to \infty.$$

This means that the proportion of structures in $\mathbf{K}_n$ such that every colour is assigned to at least $d^l n/2\delta$ elements approaches 1 as $n \to \infty$. (And in the case of proving (ii), we also use the bounds s_n and t_n, in the way indicated above, to conclude that the proportion of structures in $\mathbf{C}_n$ which can be coloured in such a way that every colour is assigned to at least $d^l n/2\delta$ elements approaches 1 as $n \to \infty$.)

Problem 9.5. Do Theorems 8.1 and 9.2 hold if the underlying pregeometry is that of a vector space over a finite field, as explained in Example 7.22, and the first and second kinds of colourings are defined as in Examples 7.22 and 7.23?

10. Proofs of Theorems 7.28, 7.29 and 7.31

Remember that Theorems 7.28 – 7.31 take place within the setting of Assumptions 7.3 and 7.10. Therefore Assumptions 7.3 and 7.10 are active throughout this section.

10.1. **Proof of Theorem 7.28.** We are assuming that $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is a set of L_0-structures and that $\mathbf{G}$ is a pregeometry. Let $k > 0$. Suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated and that $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, where $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$, accepts k-substitutions over L_0. This means that there exists a sequence of numbers $(\lambda_n : n \in \mathbb{N})$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and a polynomial $P(x)$ such that for every $n \in \mathbb{N}$:

(a) $\lambda_n \leq |G_n| \leq P(\lambda_n)$, and
(b) whenever $\mathcal{A}$ and $\mathcal{A}$ are represented, $\mathcal{A} \subset_{cl} \mathcal{B}$ and $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{G}_n$ is at least λ_n.

We must prove the following:

(i) For every $(k-1)$-extension axiom φ of $\mathbf{K}$, $\lim_{n\to\infty} \delta_n(\varphi) = 1$.
(ii) $\mathbf{K}$ polynomially k-saturated.

Part (i) will be reduced to the problem of proving that the δ_n-probability that $\mathcal{M} \in \mathbf{K}_n$ is sufficiently saturated, in the sense of Definition 10.1 below, tends to 1 as n tends to infinity.

Observe that from our assumptions (7.3 and 7.10) it follows that whenever $d, n \in \mathbb{N}$ and $\mathcal{M} \in \mathbf{K}_n{\restriction}d$, then $\mathrm{cl}_{\mathcal{M}}$ coincides with $\mathrm{cl}_{\mathcal{G}_n}$ which is the same as $\mathrm{cl}_{\mathcal{M}\restriction L_0}$ since $\mathcal{M}{\restriction}L_0 = \mathcal{G}_n$. Also, if $d \geq \rho$, then for every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}{\restriction}d = \mathcal{M}$. (See Definition 7.12 and Remark 7.13.)

In this proof, and the proof of theorems 7.29 and 7.31, we often work with $\mathbf{K} \restriction d$, for some $d \in \mathbb{N}$, and consider structures which are represented, permitted, or forbidden, *with respect to* $\mathbf{K}{\restriction}d$. Let ρ be the supremum of the arities of all relation symbols that belong to the vocabulary of L, but not to the vocabulary of L_0. Recall that δ_n is an abbreviation for $\mathbb{P}_{n,\rho}$. Essentially, the next definition just repeats point (2) from Definition 7.8 in the case of $\mathbf{K}{\restriction}d$ (instead of $\mathbf{K}$), but it will be convenient to use the terminology defined below.

Definition 10.1. (i) Let $d, m \in \mathbb{N}$ and $\mathcal{M} \in \mathbf{K}{\restriction}d$. We say that $\mathcal{M}$ is ***(m,k)-saturated with respect to*** $\mathbf{K}{\restriction}d$ if the following holds:

> whenever $\mathcal{A}$ and $\mathcal{B}$ are represented *with respect to* $\mathbf{K}{\restriction}d$, $\mathcal{A} \subset_{cl} \mathcal{B}$ and $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}$ is at least m.

(i) Since $\mathcal{M}{\restriction}\rho = \mathcal{M}$ for every $\mathcal{M} \in \mathbf{K}$, we say that $\mathcal{M} \in \mathbf{K}$ is ***(m,k)-saturated with respect to*** $\mathbf{K}$ if $\mathcal{M}$ is (m,k)-saturated with respect to $\mathbf{K}{\restriction}\rho$.

Definition 10.2. For $r \in \mathbb{N}$ we inductively we define functions $\sigma^r : \mathbb{N} \to \mathbb{N}$. Let $\sigma^0(x) = x$ for all $x \in \mathbb{N}$. Let $\sigma^{r+1}(x) = \lfloor \sqrt{\sigma^r(x)} \rfloor$ for all $x \in \mathbb{N}$.

Note that for every $r \in \mathbb{N}$, $\lim_{n\to\infty} \sigma^r(n) = \infty$. By assumption, $\lim_{n\to\infty} \lambda_n = \infty$, so for every $r \in \mathbb{N}$, $\lim_{n\to\infty} \sigma^r(\lambda_n) = \infty$; this will be used later.

Let φ be a $(k-1)$-extension axiom. In order to prove (i) we need to show that

$$\lim_{n\to\infty} \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\}\big) = 1. \tag{1}$$

By assumption, φ is the $\mathcal{B}/\mathcal{A}$-extension axiom for some $\mathcal{A} \subset \mathcal{B} \subseteq \mathcal{M}$ such that $\mathcal{M}$ is represented, both A and B are closed in $\mathcal{M}$ and $\dim_{\mathcal{B}}(B) \leq k$; in particular $\dim_{\mathcal{B}}(A) < \dim_{\mathcal{B}}(B)$. Then, letting $l = \dim_{\mathcal{B}}(B) - \dim_{\mathcal{B}}(A)$, there are closed substructures $\mathcal{B}_0, \ldots, \mathcal{B}_l$ of $\mathcal{M}$ such that $\mathcal{A} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_l = \mathcal{B}$ and $\dim_{\mathcal{B}}(B_i) + 1 = \dim_{\mathcal{B}}(B_{i+1})$ for $i = 0, \ldots, l$. By Assumption 7.10 (4), every $\mathcal{B}_i$ is represented. As noted above, $\lim_{n\to\infty} \sigma^k(\lambda_n) = \infty$. So in order to prove (1) it is sufficient to show that

$$\lim_{n\to\infty} \delta_n\big(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } (\sigma^k(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}\}\big) = 1. \tag{2}$$

For $n \in \mathbb{N}$, let

$$\mathbf{X}_n = \{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \text{ is } (\sigma^k(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}\},$$

and for $n, r \in \mathbb{N}$ let

$$\mathbf{X}_{n,r} = \{\mathcal{M} \in \mathbf{K}_n{\restriction}r : \mathcal{M} \text{ is } (\sigma^r(\lambda_n), k)\text{-saturated with respect to } \mathbf{K}{\restriction}r\}.$$

By Lemma 10.3 below, in order to prove (2) it is sufficient to prove that

$$\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1, \tag{3}$$

Lemma 10.3. *For every $n \in \mathbb{N}$, $\delta_n(\mathbf{X}_n) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{X}_{n,k})$.*

For the proof of Lemma 10.3 we need the following:

Lemma 10.4. *Let $i \in \mathbb{N}$. For every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}$ is (i,k)-saturated with respect to $\mathbf{K}$ if and only if $\mathcal{M}{\restriction}k$ is (i,k)-saturated with respect to $\mathbf{K}{\restriction}k$.*

Proof. Observe that for every $\mathcal{M} \in \mathbf{K}$ and every $A \subseteq M$ with $\dim_{\mathcal{M}}(A) \leq k$ the following holds: for any relation symbol R, of arity r, say, and every $\bar{b} \in A^r$,

$$\bar{b} \in R^{\mathcal{M}} \iff \bar{b} \in R^{\mathcal{M}{\restriction}k}.$$

In other words, $\mathcal{M}$ and $\mathcal{M}{\restriction}k$ agree on all subsets A of dimension at most k. It follows, in particular, that for every L-structure $\mathcal{A}$ such that $\mathcal{A}{\restriction}L_0 \in \mathbf{G}$ and $\mathcal{A}{\restriction}L_0$ has dimension at most k, $\mathcal{A}$ is represented with respect to $\mathbf{K}$ if and only if $\mathcal{A}$ is represented with respect to $\mathbf{K}{\restriction}k$. The lemma is now an immediate consequence of the definition of (i,k)-saturation. □

Proof of Lemma 10.3. Recall that ρ is the supremum of the arities of relation symbols which belong to the vocabulary of L but not to the vocabulary of L_0. First suppose that $\rho \leq k$. Let

$$\mathbf{Y}_n = \{\mathcal{N} \in \mathbf{K}_n{\restriction}k : \mathcal{M} \subseteq_w \mathcal{N} \text{ for some } \mathcal{M} \in \mathbf{X}_n\}.$$

By Lemma 6.5, $\mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{Y}_n)$. But $\rho \leq k$ implies that, for every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M}{\restriction}k = \mathcal{M}{\restriction}\rho = \mathcal{M}$. Hence, $\mathbf{X}_{n,k} = \mathbf{X}_n = \mathbf{Y}_n$, so $\delta_n(\mathbf{X}_n) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \mathbb{P}_{n,k}(\mathbf{X}_{n,k})$.

Now suppose that $k < \rho$. From Lemma 10.4 it follows that

$$\mathbf{X}_n = \{\mathcal{N} \in \mathbf{K}_n{\restriction}\rho : \mathcal{M} \subseteq_w \mathcal{N} \text{ for some } \mathcal{M} \in \mathbf{X}_{n,k}\}$$

By Lemma 6.5, $\mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = \mathbb{P}_{n,\rho}(\mathbf{X}_n) = \delta_n(\mathbf{X}_n)$. □

Thus, it remains to prove (3), i.e. that $\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1$. This will be done by proving, by induction on r, that for every $r = 0, \ldots, k$, $\lim_{n\to\infty} \mathbb{P}_{n,r}(\mathbf{X}_{n,r}) = 1$. In Definition 3.11 the notion of a substitution $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ of $\mathcal{A}$ for $\mathcal{B}$ inside $\mathcal{M}$ was defined. There it was assumed that the vocabulary of L is relational. However, eventual function or constant symbols in the vocabulary of L already belong to the vocabulary of $L_0 \subseteq L$, and, in what follows, we only consider substitutions when $\mathcal{A}$ and $\mathcal{B}$ agree on L_0 and on proper closed substructures (in the sense of Terminology 7.19). So in this context, substitutions $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$, according to Definition 3.11, make sense; and we will use them.

Lemma 10.5. *Let $0 \leq r < k$, $\mathcal{M} \in \mathbf{K}_n{\restriction}r{+}1$ and suppose that $\mathcal{A} \subseteq_{cl} \mathcal{M}$ and $\dim_{\mathcal{M}}(A) = r + 1$. Also assume that $\mathcal{B}$ is a represented structure with respect to $\mathbf{K}{\restriction}r + 1$ such that $\mathcal{B}$ and $\mathcal{A}$ agree on L_0 and on closed proper substructures. Then $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \in \mathbf{K}_n{\restriction}r + 1$.*

Proof. Let r, $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of the lemma, so in particular $\mathcal{A}{\restriction}L_0 = \mathcal{B}{\restriction}L_0$. Note that since $\mathcal{A}$ and $\mathcal{B}$ have dimension $r + 1$ it follows that $\mathcal{A}, \mathcal{B} \in \mathbf{K}$, because for every $\mathcal{C} \in \mathbf{K}$ with dimension at most $r + 1$ we have $\mathcal{C}{\restriction}r + 1 = \mathcal{C}$. By assumption, $\mathcal{A}$ and $\mathcal{B}$ agree on L_0 and on closed proper substructures. The assumption that $\mathbf{K}$ accepts k-substitutions over L_0 implies that there exists $\mathcal{N}' \in \mathbf{K}_n$ such that $\mathcal{N}'{\restriction}L_0 = \mathcal{N}{\restriction}L_0$, $\mathcal{N}'{\restriction}B = \mathcal{B}$ and for every $\mathcal{U} \subseteq_{cl}\subseteq \mathcal{N}'$ such that $\dim_{\mathcal{N}'}(U) \leq r + 1$ and $U \neq B$, we have $\mathcal{N}'{\restriction}U = \mathcal{N}{\restriction}U$. In particular, $\mathcal{N}'{\restriction}U = \mathcal{N}{\restriction}U$ for every U with dimension at most r.

Since $\mathcal{N}'{\restriction}r + 1 \in \mathbf{K}_n{\restriction}r + 1$ it suffices to show that $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] = \mathcal{N}'{\restriction}r + 1$. For this it is enough to show that for every closed substructure $\mathcal{C} \subseteq_{cl} \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ with dimension $r + 1$,

$$(*) \qquad \mathcal{N}'{\restriction}C = \mathcal{C}.$$

Suppose that $\mathcal{C} \subseteq_{cl} \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$. If $\mathcal{C} = \mathcal{B}$ then, by the choice of $\mathcal{N}'$, we have $\mathcal{N}'{\restriction}C = \mathcal{N}'{\restriction}B = \mathcal{B}$. If $\mathcal{C} \neq \mathcal{B}$ then, by the choice of $\mathcal{N}'$, we have $\mathcal{N}'{\restriction}C = \mathcal{N}{\restriction}C = \mathcal{C}$, where the last identity follows because $\mathcal{M} = \mathcal{N}{\restriction}r + 1$ and $\mathcal{C}$ has dimension $r + 1$; thus $(*)$ also holds in case when $\mathcal{C} \neq \mathcal{B}$. □

Lemma 10.6. *Let $0 \leq r < k$, $\mathcal{M} \in \mathbf{K}_n{\restriction}r + 1$ and suppose that $\mathcal{A} \subseteq_{cl} \mathcal{M}$ and $r < \dim_{\mathcal{M}}(A) \leq k$. Also assume that $\mathcal{B}$ represented structure with respect to $\mathbf{K}{\restriction}r + 1$ such that $\mathcal{B}{\restriction}L_0 = \mathcal{A}{\restriction}L_0$ and for every closed $U \subseteq A = B$ with dimension r, $\mathcal{A}{\restriction}U = \mathcal{B}{\restriction}U$. Then $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] \in \mathbf{K}_n{\restriction}r + 1$.*

Proof. Let r, $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of the lemma. By definition of $\mathbf{K}{\restriction}r + 1$, for every $\mathcal{N} \in \mathbf{K}{\restriction}r + 1$ and every relation symbol R which does not belong to the vocabulary of L_0, there is *no* R-relationship $\bar{a} \in R^{\mathcal{N}}$ with dimension greater than $r{+}1$. Consequently, the structure $\mathcal{M}[\mathcal{A} \triangleright \mathcal{B}]$ can be created by a finite number of substitutions of the kind considered in Lemma 10.5. More precisely: There are $\mathcal{N}_0, \ldots, \mathcal{N}_s \in \mathbf{K}_n{\restriction}r + 1$ and $\mathcal{C}_0, \ldots, \mathcal{C}_{2s}$ which dimension $r + 1$ such that

$$\mathcal{M} = \mathcal{N}_0, \; \mathcal{M}[\mathcal{A} \triangleright \mathcal{B}] = \mathcal{N}_s,$$
$$\mathcal{N}_{i+1} = \mathcal{N}_i[\mathcal{C}_{2i} \triangleright \mathcal{C}_{2i+1}], \text{ for } i = 1, \ldots, s, \text{ and}$$
$$\mathcal{C}_{2i} \text{ and } \mathcal{C}_{2i+1} \text{ agree on } L_0 \text{ and on closed proper substructures.}$$

By Lemma 10.5, $\mathcal{N}_i \in \mathbf{K}_n{\restriction}r + 1$, for $i = 0, \ldots, s$, so we are done. □

Lemma 10.7. *If $0 \leq r < k$ then for every $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ there is $\mathcal{M}' \in \mathbf{K}_n{\restriction}r + 1$ such that $\mathcal{M}'{\restriction}r = \mathcal{M}$.*

Proof. If $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ then $\mathcal{M} = \mathcal{N}{\restriction}r$ for some $\mathcal{N} \in \mathbf{K}_n$. Take $\mathcal{M}' = \mathcal{N}{\restriction}r + 1$. Then $\mathcal{M}' \in \mathbf{K}_n{\restriction}r + 1$ and $\mathcal{M}'{\restriction}r = \mathcal{N}{\restriction}r = \mathcal{M}$. □

Lemma 10.8. *For every n and very $\mathcal{M} \in \mathbf{K}_n{\restriction}0$, $\mathcal{M}$ is $\big(\lambda_n, k\big)$-saturated.*

Proof. Let $\mathcal{M} \in \mathbf{K}_n{\restriction}0$ and let $\mathcal{A} \subseteq_{cl} \mathcal{B}$ be permitted structures with respect to $\mathbf{K}{\restriction}0$ such that $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$. Suppose that $\mathcal{A}' \subseteq_{cl} \mathcal{M}$ is a copy of $\mathcal{A}$ and that $\tau : \mathcal{A}' \to \mathcal{A}$ is an isomorphism. We must show that there are $\mathcal{B}_i \subseteq_{cl} \mathcal{M}$ and isomorphisms $\tau_i : \mathcal{B}_i \to \mathcal{B}$, for $i = 1, \ldots, \lambda_n$, such that $\mathcal{A}' \subseteq_{cl} \mathcal{B}_i$, $\tau_i{\restriction}A' = \tau$ and $B_i \cap B_j = A'$ whenever $i \neq j$.

From Definition 7.12 it follows that $\mathcal{M}$ is an expansion of $\mathcal{M}{\restriction}L_0 = \mathcal{G}_n$ obtained by possibly adding some new relationships involving *only* elements in $\mathrm{cl}_{\mathcal{M}}(\emptyset)$ ($= \mathrm{cl}_{\mathcal{G}_n}(\emptyset)$). Therefore it is sufficient to find $\mathcal{B}_i \subseteq_{cl} \mathcal{M}{\restriction}L_0 = \mathcal{G}_n$ and isomorphisms $\tau_i : \mathcal{B}_i \to \mathcal{B}{\restriction}L_0$, for $i = 1, \ldots, \lambda_n$, such that $\mathcal{A}'{\restriction}L_0 \subseteq_{cl} \mathcal{B}_i$, $\tau_i{\restriction}A' = \tau$ and $B_i \cap B_j = A'$ whenever $i \neq j$. But the existence of such $\mathcal{B}_i$ and τ_i is guaranteed by (b), in the beginning of the proof. $\square$

Lemma 10.9. *Suppose that $0 \leq r < k$. For every real $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that if $n \geq n_\varepsilon$, $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated and*

$$\mathbf{E}_{r+1}(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}_n{\restriction}r+1 : \mathcal{N}{\restriction}r = \mathcal{M}\big\},$$

then the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at least $1 - \varepsilon$.

Proof. Let $0 \leq r < k$. We are assuming that $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a uniformly bounded pregeometry. Hence there is $\alpha \in \mathbb{N}$ such that if $\mathcal{A}$ is permitted with respect to $\mathbf{K}{\restriction}r+1$ and has dimension at most k, then $|A| \leq \alpha$. Suppose that $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated and let $\mathbf{E}_{r+1}(\mathcal{M}) = \big\{\mathcal{N} \in \mathbf{K}_n{\restriction}r+1 : \mathcal{N}{\restriction}r = \mathcal{M}\big\}$. We start by proving that, with the uniform probability measure on $\mathbf{E}_{r+1}(\mathcal{M})$, the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated approaches 1 as n tends to ∞. We do this by finding an upper bound (depending on n) for the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is *not* $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated; and then observe that this upper bound approaches 0 as n tends to infinity. Finally we note that the argument does not depend on which $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n{\restriction}r$ we consider; so given $\varepsilon > 0$ there is n_ε which such that for every $n \geq n_\varepsilon$ and every $\big(\sigma^r(\lambda_n), k\big)$-saturated $\mathcal{M} \in \mathbf{K}_n{\restriction}r$, the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are *not* $\big(\sigma^{r+1}(\lambda_n), k\big)$-saturated is at most ε.

Let $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ and let $\mathcal{A} \subset_{cl} \mathcal{B}$ be represented structures with respect to $\mathbf{K}{\restriction}r+1$ such that $\dim_{\mathcal{B}}(A) + 1 = \dim_{\mathcal{B}}(B) \leq k$. Suppose that $\mathcal{A}' \subseteq_{cl} \mathcal{N}$ is a copy of $\mathcal{A}$ and that $\tau : \mathcal{A}' \to \mathcal{A}$ is an isomorphism. Let $l_n = \lfloor\sqrt{\sigma^r(\lambda_n)}\rfloor = \sigma^{r+1}(\lambda_n)$. First we find an upper bound for the probability that there does not exist $\mathcal{B}_i \subseteq_{cl} \mathcal{N}$ and isomorphisms $\tau_i : \mathcal{B}_i \to \mathcal{B}$, for $i = 1, \ldots, l_n$, such that $\mathcal{A}' \subseteq_{cl} \mathcal{B}_i$, $\tau_i{\restriction}A' = \tau$, and $B_i \cap B_j = A'$ whenever $i \neq j$.

Let $l'_n = \sigma^r(\lambda_n)$. Since $\mathcal{M}$ is $\big(\sigma^r(\lambda_n), k\big)$-saturated there are $\mathcal{B}_i^- \subseteq_{cl} \mathcal{M}$, $i = 1, \ldots, l'_n$ and isomorphisms $\tau_i : \mathcal{B}_i^- \to \mathcal{B}{\restriction}r$, such that $\mathcal{A}'{\restriction}r \subseteq_{cl} \mathcal{B}_i^-$, $\tau_i{\restriction}A' = \tau$ and $B_i^- \cap B_j^- = A'$ whenever $i \neq j$. Let β be the number of represented structures with respect $\mathbf{K}{\restriction}r+1$ with universe included in $\{1, \ldots, \alpha\}$.

Lemma 10.6 implies that the probability that the map $\tau_i : B_i^- \to B$ is an isomorhpism from $\mathcal{N}{\restriction}B_i^-$ to $\mathcal{B}$ is at least $1/\beta$, independently of whether this holds for $j \neq i$. Let s be a natural number such that $0 \leq s < l_n$. The probability that for *every* $i \in \{sl_n + i, \ldots, (s+1)l_n\}$, $\tau_i : B_i^- \to B$ is *not* an isomorhpism from $\mathcal{N}{\restriction}B_i^-$ to $\mathcal{B}$ is at most

$$\big(1 - 1/\beta\big)^{l_n}.$$

Let $m_n = |G_n| = |N|$. By (a) $\lambda_n \leq m_n \leq P(\lambda_n)$ for all $n \in \mathbb{N}$, where P is a polynomial. Since, by assumption, $\lim_{n\to\infty} \lambda_n = \infty$, we have $\lim_{n\to\infty} m_n = \infty$. From the definition of l_n as $l_n = \sigma^{r+1}(\lambda_n)$ and the definition of σ^{r+1} it follows that there is a polynomial Q such that $m_n \leq Q(l_n)$. The number of ways in which we can choose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and s as above is not larger than

$$\beta^2 \cdot (m_n)^\alpha \cdot l_n \quad \leq \quad \beta^2 \cdot (Q(l_n))^\alpha \cdot l_n.$$

Moreover, for every choice of such $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$ and s, there exist, for $i = 1, \ldots, l'_n$, $\mathcal{B}_i^- \subseteq_{cl} \mathcal{M}$ and isomorphisms $\tau_i : \mathcal{B}_i^- \to \mathcal{B}$, with the properties described above. So if $\mathcal{N}$ is not $(\sigma^{r+1}(\lambda_n), k)$-saturated, then there exist $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}'$, $\mathcal{B}_i^-$, τ_i, for $i = 1, \ldots, l'_n$, and s as above such that for every $i \in \{sl_n + 1, \ldots, (s+1)l_n\}$, τ_i is not an isomorphism from $\mathcal{N} \restriction B_i^-$ to $\mathcal{B}$. Hence, the probability that a randomly chosen $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ is not $(\sigma^{r+1}(\lambda_n), k)$-saturated does *not* exceed

$$f_n = \beta^2 \cdot (Q(l_n))^\alpha \cdot l_n \cdot \big(1 - 1/\beta\big)^{l_n}.$$

Since $\lambda_n \to \infty$ as $n \to \infty$ we also have $l_n \to \infty$ as $n \to \infty$. Because $\beta^2 \cdot (Q(l_n))^\alpha \cdot l_n$ is a polynomial in l_n it follows that $f_n \to 0$ as $n \to \infty$.

Observe that the same expression for f_n works for every $(\sigma^r(\lambda_n), k)$-saturated $\mathcal{M} \in \mathbf{K}_n \restriction r$. So for every $\varepsilon > 0$ there is n_ε such that for every $n \geq n_\varepsilon$ and every $(\sigma^r(\lambda_n), k)$-saturated $\mathcal{M} \in \mathbf{K}_n \restriction r$, the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $(\sigma^{r+1}(\lambda_n), k)$-saturated is at least $1 - \varepsilon$. □

Recall that, for $r = 0, 1, \ldots, k$,

$$\mathbf{X}_{n,r} = \{\mathcal{M} \in \mathbf{K}_n \restriction r : \mathcal{M} \text{ is } (\sigma^r(\lambda_n), k)\text{-saturated}\}.$$

From Lemma 10.9 we can easily derive the following:

Lemma 10.10. *For every $r = 0, 1, \ldots, k-1$ and all sufficiently large n (take $0 < \varepsilon < 1/2$, n_ε and $n > n_\varepsilon$ so that the conclusion of Lemma 10.9 holds),*

$$\mathbf{X}_{n,r} \subseteq \{\mathcal{N} \restriction r : \mathcal{N} \in \mathbf{X}_{n,r+1}\}.$$

Proof. Suppose that $\mathcal{M} \in \mathbf{X}_{n,r}$, so $\mathcal{M}$ is $(\sigma^r(\lambda_n), k)$-saturated. By Lemma 10.9, for all sufficiently large n, $\mathbf{E}_{r+1}(\mathcal{M})$ will contain a stucture $\mathcal{N}$ which is $(\sigma^{r+1}(\lambda_n), k)$-saturated; hence $\mathcal{N} \in \mathbf{X}_{n,r+1}$ and $\mathcal{N} \restriction r = \mathcal{M}$. □

Now we can finish the proof of part (i) of Theorem 7.28 by proving (3), in other words, that $\lim_{n\to\infty} \mathbb{P}_{n,k}(\mathbf{X}_{n,k}) = 1$. Let $\varepsilon > 0$. Choose $\varepsilon' > 0$ so that $(1 - \varepsilon')^k \geq 1 - \varepsilon$. By Lemma 10.9, we can choose $n_{\varepsilon'}$ such that if $0 \leq r < k$, $n > n_{\varepsilon'}$ and $\mathcal{M} \in \mathbf{K}_n \restriction r$ is $(\sigma^r(\lambda_n), k)$-saturated, then the proportion of $\mathcal{N} \in \mathbf{E}_{r+1}(\mathcal{M})$ which are $(\sigma^{r+1}(\lambda_n), k)$-saturated is at least $1 - \varepsilon'$. By induction we show that, for $r = 0, 1, \ldots, k$ and $n > n_{\varepsilon'}$,

$$\mathbb{P}_{n,r}(\mathbf{X}_{n,r}) \geq (1 - \varepsilon')^r \geq 1 - \varepsilon.$$

The base case $r = 0$ is given by Lemma 10.8, so assume that $0 < r \leq k$ and that $\mathbb{P}_{n,r-1}(\mathbf{X}_{n,r-1}) \geq (1 - \varepsilon')^{r-1}$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_s$ be an enumeration, without repetition, of $\mathbf{X}_{n,r}$. Then let $\mathcal{M}'_1, \ldots, \mathcal{M}'_t$ be an enumeration, without repetition, of the set $\{\mathcal{M}_1 \restriction$

$r-1,\dots,\mathcal{M}_s{\restriction}r-1\}$. By the definition of $\mathbb{P}_{n,r}$, the following holds for every $n > n_{\varepsilon'}$:

$$\begin{aligned}
\mathbb{P}_{n,r}(\mathbf{X}_{n,r}) &= \mathbb{P}_{n,r}\big(\{\mathcal{M}_1,\dots,\mathcal{M}_s\}\big) = \sum_{i=1}^{s} \mathbb{P}_{n,r}(\mathcal{M}_i) \\
&= \sum_{i=1}^{s} \frac{1}{\big|\{\mathcal{N} \in \mathbf{K}_n{\restriction}r : \mathcal{N}{\restriction}r-1 = \mathcal{M}_i{\restriction}r-1\}\big|} \cdot \mathbb{P}_{n,r-1}(\mathcal{M}_i{\restriction}r-1) \\
&= \sum_{i=1}^{t} \frac{\big|\{\mathcal{N} \in \mathbf{X}_{n,r} : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|}{\big|\{\mathcal{N} \in \mathbf{K}_n{\restriction}r : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|} \cdot \mathbb{P}_{n,r-1}(\mathcal{M}'_i) \\
&= \sum_{i=1}^{t} \frac{\big|\{\mathcal{N} \in \mathbf{X}_{n,r} : \mathcal{N}{\restriction}r-1 = \mathcal{M}'_i\}\big|}{\big|\mathbf{E}_r(\mathcal{M}'_i)\big|} \cdot \mathbb{P}_{n,r-1}(\mathcal{M}'_i) \\
&\geq (1-\varepsilon') \sum_{i=1}^{t} \mathbb{P}_{n,r-1}(\mathcal{M}'_i) \qquad \text{(by the choice of } n_{\varepsilon'}) \\
&= (1-\varepsilon')\mathbb{P}_{n,r-1}\big(\{\mathcal{M}'_1,\dots,\mathcal{M}'_t\}\big) \\
&\geq (1-\varepsilon')\mathbb{P}_{n,r-1}(\mathbf{X}_{n,r-1}) \qquad \text{(by Lemma 10.10)} \\
&\geq (1-\varepsilon')(1-\varepsilon')^{r-1} = (1-\varepsilon')^{r} \qquad \text{(by the induction hypothesis).}
\end{aligned}$$

Thus (3) is proved, and hence also part (i) of Theorem 7.28.

Now we prove part (ii) of Theorem 7.28. Note that we have proved (2) above, because (3) together with Lemma 10.3 implies (2). By (2), there are, for all $n \in \mathbb{N}$, $\mathcal{M}_n \in \mathbf{K}_n$ such that $\mathcal{M}_n$ is $(\sigma^k(\lambda_n), k)$-saturated. Let $\mu_n = \sigma^k(\lambda_n)$, so $\mathcal{M}_n$ is (μ_n, k)-saturated, where $\lim_{n\to\infty} \mu_n = \infty$. From (a) and the definition of σ^k it follows that there is a polynomial Q such that $\mu_n \leq |M_n| \leq Q(\mu_n)$ for all n. Since $\mathcal{M}_n$ is (μ_n, k)-saturated, the following holds: If $\mathcal{A} \subset_{cl} \mathcal{B}$ are represented structures such that $\dim_{\mathcal{B}}(B) \leq k$, then the $\mathcal{B}/\mathcal{A}$-multiplicity of $\mathcal{M}_n$ is at least μ_n. From Assumption 7.10 (4), it follows that the sequence $(\mathcal{M}_n : n \in \mathbb{N})$ is polynomially k-saturated; and hence $\mathbf{K}$ is polynomially k-saturated. This concludes the proof of part (ii), and hence of Theorem 7.28.

10.2. **Proof of Theorem 7.29.** We still assume that, for every $k > 0$, $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated and $\mathbf{K} = \bigcup_{n\in\mathbb{N}} \mathbf{K}_n$, where $\mathbf{K}_n = \mathbf{K}(\mathcal{G}_n)$, accepts k-substitutions over L_0. We want to prove that for every L-sentence φ, either $\lim_{n\to\infty} \delta_n(\varphi) = 0$ or $\lim_{n\to\infty} \delta_n(\varphi) = 1$. The general idea of the proof follows a well-known pattern: we collect into a theory $T_{\mathbf{K}}$ all extension axioms of $\mathbf{K}$ together with sentences which express the pregeometry conditions and describe the possible isomorphism types of closed substructures of members of $\mathbf{K}$. By part (i) of Theorem 7.28, $T_{\mathbf{K}}$ is consistent. Then we show that $T_{\mathbf{K}}$ is complete by showing that it is countably categorical. From the completeness, it follows that for every L-sentence φ, either $T_{\mathbf{K}} \models \varphi$ or $T_{\mathbf{K}} \models \neg\varphi$. In the first case there is finite $\Delta \subset T_{\mathbf{K}}$ such that $\Delta \models \varphi$ and in the second case there is finite $\Delta' \subseteq T_{\mathbf{K}}$ such that $\Delta' \models \neg\varphi$. In the first case part (i) of Theorem 7.28 implies that

$$\lim_{n\to\infty} \delta_n(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \Delta\}) = 1,$$

and therefore $\lim_{n\to\infty} \delta_n(\varphi) = 1$. In the second case we get, in a similar way, that $\lim_{n\to\infty} \delta_n(\neg\varphi) = 1$, so $\lim_{n\to\infty} \delta_n(\varphi) = 0$.

Now to the details. We are assuming that $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ is a pregeometry where the closure operator all members of $\mathbf{G}$ are defined by the L_0-formulas $\theta_n(x_1,\dots,x_{n+1})$, according to Definition 7.1 and Assumption 7.10. In other words, for all m, n and all $a_1,\dots,a_{n+1} \subseteq G_m$,

(4) $$a_{n+1} \in \mathrm{cl}_{\mathcal{G}_m}(a_1,\dots,a_n) \text{ if and only if } \mathcal{G}_m \models \theta(a_1,\dots,a_{n+1}).$$

Also (by Assumption 7.10), for every m and every $\mathcal{M} \in \mathbf{K}_m = \mathbf{K}(\mathcal{G}_m)$, $\mathrm{cl}_{\mathcal{M}}$ coincides with $\mathrm{cl}_{\mathcal{G}_m}$. Moreover, the pregeometry $\mathbf{G}$ is assumed to be *uniformly* locally finite, so there is $u : \mathbb{N} \to \mathbb{N}$ such that for every every $\mathcal{M} \in \mathbf{K}$ and every $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$. We may also assume that for every $k \in \mathbb{N}$ the value $u(k)$ is minimal so that this holds.

By the ***finiteness property***, for a pregeometry (A, cl), we mean the property that for all $a \in A$ and $X \subseteq A$, $a \in \mathrm{cl}(X)$ if and only if $a \in \mathrm{cl}(Y)$ for some *finite* $Y \subseteq X$. Besides the finiteness property, all other properties of a pregeometry can, when (4) holds, be expressed for finite subsets of A by using the formulas $\theta_n(x_1, \dots, x_{n+1})$, $n \in \mathbb{N}$. Let T_{preg} be the set of sentences which express all properties of a pregeometry (for finite subsets) except the finiteness property. Then every $\mathcal{M} \in \mathbf{K}$ is a model of T_{preg}.

Note that, for every $\mathcal{M} \in \mathbf{K}$ and all $a_1, \dots, a_n \in M$, the statement "$\{a_1, \dots, a_n\}$ is a closed set (in $\mathcal{M}$)" is uniformly expressed by the first-order formula

$$\neg \exists x_{n+1} \Big(\bigwedge_{i=1}^{n} x_{n+1} \neq x_i \ \wedge \ \theta_n(x_1, \dots, x_{n+1}) \Big),$$

which we denote by $\gamma_n(x_1, \dots, x_n)$. Let $\mathcal{M}_1, \dots, \mathcal{M}_s$ be an enumeration of all isomorphism types of closed substructures with cardinality at most m of members of $\mathbf{K}$. Let $\chi_i(x_1, \dots, x_m)$ describe the isomorphism type of $\mathcal{M}_i$ in such a way that we require that all variables $x_1, \dots, x_m$ actually occur in χ_i. It means that if $\|\mathcal{M}_i\| < m$, then $\chi_i(x_1, \dots, x_m)$ must express that some variables refer to the same element, by saying '$x_k = x_l$' for some $k \neq l$. For every $k \in \mathbb{N}$ let ψ_k denote the sentence

$$\forall x_1, \dots, x_k \exists x_{k+1}, \dots, x_{u(k)} \Big(\gamma_{u(k)}(x_1, \dots, x_{u(k)}) \ \wedge \ \bigvee_{i=1}^{s} \bigvee_{\pi} \chi_i(x_{\pi(1)}, \dots, x_{\pi(u(k))}) \Big),$$

where the second disjunction ranges over all permutations π of $\{1, \dots, u(k)\}$. If $k = 0$ and $u(k) > 0$, then the universal quantifiers do not occur so ψ_0 is an existential formula. If $u(0) = 0$, then, by convention, ψ_0 is $\forall x(x = x)$. If $u(k) = k$, then the existential quantifiers do not occur and ψ_k is a universal formula. Note that for every $k \in \mathbb{N}$ and every $\mathcal{M} \in \mathbf{K}$, $\mathcal{M} \models \psi_k$. Let $T_{iso} = \{\psi_k : k \in \mathbb{N}\}$ so every $\mathcal{M} \in \mathbf{K}$ is a model of T_{iso}.

Finally, let T_{ext} consist (exactly) of all extension axioms of $\mathbf{K}$ and let

$$T_{\mathbf{K}} = T_{preg} \cup T_{iso} \cup T_{ext}.$$

By Theorem 7.28 and compactness, $T_{\mathbf{K}}$ is consistent. Note that every model of $T_{\mathbf{K}}$ is infinite, because we assume that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated (for every $k > 0$), which implies that for some sequence $(\lambda_n : n \in \mathbb{N})$ which tends to infinity as $n \to \infty$, $\mathcal{G}_n$ contains at least λ_n different elements.

Lemma 10.11. *Suppose that $\mathcal{M} \models T_{\mathbf{K}}$ and define $\mathrm{cl}_{\mathcal{M}}$ as follows:*

(a) *for all $n \in \mathbb{N}$ and all $a_1, \dots, a_{n+1} \in M$, $a_{n+1} \in \mathrm{cl}_{\mathcal{M}}(a_1, \dots, a_n) \iff \mathcal{M} \models \theta_n(a_1, \dots, a_{n+1})$.*

(b) *for all $X \subseteq M$ and all $a \in M$, $a \in \mathrm{cl}_{\mathcal{M}}(X) \iff$ for some finite $Y \subseteq X$, $a \in \mathrm{cl}_{\mathcal{M}}(Y)$.*

Then $(M, \mathrm{cl}_{\mathcal{M}})$ is a pregeometry such that for every finite $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$.

Proof. Suppose that $\mathcal{M} \models T_{\mathbf{K}}$. Since $T_{preg} \subseteq T_{\mathbf{K}}$, it follows from part (a) that $\mathrm{cl}_{\mathcal{M}}$ satisfies all properties of a pregeometry on finite subsets of M. But (b) guarantees that $\mathrm{cl}_{\mathcal{M}}$ has the finiteness property, and then all other properties follow for all subsets of M. So $(M, \mathrm{cl}_{\mathcal{M}})$ is a pregeometry. Since $T_{iso} \subset T_{\mathbf{K}}$ it follows that, for every $X \subseteq M$, $|\mathrm{cl}_{\mathcal{M}}(X)| \leq u(\dim_{\mathcal{M}}(X))$. □

To complete the proof of Theorem 7.29 we only need to prove:

Lemma 10.12. *$T_{\mathbf{K}}$ is countably categorical and hence complete.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be countable models of $T_{\mathbf{K}}$. We show that $\mathcal{M} \cong \mathcal{N}$, by a back-and-forth argument. By symmetry it is sufficient to show the following:

> Suppose that $\mathcal{A}$ is a closed finite substructure of $\mathcal{M}$ (or $A = \emptyset$), that $\mathcal{B}$ is a closed finite substructure of $\mathcal{N}$ (or $B = \emptyset$), that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism (if A and B are nonempty) and that $a \in M - A$. Then there are a closed $\mathcal{B}' \subseteq M$ such that $B \subset B'$ and an isomorphism $g : \mathrm{cl}_{\mathcal{M}}(A \cup \{a\}) \to \mathcal{B}'$ which extends f.

So suppose that $\mathcal{A}$ is a closed finite substructure of $\mathcal{M}$, that $\mathcal{B}$ is a closed finite substructure of $\mathcal{N}$, that $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism and that $a \in M - A$. Since $\mathcal{M} \models T_{\mathbf{K}} \supset T_{iso}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathrm{cl}_{\mathcal{M}}(A \cup \{a\})$ are isomorphic with closed substructures of members of $\mathbf{K}$. Since $\mathcal{N} \models T \supset T_{ext}$, it follows that $\mathcal{N}$ satisfies the $\mathrm{cl}_{\mathcal{M}}(A \cup \{a\})/\mathcal{A}$-extension axiom, and as $\mathcal{B} \cong \mathcal{A}$ there is a closed $\mathcal{B}' \subset \mathcal{N}$ such that $\mathcal{B} \subset \mathcal{B}'$ and an isomorphism $g : \mathrm{cl}_{\mathcal{M}}(A \cup \{a\}) \to \mathcal{B}'$ which exends f. Recall the convention that for every structure $\mathcal{P}$ which is isomorphic with a closed substructure of a member of $\mathbf{K}$, the statement "there exists a closed copy of $\mathcal{P}$" is an extension axiom, called the $\mathcal{P}/\emptyset$-extension axiom; this takes care of the case $A = B = \emptyset$. □

10.3. **Proof of Theorem 7.31.** Let $\mathbf{G} = \{\mathcal{G}_n : n \in \mathbb{N}\}$ be a set of L_0-structures which form a uniformly bounded pregeometry, and suppose that $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially k-saturated for every $k \in \mathbb{N}$. Assume that there is, up to isomorphism, a unique represented structure with dimension 0; hence $\mathbf{K}$ accepts 0-substitutions over L_0. Suppose that k is *minimal* such that $\mathbf{K}$ does *not* accept k-substitutions over L_0; hence $k > 0$ and $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0. Moreover assume that there are represented structures, with respect to $\mathbf{K}$, $\mathcal{A}$ and $\mathcal{A}'$ such that

- $\mathcal{A}$ and $\mathcal{A}'$ have dimension k,
- $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 and on closed proper substructures,
- $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \rhd \mathcal{A}]$ over L_0, but
- $\mathbf{K}$ does *not* accept the substitution $[\mathcal{A} \rhd \mathcal{A}']$ over L_0.

Let ρ be the supremum of the arities of all relation symbols which belong to the vocabulary of L but not to the vocabulary of L_0. By Remark 7.21, $0 < k \leq \rho$.

In order to prove Theorem 7.31, we assume that $\mathbf{K}$ has the independent amalgamation property and show that there are extension axioms φ and ψ such that $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$. We start with the following, which is straightforward to verify:

Observation 10.13. *For every L-structure $\mathcal{M}$ and $d \in \mathbb{N}$, $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}d$ if and only if there is $\mathcal{M}'$ such that $\mathcal{M}'$ is represented with respect to $\mathbf{K}$ and $\mathcal{M} = \mathcal{M}'{\restriction}d$.*

Note that the notion of 'acceptance of l-substitutions over L_0', which was defined for $\mathbf{K}$, can equally well be defined for $\mathbf{K}{\restriction}r$ for any r; the only difference is that the notion 'represented' is in this case with respect to $\mathbf{K}{\restriction}r$. By assumption, $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0. From Observation 10.13 it follows that $\mathbf{K}{\restriction}(k-1)$ accepts $(k-1)$-substitutions over L_0. Note that for every $\mathcal{M} \in \mathbf{K}{\restriction}k-1$ and every relation symbol R in the vocabulary of L but not in the vocabulary of L_0, $\mathcal{M}$ does not have any R-relationship with dimension greater than $k-1$. From this and the assumption that $\mathbf{K}$ accepts $(k-1)$-substitutions over L_0 it follows that

(5) $\mathbf{K}{\restriction}(k-1)$ accepts l-substitutions over L_0 for *every* $l \in \mathbb{N}$.

By assumption, $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \rhd \mathcal{A}]$, and by Observation 10.13 it follows that $\mathbf{K}{\restriction}k$ accepts the substitution $[\mathcal{A}' \rhd \mathcal{A}]$.

Since $\mathcal{A}$ and $\mathcal{A}'$ agree on L_0 it makes sense to speak about the substitution $\mathcal{M}[\mathcal{A} \rhd \mathcal{A}]$ if $\mathcal{A} \subseteq_{cl} \mathcal{M}$, or $\mathcal{M}[\mathcal{A}' \rhd \mathcal{A}]$ if $\mathcal{A}' \subseteq_{cl} \mathcal{M}$, as was explained in the paragraph before

Lemma 10.5. Since $\mathcal{A}$ and $\mathcal{A}'$ have dimension k and $\mathbf{K}$ accepts the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$ over L_0, it follows from that if $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}k$, and $\mathcal{A}' \subseteq_{cl} \mathcal{M}$, then $\mathcal{M}[\mathcal{A}' \triangleright \mathcal{A}]$ is represented with respect to $\mathbf{K}{\restriction}k$. In other words, $\mathbf{K}{\restriction}k$ *admits* the substitution $[\mathcal{A}' \triangleright \mathcal{A}]$. By assumption, $\mathbf{K}$ does not accept the substitution $[\mathcal{A} \triangleright \mathcal{A}']$. Therefore we can argue similarly as we just did for the substitution $[\mathcal{A} \triangleright \mathcal{A}']$ to conclude that there is $\mathcal{P}$ such that $\mathcal{P}$ is represented with respect to $\mathbf{K}{\restriction}k$, $\mathcal{A} \subset_{cl} \mathcal{P}$ and $\mathcal{P}[\mathcal{A} \triangleright \mathcal{A}']$ is forbidden with respect to $\mathbf{K}{\restriction}k$.

Since the core of the argument (the proof of Lemma 10.16 below) is an adaptation of the proof of Theorem 3.17 to the present context, we introduce the same notation as in Section 5. We rename $\mathcal{A}$ and $\mathcal{A}'$ with $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$, so in particular $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}$ have dimension k. As concluded above, $\mathbf{K}{\restriction}k$ *admits* the substitution $[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$, in the sense that whenever $\mathcal{M}$ is represented with respect to $\mathbf{K}{\restriction}k$, then $\mathcal{M}[\mathcal{S}_\mathcal{F} \triangleright \mathcal{S}_\mathcal{P}]$ is represented with respect to $\mathbf{K}{\restriction}k$. Moreover, there is $\mathcal{P}$ such that $\mathcal{P}$ is represented with respect to $\mathbf{K}{\restriction}k$, $\mathcal{S}_\mathcal{P} \subseteq_{cl} \mathcal{P}$ and $\mathcal{F} = \mathcal{P}[\mathcal{S}_\mathcal{P} \triangleright \mathcal{S}_\mathcal{F}]$ is *forbidden* with respect to $\mathbf{K}{\restriction}k$. This implies that the dimension of $\mathcal{P}$ is strictly larger than the dimension of $\mathcal{S}_\mathcal{P}$ which is k.

By Observation 10.13, there is $\widehat{\mathcal{P}}$ which is represented with respect to $\mathbf{K}$ and such that $\widehat{\mathcal{P}}{\restriction}k = \mathcal{P}$. We are assuming that $\mathbf{K}$ has the independent amalgamation property. Hence, there are a represented $\mathcal{C}$, with respect to $\mathbf{K}$, and embeddings $\tau_i : \widehat{\mathcal{P}} \to \mathcal{C}$, for $i = 1, 2$, such that $\tau_1{\restriction}|\mathcal{S}_\mathcal{P}| = \tau_2{\restriction}|\mathcal{S}_\mathcal{P}|$ and $|\mathcal{S}_\mathcal{P}| = \tau_1(|\mathcal{P}|) \cap \tau_2(|\mathcal{P}|)$; so in particular $\mathcal{S}_\mathcal{P} \subset_{cl} \mathcal{C}$. By replacing $\mathcal{C}$ with the closure of $\tau_1(|\widehat{\mathcal{P}}|) \cup \tau_2(|\widehat{\mathcal{P}}|)$ in $\mathcal{C}$, we may assume that $\dim_\mathcal{C}(|\mathcal{C}|) = 2\dim_\mathcal{P}(|\mathcal{P}|) - \dim_{\mathcal{S}_\mathcal{P}}(|\mathcal{S}_\mathcal{P}|) = 2\dim_\mathcal{P}(|\mathcal{P}|) - k$. Let $c = \dim_\mathcal{C}(|\mathcal{C}|)$. Since $\dim_\mathcal{P}(|\mathcal{P}|) > k > 0$ (as noted above), we have $c > \dim_\mathcal{P}(|\mathcal{P}|) > k > 0$, so $c \geq 3$.

If $k = 1$ then let $\mathcal{U}$ be the unique closed proper substructure of $\mathcal{S}_\mathcal{F}$ with dimension 0. If $k > 1$ then let $\mathcal{U}$ be any closed proper substructure of $\mathcal{S}_\mathcal{F}$ with dimension 1. In both cases $\mathcal{U}$ is represented with respect to $\mathbf{K}$, with respect to $\mathbf{K}{\restriction}k$, and with respect to $\mathbf{K}{\restriction}k-1$.

Let φ denote the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and let ψ denote the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom. We prove that $\lim_{n\to\infty} \delta_n(\varphi \wedge \psi) = 0$. Let $\mathcal{C}' = \mathcal{C}{\restriction}k$, so $\mathcal{C}'$ is represented with respect to $\mathbf{K}{\restriction}k$, and note that since the dimension of $\mathcal{S}_\mathcal{P}$ and of $\mathcal{S}_\mathcal{F}$ is k and $\mathcal{U} \subset_{cl} \mathcal{S}_\mathcal{F}$ we have $\mathcal{U}{\restriction}k = \mathcal{U}$, $\mathcal{S}_\mathcal{P}{\restriction}k = \mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}{\restriction}k = \mathcal{S}_\mathcal{F}$. The next lemma shows that instead of working with $\mathbf{K}$, φ and ψ we can work with $\mathbf{K}{\restriction}k$, the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom.

Lemma 10.14. *Let p be the probability, with the measure δ_n, that a structure in $\in \mathbf{K}_n$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom ($= \varphi$) and the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom ($= \psi$). Let q be the probability, with the measure $\mathbb{P}_{n,k}$, that a structure in $\mathbf{K}_n{\restriction}k$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom. Then $p \leq q$.*

Proof. Recall that $k \leq \rho$. By the definitions of $\mathbb{P}_{n,k}$ and δ_n, for every $\mathcal{M} \in \mathbf{K}_n{\restriction}k$,

$$\mathbb{P}_{n,k}(\mathcal{M}) = \delta_n\big(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N}{\restriction}k = \mathcal{M}\}\big).$$

As mentioned above, $\mathcal{S}_\mathcal{P}{\restriction}k = \mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{F}{\restriction}k = \mathcal{S}_\mathcal{F}$. So whenever $\mathcal{N} \in \mathbf{K}_n$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom, then $\mathcal{N}{\restriction}k$ satisfies the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom. And whenever $\mathcal{N} \in \mathbf{K}_n$ satisfies the $\mathcal{C}/\mathcal{S}_\mathcal{P}$-extension axiom, then $\mathcal{N}{\restriction}k$ satisfies the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom. Therefore p cannot exceed q. □

By Lemma 10.14 it suffices to prove that

(6) there is $\beta < 1$ such that for all sufficiently large n the probability, with the measure $\mathbb{P}_{n,k}$, that a structure in $\mathbf{K}_n{\restriction}k$ satisfies both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom does not exceed β; and if $k > 1$, then this probability tends to 0 as $n \to \infty$.

The claim (6) follows from the next two lemmas and the definition of the measures $\mathbb{P}_{n,r}$, $r \in \mathbb{N}$. Remember that c is the dimension of $\mathcal{C}$ (and of $\mathcal{C}'$).

Lemma 10.15. *The probability, with the measure $\mathbb{P}_{n,k-1}$, that a structure in $\mathbf{K}_n \restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K} \restriction k-1$, tends to 1 as $n \to \infty$.*

Lemma 10.16. *Let α be the number of represented structures with universe $|\mathcal{S}_\mathcal{F}|$. Suppose that $\mathcal{M} \in \mathbf{K}_n \restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated with respect to $\mathbf{K} \restriction k-1$ and let*

$$\mathbf{E}_k(\mathcal{M}) = \{\mathcal{N} \in \mathbf{K} \restriction k : \mathcal{N} \restriction k-1 = \mathcal{M}\}.$$

(i) The proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom never exceeds $1 - 1/(1+\alpha)$.
(ii) If $k > 1$ then the proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom never exceeds $\alpha \|\mathcal{S}_\mathcal{F}\| / \sigma^c(\lambda_n)$. Note that this expression does not depend on $\mathcal{M}$ and approaches 0 as $n \to \infty$.

Proof of Lemma 10.15 Note that when saying that $\mathbf{K} \restriction k-1$ accepts r-substitutions over L_0 we only consider substitutions of the form $[\mathcal{A} \triangleright \mathcal{A}']$ where $\mathcal{A}$ and $\mathcal{A}'$ are represented with respect to $\mathbf{K} \restriction k-1$.

Let $\mathbf{K}'_n = \mathbf{K}_n \restriction k-1$ and $\mathbf{K}' = \mathbf{K} \restriction k-1$. Let $\mathbb{P}'_{n,0}$ be the uniform measure on $\mathbf{K}'_n \restriction 0$ and for positive $r \in \mathbb{N}$, let $\mathbb{P}'_{n,r}$ be the $(\mathbf{K}'_n \restriction 0, \ldots, \mathbf{K}'_n \restriction r-1)$-conditional measure on $\mathbf{K}'_n \restriction r$. Observe that we have the following:

For $r \leq k-1$, $\mathbf{K}'_n \restriction r = \mathbf{K}_n \restriction r$ and $\mathbb{P}'_{n,r}$ coincides with $\mathbb{P}_{n,r}$

For $r \geq k-1$, $\mathbf{K}'_n \restriction r = \mathbf{K}'_n = \mathbf{K}_n \restriction k-1$ and $\mathbb{P}'_{n,r}$ coincides with $\mathbb{P}'_{n,k-1}$

As $c > k-1$, we in particular have

$\mathbf{K}'_n \restriction c = \mathbf{K}'_n = \mathbf{K}_n \restriction k-1$

and $\mathbb{P}'_{n,c}$ coincides with $\mathbb{P}'_{n,k-1}$ which in turn coincides with $\mathbb{P}_{n,k-1}$.

So $\mathbb{P}'_{n,c}$ and $\mathbb{P}_{n,k-1}$ are the same measure on $\mathbf{K}'_n \restriction c = \mathbf{K}_n \restriction k-1$. Thus, in order to prove Lemma 10.15 it suffices to show that the probability, with the measure $\mathbb{P}'_{n,c}$, that a structure in $\mathbf{K}'_n \restriction c$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}' \restriction c$, tends to 1 as $n \to \infty$.

But as mentioned in the beginning of the proof, $\mathbf{K}'$ ($= \mathbf{K} \restriction k-1$) accepts r-substitutions over L_0 for every $r \in \mathbb{N}$, so in particular for $r = c$. By assumption, $(\mathcal{G}_n : n \in \mathbb{N})$ is polynomially c-saturated. Consequently, if

$$\mathbf{X}'_{n,c} = \{\mathcal{M} \in \mathbf{K}'_n \restriction c : \mathcal{M} \text{ is } (\sigma^c(\lambda_n), c)\text{-saturated}\},$$

then, by the proof of (3) in the proof of Theorem 7.28, applied to $\mathbb{P}'_{n,c}$ and $\mathbf{X}'_{n,c}$ instead of $\mathbb{P}_{n,k}$ and $\mathbf{X}_{n,k}$, we get

$$\lim_{n \to \infty} \mathbb{P}'_{n,c}(\mathbf{X}'_{n,c}) = 1$$

which is what we needed to prove. □

Proof of Lemma 10.16. Suppose that $\mathcal{M} \in \mathbf{K}_n \restriction k-1$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K} \restriction k-1$, and let

$$\mathbf{E}_k(\mathcal{M}) = \{\mathcal{N} \in \mathbf{K} \restriction k : \mathcal{N} \restriction k-1 = \mathcal{M}\}.$$

Let α be the number of represented structures, with respect to $\mathbf{K} \restriction k$, with universe $|\mathcal{S}_\mathcal{P}|$. It suffices to show that the proportion of structures in $\mathbf{E}_k(\mathcal{M})$ which satisfy both the $\mathcal{S}_\mathcal{F}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_\mathcal{P}$-extension axiom does not exceed $1 - 1/(1+\alpha)$; and if $k > 1$ then this proportion approaches 0 as $n \to \infty$. We will consider the cases $k = 0$ and $k > 0$ one by one.

First assume that $k = 1$. Then, by the choice of $\mathcal{U}$, $\mathcal{U}$ has dimension 0 and is represented, since it is a closed substructure of a represented structure. By assumption there is a unique, up to isomorphism, represented structure of dimension 0. Hence, every represented structure (with respect to $\mathbf{K}$, $\mathbf{K}{\restriction}k$ or $\mathbf{K}{\restriction}k-1$) contains a copy of $\mathcal{U}$. Therefore every $\mathcal{M} \in \mathbf{K}{\restriction}k$ which satisfies the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom contains a copy of $\mathcal{S}_{\mathcal{F}}$. Note that if $\mathcal{N} \in \mathbf{K}{\restriction}k$ satisfies the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom, then the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2. Now we can argue as in Section 5. More precisely, the proofs of lemmas 5.2, 5.4 and 5.6 as well as the proof of part (i) of Theorem 3.17 carry over to the present context if we have the following in mind: The structures $\mathcal{S}_{\mathcal{P}}$, $\mathcal{S}_{\mathcal{F}}$, $\mathcal{P}$ and $\mathcal{F}$ play the same roles in the present context as in Section 5; in the present context 'closed substructures' play the role of 'substructures' in Section 5; dimension plays the role here that cardinality had in that section; and $\mathbf{E}_k(\mathcal{M})$ plays the role here that '$\mathbf{K}_n$' had in that section. In this way we can conclude that the proportion of $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ which contain a copy of $\mathcal{S}_{\mathcal{F}}$ and satisfy the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom never exceeds $1 - 1/(1+\alpha)$.

Now suppose that $k > 1$. Again, the reasoning from Section 5 carries over to the present context. Since we assume $k > 1$, $\mathcal{U}$ has dimension 1 and $\mathcal{U} \subset_{cl} \mathcal{S}_{\mathcal{F}}$. As noted earlier, $c > k > 1$. Since $\mathcal{M}$ is $(\sigma^c(\lambda_n), c)$-saturated, with respect to $\mathbf{K}{\restriction}k-1$, $\mathcal{M}$ contains at least $\sigma^c(\lambda_n)$ distinct copies of $\mathcal{U}$. Since $\mathcal{M}$ and every $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ agree on all substructures of dimension at most $k-1 \geq 1$, it follows that every $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ contains at least $\sigma^c(\lambda_n)$ distinct copies of $\mathcal{U}$. Suppose that $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ satisfies both the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom and the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom. First we notice that the satisfaction of the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom implies that $\mathcal{N}$ contains at least $\sigma^c(\lambda_n)/\left\|\mathcal{S}_{\mathcal{F}}\right\|$ distinct copies of $\mathcal{S}_{\mathcal{F}}$ (the copies may *partially* overlap, but this poses no problem). Secondly, the satisfaction of the $\mathcal{C}'/\mathcal{S}_{\mathcal{P}}$-extension axiom implies that the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2.

As in the previous case (when $k = 1$) the proofs of lemmas 5.2, 5.4 and 5.6 carry over – with the already mentioned provisos – to this context. But we are now able to continue the argument similarly as in the proof of part (iii) of Theorem 3.17. The number $\sigma^c(\lambda_n)$ plays the same role here as the number 'm_n' did in the proof of part (iii) of Theorem 3.17. In a similar way as in that proof we can now derive that '$\alpha \left\|\mathcal{S}_{\mathcal{F}}\right\| / \sigma^c(\lambda_n)$' (instead of '$k\alpha/m_n$' as in the proof of part (iii) of Theorem 3.17) is an upper bound of the proportion of $\mathcal{N} \in \mathbf{E}_k(\mathcal{M})$ such that $\mathcal{N}$ satisfies the $\mathcal{S}_{\mathcal{F}}/\mathcal{U}$-extension axiom and the $\mathcal{P}/\mathcal{S}_{\mathcal{P}}$-multiplicity of $\mathcal{N}$ is at least 2. □

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