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Countable Ideals Models for **CZF**

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Abstract

In [AW, AFW] S. Awodey, H. Forssell and M. Warren have studied predicative algebraic set theory, i.e. the categorical semantics of predicative set theories. A drawback of their models is that they validate induction over the set ω of natural numbers only for bounded predicates unlike in P. Aczel's **CZF** [Ac1] where induction holds for arbitrary predicates.

This is remedied in the current paper where among other things we construct categorical models for **CZF** refuting both the Powerset axiom and the Full Separation schema. This is obtained by associating with every Π-pretopos $\mathcal{E}$ with disjoint and stable countable sums a category $\mathsf{Idl}_\infty(\mathcal{E})$ of countable ideals in $\mathcal{E}$ and taking as small maps the representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$. A class V_A of sets over an object $A \in \mathsf{Idl}_\infty(\mathcal{E})$ of atoms is obtained as initial fixpoint of the functor $A + \mathcal{P}_s(-)$.

1 Introduction

At the end of the 1970ies Peter Aczel introduced the predicative constructive set theory **CZF** based on previous work by J. Myhill [Myh] but motivated by a translation to Martin-Löf type theory [Ac1]. The system **CZF** is obtained from Zermelo-Fraenkel set theory **ZF** by

1. dropping the classical principle of excluded middle
2. reformulating regularity as $\in$-induction
3. restricting the separation scheme to bounded formulas
4. postulating instead of the replacement scheme the scheme of

   (Strong) Collection $\quad (\forall x{\in}a)(\exists y)\,\phi \Rightarrow (\exists b)\;\mathsf{match}(x{:}a, y{:}b)\,\phi$

   where $\mathsf{match}(x{:}a, y{:}b)\,\phi(x, y, \dots)$ stands for

$$(\forall x{\in}a)(\exists y{\in}b)\phi(x, y, \dots) \wedge (\forall y{\in}b)(\exists x{\in}a)\phi(x, y, \dots)$$

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5. postulating instead of the powerset axiom the scheme of

   Subset Collection $(\exists c)(\forall \vec{z})\big((\forall x{\in}a)(\exists y{\in}b)\phi\big) \Rightarrow (\exists d{\in}c)\mathsf{match}(x{:}a, y{:}d)\,\phi$

   where $\vec{z}$ is the list of free variables of ϕ different from x and y.

As shown in [Ac1] the somewhat unpalatable Subset Collection scheme is equivalent to the following axiom of

Fullness $(\forall a, b)(\exists c{\subseteq}\mathsf{mv}(a, b))(\forall r{\in}\mathsf{mv}(a, b))(\exists s{\in}c)\; s \subseteq r$

where $\mathsf{mv}(a, b)$ stands for the class of all total relations from a to b.

In **ZF** one can prove the strong collection schema from replacement but this is not the case in a constructive setting. The main consequence of the fullness axiom is that it entails the exponentiation axiom stating that for sets a and b the class b^a of all functions from a to b is a set. We write $\mathbf{CZF_{Exp}}$ for the system where instead of the fullness axiom one just postulates the exponentiation axiom.

A possibly more well known system is the impredicative system **IZF** which is obtained from **ZF** by dropping the principle of excluded middle, reformulating regularity as $\in$-induction and strengthening replacement to strong collection. It is well known that all sheaf and realizability toposes host models of **IZF**. In this paper we will among other things construct submodels of certain sheaf models which validate **CZF** but not the Full Separation scheme. Moreover, in even more special but very natural situations also the Powerset axiom will get refuted.

We will also consider the system **CZFA** whose language besides equality $=$ and elementhood $\in$ contains a further unary predicate symbol S where $\mathsf{S}(a)$ intuitively means that "a is a set". Thus **CZFA** allows also the existence of elements which are nots sets, so called *atoms*. The axioms of **CZFA** are obtained from those for **CZF** by appropriately replacing all set quantifiers $(\forall x)\ldots$ by $(\forall x)\mathsf{S}(x) \to \ldots$. Similarly, we write $\mathbf{CZFA_{Exp}}$ for **CZFA** with exponentiation axiom instead of the fullness axiom.

2 A Recap of Awodey and Warren's Work

In [AW] S. Awodey and M. Warren have studied a predicative set theory **CST** and its categorical semantics. We first recall their notion of model for **CST**.

Definition 2.1 *Let $\mathcal{C}$ be a Heyting category with stable and disjoint finite sums. A* **class of small maps** *in $\mathcal{C}$ is a class $\mathcal{S}$ of maps in $\mathcal{C}$ such that*

(S1) *$\mathcal{S}$ contains all isomorphisms of $\mathcal{C}$ and is closed under composition*

(S2) *$\mathcal{S}$ is stable under pullbacks in $\mathcal{C}$*

(S3) *$\mathcal{S}$ contains all regular monomorphisms of $\mathcal{C}$*

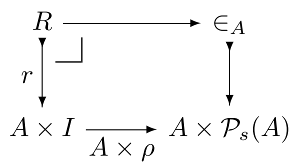
(S4) *if f and $f \circ e$ are in $\mathcal{S}$ an e is a regular epi then f is in $\mathcal{S}$*

(S5) *if* $a : A \to I$ *and* $b : B \to I$ *are in* $\mathcal{S}$ *then* $[a, b] : A + B \to I$ *is also in* $\mathcal{S}$.

An object A *of* $\mathcal{C}$ *is called* small *iff its terminal projection* $A \to 1$ *is in* $\mathcal{S}$. *We write* $\mathcal{C}_{\mathcal{S}}$ *for the full subcategory of* $\mathcal{C}$ *on small objects. We call* $\mathcal{C}_{\mathcal{S}}$ *the* small part of $(\mathcal{C}, \mathcal{S})$.

A relation $r : R \rightarrowtail A \times I$ *is called* small *iff* $\pi_2 \circ r : R \to I$ *is in* $\mathcal{S}$. *A* **basic class structure** *on* $\mathcal{C}$ *is given by a class* $\mathcal{S}$ *of small maps satisfying the condition*

(P) *for all* $A \in \mathcal{C}$ *there exists a small relation* $\in_A \rightarrowtail A \times \mathcal{P}_s(A)$ *such that for all small relations* $r : R \rightarrowtail A \times I$ *there exists a unique map* $\rho : B \to \mathcal{P}_s(A)$ *with*



called classifying map for r.

A **predicative class structure** *on* $\mathcal{C}$ *is given by a basic class structure* $\mathcal{S}$ *on* $\mathcal{C}$ *satisfying the condition*

(E) *for every* $f : J \to I$ *in* $\mathcal{S}$ *the functor* $f^* : \mathcal{C}/I \to \mathcal{C}/J$ *has a right adjoint* $\Pi_f : \mathcal{C}/J \to \mathcal{C}/I$.

A universe *is an object* U *of* $\mathcal{C}$ *together with a mono* $i_U : \mathcal{P}_s(U) \rightarrowtail U$. *A* **predicative category of classes** *is a Heyting category* $\mathcal{C}$ *with stable and disjoint finite sums together with a predicative class structure* $\mathcal{S}$ *and a universe* U. *Its* small part *is the full subcategory* $\mathcal{C}_{\mathcal{S},U}$ *of* $\mathcal{C}$ *on those small objects which appear as subobjects of* U.

Given a predicative category of classes $(\mathcal{C}, \mathcal{S}, U)$ one can interpret the set theory **CST** (as described in [AW]) as follows: the unary predicate S (expressing sethood) is interpreted by the mono $i_U : \mathcal{P}_s(U) \rightarrowtail U$ and the relation $\in$ by the mono $\in_U \rightarrowtail U \times \mathcal{P}_s(U) \overset{U \times i_S}{\rightarrowtail} U \times U$.

It has been shown in [AW] that if $(\mathcal{C}, \mathcal{S})$ is a predicative class structure then its small part $\mathcal{C}_{\mathcal{S}}$ is a Π-pretopos, i.e. a locally cartesian closed pretopos, and also that if $(\mathcal{C}, \mathcal{S}, U)$ is a predicative category of classes then its small part $\mathcal{C}_{\mathcal{S},U}$ is again a Π-pretopos. One of the main results in [AW] was that every Π-pretopos $\mathcal{E}$ can be obtained as the small part of a predicative category of classes.

In order to formulate their result we need to recall Grothendieck's notion of representable morphism.

Definition 2.2 *Let* $\mathbb{C}$ *be a small category. Then a map* $f : B \to A$ *in the presheaf topos* $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathrm{op}}}$ *is called* representable *iff for all* $a : \mathsf{y}(I) \to A$ *there* *exist a pullback diagram*

$$\begin{array}{ccc} \mathsf{y}(J) & \longrightarrow & B \\ {\scriptstyle a^*u}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathsf{y}(I) & \underset{a}{\longrightarrow} & A \end{array}$$

where $u : J \to I$ *is a map in* $\mathbb{C}$*.*

If one thinks of "small" as "representable" (as suggested in [Bén]) then representable morphism are those families of types all whose items are small.

Definition 2.3 *A presheaf* $A \in \widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ *is called* separated *iff the diagonal map* $\delta_A = \langle id_A, id_A \rangle : A \times A \to A$ *is representable, i.e. iff for* $x, y \in A(I)$ *the sieve* $\{u : J \to I \mid x{\cdot}u = y{\cdot}u\}$ *is representable.*

The following proposition from [AW] ensures that every Π-pretopos is equivalent to the small part of some predicative category of classes.

Proposition 2.1 *Let* $\mathcal{E}$ *be a* Π*-pretopos and* $\mathsf{Sh}(\mathcal{E})$ *the topos of coherent sheaves over* $\mathcal{E}$*.*[1] *Let* $\mathsf{Idl}(\mathcal{E})$ *be the full subcategory of* $\mathsf{Sh}(\mathcal{E})$ *on separated objects. Then* $\mathsf{Idl}(\mathcal{E})$ *is a Heyting category with stable and disjoint finite sums inheriting this structure from* $\mathsf{Sh}(\mathcal{E})$ *and the class* $\mathcal{S}_\mathcal{E}$ *of representable morphisms in* $\mathsf{Idl}(\mathcal{E})$ *gives rise to a predicative class structure on* $\mathsf{Idl}(\mathcal{E})$*.*

The sum $\mathsf{At}_\mathcal{E} = \coprod_{I \in \mathsf{Ob}(\mathcal{E})} \mathsf{y}(I)$ *in* $\widehat{\mathbb{C}}$ *is an object of* $\mathsf{Idl}(\mathcal{E})$*. In* $\mathsf{Idl}(\mathcal{E})$ *there exists an initial fixpoint* $U_\mathcal{E}$ *of the endofunctor* $\mathsf{At}_\mathcal{E} + \mathcal{P}_s(-)$ *on* $\mathsf{Idl}(\mathcal{E})$*. Thus* $(\mathsf{Idl}(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ *is a predicative category of classes whose small part is equivalent to* $\mathcal{E}$*.*

We recall that by Yoneda the small power object $\mathcal{P}_s(A)$ in $\mathsf{Idl}(\mathcal{E})$ is given by

$$\mathcal{P}_s(A)(I) \cong \{R \rightarrowtail A \times \mathsf{y}(I) \mid R \text{ representable}\}$$

since a relation $r : R \rightarrowtail A \times \mathsf{y}(I)$ is small iff $\pi_2 \circ r : R \to \mathsf{y}(I)$ is in $\mathcal{S}$ iff R is representable.

For later reference we also recall the following characterisation of separated objects in $\mathsf{Sh}(\mathcal{E})$ from [AW] originally suggested by A. Joyal.

Proposition 2.2 *For* $A \in \mathsf{Sh}(\mathcal{E})$ *the following conditions are equivalent*

(1) $A \in \mathsf{Idl}(\mathcal{E})$

(2) *for every* $f : \mathsf{y}(I) \to A$ *its image in* $\mathsf{Sh}(\mathcal{E})$ *is representable*

(3) A *arises as colimit in* $\widehat{\mathcal{E}} = \mathbf{Set}^{\mathcal{E}^{\mathsf{op}}}$ *of some directed diagram* $D : (I, \leq) \to \mathsf{Sh}(\mathcal{E})$ *where all* $D(i)$ *are representable and all* $D(i \leq j)$ *are monic.*

Directed colimits of monos of representables are called *ideal colimits*. It follows from Proposition 2.2 that $\mathsf{Idl}(\mathcal{E})$ is closed under ideal colimits and those are computed as in $\widehat{\mathcal{E}}$.

[1] The coherent topology on $\mathcal{E}$ is the one generated by finite jointly epic families in $\mathcal{E}$.

3 Categorical Semantics of CZFA

In [AW] only a restricted form of the Infinity axiom is considered (see p.11 of [AW]), namely a set which is a natural numbers object *within sets* and *not* within classes. The reason is that if $\mathcal{E}$ is a Π-pretopos with a natural numbers object N then $\mathsf{y}(N)$ is in general *not* a natural numbers object in $\mathsf{Idl}(\mathcal{E})$. This has the consequence that unlike in **CZF** induction over N is admissible only for bounded formulas. One of the main motivations for this paper is too overcome this restricted form of infinity. We first introduce sufficiently strong notion(s) of infinity and in the next section construct models for it.

For predicative class structures $(\mathcal{C}, \mathcal{S})$ the most natural notion of infinity is given by the axiom

(I) there exists a natural numbers object (nno) N in $\mathcal{C}$ which is small, i.e. $N \to 1$ in $\mathcal{S}$.

As an extension of Theorem 3.40 in [AW] we obtain that

Theorem 3.1 *Every predicative category of classes* $(\mathcal{C}, \mathcal{S}, U)$ *satisfying the axiom* (I) *validates* $\mathbf{CZFA_{Exp}}$, *i.e.* **CZFA** *with Exponentiation instead of the Fullness axiom.*

Proof: For validity of axioms other than Infinity we refer to the soundness proof in [AW] for **CST** which is equivalent to **CZFA** without Infinity.

For verifying the Infinity axiom we first construct an appropriate global element ω of U. Let $\mathsf{succ} : U \to U$ be defined as $\mathsf{succ}(a) = a \cup \{a\}$. Let N be a nno in $\mathcal{C}$. Then there exists a unique map $f : N \to U$ in $\mathcal{C}$ with $f(0) = \emptyset$ and $(\forall n{:}N) f(n+1) = \mathsf{succ}(f(n))$. Since N is small the image of f gives rise to a small subobject of U, i.e. a global element $\omega : 1 \to \mathcal{P}_s(U) \rightarrowtail U$. It is straightforward to verify that ω validates the Infinity axiom. $\Box$

As usual one can show by a term model construction that **CZFA** is also complete w.r.t. predicative categories of classes satisfying (I). However, as already mentioned above there arises the problem that the ideal approach of Proposition 2.1 does not extend to (I) since even if N is a nno in $\mathcal{E}$ then $\mathsf{y}(N)$ will in general not be a nno in $\mathsf{Idl}(\mathcal{E})$. Thus it seems to be a hard problem to characterise those Π-pretoposes with nno which are equivalent to the small part of predicative categories of classes satisfying the infinity axiom (I).

In order to avoid this problem we consider instead the following stronger infinity axiom for predicative class structures $(\mathcal{C}, \mathcal{S})$

(I_ω) $\mathcal{C}$ has stable and disjoint countable sums and for any countable family $(A_i \to C)_{i \in I}$ in $\mathcal{S}$ its source tupling $\coprod_{i \in I} A_i \to C$ is again in $\mathcal{S}$

which entails that small objects are closed under countable sums. Thus (I_ω) entails (I) since an appropriate natural numbers object is given by $\coprod_{n \in \omega} 1_{\mathcal{C}}$. One easily shows that if $(\mathcal{C}, \mathcal{S})$ is a predicative class structure satisfying (I_ω) then $\mathcal{C}_{\mathcal{S}}$ is a Π-pretopos with stable and disjoint countable sums. In the next section we will show that every such Π-pretopos arises as the small part of some predicative category of classes satisfying (I_ω).

4 Countable Ideals Models

Let $\mathcal{E}$ be a Π-pretopos with stable and disjoint countable sums. Let us assume that **Set** is big enough for $\mathcal{E}$ being internal to **Set**. We consider $\mathcal{E}$ as endowed with the *countable cover* topology where a sieve S on I covers iff S contains a countable jointly epic family of morphisms. We write $\mathsf{Sh}_\infty(\mathcal{E})$ for the category of sheaves on $\mathcal{E}$ w.r.t. the countable cover topology. Since any coherent cover is in particular a countable cover the category $\mathsf{Sh}_\infty(\mathcal{E})$ is a full subcategory of $\mathsf{Sh}(\mathcal{E})$ and the inclusion has a finite limit preserving left adjoint a called "sheafification".

Since all representable objects are sheaves w.r.t. the countable cover topology we have $\mathsf{y} : \mathcal{E} \to \mathsf{Sh}_\infty(\mathcal{E})$ which is known to preserve countable sums. Thus for $N = \coprod_{n\in\omega} 1_\mathcal{E}$ we have $\mathsf{y}(N) \cong \coprod_{n\in\omega} \mathsf{y}(1_\mathcal{E}) \cong \coprod_{n\in\omega} 1_{\mathsf{Sh}_\infty(\mathcal{E})}$ and, accordingly, $\mathsf{y}(N)$ is a natural numbers object in $\mathsf{Sh}_\infty(\mathcal{E})$.

Next we show that $\mathsf{Sh}_\infty(\mathcal{E})$ is closed under a particular kind of colimits, so called "∞-ideal colimits".

Definition 4.1 *A poset $(I, \leq)$ is* ω_1-directed *iff every countable subset of I has an upper bound in I. An ∞-ideal diagram in a category $\mathcal{C}$ is a mono preserving functor $D : (I, \leq) \to \mathcal{C}$ for some ω_1-directed poset $(I, \leq)$ considered as a category. An* ∞-ideal colimit *is a colimit of an ∞-ideal diagram.*

It is a straightforward exercise to show that

Proposition 4.1 *The category $\mathsf{Sh}_\infty(\mathcal{E})$ is closed under ∞-ideal colimits taken in $\widehat{\mathcal{E}}$.*

Moreover, we have the following characterisation of separated objects in $\mathsf{Sh}_\infty(\mathcal{E})$.

Proposition 4.2 *For $A \in \mathsf{Sh}_\infty(\mathcal{E})$ the following conditions are equivalent*

(1) *$A \in \mathsf{Idl}(\mathcal{E})$, i.e. is separated*

(2) *for every $f : \mathsf{y}(I) \to A$ its image in $\mathsf{Sh}_\infty(\mathcal{E})$ is representable*

(3) *A arises as ∞-ideal colimit of representable objects in $\widehat{\mathcal{E}}$.*

Proof: Analogous to the proof of Proposition 2.2 which can be found in [AF] as proof of Theorem 3.7 in *loc.cit.* □

In order to construct an appropriate model for **CZFA** from $\mathcal{E}$ we consider the following category $\mathsf{Idl}_\infty(\mathcal{E})$ of *countable ideals* in $\mathcal{E}$.

Definition 4.2 *Let $\mathsf{Idl}_\infty(\mathcal{E})$ be the full subcategory of $\mathsf{Sh}_\infty(\mathcal{E})$ on separated objects, i.e. $\mathsf{Idl}_\infty(\mathcal{E}) = \mathsf{Sh}_\infty(\mathcal{E}) \cap \mathsf{Idl}(\mathcal{E})$.*

From Propositions 4.1 and 4.2 (3) it follows that

Proposition 4.3 $\mathsf{Idl}_\infty(\mathcal{E})$ *is closed under* ∞*-ideal colimits taken in* $\widehat{\mathcal{E}}$.

The following lemma will be crucial for verifying that the class $\mathcal{S}_\mathcal{E}$ of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ gives rise to a predicative class structure.

Lemma 4.1 *The adjunction* $\mathsf{a} \dashv \mathsf{i} : \mathsf{Sh}(\mathcal{E}) \hookrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ *restricts to an adjunction* $\mathsf{a} \dashv \mathsf{i} : \mathsf{Idl}(\mathcal{E}) \hookrightarrow \mathsf{Idl}_\infty(\mathcal{E})$ *which is a localisation, i.e. the left adjoint preserves finite limits. Moreover, the category* $\mathsf{Idl}_\infty(\mathcal{E})$ *is regular and, therefore, an object* $A \in \mathsf{Idl}(\mathcal{E})$ *is in* $\mathsf{Idl}_\infty(\mathcal{E})$ *iff* $\mathsf{Idl}(\mathcal{E})(m, A)$ *is a bijection for all monos* m *in* $\mathsf{Idl}(\mathcal{E})$ *inverted by* a.

Proof: Since the sheafification functor $\mathsf{a} \dashv \mathsf{i} : \mathsf{Sh}_\infty(\mathcal{E}) \hookrightarrow \mathsf{Sh}(\mathcal{E})$ preserves finite limits and representable objects it also preserves separated objects. Thus, the sheafification functor a restricts to a functor from $\mathsf{Idl}_\infty(\mathcal{E})$ to $\mathsf{Idl}(\mathcal{E})$ left adjoint to the inclusion $\mathsf{i} : \mathsf{Idl}_\infty(\mathcal{E}) \hookrightarrow \mathsf{Idl}(\mathcal{E})$. The functor $\mathsf{a} : \mathsf{Idl}(\mathcal{E}) \to \mathsf{Sh}_\infty(\mathcal{E})$ preserves finite limits since $\mathsf{Idl}_\infty(\mathcal{E})$ inherits finite limits from $\mathsf{Sh}(\mathcal{E})$. Thus $\mathsf{a} \dashv \mathsf{i} : \mathsf{Idl}(\mathcal{E}) \hookrightarrow \mathsf{Idl}_\infty(\mathcal{E})$ is a localisation. Thus, since $\mathsf{Idl}(\mathcal{E})$ is regular the category $\mathsf{Idl}_\infty(\mathcal{E})$ is also regular. It follows from Proposition 5.6.4 of vol.1 of [Bor] that an object $A \in \mathsf{Idl}(\mathcal{E})$ is in $\mathsf{Idl}_\infty(\mathcal{E})$ iff $\mathsf{Idl}(\mathcal{E})(m, A)$ is a bijection for all monos m in $\mathsf{Idl}(\mathcal{E})$ inverted by a. □

Proposition 4.4 $\mathsf{Idl}_\infty(\mathcal{E})$ *is a Heyting category with stable and disjoint sums.*

Proof: $\mathsf{Idl}_\infty(\mathcal{E})$ is a Heyting category since by Proposition 2.1 $\mathsf{Idl}(\mathcal{E})$ is a Heyting category and this property is stable under localisation.

Notice that in $\widehat{\mathcal{E}}$ separated objects are closed under small sums. Thus, since $\mathsf{a} : \widehat{\mathcal{E}} \to \mathsf{Sh}_\infty(\mathcal{E})$ is a left adjoint preserving finite limits and representable objects it follows that separated objects in $\mathsf{Sh}_\infty(\mathcal{E})$ are closed under small sums in $\mathsf{Sh}_\infty(\mathcal{E})$ which are stable and disjoint since $\mathsf{Sh}_\infty(\mathcal{E})$ is a Grothendieck topos and the initial object of $\mathsf{Sh}_\infty(\mathcal{E})$ is separated. □

Proposition 4.5 *The class* $\mathcal{S}_\mathcal{E}$ *of representable morphisms in* $\mathsf{Idl}_\infty(\mathcal{E})$ *is a class of small maps, i.e. validates the axioms* (S1)-(S5).

Proof: Immediate from the fact that $\mathsf{a} : \mathsf{Sh}(\mathcal{E}) \to \mathsf{Sh}_\infty(\mathcal{E})$ is a left adjoint preserving finite limits and representable objects. □

Using Proposition 4.2 and results from [AW] one can show that

Proposition 4.6 *The functor* $\mathcal{P}_s : \mathsf{Idl}(\mathcal{E}) \to \mathsf{Idl}(\mathcal{E})$ *preserves* $\mathsf{Idl}_\infty(\mathcal{E})$*, i.e. restricts to a functor* $\mathcal{P}_s : \mathsf{Idl}_\infty(\mathcal{E}) \to \mathsf{Idl}_\infty(\mathcal{E})$.

Proof: From [AW] it follows that $\mathcal{P}_s$ commutes with ideal colimits and thus with ∞-ideal colimits and $\mathcal{P}_s$ preserves separatedness. Thus it suffices to show that $\mathcal{P}_s(A) \in \mathsf{Sh}_\infty(\mathcal{E})$ for every $A \in \mathcal{E}$.

For that purpose suppose $(u_n : I_n \to I)$ is a countable cover of I and $\big(S_n \in \mathcal{P}_s(A)(I_n)\big)$ is a family compatible in the sense that whenever $u_n v = u_m w$ for some arrows v and w with source J then $(v\times A)^*S_n \cong (w\times A)^*S_m$ as subobjects of $J\times A$. Then, due to the assumptions on $\mathcal{E}$ the subobject $S = \bigvee(u_n\times A)[S_n]$ of $I\times A$ is the unique $S \in \mathcal{P}_s(A)(I)$ with $S{\cdot}u_n \cong S_n$ for all n. □

Proposition 4.7 *Representable morphisms in* $\mathsf{Idl}_\infty(\mathcal{E})$ *validate axiom* (E).

Proof: In Proposition 4.26 of [AW] it has been shown that $\mathsf{Idl}(\mathcal{E})$ validates axiom (E). Since $\mathsf{Idl}_\infty(\mathcal{E})$ appears as localisation of $\mathsf{Idl}(\mathcal{E})$ property (E) is preserved since the inlusion of $\mathsf{Idl}_\infty(\mathcal{E})$ into $\mathsf{Idl}(\mathcal{E})$ preserves dependent products. □

Proposition 4.8 *Representable morphisms in* $\mathsf{Idl}_\infty(\mathcal{E})$ *validate axiom* (I_ω).

Proof: Suppose $(a_i : A_i \to C)_{i\in I}$ is a countable family of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$. Let $a : \coprod_{i\in I} A_i \to C$ be the source tupling of the a_i. Let $c : \mathsf{y}(X) \to C$ and $b_i = c^*a_i : \mathsf{y}(Y_i) \to \mathsf{y}(X)$ for $i \in I$. Then c^*a is isomorphic to the source tupling $b : \coprod_{i\in I} \mathsf{y}(Y_i) \to \mathsf{y}(X)$ of the b_i. But since y preserves countable products the source of c^*a is isomorphic to $\mathsf{y}(\coprod_{i\in I} Y_i)$, i.e. representable. Thus a is a representable morphism as desired. □

Summarising these results we observe that

Theorem 4.1 *The representable morphisms give rise to a predicative class structure on* $\mathsf{Idl}_\infty(\mathcal{E})$ *validating axiom* (I_ω).

Proof: Immediate from Propositions 4.4, 4.5, 4.6, 4.7 and 4.8. □

In order to construct universes in $\mathsf{Idl}_\infty(\mathcal{E})$ we need the following two preparatory lemmas.

Lemma 4.2 $\mathsf{Idl}_\infty(\mathcal{E})$ *is closed under ideal colimits.*

Proof: Since $\mathsf{Idl}(\mathcal{E})$ is closed under ideal colimits and $\mathsf{a} : \mathsf{Idl}(\mathcal{E}) \to \mathsf{Idl}_\infty(\mathcal{E})$ preserves them the claim follows. □

Lemma 4.3 *The functor* $\mathcal{P}_s : \mathsf{Idl}_\infty(\mathcal{E}) \to \mathsf{Idl}_\infty(\mathcal{E})$ *preserves* ∞*-ideal colimits.*

Proof: In [AW] it has been shown that $\mathcal{P}_s : \mathsf{Idl}(\mathcal{E}) \to \mathsf{Idl}(\mathcal{E})$ preserves ideal colimits and thus in particular ∞-ideal colimits. From Proposition 4.3 we know that ∞-ideal colimits exist in $\mathsf{Idl}_\infty(\mathcal{E})$ and are computed as in $\widehat{\mathcal{E}}$ and, therefore, as in $\mathsf{Idl}(\mathcal{E})$. Thus $\mathcal{P}_s$ preserves ∞-ideal colimits in $\mathsf{Idl}_\infty(\mathcal{E})$. □

Theorem 4.2 *For every object* A *of* $\mathsf{Idl}_\infty(\mathcal{E})$ *the functor* $A + \mathcal{P}_s(-)$ *has an initial fixpoint* $U_A \cong A + \mathcal{P}_s(U_A)$.

Let $\mathsf{At}_\mathcal{E} = \coprod_{A\in\mathsf{Ob}(\mathcal{E})} \mathsf{y}(A)$. *Then* $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_{\mathsf{At}_\mathcal{E}})$ *is a predicative category of classes whose small part is equivalent to* $\mathcal{E}$.

Proof: In [AW] it has been shown that $\mathcal{P}_s$ preserves monos and thus the functor $F_A = A+\mathcal{P}_s(-)$ also preserves monos. Since $A+(-)$ preservs colimits it follows from Lemma 4.3 that F_A preserves ∞-ideal colimits.

Thus, since by Lemma 4.2 $\mathsf{Idl}_\infty(\mathcal{E})$ is closed under ideal colimits we can define the ∞-ideal diagram $(F_A^\alpha(0))_{\alpha<\omega_1}$ where $F_A^{\alpha+1}(0) = F_A(F_A^\alpha(0))$ and $F_A^\lambda(0) = \mathrm{colim}_{\alpha<\lambda} F_A^\alpha(0)$ for limit ordinals $\lambda < \omega_1$. By Proposition 4.3 the ∞-ideal colimit $U_A = \mathrm{colim}_{\alpha<\omega_1}$ exists. Since F_A preserves ∞-ideal colimits it follows that U_A is an initial fixpoint of F_A.

Due to Proposition 4.4 the sum $\mathsf{At}_\mathcal{E} = \coprod_{A\in\mathsf{Ob}(\mathcal{E})} \mathsf{y}(A)$ exists in $\mathsf{Idl}_\infty(\mathcal{E})$. Since $U_{\mathsf{At}_\mathcal{E}} \cong \mathsf{At}_\mathcal{E} + \mathcal{P}_s(U_{\mathsf{At}_\mathcal{E}})$ we have $\mathcal{P}_s(U_{\mathsf{At}_\mathcal{E}}) \rightarrowtail U_{\mathsf{At}_\mathcal{E}}$ and thus $U_{\mathsf{At}_\mathcal{E}}$ gives rise to a universe in the predicative class structure $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E})$. Since $\mathsf{At}_\mathcal{E}$ is a subobject of $U_{\mathsf{At}_\mathcal{E}}$ and all representable objects appear as subobjects of $\mathsf{At}_\mathcal{E}$ it follows that the small part of $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_{\mathsf{At}_\mathcal{E}})$ is equivalent to $\mathcal{E}$. □

Theorem 4.3 *For every object A in $\mathsf{Idl}_\infty(\mathcal{E})$ the predicative category of classes $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_A)$ validates* $\mathbf{CZFA_{Exp}}$. *If $U_\mathcal{E}$ is an initial fixpoint of $\mathcal{P}_s$ then $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ validates* $\mathbf{CZF_{Exp}}$.

Proof: The first claim is immediate from Theorems 3.1, 4.1 and 4.2. The second claim follows by instantiating A with the initial object 0 and observing that from $U_\mathcal{E} \cong \mathcal{P}_s(U_\mathcal{E})$ it follows that all elements of $U_\mathcal{E}$ are sets. □

Whereas in $\mathsf{Idl}(\mathcal{E})$ the initial fixpoint of $\mathcal{P}_s$ stabilizes at ω and thus contains only "hereditarily finite" sets the fixpoint of $\mathcal{P}_s$ in $\mathsf{Idl}_\infty(\mathcal{E})$ stabilizes at ω_1 and thus contains in particular the set of van Neumann natural numbers as constructed in the proof of Theorem 3.1.

5 Extension to Fullness

In [vdB, BM1, BM2] van den Berg and Moerdijk have identified a condition (F) on classes of small maps ensuring that the model arising from the initial fixpoint of $\mathcal{P}_s$ validates the set-theoretic fullness axiom.

In order to formulate (F) we have to introduce the following notation. For morphism $a : A \to X$ and $b : B \to X$ let $M_X(a, b)$ denote the poset of relations $R \rightarrowtail A\times B$ where $\pi_1 r : R \to A$ is a regular epi. Since such spans are preserved by pullbacks every morphism $f : Y \to X$ induces a monotone map $f^* : M_X(a, b) \to M_Y(f^*a, f^*b)$. Now van den Berg and Moerdijk's axiom is

(F) For any two small maps $a : A \to X$ and $b : B \to X$ there exist a regular epi $\widetilde{e} : \widetilde{X} \twoheadrightarrow X$, a small map $c : C \to \widetilde{X}$ and an $R \in M_C(c^*\widetilde{e}^*a, c^*\widetilde{e}^*b)$ such that for every $d : d \to \widetilde{X}$ and $S \in M_D(d^*\widetilde{e}^*a, d^*\widetilde{e}^*b)$ there exists a regular epi $e : E \twoheadrightarrow D$ and a map $f : E \to C$ with $de = cf$ and $g^*R \leq e^*S$.

which, although quite complicated, is obtained as the Kripke-Joyal translation of the statement that for small objects A and B there exists a small collection

C of total relations from A to B such that every total relation from A to B contains one in C.

In [BM1, BM2] it has been shown that condition (F) for a class of small maps ensures that for an initial fixpoint $U \cong \mathcal{P}_s(U)$ the corresponding interpretation of the language of set theory validates the set-theoretic fullness axiom.

In the next definition we give a condition, the *type-theoretic fulness axiom*, on a Π-pretopos $\mathcal{E}$ with stable and disjoint countable sums which will ensure that representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ validate condition (F).

Definition 5.1 *Let $\mathcal{E}$ be a pretopos. For maps $a : A \to I$ and $b : B \to I$ in $\mathcal{E}$ we write $M_I(a,b)$ for the collection of all subobjects $r : R \rightarrowtail A \times B$ such that $\pi_1 r : R \to A$ is a regular epi.*

The pretopos $\mathcal{E}$ validates the type-theoretic fullness axiom *iff for all $a : A \to I$ and $b : B \to I$ in $\mathcal{E}$ there exist a cover $\widetilde{e} : \widetilde{I} \twoheadrightarrow I$, a map $c : C \to \widetilde{I}$ and $R \in M_C(c^*\widetilde{e}^*A, c^*\widetilde{e}^*B)$ such that for every $d : D \to \widetilde{I}$ and $S \in M_D(d^*\widetilde{e}^*A, d^*\widetilde{e}^*B)$ there exists a cover $e : E \twoheadrightarrow D$ and a map $f : E \to C$ with $de = cf$ and $g^*R \subseteq e^*S$.*

Intuitively, it says that (in every context) for all types A and B there exists a type C and a C-indexed family $(R_c)_{c \in C}$ of total relations from A to B such that for every total relation S from A to B there is a $c \in C$ with $R_c \subseteq S$.

Theorem 5.1 *If $\mathcal{E}$ is a Π-pretopos with stable and disjoint countable sums and satisfies the type-theoretic fullness axiom then $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ in $U(\mathcal{E})$ validates* **CZF**.

Proof: First observe that one can show using the Kripke-Joyal translation that the type theoretic fullness axiom for $\mathcal{E}$ guarantees that the class $\mathcal{S}_\mathcal{E}$ of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ validates axiom (F). Then by Proposition 7.2(4) of [BM2] it follows that $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ validates the set-theoretic fullness axiom. This together with Theorem 4.3 establishes the claim that $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ gives rise to a model of **CZF**. □

6 Failure of Full Separation and Powerset

First we show that the countable ideals models of $\mathbf{CZF_{Exp}}$ from Theorem 4.3 **never validate Full Separation**. As a preparation we need the following

Lemma 6.1 *Let $\mathcal{E}$ be a Π-pretopos with stable and disjoint countable sums and let $U_\mathcal{E}$ be the initial fixpoint of $\mathcal{P}_s$ in $\mathsf{Idl}_\infty(\mathcal{E})$. Then the image of any countable family of global element of $U_\mathcal{E}$ is a small subobject of $U_\mathcal{E}$ and thus a set.*

Proof: Suppose $(u_n)_{n\in\omega}$ is a family of global elements of $U_\mathcal{E}$. Let $u : N \to U_\mathcal{E}$ be the source tupling of the family (u_n) (where $N = \coprod_{n\in\omega} 1$). Since N is small it follows from (S4) that the image of u is a small subobject of $U_\mathcal{E}$ and thus a set, i.e. an element of $\mathcal{P}_s(U_\mathcal{E}) \cong U_\mathcal{E}$. □

Theorem 6.1 *For a* Π*-pretopos* $\mathcal{E}$ *with stable and disjoint countable sums the model* $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ *of* $\mathbf{CZF_{Exp}}$ *does not validate the Full Separation scheme.*

Thus, if $\mathcal{E}$ *validates also the type-theoretic fullness axiom then* $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ *is a model of* **CZF** *not validating the Full Separation scheme.*

Proof: In $\mathbf{CZF_{Exp}}$ we may define Brouwer's 2nd Number Class W_1 inductively as a subclass of the set $\omega^{(\omega^\omega)}$, namely as the least C such that

(1) $\lambda f.0 \in C$

(2) if $F \in C$ then the functional

$$\mathrm{succ}(F)(f) = \begin{cases} 1 & \text{if } f(0) = 0 \\ F(\lambda n.f(n{+}1)) & \text{otherwise} \end{cases}$$

is in C as well

(3) if $(F_n)_{n\in\omega}$ is a sequence in C then the functional

$$\big(\sup_{n\in\omega} F_n\big)(f) = \begin{cases} 2 & \text{if } f(0) = 0 \\ F_{f(0)-1}(\lambda n.f(n{+}1)) & \text{otherwise} \end{cases}$$

is in C as well.

By transfinite recursion over W_1 we can define a class function $E : W_1 {\to} U(\mathcal{E})$ with

$$E(t) = \begin{cases} \emptyset & \text{if } t = 0 \\ E(t') \cup \{E(t')\} & \text{if } t = \mathrm{succ}(t') \\ \bigcup_{n\in\omega} E(t_n) & \text{if } t = \sup_{n\in\omega} t_n \end{cases}$$

If $U_\mathcal{E}$ validates the full separation scheme Sep then W_1 is a set from which it follows by replacement (i.e. Strong Collection) that $E[W_1]$ is a set as well. With every external ordinal $\alpha < \omega_1$ we may associate a global element $\widehat{\alpha}$ of $E[W_1] \subseteq U_\mathcal{E}$ as follows: $\widehat{\alpha{+}1} = \widehat{\alpha} \cup \{\widehat{\alpha}\}$ for $\alpha < \omega_1$ and (using Lemma 6.1) $\widehat{\lambda} = \bigcup_{\alpha\in\lambda} \widehat{\alpha}$ for limit ordinals $\lambda < \omega_1$. One easily checks by external induction on ω_1 that for all $\alpha < \omega_1$ the least ordinal β with $\widehat{\alpha} \in \mathcal{P}_s^\beta(0)$ is $\alpha{+}1$. But since $E[W_1]$ is a set we have $E[W_1] \subseteq \mathcal{P}_s^\alpha(0)$ for some $\alpha < \omega_1$ and thus $\widehat{\alpha} \in E[W_1] \subseteq \mathcal{P}_s^\alpha(0)$ which is impossible since $\alpha{+}1$ is the least β with $\widehat{\alpha{+}1} \subseteq \mathcal{P}_s^\beta(0)$. □

It is well known that relative to $\mathbf{CZF_{Exp}}$ the Power Set axiom is equivalent to $\mathcal{P}(1)$ being a set.

Lemma 6.2 *Let* $\mathcal{E}$ *be a* Π*-pretopos with stable and disjoint countable sums. Then* $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ *validates the Power Set axiom if and only if* $\mathcal{E}$ *is a topos.*

Proof: The object $\mathcal{P}_s(1)$ in $\mathsf{Idl}_\infty(\mathcal{E})$ is given by the presheaf $\mathrm{Sub}_\mathcal{E}$ sending object A of $\mathcal{E}$ to the lattice of subobjects of A in $\mathcal{E}$ and whose morphism part is given by pulling back subobjects along morphisms in $\mathcal{E}$. Thus $\mathcal{P}_s(1)$ is a set iff $\mathrm{Sub}_{\mathcal{E}}$ is representable iff $\mathcal{E}$ has a subobject classifier iff $\mathcal{E}$ is a topos. $\square$

Thus there exist plenty of models of **CZF** which validate the Power Set axiom but not the Full Separation scheme.

Theorem 6.2 *If $\mathcal{E}$ is a Grothendieck topos then $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ is a model of* $\mathbf{CZF} + \mathrm{Pow}$ *not validating the Full Separation scheme.*

Proof: A Grothendieck topos is in particular locally cartesian closed and thus a Π-pretopos. Since a Grothendieck topos has also stable and disjoint small sums it has *a fortiori* stable and disjoint countable sums. Thus, by Theorem 4.3, $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ is a model for $\mathbf{CZF_{Exp}}$. From Lemma 6.2 it follows that $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ validates the Power Set axiom and from Lemma 6.1 that it does not validate the Full Separation scheme.

Since every topos validates the type-theoretic fullness axiom it follows from Theorem 5.1 that $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ is a also a model for **CZF**. $\square$

Another consequence of Lemma 6.2 is that it opens up the possibility of constructing models for **CZF** which refute both the Power Set axiom and the Full Separation scheme *without assuming a predicative metatheory.*[2] Namely, if $\mathcal{E}$ is a Π-pretopos with stable and disjoint countable sums but without a subobject classifier then $(\mathsf{Idl}_\infty(\mathcal{E}), \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ gives rise to a model of $\mathbf{CZF_{Exp}}$ which refutes the Power Set axiom by Lemma 6.2 and the Full Separation schema by Theorem 6.1.

Examples of such pretoposes are

(1) the exact completion of the category $\omega\mathsf{Top}_0$ of countably based T_0 spaces

(2) the ex/reg-completions of $\mathsf{Mod}(\mathcal{P}\omega)$ and $\mathsf{Mod}(K_2)$, respectively

(3) the ex/reg-completions of $\mathsf{Asm}(\mathcal{T}_{\mathcal{P}\omega})$ and $\mathsf{Asm}(\mathcal{T}_{K_2})$, respectively, where $\mathcal{T}_{\mathcal{P}\omega}$ and $\mathcal{T}_{K_2}$ are the typed partial combinatory algebras arising from $\mathsf{Mod}(\mathcal{P}\omega)$ and $\mathsf{Mod}(K_2)$, respectively (see e.g. [LiSt])

where $\mathcal{P}\omega$ is the partial combinatory algebra (pca) arising from Scott's graph model and K_2 is the pca for function realizability. Notice that the regular completion of $\omega\mathsf{Top}_0$ can be characterized as the category QCB_0 of T_0 quotients of countably based T_0 spaces and thus the exact completion of $\omega\mathsf{Top}_0$ is equivalent to the ex/reg completion of QCB_0. Since QCB_0 can be described as the full subcategory of $\mathsf{Mod}(\mathcal{P}\omega)$ or $\mathsf{Mod}(K_2)$ on Σ-separated objects, i.e. objects X for which $\eta_X : X \to \Sigma^{\Sigma^X}$ is a regular mono, examples (1) and (2) are quite similar in character. From well known results in the theory of exact completions it follows that all examples are Π-pretoposes but not toposes since they lack a

[2] Models of **CZF** refuting just the Power Set axiom have been constructed in [vdB, Lub, Str2]. M. Rathjen has considered various model constructions which give rise to models for **IZF** relative to an impredicative metatheory but models only for **CZF** relative to a predicative metatheory modeled by an appropriate initial segment of Gödel's L.

"weak proof classifier" (see [Men]). One easily verifies that QCB_0, $\mathsf{Mod}(\mathcal{P}\omega)$, $\mathsf{Mod}(K_2)$, $\mathsf{Asm}(\mathcal{T}_{\mathcal{P}\omega})$ and $\mathsf{Asm}(\mathcal{T}_{K_2})$ have stable and disjoint countable sums[3] and these are preserved by ex/reg completion. Thus all the categories in (1)-(3) are Π-pretoposes with stable and disjoint countable sums but without a subobject classifier.

One can verify that the categories QCB_0, $\mathsf{Mod}(\mathcal{P}\omega)$, $\mathsf{Mod}(K_2)$, $\mathsf{Asm}(\mathcal{T}_{\mathcal{P}\omega})$ and $\mathsf{Asm}(\mathcal{T}_{K_2})$ all validate the type-theoretic fullness axiom. The idea is that when given objects A and B a sufficiently big object of total relations from A to B is parameterized by the set of all realizers of total relations from A to B. In [BM2] it has been shown that the type-theoretic fullness axiom is preserved by ex/reg completion. Thus the examples above all validate the type-theoretic fullness axiom and, accordingly, the associated countable ideals models validate the fullness axiom of **CZF**.

Finally, it follows from (the proof of) Theorem 6.1 that countable ordinals do not form a set in countable ideals models for $\mathbf{CZF_{Exp}}$ and thus the Regular Extension Axiom cannot hold in these models.

[3]This is *not* the case for K_1 instead of K_2!

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