A Notion of Homotopy for the Effective Topos

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Abstract

We define a notion of homotopy in the effective topos.

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Introduction

This is a preliminary report on some findings by the author while he spent the Fall of 2009 at the Mittag-Leffler Institute in Stockholm, in the program Set theory and Model theory. The author is deeply grateful to both the organizers of this programme (for having invited him, although he is neither a set theorist nor a model theorist) and to the dedicated staff of the MLI (for making his stay pleasant as well as fruitful).

This paper deals with a notion of 'homotopy' one can define in the *effective topos* of Martin Hyland ([7]).

In recent years, there has been an upsurge of interest in connections between abstract homotopy theories and type theory: see [1, 11, 4, 13, 14]. A prominent area of focus is that of *(closed) model categories*, first defined by D.G. Quillen ([9]; for modern expositions see [3, 6, 5]). A closed model structure on a category defines a notion of homotopy, and this is used in order to model the *identity types* of Martin-Löf's Type Theory.

In this paper, the focus is rather different. It may be the case that the structure described here is part of a closed model structure; but the author has been unable to establish this (the structure does *almost* form a 'fibration category' in the sense of [2]; see the end of this paper). It may also turn out that the effective topos harbors interesting interpretations of Martin-Löf's Type Theory, but the author has not investigated this.

Instead, his main motivation has been to find ways to understand the effective topos in geometric (topological) terms.

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In this paper, it is shown that a sensible notion of 'path object', for every object of the topos, arises out of the so-called 'discrete reflection'. In the effective topos, the full subcategory of *discrete objects* has been thoroughly studied (see, e.g., [8]). I remind the reader that an object of the effective topos is called 'discrete' if it is a quotient of a subobject of the natural numbers object. There are several equivalent characterizations of the discrete objects, among which:

> An object X is discrete if and only if the canonical map from X to the exponential $X^{\mathcal{P}(N)}$ is an isomorphism, where $\mathcal{P}(N)$ denotes the power object of the natural numbers.

Easy proofs of this and related facts may be found in [12]. The fact above means that the discrete objects are defined by an orthogonality property, and hence enjoy good completeness properties relative to the effective topos. In particular, the inclusion of the discrete objects as full subcategory of the effective topos has a left adjoint, which is called the 'discrete reflection'.

Part of the significance of the present paper is to provide a new and very concrete, intuitive meaning to the word 'discrete'. Indeed, my notion of 'path' will be such that:

1) An object X is discrete if and only if there are no nontrivial (= nonconstant) paths in X

2) The discrete reflection of X is (internally) the set of path components of X

This should lead, hopefully, to a study of 'topological' properties of non-discrete objects. For, the non-discrete part of the effective topos has not been studied nearly as extensively as the discrete part (for example, the following question has, to my knowledge, not been answered: which objects Y have the property that $X \to X^Y$ is an isomorphism, for every discrete object X?).

Note, by the way, that the properties of 'paths' above imply that there is no connection to the real line. Indeed, in the effective topos the object of real numbers is discrete!

We obtain sensible interpretations of standard notions from topology. For example, an object is *path connected* if and only if its discrete reflection is isomorphic to the terminal object; an object is *simply connected* if it is path connected and its fundamental group is trivial. We shall see that these notions do not coincide: we have 'circles' whose fundamental group is isomorphic to $\mathbb{Z}$.

Preliminaries. For the sake of self-containedness, I give the definition of the effective topos as a category. From now on, it will be denoted by $\mathcal{E}ff$. You are referred to [7] and [12] for further information.

The category $\mathcal{E}ff$ is built on the notion of a *computable function* on the natural numbers.

An object is a pair $(X, \sim)$ where X is a set and $\sim$ is a function from $X \times X$ to the set $P(\mathbb{N})$ of subsets of $\mathbb{N}$, usually written as $x, y \mapsto [x \sim y]$. This

function $\sim$ has to satisfy the requirement that there exist computable functions s (symmetry) and t (transitivity), such that the following hold:

- if $a \in [x \sim y]$ then $\mathsf{s}(a) \in [y \sim x]$

- if $a \in [x \sim y]$ and $b \in [y \sim z]$ then $\mathsf{t}(a,b) \in [x \sim z]$

The set $[x \sim y]$ is called the *equality* of x and y.

Given two objects $(X, \sim)$ and $(Y, \approx)$, a morphism $(X, \sim) \to (Y, \approx)$ is represented by a function $F : X \times Y \to P(\mathbb{N})$ for which there exist computable functions st_X, st_Y (strictness), rel_X, rel_Y (relationality), tl (totality) and sv (single-valuedness), satisfying:

- if $a \in F(x,y)$ then $\mathsf{st}_X(a) \in [x \sim x]$ and $\mathsf{st}_Y(a) \in [y \approx y]$

- if $a \in F(x,y)$, $b \in [x' \sim x]$ and $c \in [y \approx y']$ then $\mathsf{rel}_X(b,a) \in F(x',y)$ and $\mathsf{rel}_Y(a,c) \in F(x,y')$

- if $a \in [x \sim x]$ then $\mathsf{tl}(a) \in \bigcup_{y \in Y} F(x,y)$

- if $a \in F(x,y)$ and $b \in F(x,y')$ then $\mathsf{sv}(a,b) \in [y \approx y']$

Two such functions F, G determine the same morphism $(X, \sim) \to (Y, \approx)$ if there is a computable function ϕ such that whenever $a \in F(x,y)$, $\phi(a) \in G(x,y)$ (this is an equivalence relation on functions which represent a morphism).

We shall write $\langle a_1, \ldots, a_n \rangle$ for the natural number which codes the finite tuple of natural numbers $(a_1, \ldots, a_n)$ in some coding for which all the operations one wishes to perform are given by computable functions: e.g., determining the length of a sequence coded by x, computing the code of the sequence which is the concatenation of the sequences coded by x and y, determining the i'th element of the sequence coded by x, etc.

For an object $(X, \sim)$ and $x \in X$ we shall often write $E(x)$ for $[x \sim x]$. $E(x)$ is called the *existence* of x.

Example 1. Suppose $(X, \sim)$ and $(Y, \approx)$ are objects of $\mathcal{E}ff$ and $f : X \to Y$ is a function such that there exist computable functions ϕ and ψ satisfying:

- whenever $a \in E(x)$, $\phi(a) \in E(f(x))$

- whenever $a \in [x \sim x']$, $\psi(a) \in [f(x) \approx f(x')]$

Then the following function F represents a morphism $(X, \sim) \to (Y, \approx)$:

$$F(x,y) \;=\; \bigcup_{x' \in X} \{ \langle a, b \rangle \mid a \in [x \sim x'], \, b \in [f(x') \approx y] \}$$

Let us say that this morphism is *induced* by the function f.

Example 2. If $(X, \sim)$ is an object of $\mathcal{E}ff$, then a subobject of $(X, \sim)$ is determined by a function $F : X \to P(\mathbb{N})$ for which there exist computable functions ϕ and ψ satisfying

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- if $a \in F(x)$ then $\phi(a) \in E(x)$

- if $a \in F(x)$ and $b \in [x \sim x']$ then $\psi(a, b) \in F(x')$

The subobject determined by F is then represented by the object $(X, \sim')$ where

$$[x \sim' y] \;=\; \{\langle a, b \rangle \mid a \in F(x),\, b \in [x \sim y]\}$$

Example 3. An *assembly* is an object $(X, \sim)$ which is such that $[x \sim y] = \emptyset$ if $x \neq y$. Hence, an assembly is given by a map $E : X \to P(\mathbb{N})$. The assemblies are (up to isomorphism) the $\neg\neg$-separated objects of $\mathcal{E}ff$. An assembly is called *canonically projective* if $E(x)$ is a singleton for each x. These are (up to isomorphism) the projective objects of $\mathcal{E}ff$.

Every morphism into an assembly is induced by a unique function on the level of sets, as in Example 1.

1 Intervals and Paths

Since, in $\mathcal{E}ff$, the object of real numbers is discrete, it will come as no surprise that the 'intervals' defined here have nothing to do with the unit interval $[0, 1]$ of real numbers.

Definition 1.1 Let $A = \{\alpha_0, \alpha_1, \ldots\}$ and $B = \{\beta_0, \beta_1, \ldots\}$ be two disjoint countable sets. The *generic interval of length n* is the object $(X, \sim)$ where $X = \{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\}$ and $\sim$ is given by:

$$
\begin{aligned}
[\alpha_i \sim \alpha_i] &= \{i\} & [\alpha_i \sim \alpha_j] &= \emptyset \text{ if } i \neq j \\
[\beta_i \sim \beta_i] &= \{i+1\} & [\beta_i \sim \beta_j] &= \emptyset \text{ if } i \neq j \\
[\alpha_i \sim \beta_i] &= \{\langle i, i+1 \rangle\} & [\alpha_i \sim \beta_j] &= \emptyset \text{ if } i \neq j \\
[\beta_i \sim \alpha_i] &= \{\langle i+1, i \rangle\} & [\beta_i \sim \alpha_j] &= \emptyset \text{ if } i \neq j
\end{aligned}
$$

This object is denoted I_n.

I prefer to visualize the object I_n in the following way:

$$
\alpha_n \xrightarrow[\langle n, n+1 \rangle]{\langle n+1, n \rangle} \beta_n
$$

$$\vdots$$

$$
\alpha_2 \xrightarrow[\langle 2, 3 \rangle]{\langle 3, 2 \rangle} \cdots
$$

$$
\alpha_1 \xrightarrow[\langle 1, 2 \rangle]{\langle 2, 1 \rangle} \beta_1
$$

$$
\alpha_0 \xrightarrow[\langle 0, 1 \rangle]{\langle 1, 0 \rangle} \beta_0
$$

$$
0 \qquad 1 \qquad 2 \qquad \cdots \qquad n \qquad n+1
$$

That is, nontrivial *equalities* are given by labelled horizontal lines; elements on the same vertical line have the same *existence*.

The following facts about I_n are easily established and left to the reader.

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Proposition 1.2

1) Let K_n be the canonically projective object $(\{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\}, E)$ with $E(\alpha_i) = \{i\}$ and $E(\beta_i) = \{i+1\}$. Let $[n+1]$ be the canonically projective object (finite cardinal) $(\{0, \ldots, n\}, E)$ with $E(i) = \{i\}$. Then there are morphisms $f, g : [n+1] \to K_n$, given by $f(i) = \alpha_i$, $g(i) = \beta_i$, such that there is a coequalizer diagram

$$[n+1] \underset{g}{\overset{f}{\rightrightarrows}} K_n \longrightarrow I_n$$

2) The object I_n is $\neg\neg$-separated, and isomorphic to the assembly $\tilde{I}_n = (\{0, \ldots, n\}, E)$ with $E(i) = \{i, i+1\}$.

Fact i) of proposition 1.2 makes it easy to describe the exponentials $(X, \sim)^{I_n}$. For, we have an equalizer diagram

$$(X, \sim)^{I_n} \longrightarrow (X, \sim)^{K_n} \rightrightarrows (X, \sim)^{[n+1]}$$

and hence, by the well-known formation of exponentials in the case that the exponent is canonically projective, we see that $(X, \sim)^{I_n}$ has as underlying set the set of functions from $\{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\}$ to X, where the existence of such a function f is the set of coded pairs $\langle \sigma, \tau \rangle$ such that σ is a coded tuple $\langle c_0, \ldots, c_{n+1} \rangle$ satisfying $c_i \in E(f(\alpha_i))$ for $0 \le i \le n$ and $c_i \in E(f(\beta_{i-1}))$ for $1 \le i \le n+1$; and τ is a coded tuple $\langle a_0, \ldots, a_n \rangle$ such that $a_i \in [f(\alpha_i) \sim f(\beta_i)]$ for $0 \le i \le n$. The equality between two such functions f and g is the set of coded tuples $\nu = \langle d_0, \ldots, d_{n+1} \rangle$ such that $d_i \in [f(\alpha_i) \sim g(\alpha_i)]$ for $0 \le i \le n$, and $d_i \in [f(\beta_{i-1}) \sim g(\beta_{i-1})]$ for $1 \le i \le n+1$.

Definition 1.3 A morphism $I_n \overset{F}{\to} I_m$ is called *order and endpoint preserving* if the unique function $f : \{0, \ldots, n\} \to \{0, \ldots, m\}$ which induces the corresponding arrow $\tilde{I}_n \to \tilde{I}_m$ (via the isomorphism of 1.2 2)), is order-preserving and satisfies $f(0) = 0$, $f(n) = m$.

Proposition 1.4 *Every order and endpoint preserving map $I_n \to I_m$ is surjective; hence such maps exist if and only if $n \ge m$.*

Proof. Consider $f : \{0, \ldots, n\} \to \{0, \ldots, m\}$. If $i \in \{0, \ldots, m\}$ is not in the image of f then $i \ne 0$ and $i \ne m$ (since f is endpoint preserving) so there must be an element $j \in \{0, \ldots, n\}$ such that $f(j) < i$ and $f(j+1) > i$. But then, in $\tilde{I}_m$, $E(f(j)) \cap E(f(j+1)) = \emptyset$ whereas in $\tilde{I}_n$, $j+1 \in E(j) \cap E(j+1)$. So f cannot induce a morphism in $\mathcal{E}ff$. ∎

Definition 1.5 Let $(X, \sim)$ be an object of $\mathcal{E}ff$. The *path object* of $(X, \sim)$, denoted $\mathsf{P}(X, \sim)$, is defined as follows.

Its underlying set is the set of all pairs (n, f) with $n \in \mathbb{N}$ and f is a function from $\{\alpha_0, \beta_0, \ldots, \alpha_n, \beta_n\}$ to X.

Given two such pairs (n, f) and (m, g), the equality $[(n, f) \sim (m, g)]$ between them is the set of all coded triples $\langle a, s, b \rangle$ satisfying:

a) $a \in E(f)$ as element of $(X, \sim)^{I_n}$

b) $b \in E(g)$ as element of $(X, \sim)^{I_m}$

c) there is an order and endpoint preserving morphism $\sigma : I_n \to I_m$ such that $s \in [f \sim g\sigma]$ in the sense of $(X, \sim)^{I_n}$; or there is an order and endpoint preserving morphism $\sigma : I_m \to I_n$ such that $s \in [f\sigma \sim g]$ in the sense of $(X, \sim)^{I_m}$.

Proposition 1.6

i) *The construction of $\mathsf{P}(X, \sim)$ extends to a functor $\mathsf{P} : \mathcal{E}ff \to \mathcal{E}ff$ which preserves products.*

ii) *The object $\mathsf{P}(X, \sim)$ comes equipped with well-defined maps:*

  s *(source),* t *(target):* $\mathsf{P}(X, \sim) \to (X, \sim)$

  c *(trivial path):* $(X, \sim) \to \mathsf{P}(X, \sim)$

  $*$ *(composition of paths):* $\mathsf{P}(X, \sim) \times_{(X,\sim)} \mathsf{P}(X, \sim) \to \mathsf{P}(X, \sim)$, *where the domain is the pullback*

$$
\begin{array}{ccc}
\mathsf{P}(X, \sim) \times_{(X,\sim)} \mathsf{P}(X, \sim) & \longrightarrow & \mathsf{P}(X, \sim) \\
\downarrow & & \downarrow{\scriptstyle \mathsf{s}} \\
\mathsf{P}(X, \sim) & \xrightarrow[\mathsf{t}]{} & (X, \sim)
\end{array}
$$

  $\widetilde{(\cdot)}$ *(converse path):* $\mathsf{P}(X, \sim) \to \mathsf{P}(X, \sim)$

  With these data, $\mathsf{P}(X, \sim)$ is an internal category in $\mathcal{E}ff$ which has a contravariant involution which is the identity on objects.

Proof. I leave most of this to the reader. That P preserves products is a consequence of the equality relation we defined on $\mathsf{P}(X, \sim)$, which ensures that we "can assume that two paths are of the same length", i.e. for (n, f) representing an element of $\mathsf{P}(X, \sim)$ and $a \in E(n, f)$ we can, for each $m \geq n$, find an element (m, f') and an element of $[(n, f) \sim (m, f')]$.

For functions $f : \{\alpha_0, \beta_0, \ldots, \alpha_n, \beta_n\} \to X$ and $g : \{\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m\} \to X$ let $f * g : \{\alpha_0, \beta_0, \ldots, \alpha_{n+m}, \beta_{n+m}\} \to X$ be the function defined by

$$
\begin{aligned}
f * g(\alpha_i) &= \begin{cases} f(\alpha_i) & i \leq n \\ g(\alpha_{i-n}) & i > n \end{cases} \\
f * g(\beta_i) &= \begin{cases} f(\beta_i) & i < n \\ g(\beta_{i-n}) & i \geq n \end{cases}
\end{aligned}
$$

Then composition of paths is represented by the function which sends a triple $(n, f), (m, g), (k, h)$ to the set of coded 4-tuples $\langle a, b, c, d \rangle$ such that $a \in E(f)$, $b \in E(g)$, $c \in [f(\beta_n) \sim g(\alpha_0)]$ and $d \in [(k, h) \sim (n + m, f * g)]$. Here $E(f), E(g)$ refer to the existence of $(X, \sim)^{I_n}$, $(X, \sim)^{I_m}$ respectively.

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Composition is strictly associative. It is another consequence of the particular equality on $\mathsf{P}(X,\sim)$ that the trivial (constant) paths are strict identities for composition. ∎

For the following proposition, I find it convenient to visualise paths in $(X,\sim)$ in the way of the picture for I_n; so if $f : \{\alpha_0, \beta_0, \ldots, \alpha_n, \beta_n\} \to X$ represents such a path, $x_i = f(\alpha_i)$ and $y_i = f(\beta_i)$ and $\langle \sigma, \tau \rangle \in E(f)$ in the sense of $(X,\sim)^{I_n}$, so $\sigma = \langle c_0, \ldots, c_{n+1} \rangle$, $\tau = \langle a_0, \ldots, a_n \rangle$, we picture f as

$$
\begin{array}{ccccccc}
& & & & x_n & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_n \\
& & & & & {\scriptstyle a_n} & \\
& & & & & \vdots & \\
& x_1 & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_1 & & & \\
& & {\scriptstyle a_1} & & & & \\
x_0 & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_0 & & & & \\
& {\scriptstyle a_0} & & & & & \\
c_0 & c_1 & c_2 & \cdots & c_n & c_{n+1} &
\end{array}
$$

Such a path proceeds (in X) by alternately taking a *horizontal* and a *vertical* step: a horizontal step involves an equality in $(X,\sim)$, a vertical step involves an element in the intersection of the existences.

Proposition 1.7 *There is a morphism $\mathsf{P}(X,\sim) \times \mathsf{P}(X,\sim) \overset{L}{\to} \mathsf{PP}(X,\sim)$ such that, internally in $\mathscr{E}\!f\!f$, $\mathsf{s}(L(f,g)) = f * g$ and $\mathsf{t}(L(f,g)) = g$.*

Proof. We do this for the special case that g is constant; the generalization to the statement in the proposition is straightforward and left to the reader.

So, we wish to show that there is $L : \mathsf{P}(X,\sim) \to \mathsf{PP}(X,\sim)$ such that $\mathsf{s}L = \mathrm{id}_{\mathsf{P}(X,\sim)}$ and $\mathsf{t}L = \mathsf{ct}$:

$$
\begin{array}{ccc}
\mathsf{P}(X,\sim) \xrightarrow{\ L\ } \mathsf{PP}(X,\sim) & & \mathsf{P}(X,\sim) \xrightarrow{\ L\ } \mathsf{PP}(X,\sim) \\
 \searrow_{\mathrm{id}} \quad \downarrow {\scriptstyle \mathsf{s}} & & \quad {\scriptstyle \mathsf{t}}\downarrow \qquad\qquad \downarrow {\scriptstyle \mathsf{t}} \\
\mathsf{P}(X,\sim) & & \mathsf{P}(X,\sim) \xrightarrow[\ \mathsf{c}\ ]{} \mathsf{P}(X,\sim)
\end{array}
$$

First we show that there is a map $L_k : (X,\sim)^{I_k} \to ((X,\sim)^{I_k})^{I_k}$ these properties. It induced by the function ℓ_k described now:

Suppose f represents an element of $(X,\sim)^{I_k}$:

$$
\begin{array}{ccccccc}
& & & & x_n & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_n \\
& & & & & {\scriptstyle a_n} & \\
& & & & & \vdots & \\
& x_1 & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_1 & & & \\
& & {\scriptstyle a_1} & & & & \\
x_0 & \!\!\rule[0.3ex]{2em}{0.4pt}\!\! & y_0 & & & & \\
& {\scriptstyle a_0} & & & & & \\
c_0 & c_1 & c_2 & \cdots & c_n & c_{n+1} &
\end{array}
$$

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Then $\ell_k(f)$ is the path $(f_0, p_0, \ldots, f_k, p_k)$ defined inductively. Let $f_0 = f$. Now suppose inductively that f_i is the path

$$x_k \xrightarrow[a_k]{} y_k$$

$$\vdots$$

$$x_{i+1} \quad \cdots$$

$$x_i \xrightarrow[a_i]{} y_i$$

$$\vdots$$

$$x_i \xrightarrow[c_i]{} x_i \quad \cdots$$

$$x_i \xrightarrow[c_i]{} x_i$$

$$c_i \quad c_i \quad c_i \quad \cdots \quad c_i \quad c_{i+1} \quad \cdots \quad c_k \quad c_{k+1}$$

Then we can take a horizontal step, obtaining p_i, which is

$$x_k \xrightarrow[a_k]{} y_k$$

$$\vdots$$

$$x_{i+1} \quad \cdots$$

$$y_i \xrightarrow[c_{i+1}]{} y_i$$

$$\vdots$$

$$y_i \xrightarrow[c_{i+1}]{} y_i \quad \cdots$$

$$y_i \xrightarrow[c_{i+1}]{} y_i$$

$$c_{i+1} \quad c_{i+1} \quad c_{i+1} \quad \cdots \quad c_{i+1} \quad c_{i+1} \quad \cdots \quad c_k \quad c_{k+1}$$

We have $\langle a_i, \ldots, a_i, c_{i+1}, \ldots, c_{k+1} \rangle \in [f_i \sim p_i]$. Subsequently, we take a vertical step, obtaining f_{i+1} from p_i. Clearly, p_k is the constant path at y_k. This defines the function ℓ_k.

In order to see that ℓ_k really induces a morphism $(X, \sim)^{I_k} \to ((X, \sim)^{I_k})^{I_k}$, suppose f, g represent elements of $(X, \sim)^{I_k}$ and $\gamma = \langle \gamma_0, \ldots, \gamma_{k+1} \rangle \in [f \sim g]$. If f is the sequence $(x_0, y_0, \ldots, x_k, y_k)$ and g is $(z_0, w_0, \ldots, z_k, w_k)$, then $\gamma_0 \in [x_0 \sim z_0]$, $\gamma_i \in [x_i \sim z_i] \cap [y_{i-1} \sim w_{i-1}]$ for $1 \le i \le k$, and $\gamma_{k+1} \in [y_k \sim w_k]$.

Then if $\ell_k(f) = (f_0, p_0, \ldots, f_k, p_k)$ and $\ell_k(g) = (g_0, h_0, \ldots, g_k, h_k)$, one sees by induction that $\gamma \in [f_0 \sim g_0]$, that $\langle \gamma_i, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_{k+1} \rangle$ is an element of $[f_i \sim g_i] \cap [p_{i-1} \sim h_{i-1}]$ for $1 \le i \le k$, and $\langle \gamma_{k+1}, \ldots, \gamma_{k+1} \rangle$ is in $[p_k \sim h_k]$. So if $\rho_i = \langle \gamma_i, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_{k+1} \rangle$, then

$$\rho = \langle \rho_0, \ldots, \rho_{k+1} \rangle \in [\ell_k(f) \sim \ell_k(g)]$$

So we have a well-defined morphism $L_k : (X, \sim)^{I_k} \to ((X, \sim)^{I_k})^{I_k}$.

The next step is to see that the maps L_k extend to a map $L : \mathsf{P}(X, \sim) \to \mathsf{PP}(X, \sim)$: that is, that they interact well with the equality relation on $\mathsf{P}(X, \sim)$.

Let ℓ be the function which sends (k, f) (where f represents an element of $(X, \sim)^{I_k}$) to $(k, \ell_k(f))$. We have to see that there is a computable function ϕ such that whenever $a \in [(k, f) \sim (m, g)]$ then $\phi(a) \in [\ell(k, f) \sim \ell(m, g)]$. From a we can computably extract k and m and hence their maximum; say this is m. Then we only have to consider order and endpoint preserving maps $I_m \to I_k$. Such a map is a composition of maps σ_j induced by functions $t_j : \{0, \dots, k+1\} \to \{0, \dots, k\}$ of the form: $t_j(i) = i$ for $i \le j$, and $t_j(i) = i - 1$ otherwise. Then the resulting map $(X, \sim)^{\sigma_j} : (X, \sim)^{I_k} \to (X, \sim)^{I_{k+1}}$ is induced by the function s_j which sends

$$f = (x_0, y_0, \dots, x_k, y_k)$$

to

$$s_j(f) = (x_0, y_0, \dots, x_j, x_j, x_j, y_j, x_{j+1}, \dots, x_k, y_k)$$

Suppose $a \in [(k, f) \sim (k+1, g)]$ so (ignoring irrelevant information) $a \in [s_j(f) \sim g]$ in $(X, \sim)^{I_{k+1}}$. From a we find elements $a_1 \in E(s_j(f))$, $a_2 \in E(f)$ and $a_3 \in [\ell_{k+1}(s_j(f)) \sim \ell_{k+1}(g)]$ (the last since ℓ_{k+1} induces a morphism as we have seen). It suffices therefore to find an element of

$$[(k, \ell_k(f)) \sim \ell_{k+1}(s_j(f))]$$

which does not depend on j.

Now from the definition of ℓ_k it is clear that if $\ell_k(f) = (f_0, g_0, \dots, f_k, g_k)$ then

$$
\begin{aligned}
\ell_{k+1}(s_j(f)) &= (s_j(f_0), s_j(g_0), \dots, s_j(f_j), s_j(g_j), s_j(f_j), s_j(g_j), s_j(f_{j+1}), \dots, s_j(g_k)) \\
&= s_j(s_j(f_0), s_j(g_0), \dots, s_j(f_k), s_j(g_k))
\end{aligned}
$$

So from a_3 we find an element of

$$[\ell_{k+1}(s_j(f)) \sim (s_j(f_0), \dots, s_j(g_k))]$$

and successively elements of $[s_j(f_i) \sim f_i]$, hence (by the equality of function spaces) an element of

$$[(s_j(f_0), \dots, s_j(g_k)) \sim (f_0, \dots, g_k)]$$

Combining, we get the desired element of $[\ell_{k+1}(s_j(f)) \sim \ell_k(f)]$. ∎

2   Discrete and Path connected objects

In the effective topos, an object is called discrete if it is a subquotient of the object of natural numbers N. The object N is the canonically projective object with underlying set $\mathbb{N}$, and $E(n) = \{n\}$. The following characterization of the discrete objects is taken from [12], 3.2.20.

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Proposition 2.1 *An object $(X, \sim)$ of $\mathcal{E}\!f\!f$ is discrete if and only if there is a computable function ϕ such that for all $x, x' \in X$ we have: if $n \in E(x) \cap E(x')$ then $\phi(n) \in [x \sim x']$. This is the case if and only if $(X, \sim)$ is isomorphic to an object $(Y, \sim)$ for which we have $E(y) \cap E(y') = \emptyset$ whenever $y \neq y'$.*

Let $\mathcal{E}\!f\!f_d$ be the full subcategory of $\mathcal{E}\!f\!f$ on the discrete objects. $\mathcal{E}\!f\!f_d$ is a very rich category: it contains all of 'constructive mathematics'.

The inclusion functor $\mathcal{E}\!f\!f_d \rightarrow \mathcal{E}\!f\!f$ has a left adjoint which preserves products: the *discrete reflection*.

Definition 2.2 ([12], 3.2.19iii)) The discrete reflection of an object $(X, \sim)$, denoted $(X, \sim)_d$, is the object with underlying set $\bigcup_{x \in X} E(x)$, and with the following equality: $[n \sim m]$ consists of all coded sequences

$$\sigma = \langle n_0, a_0, n_1, \ldots, n_k, a_k, n_{k+1} \rangle$$

satisfying: for each $i \leq k$ there are $x_i, y_i \in X$ such that $n_i \in E(x_i)$, $n_{i+1} \in E(y_i)$ and $a_i \in [x_i \sim y_i]$.

The universal arrow $\eta : (X, \sim) \rightarrow (X, \sim)_d$ is represented by the function

$$H(x, m) = \bigcup_{n \in E(x)} [n \sim m]$$

It is easily verified that η is always an epimorphism. Now from the definition of $\mathsf{P}(X, \sim)$ in the previous section it should be obvious that the kernel pair of η is just the image of the map $(\mathsf{s}, \mathsf{t}) : \mathsf{P}(X, \sim) \rightarrow (X, \sim) \times (X, \sim)$. Furthermore, since $\mathcal{E}\!f\!f_d$ is a full subcategory of $\mathcal{E}\!f\!f$ we have that η is an isomorphism if and only if $(X, \sim)$ is discrete. Summarizing:

Proposition 2.3

i) *An object of $\mathcal{E}\!f\!f$ is discrete if and only if it is internally true that there are no nonconstant paths*

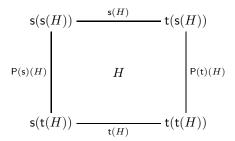ii) *The discrete reflection is, internally, the set of path components.*

We shall therefore call an object *path connected* if its discrete reflection is isomorphic to 1, the terminal object of $\mathcal{E}\!f\!f$.

3 Homotopy and Fundamental Group(oid)

Two maps $f, g : (X, \sim) \rightarrow (Y, \sim)$ are *homotopic* if there exists a *homotopy* from f to g, that is: a map $H : (X, \sim) \rightarrow \mathsf{P}(Y, \sim)$ such that $\mathsf{s}H = f$ and $\mathsf{t}H = g$. By composition of paths, converse paths and constant paths, homotopy is an equivalence relation.

For paths in $(X, \sim)$ there is the further notion of *endpoint preserving homotopy*. This is an element H of $\mathsf{PP}(X, \sim)$ such that both $\mathsf{P}(\mathsf{s})(H)$ and $\mathsf{P}(\mathsf{t})(H)$

are constant paths. The picture is:

$$
\begin{array}{ccc}
\mathsf{s}(\mathsf{s}(H)) & \xrightarrow{\ \mathsf{s}(H)\ } & \mathsf{t}(\mathsf{s}(H)) \\
{\scriptstyle \mathsf{P}(\mathsf{s})(H)}\Big| & H & \Big|{\scriptstyle \mathsf{P}(\mathsf{t})(H)} \\
\mathsf{s}(\mathsf{t}(H)) & \xrightarrow[\ \mathsf{t}(H)\ ]{} & \mathsf{t}(\mathsf{t}(H))
\end{array}
$$

Composition of homotopies is "vertical composition":

$$
\begin{array}{ccc}
\mathsf{s}(\mathsf{s}(H)) & \xrightarrow{\ \mathsf{s}(H)\ } & \mathsf{t}(\mathsf{s}(H)) \\
{\scriptstyle \mathsf{P}(\mathsf{s})(H)}\Big| & H & \Big|{\scriptstyle \mathsf{P}(\mathsf{t})(H)} \\
\mathsf{s}(\mathsf{t}(H)) & \xrightarrow[\ \mathsf{t}(H)=\mathsf{s}(G)\ ]{} & \mathsf{t}(\mathsf{t}(H)) \\
{\scriptstyle \mathsf{P}(\mathsf{s})(G)}\Big| & G & \Big|{\scriptstyle \mathsf{P}(\mathsf{t})(G)} \\
\mathsf{s}(\mathsf{t}(G)) & \xrightarrow[\ \mathsf{t}(G)\ ]{} & \mathsf{t}(\mathsf{t}(G))
\end{array}
\quad\Rightarrow\quad
\begin{array}{ccc}
\mathsf{s}(\mathsf{s}(H)) & \xrightarrow{\ \mathsf{s}(H)\ } & \mathsf{t}(\mathsf{s}(H)) \\
\Big| & H * G & \Big| \\
\mathsf{s}(\mathsf{t}(G)) & \xrightarrow[\ \mathsf{t}(G)\ ]{} & \mathsf{t}(\mathsf{t}(G))
\end{array}
$$

We also have "horizontal composition": if $\mathsf{P}(\mathsf{t})(H) = \mathsf{P}(\mathsf{s})(G)$ then the pair (H, G) is actually a path in the object $\mathsf{P}(X, \sim) \times_{(X,\sim)} \mathsf{P}(X, \sim)$ and we have $H \circ G = \mathsf{P}(*)(H, G)$:

$$
\begin{array}{cc}
\overset{\mathsf{s}(H)}{\boxed{\ H\ }} & \overset{\mathsf{s}(G)}{\boxed{\ G\ }} \\
{\scriptstyle \mathsf{t}(H)} & {\scriptstyle \mathsf{t}(G)}
\end{array}
\quad\Rightarrow\quad
\begin{array}{c}
\overset{\mathsf{s}(H)*\mathsf{s}(G)}{\boxed{\ H \circ G\ }} \\
{\scriptstyle \mathsf{t}(H)*\mathsf{t}(G)}
\end{array}
$$

which is a homotopy from $\mathsf{s}(H) * \mathsf{s}(G)$ to $\mathsf{t}(H) * \mathsf{t}(G)$.

It is easy to see that the map L from proposition 1.7, satisfying $\mathsf{s}L(f) = f$ and $\mathsf{t}L(f) = \mathsf{c}\mathsf{t}(f)$, preserves the endpoint: $\mathsf{P}(\mathsf{t})(L(f)) = \mathsf{c}(\mathsf{t}(f))$.

Using this map L and horizontal composition, one readily verifies that there is, for each $f \in \mathsf{P}(X, \sim)$, and endpoint preserving homotopy from $\mathsf{c}(\mathsf{s}(f))$ to $f * \tilde{f}$

Definition 3.1 Let $(X, \sim)$ be an object of $\mathcal{E}\!f\!f$ and $x : 1 \to (X, \sim)$ a base point. The *fundamental group of $(X, \sim)$ with base point x*, or $\pi_1((X, \sim), x)$, is the set of enpoint preserving homotopy classes of paths from x to x, with composition of paths as operation.

The fundamental group is in fact a group: composition is well defined on homotopy classes by horizontal composition, and strictly associative, the constant path on x is the unit, and $\tilde{f}$ is a two-sided inverse for f. As usual, if $(X, \sim)$ is path connected the fundamental group does not really depend on the base point and we can omit it.

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Definition 3.2 An object is *simply connected* if it is path connected and its fundamental group is trivial.

We can also define, for each $(X, \sim)$, the fundamental groupoid of $(X, \sim)$: its object of objects is $(X, \sim)$, its object of arrows is the object of endpoint preserving homotopy classes of paths.

As an example, I now briefly discuss "circles". A circle, naturally, is constructed by identifying the endpoints of an interval.

Definition 3.3 The circle C_n is defined by the coequalizer diagram

$$1 \underset{n}{\overset{0}{\rightrightarrows}} \tilde{I}_n \longrightarrow C_n$$

Proposition 3.4 *For $n \neq 2$, the object C_n is separated. In fact, $C_0 \cong C_1 \cong 1$ and for $n > 2$, the object C_n is isomorphic to the assembly $(\{0, \ldots, n-1\}, E)$ with*

$$E(0) = \{0, 1\}, \ldots, E(n-2) = \{n-2, n-1\}, E(n-1) = \{n-1, 0\}$$

Proof. Straightforward calculation. ∎

One can show that the object C_2, which has 2 points, is not separated.

Proposition 3.5 *For each $n > 2$, $\pi_1(C_n) \cong \mathbb{Z}$.*

Proof. This follows standard lines. First one defines the "line" R: it is the object $(\mathbb{Z}, E)$ with

$$
\begin{array}{rcll}
E(n) & = & \{2n-1, 2n+1\} & n > 0 \\
E(0) & = & \{0, 1\} & \\
E(n) & = & \{2n, 2n+2\} & n < 0
\end{array}
$$

One observes that R is simply connected and that the map $R \to C_n$ given by $m \mapsto m \bmod n$ is a "universal covering": it has the unique path lifting and homotopy lifting properties.

Consider both R and C_n as equipped wit the base point 0. The map from $\pi_1(C_n, 0)$ to R, sending a homotopy class of a path to the target of its lifting, is well-defined and gives a bijection from $\pi_1(C_n, 0)$ to $\mathbb{Z}$. Composition of paths corresponds under this map to addition, so the required isomorphism is there. ∎

Of course, in $\mathcal{Eff}$ we also have 'tori' $T_n = C_n \times C_n$ and since we can prove that π_1 is a functor from $\mathcal{Eff}_*$ (the category of objects of $\mathcal{Eff}$ with a base point, and base point preserving maps) to the category of groups and preserves products, we have

$$\pi_1(T_n) = \mathbb{Z} \times \mathbb{Z}$$

for $n > 2$.

Simlarly, we can discuss 'wedges of circles', etcetera.

4 Homotopy equivalences and Hurewicz fibrations

Definition 4.1

a) An arrow $f : (X, \sim) \to (Y, \sim)$ *embeds* $(X, \sim)$ *as strong deformation retract into* $(Y, \sim)$ if there is a map $F : (Y, \sim) \to \mathsf{P}(Y, \sim)$ with the properties:

    i) $\mathsf{s}F = \mathrm{id}_{(Y,\sim)}$

    ii)   the diagram

$$
\begin{array}{ccc}
(X, \sim) & \xrightarrow{\ \mathsf{c}\ } & \mathsf{P}(X, \sim) \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathsf{P}(f)} \\
(Y, \sim) & \xrightarrow[F]{} & \mathsf{P}(Y, \sim)
\end{array}
$$

         commutes, i.e. for $x \in (X, \sim)$, $F(f(x))$ is the constant path on $f(x)$

    iii) the map $\mathsf{t}F$ factors through f

b) An arrow $f : (X, \sim) \to (Y, \sim)$ is a *homotopy equivalence* if there is an arrow $g : (Y, \sim) \to (X, \sim)$ such that fg is homotopic to $\mathrm{id}_{(Y,\sim)}$ and gf is homotopic to $\mathrm{id}_{(X,\sim)}$.

Clearly, every embedding as strong deformation retract is a homotopy equivalence.

Proposition 4.2 *The map* $\mathsf{c} : (X, \sim) \to \mathsf{P}(X, \sim)$ *embeds* $(X, \sim)$ *as strong deformation retract into* $\mathsf{P}(X, \sim)$.

Proof. Use L.   ■

Definition 4.3 Given $f : (X, \sim) \to (Y, \sim)$, denote by C_f the object defined by the pullback square

$$
\begin{array}{ccc}
C_f & \xrightarrow{\hspace{1cm}} & (X, \sim) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
\mathsf{P}(Y, \sim) & \xrightarrow[\mathsf{s}]{} & (Y, \sim)
\end{array}
$$

So, $C_f = \{(\alpha, x) \in \mathsf{P}(Y, \sim) \times (X, \sim) \,|\, \mathsf{s}(\alpha) = f(x)\}$.

Let $\nu_f : \mathsf{P}(X, \sim) \to C_f$ be the map defined by the commutative diagram

$$
\begin{array}{ccc}
\mathsf{P}(X, \sim) & \xrightarrow{\ \mathsf{s}\ } & (X, \sim) \\
\Big\downarrow{\scriptstyle \mathsf{P}(f)} & & \Big\downarrow{\scriptstyle f} \\
\mathsf{P}(Y, \sim) & \xrightarrow[\mathsf{s}]{} & (Y, \sim)
\end{array}
$$

i.e., $\nu_f(\omega) = (\mathsf{P}(f)(\omega), \mathsf{s}(\omega))$.

The map f is called a *Hurewicz fibration* if the map ν_f has a section.

Our definition of 'Hurewicz fibration' is equivalent to the standard one for topological spaces, and is a rewrite in a language with no interval, but only path objects.

Proposition 4.4 *Every arrow $f : (X, \sim) \to (Y, \sim)$ in $\mathcal{E}ff$ factors as an embedding as strong deformation retract, followed by a Hurewicz fibration.*

Proof. This follows the usual definition for topological spaces. Factor f as

$$(X, \sim) \xrightarrow{\ \iota\ } C_p \xrightarrow{\ \pi\ } (Y, \sim)$$

where $\iota(x) = (x, \mathsf{c}(f(x)))$ and $\pi(x, \omega) = \mathsf{t}(\omega)$. Analogously to the embedding $(X, \sim) \to \mathsf{P}(X, \sim)$, ι is easily seen to be an embedding as strong deformation retract. For the other map π, we have to show that the map

$$\mathsf{P}(C_f) \overset{(\mathsf{s}, \mathsf{P}(\pi))}{\to} C_\pi$$

has a section.

The object C_π is internally defined as

$$C_\pi \ = \ \{((x, \omega), \eta) \in C_f \times \mathsf{P}(Y, \sim) \mid f(x) = \mathsf{s}(\omega), \mathsf{t}(\omega) = \mathsf{s}(\eta)\}$$

A typical element of $\mathsf{P}(C_f)$ is a pair (α, H) where α is a path in $(X, \sim)$ and H is a path in $\mathsf{P}(Y, \sim)$ with $\mathsf{P}(\mathsf{s})(H) = \mathsf{P}(f)(\alpha)$, so $f(\mathsf{s}(\alpha)) = \mathsf{s}(\mathsf{s}(H))$.

Now for $((x, \omega), \eta) \in C_\pi$ we have to find (α, H) such that $\mathsf{s}(H) = \omega$ and $\mathsf{P}(\mathsf{t})(H) = \eta$. But we can use proposition 1.7 again: let α be $\mathsf{c}(f(x))$ and H be the path from ω to $\omega * \eta$ such that $\mathsf{P}(\mathsf{s})(H)$ is constant on $f(x)$ and $\mathsf{P}(\mathsf{t})(H) = \eta$. So indeed, π is a Hurewicz fibration. ∎

Remarks

1) In an old paper by Arne Strøm ([10]) it is shown that there is a closed model structure on the category of topological spaces in which the fibrations are the Hurewicz fibrations, the weak equivalences are the homotopy equivalences and the cofibrations are the 'Hurewicz cofibrations', that is in our terminology: all maps which have the left lifting property with respect to all maps $\mathsf{P}(X) \overset{\mathsf{s}}{\to} X$. I have not been able to find good factorizations as Hurewicz cofibrations followed by trivial fibrations, and I doubt they exist.

2) H.J. Baues ([2]) proposes the following relaxation of the notion of closed model structure: the notion of a *fibration category*. It is a category together with two classes of maps, the *fibrations* and the *weak equivalences* satisfying the following requirements (in stating them, I assume that we work in a category with pullbacks):

    F1 Every isomorphism is both a fibration and a weak equivalence; fibrations are closed under composition and weak equivalences satisfy the 2-out-of-3 property

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F2 Fibrations are stable under pullback and weak equivalences are stable under pullback along fibrations. Also, trivial fibrations (that is: fibrations which are also weak equivalences) are stable under pullback

F3 Every arrow factors as a weak equivalence followed by a fibration

F4 Call an object *cofibrant* if every trivial fibration into it has a section. For each object X there is a cofibrant object QX and a trivial fibration $QX \to X$.

The structure we desribed on $\mathcal{E}ff$ with homotopy equivalences as weak equivalences, and Hurewicz fibrations as fibrations, satisfies at least F1,F3 and F4 (due to time pressure when writing this paper the author was not able to check F2). That F4 holds is an easy consequence of the observation that every Hurewicz fibration which is also a homotopy equivalence, must have a section, so every object is cofibrant.

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