THE ROYAL SWEDISH ACADEMY OF SCIENCES

INSTITUT MITTAG-LEFFLER

Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

Notions of Independence for Metric Abstract Elementary Classes

A. Villaveces and P. Zambrano

REPORT No. 27, 2009/2010, fall

ISSN 1103-467X
ISRN IML-R- -27-09/10- -SE+fall

NOTIONS OF INDEPENDENCE FOR METRIC ABSTRACT ELEMENTARY CLASSES

ANDRÉS VILLAVECES
PEDRO ZAMBRANO

Abstract. We study notions of independence appropriate for a stability theory of metric abstract elementary classes. We build on previous notions used in the discrete case, and adapt definitions to the metric case. In particular, we study notions that behave well under superstability-like assumptions.

Introduction

In the study of the Stability Theory of Abstract Elementary Classes (for short, AEC, in this paper), various versions of independence linked to *splitting* (introduced originally by Shelah in the discrete AEC case [Sh 394]) have played an important rôle. Various categoricity transfer results, as well as the development of stability theory in AEC have so far used non-splitting as one of the main independence notions.

In the metric continuous case (a generalization of both usual, or "discrete" AEC *and* "First Order" Continuous Model Theory), notions of independence have been used with some success in a strongly homogeneous ω-stable, (Löwenheim-Skolem number $\aleph_0$) case by Åsa Hirvonen [Hi].

We focus here in a notion of independence, called *r-independence* (see 2.2), that generalizes non-splitting to the metric context (in the stable case), and works well under the existence of various sorts of limit models. We study conditions under which r-independence satisfies appropriate variants of transitivity, stationarity, extension, existence (Section 2). We also study the continuity of this independence notion (see 2.13).

Applications of these techniques include the study of "superstable" metric AEC (limit models and r-towers [ViZa2]), and steps towards a generalization of the main theorem in [GrVaVi] (Uniqueness of Limit Models

Date: Mar 15, 2010.

Support by the Institut Mittag-Leffler (Djursholm, Sweden) is gratefully acknowledged. The second author wants to thank the first author for the time devoted to advising him for his Ph.D.'s thesis, one of whose fruits is this paper. He also thanks Tapani Hyttinen for helpful discussions in Helsinki during November 2008. Both authors were partially supported by Colciencias.

in AEC under superstability-like assumptions). They also include notions of domination appropriate for both discrete and metric continuous (superstable) AEC.

1. Some basic definitions and results

Definition 1.1. The *density character* of a topological space is the smallest cardinality of a dense subset of the space. If X is a topological space, we denote its density character by $dc(X)$. If A is a subset of a topological space X, we define $dc(A) := dc(\overline{A})$.

We consider a natural adaptation of the notion of *Abstract Elementary Class* (see [Gr] and [Ba]), but work in a context of Continuous Logic that generalizes the "First Order Continuous" setting of [BeBeHeUs] by removing the assumption of uniform continuity[1]. We follow the definitions given by Åsa Hirvonen and Tapani Hyttinen (see [Hi]).

Definition 1.2. Let $\mathcal{K}$ be a class of L-structures (in the context of Continuous Logic) and $\prec_{\mathcal{K}}$ be a binary relation defined in $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is a *Metric Abstract Elementary Class* (shortly *MAEC*) if:

1. $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism.
2. $\prec_{\mathcal{K}}$ is a partial order in $\mathcal{K}$.
3. If $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
4. (*Completion of Union of Chains*) If $(M_i : i < \lambda)$ is a $\prec_{\mathcal{K}}$-increasing chain then
   (a) the function symbols in L can be uniquely interpreted on the completion of $\bigcup_{i<\lambda} M_i$ in such a way that $\overline{\bigcup_{i<\lambda} M_i} \in \mathcal{K}$
   (b) for each $j < \lambda$, $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\lambda} M_i}$
   (c) if each $M_i \in \mathcal{K} \in N$, then $\overline{\bigcup_{i<\lambda} M_i} \prec_{\mathcal{K}} N$.
5. (*Coherence*) if $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, then $M_1 \prec_{\mathcal{K}} M_2$.
6. (DLS) There exists a cardinality $LS^d(K)$ (which is called the *metric Löwenheim-Skolem number*) such that if $M \in \mathcal{K}$ and $A \subseteq M$, then there exists $N \in \mathcal{K}$ such that $dc(N) \leq dc(A) + LS^d(K)$ and $A \subseteq N \prec_{\mathcal{K}} M$.

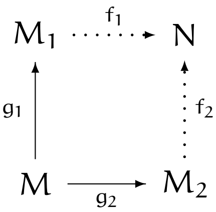
Definition 1.3. We call a function $f : M \to N$ a $\mathcal{K}$-*embedding* if

1. For every k-ary function symbol F of L, we have $f(F^M(a_1 \cdots a_k)) = F^N(f(a_1) \cdots f(a_k))$.
2. For every constant symbol c of L, $f(c^M) = c^N$.

[1] Uniform continuity guarantees logical compactness in their formalization, but we drop compactness in AEC-like settings.

(3) For every m-ary relation symbol R of L, for every $\bar{a} \in M^m$, $d(\bar{a}, R^M) = d(f(\bar{a}), R^N)$.
(4) $f[M] \prec_{\mathcal{K}} N$.

Definition 1.4 (Amalgamation Property, AP). Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies *Amalgamation Property* (for short *AP*) if and only if for every $M, M_1, M_2 \in \mathcal{K}$, if $g_i : M \to M_i$ is a $\mathcal{K}$-embedding $(i \in \{1,2\})$ then there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$ such that $f_1 \circ g_1 = f_2 \circ g_2$.



Definition 1.5 (Joint Embedding Property, JEP). Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies *Joint Embedding Property* (for short *JEP*) if and only if for every $M_1, M_2 \in \mathcal{K}$ there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$.

Remark 1.6. Notice that if $\mathcal{K}$ has a prime model, then AP implies JEP.

Remark 1.7 (Monster Model). If $\mathcal{K}$ is an MAEC which satisfies AP and JEP, then we can construct a large enough model $\mathbb{M}$ (which we call a *Monster Model*) which is homogeneous –i.e., every isomorphism between two $\mathcal{K}$-substructures of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$– and also universal –i.e., every model with density character $< dc(\mathbb{M})$ can be $\mathcal{K}$-embedded into $\mathbb{M}$.

Definition 1.8 (Galois type). Under the existence of a monster model $\mathbb{M}$ as in remark 1.7, for all $\overline{a} \in \mathbb{M}$ and $N \prec_{\mathcal{K}} \mathbb{M}$, we define ga-tp$(\overline{a}/N)$ (the *Galois type of* $\overline{a}$ *over* N) as the orbit of $\overline{a}$ under $Aut(\mathbb{M}/N) := \{f \in Aut(\mathbb{M}) : f \upharpoonright N = id_N\}$. We denote the space of Galois types over a model $M \in \mathcal{K}$ by ga-S(M).

Throughout this paper, we assume the existence of a homogenous and universal monster model as in remark 1.7.

Definition 1.9 (Distance between types). Let $p, q \in$ ga-S(M). We define $d(p,q) := \inf\{d(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \mathbb{M}, \overline{a} \models p, \overline{b} \models q\}$, where $\lg(\overline{a}) = \lg(\overline{b}) =: n$ and $d(\overline{a}, \overline{b}) := \max\{d(a_i, b_i) : 1 \leq i \leq n\}$.

Definition 1.10 (Continuity of Types). Let $\mathcal{K}$ be an MAEC and consider $(a_n) \to a$ in $\mathbb{M}$. We say that $\mathcal{K}$ satisfies *Continuity of Types Property*[2] (for

[2]This property is also called *Perturbation Property in* [Hi]

short, *CTP*), if and only if, if $\text{ga-tp}(a_n/M) = \text{ga-tp}(a_0/M)$ for all $n < \omega$ then $\text{ga-tp}(a/M) = \text{ga-tp}(a_0/M)$.

Remark 1.11. In general, distance between types d (see Definition 1.9) is just a pseudo-metric. But it is straightforward to see that the fact that d is a metric is equivalent to CTP.

Throughout this paper, we also assume CTP (so, distance between types is in fact a metric).

Definition 1.12 (Universality). Let $\mathcal{K}$ be an MAEC and $N \prec_{\mathcal{K}} M$. We say that M is λ-*universal* over N iff for every $N' \succ_{\mathcal{K}} N$ with density character λ there exists a $\mathcal{K}$-embedding $f : N' \to M$ such that $f \upharpoonright N = id_N$. We say that M is *universal* over N if M is $dc(M)$-universal over N.

Examples 1.13.
1. Any continuous elementary class (see [BeBeHeUs]) with the usual elementary substructure relation is an MAEC. Important cases include
   (a) Hilbert spaces with a unitary operator (Argoty and Berenstein, see [ArBe]).
   (b) Nakano spaces with compact essential rank (Poitevin, see [Po] and forthcoming [PoZa]).
   (c) Probability Spaces.
   (d) Compact Abstract Theories, see [Be1, Be2]
2. Nakano spaces whose essential rank is not necessarily compact (see forthcoming [PoZa]).
3. Hilbert Spaces, with various classes of unbounded operators.
4. A subclass of completions of metric spaces which satisfy approximately a positive bounded theory, where $\prec_{\mathcal{K}}$ is interpreted by the approximate elementary submodel relation (see [HeIo]).
5. Various classes of Banach spaces, where $\prec_{\mathcal{K}}$ is interpreted by the closed subspace relation[3] (see [Hi]).

Lemma 1.14. *Let $\mathcal{K}$ be a MAEC. If $f_i : M_i \to M$ ($i < \mu$) is a $\subseteq$-increasing and continuous (in the metric sense) chain of $\mathcal{K}$-embeddings, then there exists a $\mathcal{K}$-embedding $f : \overline{\bigcup_{i<\mu} M_i} \to M$ which extends $g := \bigcup_{i<\mu} f_i : \bigcup_{i<\mu} M_i \to M$.*

Proof. Let $a \in \overline{\bigcup_{i<\mu} M_i}$, so there exist elements $a_n \in \bigcup_{i<\mu} M_i$ for $n < \omega$, such that $(a_n)_{n<\omega} \to a$. As $(a_n)_{n<\omega}$ is a Cauchy sequence, $(g(a_n))_{n<\omega}$ is also a Cauchy sequence (since g is an isometry). So, there exists $b \in M$ such that $(g(a_n))_{n<\omega} \to b$. Define $f(a) := b$. Proceed in a similar

[3]Notice that several of these classes fall under case (1) — however, in general, natural classes of Banach spaces are not axiomatizable in the context of [BeBeHeUs].

way for every $a \in \overline{\bigcup_{i<\mu} M_i}$. The function f is well-defined: if we take $(a'_n)_{n<\omega}$ a sequence in $\bigcup_{i<\mu} M_i$ such that $(a'_n)_{n<\omega} \to a$, let $b' \in M$ be such that $(g(a'_n))_{n<\omega} \to b'$. We will prove that $b = b'$. Otherwise, let $\varepsilon := d(b, b') > 0$.

Claim 1.15. *Given $\varepsilon' > 0$, there exists $N < \omega$ such that for all $n \geq N$ $d(g(a_n), g(a'_n)) < \varepsilon'$.*

Proof. As $(a_n)_{n<\omega} \to a$ and $(a'_n)_{n<\omega} \to a$, there exists $N < \omega$ such that for all $n \geq N$ we have that $d(a_n, a) < \varepsilon'/2$ and $d(a'_n, a) < \varepsilon'/2$, so for all $n \geq N$ we have that $d(a_n, a'_n) \leq d(a_n, a) + d(a, a'_n) < \varepsilon'$. As g is an isometry, for all $n \geq N$ we have that $d(g(a_n), g(a'_n)) < \varepsilon'$. □

As $(g(a_n))_{n<\omega} \to b$, $(g(a'_n))_{n<\omega} \to b'$ and by claim 1.15, there exists $M < \omega$ such that for all $n \geq M$ we have that $d(g(a_n), b) < \varepsilon/3$, $d(g(a'_n), b') < \varepsilon/3$ and $d(g(a_n), g(a'_n)) < \varepsilon/3$. So, for all $n \geq M$ we have that $d(b, b') \leq d(b, g(a_n)) + d(g(a_n), g(a'_n)) + d(g(a'_n), b') < \varepsilon = d(b, b')$ (contradiction).

Therefore $b = b'$ and so f is well-defined.

We have that f extends g: let $a \in \bigcup_{i<\omega} M_i$, so taking $a_n := a$ $(n < \omega)$ we have that $(a_n)_{n<\omega} \to a$ and $(g(a_n))_{n<\omega}$ is also a constant sequence. So, $f(a) := \lim_{n<\omega} g(a_n) = g(a)$.

Let $c \in f[\overline{\bigcup_{i<\mu} M_i}]$, so there exists $a \in \overline{\bigcup_{i<\mu} M_i}$ such that $f(a) = c$, so there exists $(a_n)_{n<\omega}$ a sequence in $\bigcup_{i<\mu} M_i$ such that $(a_n)_{n<\omega} \to a$ and $c := \lim_{n<\omega} g(a_n)$. Therefore $c \in \overline{g[\bigcup_{i<\mu} M_i]} = \overline{\bigcup_{i<\mu} f_i[M_i]}$, so $f[\overline{\bigcup_{i<\mu} M_i}] \subseteq \overline{\bigcup_{i<\mu} f_i[M_i]}$. Take $c \in \overline{\bigcup_{i<\mu} f_i[M_i]} = \overline{g[\bigcup_{i<\mu} M_i]}$, so there exists a sequence $(b_n)_{n<\omega}$ in $\bigcup_{i<\mu} M_i$ such that $(g(b_n))_{n<\omega} \to c$. As $(g(b_n))_{n<\omega}$ is a Cauchy sequence and g is an isometry, we have that $(b_n)_{n<\omega}$ is also a Cauchy sequence. So, there exists $a \in \overline{\bigcup_{i<\mu} M_i}$ such that $(b_n)_{n<\omega} \to a$, and therefore $f(a) := \lim_{n<\omega} g(b_n) = c$, hence $c \in f[\overline{\bigcup_{i<\mu} M_i}]$. So, $f[\overline{\bigcup_{i<\mu} M_i}] = \overline{\bigcup_{i<\mu} f_i[M_i]}$. As $(f_i : i < \mu)$ is a $\subseteq$-increasing and continuous chain of $\mathcal{K}$-embeddings, $f_i[M_i] \prec_{\mathcal{K}} M$, so by Łoś-Tarski and coherence MAEC axioms and we have that $f[\overline{\bigcup_{i<\mu} M_i}] = \overline{\bigcup_{i<\mu} f_i[M_i]} \prec_{\mathcal{K}} M$. Furthermore, for every symbol σ of $L(\mathcal{K})$, f is compatible with the interpretation of σ in $\overline{\bigcup_{i<\mu} M_i}$: f is a limit of $\mathcal{K}$-embeddings – function symbols on these limits are uniquely interpreted by Axiom 4(a), and f being a limit of $\mathcal{K}$-embeddings, distances to interpretations of predicates are preserved. Therefore f is a $\mathcal{K}$-embedding which extends g. □

Fact 1.16 (Hyttinen-Hirvonen). *Given* $\varepsilon > 0$ *and* $a \models p$*, there exists* $b \models q$ *such that* $d(a, b) \leq \mathbf{d}(p, q) + \varepsilon$

Proof. Fix $\varepsilon > 0$. By the definition of d, there exist realizations $c \models p$ and $c' \models q$ such that $d(c, c') \leq \mathbf{d}(p, q) + \varepsilon$. As $a, c \models p$ then there exists $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(c) = a$. Note that $d(a, f(c')) = d(f(c), f(c')) = d(c, c') \leq \mathbf{d}(p, q) + \varepsilon$, where $f(c') \models q$, so $f(c')$ is the required b. □

Corollary 1.17. *Given* $\varepsilon > 0$ *and* $p, q \in$ ga-S(M) *such that* $\mathbf{d}(p, q) < \varepsilon$ *and* $b \models q$*, then there exists* $a_\varepsilon \models p$ *such that* $d(a_\varepsilon, b) < 2\varepsilon$*.*

Proof. By fact 1.16, there exists $a_\varepsilon \models p$ such that $d(a_\varepsilon, b) \leq \mathbf{d}(p, q) + \varepsilon$, therefore $d(a_\varepsilon, b) \leq \mathbf{d}(p, q) + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon$. □

The following lemma is useful for later constructions — usually, it is easier in the metric case to realize *dense* subsets of typespaces ga-S(M); the lemma provides a criterion for relative metric Galois saturation.

Lemma 1.18. *Suppose that we have an increasing* $\prec_{\mathcal{K}}$*-chain of models* $(N_n : n < \omega)$ *such that* N_{n+1} *realizes a dense subset of* ga-S(N_n)*. Then, every type in* ga-S(N_0) *is realized in* $N_\omega := \overline{\bigcup_{n<\omega} N_n}$*.*

Proof. Given $p :=$ ga-tp(b/N_0) there exists $q_0 \in$ ga-S(N_0) which is realized in N_1 (by assumption) and $\mathbf{d}(p, q_0) < \frac{1}{2(0+1)^2} = \frac{1}{2}$. Let a_0 be a realization of q_0. By corollary 1.17 there exists $b_0 \models p$ such that $d(b_0, a_0) < 2(\frac{1}{2}) = 1$.

The key idea is to build two Cauchy sequences $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ such that $a_n \in N_{n+1}$, ga-tp$(b_n/N_0) =$ ga-tp(b/N_0) for every $n < \omega$ and also a_n and b_n are closed enough, so if $c := \lim_{n<\omega} b_n = \lim_{n<\omega} a_n$ then by CTP (Definition 1.10) we have that ga-tp$(c/N_0) =$ ga-tp$(b_0/N_0) = p$. Since $c = \lim_{n<\omega} a_n$, then $c \in N_\omega := \overline{\bigcup_{n<\omega} N_n}$, and so p is realized in N_ω.

<u>The construction</u>: Consider $n > 0$. Since N_{n+1} realizes a dense subset of ga-S(N_n), take $a_n \in N_{n+1}$ a realization of a type $q_n \in$ ga-S(N_n) which satisfies $\mathbf{d}(\text{ga-tp}(b_{n-1}/N_n), q_n) < \frac{1}{2n^2}$. By corollary 1.17, take $b_n \models$ ga-tp(b_{n-1}/N_n) such that $d(b_n, a_n) < 2(\frac{1}{2n^2}) = \frac{1}{n^2}$.

We have that $(a_n)_{n<\omega}$ is a Cauchy sequence: as $b_{n+1} \models$ ga-tp(b_n/N_{n+1}), there exists $g \in \mathrm{Aut}(\mathbb{M}/N_{n+1})$ such that $g(b_n) = b_{n+1}$. Since g is an isometry and $a_n \in N_{n+1}$, then $d(b_{n+1}, a_n) = d(g(b_n), g(a_n)) = d(b_n, a_n) < \frac{1}{n^2}$. Therefore, $d(a_{n+1}, a_n) \leq d(a_{n+1}, b_{n+1}) + d(b_{n+1}, a_n) <$

$\frac{1}{(n+1)^2} + \frac{1}{n^2} < \frac{2}{n^2}$, so we have that $(a_n : n < \omega)$ is a Cauchy sequence.

Therefore, there exists $c := \lim_{n<\omega} a_n$, $c \in N_\omega$ and also $c = \lim_{n<\omega} b_n$. So, we are done. $\square$

2. Relative independence in MAECs

Throughout this section, every model has density cardinal μ (unless we specify a different density).

Definition 2.1 (ε-splitting and $\mathop{\smile\kern-0.9em\mid}^{\varepsilon}$). Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) *ε-splits* over N iff there exist N_1, N_2 with $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \cong_N N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1))) \geq \varepsilon$. We use $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N} M$ to denote the fact that ga-tp(a/M) does not ε-split over N,

Definition 2.2. Let $N \prec_{\mathcal{K}} M$. Fix $\mathcal{N} := \langle N_i : i < \sigma \rangle$ a resolution of N. We say that a is *r-independent* from M over N relative to $\mathcal{N}$ (denoted by $a \mathop{\smile\kern-0.9em\mid}^{\mathcal{N}}_{N} M$) iff for every $\varepsilon > 0$ there exists $i_\varepsilon < \sigma$ such that $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N_{i_\varepsilon}} M$.

Notation 2.3. Let p be a Galois-type over M, N a $\mathcal{K}$-submodel of M and $\mathcal{N}$ a resolution of N. We denote by $p \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N} M$ ($p \mathop{\smile\kern-0.9em\mid}^{\mathcal{N}}_{N} M$) iff for any realization $a \models p$ we have that $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N} M$ ($a \mathop{\smile\kern-0.9em\mid}^{\mathcal{N}}_{N} M$).

Proposition 2.4 (Monotonicity of r-independence). *Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. Fix $\mathcal{M}_k := \langle M^k_i : i < \sigma_k \rangle$ a resolution of M_k ($k = 0, 1$), where $\mathcal{M}_0 \subseteq \mathcal{M}_1$. If $a \mathop{\smile\kern-0.9em\mid}^{\mathcal{M}_0}_{M_0} M_3$ then $a \mathop{\smile\kern-0.9em\mid}^{\mathcal{M}_1}_{M_1} M_2$.*

Proof. Let $\varepsilon > 0$. Since $a \mathop{\smile\kern-0.9em\mid}^{\mathcal{M}_0}_{M_0} M_3$, there exists $i_\varepsilon < \sigma_0$ such that $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{M^0_{i_\varepsilon}} M_3$. But $\mathcal{M}_0 \subseteq \mathcal{M}_1$, then there exists $j_\varepsilon < \sigma_1$ such that $M^0_{i_\varepsilon} = M^1_{j_\varepsilon}$. Therefore, for every $M^1_{j_\varepsilon} \prec_{\mathcal{K}} N_1 \overset{h}{\cong}_{M^1_{j_\varepsilon}} N_2 \prec_{\mathcal{K}} M_3$ (in particular if $N_1, N_2 \prec_{\mathcal{K}} M_2$) we have that $d(\text{ga-tp}(a/N_2), \text{ga-tp}(h(a)/N_2)) < \varepsilon$. Then $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{M^1_{j_\varepsilon}} M_2$. Since this holds for every $\varepsilon > 0$, then $a \mathop{\smile\kern-0.9em\mid}^{\mathcal{M}_1}_{M_1} M_2$. $\square$

Proposition 2.5 (Monotonicity of non-ε-splitting). *Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. If $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{M_0} M_3$ then $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{M_1} M_2$.*

Lemma 2.6 (Stationarity (1)). *Suppose that $N_0 \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ and N_1 is universal over N_0. If ga-tp(a/N_1) = ga-tp(b/N_1), $a \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N_0} N_2$ and $b \mathop{\smile\kern-0.9em\mid}^{\varepsilon}_{N_0} N_2$, then $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2) < 2\varepsilon$.*

Proof. Since N_1 is universal over N_0, then there exists a $\mathcal{K}$-embedding $g : N_2 \to_{N_0} N_1$. So, $N_0 \prec_{\mathcal{K}} g[N_2] \prec_{\mathcal{K}} N_1$.

Since $N_0 \prec_{\mathcal{K}} g[N_2], N_2 \prec_{\mathcal{K}} N_2$, $g^{-1} \upharpoonright g[N_2] : g[N_2] \cong_{N_0} N_2$ and

$a \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{N_0} N_2$, then $\mathbf{d}(\text{ga-tp}(g^{-1}(a)/N_2), \text{ga-tp}(a/N_2)) < \varepsilon$.

Doing a similar argument, it is easy to prove that $\mathbf{d}(\text{ga-tp}(g^{-1}(b)/N_2), \text{ga-tp}(b/N_2)) < \varepsilon$.

Also, since $\text{ga-tp}(a/N_1) = \text{ga-tp}(b/N_1)$ and $g[N_2] \prec_{\mathcal{K}} N_1$, then $\text{ga-tp}(a/g[N_2]) = \text{ga-tp}(b/g[N_2])$, so $\text{ga-tp}(g^{-1}(a)/N_2) = \text{ga-tp}(g^{-1}(b)/N_2)$.

Therefore,

$$\begin{aligned}
\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2)) &\leq \mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(g^{-1}(a)/N_2)) \\
&\quad + \mathbf{d}(\text{ga-tp}(g^{-1}(a)/N_2), \text{ga-tp}(g^{-1}(b)/N_2)) \\
&\quad + \mathbf{d}(\text{ga-tp}(g^{-1}(b)/N_2), \text{ga-tp}(b/N_2)) \\
&< \varepsilon + 0 + \varepsilon \\
&= 2\varepsilon
\end{aligned}$$

□

Proposition 2.7 (Extension of $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}$ over universal models). *If* $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, $\mathcal{N} := \langle N_i : i < \sigma \rangle$ *is a resolution of* N, M *is universal over* N *and* $p := \text{ga-tp}(a/M) \in \text{ga-S}(M)$ *is a Galois type such that* $a \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N} M$, *then there exists* b *such that* $\text{ga-tp}(b/M) = \text{ga-tp}(a/M)$ *and* $b \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N} M'$.

Proof. Since M is universal over N, there exists a $\mathcal{K}$-embedding $h' : M' \to_N M$. Extend h' to an automorphism $h \in Aut(\mathbb{M}/N)$. Since $a \mathop{\smile\!\!\!\!\!\mid}_{N} M$ and $h[M'] \prec_{\mathcal{K}} M$, by monotonicity of $\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}$ we have that $a \mathop{\smile\!\!\!\!\!\mid}_{N} h[M']$. By invariance, we have that $h^{-1}(a) \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N} M'$.

Claim 2.8. $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$.

Proof. Take $N_1 := h^{-1}[M]$ and $N_2 := M$. Notice that $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} h^{-1}[M]$ and $h \restriction N_1 : N_1 \cong_N N_2$. Since $a \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N} M$, by invariance we have that $h^{-1}(a) \mathop{\smile\!\!\!\!\!\mid}^{\mathcal{N}}_{N} h^{-1}[M]$. So, given $n < \omega$ there exists $i_n < \sigma$ such that $h^{-1}(a) \mathop{\smile\!\!\!\!\!\mid}^{\frac{1}{n+1}}_{N_{i_n}} h^{-1}[M]$.

By monotonicity of non-ε-splitting (Proposition 2.5), we may conclude that $h^{-1}(a) \mathop{\smile\!\!\!\!\!\mid}^{\frac{1}{n+1}}_{N} h^{-1}[M]$ for every $n < \omega$.

Since $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} h^{-1}[M]$, we have that for every $n < \omega$ $\mathbf{d}(\text{ga-tp}(h^{-1}(a)/N_2), \text{ga-tp}((h \circ h^{-1})(a)/N_2)) < \frac{1}{n+1}$

Since $N_2 := M$, we have that $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$. This finishes the proof of claim 2.8 □

Since $\text{ga-tp}(a/M) = \text{ga-tp}(h^{-1}(a)/M)$, there exists $g \in Aut(\mathbb{M}/M)$ such that $g(h^{-1}(a)) = a$. Recall that $h^{-1}(a) \downarrow_N^{\mathcal{N}} M'$, so by invariance we have that $g(h^{-1}(a)) \downarrow_N^{\mathcal{N}} g[M']$, i.e.: $a \downarrow_N^{\mathcal{N}} g[M']$. Applying invariance again, we have that $g^{-1}(a) \downarrow_N^{\mathcal{N}} M'$. Take $b := g^{-1}(a)$. This now ends the proof of Proposition 2.7. □

Proposition 2.9 (Stationarity (2)). *If* $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, M *is universal over* N, $\mathcal{N} := \langle N_i : i < \sigma \rangle$ *a resolution of* N *and* $p := \text{ga-tp}(a/M) \in \text{ga-S}(M)$ *is a Galois type such that* $a \downarrow_N^{\mathcal{N}} M$, *then there exists an unique extension* $p^* \supset p$ *over* M' *which is independent (relative to* $\mathcal{N}$*) from* M' *over* N.

Proof. By proposition 2.7, there exists at least an extension $p^* := \text{ga-tp}(b/M')$ of p with the desired property.

Let $q^* := \text{ga-tp}(c/M') \supset p$ be another extension with satisfies the desired property. So, $p^* \upharpoonright M = q^* \upharpoonright M$, $b \downarrow_N^{\mathcal{N}} M'$ and $c \downarrow_N^{\mathcal{N}} M'$.

Let $\varepsilon > 0$. So, there exist $i_\varepsilon^a, i_\varepsilon^b < \sigma$ such that $a \downarrow_{N_{i_\varepsilon^a}}^{\varepsilon} M'$ and $b \downarrow_{N_{i_\varepsilon^b}}^{\varepsilon} M'$. Taking $i := \max\{i_\varepsilon^a, i_\varepsilon^b\}$, by monotonicity of non-ε-splitting we have that $a \downarrow_{N_i}^{\varepsilon} M'$ and $b \downarrow_{N_i}^{\varepsilon} M'$.

Since M is universal over N_i (because M is universal over N), $a \downarrow_{N_i}^{\varepsilon} M'$, $b \downarrow_{N_i}^{\varepsilon} M'$ and $p^* \upharpoonright M = q^* \upharpoonright M$, by lemma 2.6 we have that $\mathbf{d}(p^*, q^*) < 2\varepsilon$. Therefore $p^* = q^*$. □

Proposition 2.10 (Locality of non-ε-splitting). *Let* $\mathcal{K}$ *be a* μ-d-*stable MAEC and* $\varepsilon > 0$. *For every* $p \in \text{ga-S}(N)$ *with* N *of density character* $> \mu$ *there exists* $M \prec_{\mathcal{K}} N$ *with density character* μ *such that* $p \downarrow_M^{\varepsilon} N$

Proof. Suppose that there exists some $p := \text{ga-tp}(\overline{a}/N)$ such that $p \not\downarrow_M^{\varepsilon} N$ for every $M \prec_{\mathcal{K}} N$ with density character μ. If $\overline{a} \in N$, it is straightforward to see that p does not ε-split over its domain. Then, suppose that $\overline{a} \notin N$.

Define $\chi := \min\{\kappa : 2^\kappa > \mu\}$. So, $\chi \le \mu$ and $2^{<\chi} \le \mu$.

We will construct a sequence of models $\langle M_\alpha, N_{\alpha,1}, N_{\alpha,2} : \alpha < \chi \rangle$ in the following way: First, take $M_0 \prec_{\mathcal{K}} N$ as any submodel of density character μ.

Suppose $\alpha := \gamma + 1$ and that M_γ (with density character μ) has been constructed. Then p ε-splits over M_γ. Then there exist $M_\gamma \prec_{\mathcal{K}} N_{\gamma,1}, N_{\gamma,2} \prec_{\mathcal{K}} N$ with density character μ and $F_\gamma : N_{\gamma,1} \cong_{M_\gamma} N_{\gamma,2}$ such that $d(F_\gamma(p \upharpoonright$

$N_{\gamma,1}), p \restriction N_{\gamma,2}) \geq \varepsilon$. Take $M_{\gamma+1} \prec_{\mathcal{K}} N$ a submodel of size μ which contains $|N_{\gamma,1}| \cup |N_{\gamma,2}|$. At limit stages α, take $M_\alpha := \overline{\bigcup_{\gamma<\alpha} M_\gamma}$.

Let us construct a sequence $\langle M^*_\alpha : \alpha \leq \chi \rangle$ of models and a tree $\langle h_\eta : \eta < \alpha \rangle$ $(\alpha \leq \chi)$ of $\mathcal{K}$-embeddings such that:

(1) $\gamma < \alpha$ implies $M^*_\gamma \prec_{\mathcal{K}} M^*_\alpha$.
(2) $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ if α is limit.
(3) $\gamma < \alpha$ and $\eta \in {}^{\alpha}2$ imply that $h_{\eta\restriction\gamma} \subset h_\eta$.
(4) $h_\eta : M_\alpha \to M^*_\alpha$ for every $\eta \in {}^{\alpha}2$.
(5) If $\eta \in {}^{\gamma}2$ then $h_{\eta^\frown 0}(N_{\gamma,1}) = h_{\gamma^\frown 1}(N_{\gamma,2})$

Take $M^*_0 := M_0$ and $h_{\langle\rangle} := id_{M_0}$.

If α is limit, take $M^*_\alpha := \overline{\bigcup_{\gamma<\alpha} M^*_\gamma}$ and if $\eta \in {}^{\alpha}2$ define $h_\eta := \overline{\bigcup_{\gamma<\alpha} h_{\eta\restriction\gamma}}$, the unique extension of $\bigcup_{\gamma<\alpha} h_{\eta\restriction\gamma}$ to $M_\alpha = \overline{\bigcup_{\gamma<\alpha} M_\gamma}$.

If $\alpha := \gamma + 1$, let $\eta \in {}^{\gamma}2$. Take $\overline{h_\eta} \supset h_\eta$ any automorphism of the monster model $\mathbb{M}$ (this is possible because $\mathbb{M}$ is homogeneous).

Notice that $\overline{h_\eta} \circ F_\gamma(N_{\gamma,1}) = \overline{h_\eta}(N_{\gamma,2})$. Define $h_{\eta^\frown 0}$ as any extension of $\overline{h_\eta} \circ F_\gamma$ to $M_{\gamma+1}$ and $h_{\eta^\frown 1}$ as $\overline{h_\eta} \restriction M_{\gamma+1}$. Take $M^*_{\gamma+1} \prec_{\mathcal{K}} N$ as any model with density character μ which contains $h_{\eta^\frown l}(M_{\gamma+1})$ for any $\eta \in {}^{\gamma}2$ and $l = 0, 1$.

Now, for every $\eta \leq {}^{\chi}2$, let H_η be an automorphism of $\mathbb{M}$ which extends h_η,

Claim 2.11. *If* $\eta \neq \nu \in {}^{\chi}2$ *then* $d(\text{ga-tp}(H_\eta(\bar{a})/M^*_\chi), \text{ga-tp}(H_\nu(\bar{a})/M^*_\chi)) \geq \varepsilon$.

Proof. Suppose not, then $d(\text{ga-tp}(H_\eta(\bar{a})/M^*_\chi), \text{ga-tp}(H_\nu(\bar{a})/M^*_\chi)) < \varepsilon$. Let $\rho := \eta \wedge \nu$. Without loss of generality, suppose that $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$. Let $\gamma := lg(\rho)$. Since $h_{\rho^\frown 0}(N_{\gamma,1}) = h_{\rho^\frown 1}(N_{\gamma,2}) \prec_{\mathcal{K}} M^*_\chi$, then $d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) < \varepsilon$. Also[4]

$$\begin{aligned} d(\text{ga-tp}(H_\nu^{-1} \circ H_\eta(\bar{a})/F_\gamma(N_{\gamma,1})), \text{ga-tp}(\bar{a}/N_{\gamma,2})) &= \\ d(\text{ga-tp}(H_\eta(\bar{a})/h_{\rho^\frown 0}(N_{\gamma,1})), \text{ga-tp}(H_\nu(\bar{a})/h_{\rho^\frown 1}(N_{\gamma,2})) &< \varepsilon \end{aligned}$$

(as H_ν is an isometry, $h_{\rho^\frown 0} = h_\rho \circ F_\gamma$, $\rho < \nu$, $\rho^\frown 0 \leq \eta$ and $\rho^\frown 1 \leq \nu$). Since $H_\nu^{-1} \circ H_\eta \supset F_\gamma$, then $d(F_\gamma(p \restriction N_{\gamma,1}), p \restriction N_{\gamma,2}) < \varepsilon$, which contradicts the choice of $N_{\gamma,1}$, $N_{\gamma,2}$ and F_γ. This finishes the proof of claim 2.11 □

[4] This distance between Galois types makes sense, as $h_{\rho^\frown 0}(N_{\gamma,1}) = h_{\rho^\frown 1}(N_{\gamma,2})$.

We have that $dc(M^*_\chi) = \mu$, but claim 2.11 says that there are at least $2^\chi > \mu$ many types mutually at distance at least ε. Therefore $dc(\text{ga-S}(M^*_\chi)) > \mu$, which contradicts μ-d-stability. □

Proposition 2.12 (Existence). *Let $\mathcal{K}$ be a μ-d-stable MAEC. Then, for every $\overline{a} \in \mathbb{M}$ and every $N \in \mathcal{K}$ there exists $M \prec_{\mathcal{K}} N$ with density character μ and a resolution $\mathcal{M} := \langle M_i : i < \omega\rangle$ of M such that $a \downarrow^{\mathcal{M}}_{M} N$.*

Proof. Let $n < \omega$. By proposition 2.10, there exists $M_n \prec_{\mathcal{K}} N$ with density character μ such that $\overline{a} \downarrow^{\frac{1}{n+1}}_{M_n} N$. By monotonicity, without loss of generality we can assume that $m < n < \omega$ implies $M_m \prec_{\mathcal{K}} M_n$. Take $M := \overline{\bigcup_{n<\omega} M_n}$. Notice that $dc(M) = \mu$.

It is straightforward to see that $\overline{a} \downarrow^{\mathcal{M}}_{M} N$. □

Lemma 2.13 (Continuity of independence). *Let $(b_n)_{n<\omega}$ be a convergent sequence and $b := \lim_{n<\omega} b_n$. If $b_n \downarrow^{\mathcal{N}}_{N} M$ for every $n < \omega$, then $b \downarrow^{\mathcal{N}}_{N} M$.*

Proof. Since $b_n \downarrow^{\mathcal{N}}_{N} M$ ($n < \omega$), for every $\varepsilon > 0$ there exists $i_{n,\varepsilon} < \sigma$ such that for every $N_{i_{n,\varepsilon}} \prec_{\mathcal{K}} N^1 \overset{h}{\cong}_{N^n_\varepsilon} N^2 \prec_{\mathcal{K}} M$, therefore we have that $d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(h(b_n)/N^2)) < \varepsilon/3$.

Let $K < \omega$ be such that for every $n \geq K$ we have that $d(b_n, b) < \varepsilon/3$. Therefore, $d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(b/N^2)) < \varepsilon/3$ for every $n \geq K$.

Since h is an isometry, we have that $(h(b_n)) \to h(b)$ and also for every $n \geq K$ we have that $d(h(b_n), h(b)) < \varepsilon/3$ (and therefore $d(\text{ga-tp}(h(b_n)/N^2), \text{ga-tp}(h(b)/N^2)) < \varepsilon/3$).

Therefore, for any $n \geq K$ we have that

$$\begin{aligned} d(\text{ga-tp}(h(b)/N^2), \text{ga-tp}(b/N^2)) &\leq d(\text{ga-tp}(h(b)/N^2), \text{ga-tp}(h(b_n)/N^2)) + \\ &\quad d(\text{ga-tp}(h(b_n)/N^2), \text{ga-tp}(b_n/N^2)) + \\ &\quad d(\text{ga-tp}(b_n/N^2), \text{ga-tp}(b/N^2)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore, $b \downarrow^{\varepsilon}_{N_{i_{n,\varepsilon}}} M$ and so, $b \downarrow^{\mathcal{N}}_{N} M$. □

Proposition 2.14 (stationarity (3)). *Let $M_0 \prec_{\mathcal{K}} M \prec_{\mathcal{K}} N$ be such that M is a (μ, σ)-limit model over M_0, where $\mathcal{M} := \{M_i : i < \sigma\}$ witnesses that M is (μ, σ)-limit over M_0. If $a, b \downarrow^{\mathcal{M}}_{M} N$ and $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$, then $\text{ga-tp}(a/N) = \text{ga-tp}(b/N)$.*

Proof. Let $\varepsilon > 0$. Since $a, b \downarrow^{\mathcal{M}}_{M} N$, there exists $i < \sigma$ such that $a, b \downarrow^{\varepsilon}_{M_i} N$ (by definition and monotonicity of non-ε-splitting). Since M_{i+1} is universal over M_i and $M_i \prec_{\mathcal{K}} N$, there exists and $\prec_{\mathcal{K}}$-embedding $f : N \to_{M_i} M_{i+1}$. Also, since $M_i \prec_{\mathcal{K}} f[N] \overset{f^{-1}}{\cong}_{M_i} N \prec_{\mathcal{K}} N$ and $a \downarrow^{\varepsilon}_{M_i} N$, therefore $d(\text{ga-tp}(a/N), \text{ga-tp}(f^{-1}(a)/N)) < \varepsilon$.

Doing a similar argument, we have that $d(\text{ga-tp}(b/N), \text{ga-tp}(f^{-1}(b)/N)) < \varepsilon$.

On the other hand, we have that $\text{ga-tp}(a/f[N]) = \text{ga-tp}(b/f[N])$ (since $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$ and $f[N] \prec_{\mathcal{K}} M_{i+1} \prec_{\mathcal{K}} M$), therefore we have that $\text{ga-tp}(f^{-1}(a)/N) = \text{ga-tp}(f^{-1}(a)/N)$.

Hence

$$\begin{aligned} d(\text{ga-tp}(a/N), \text{ga-tp}(b/N)) &\le \text{ga-tp}(a/N) + \text{ga-tp}(f^{-1}(a)/N) \\ &+ d(\text{ga-tp}(f^{-1}(a)/N), \text{ga-tp}(f^{-1}(b)/N)) \\ &+ d(\text{ga-tp}(f^{-1}(b)/N), \text{ga-tp}(b/N)) \\ &< \varepsilon + 0 + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

Therefore, $\text{ga-tp}(a/N) = \text{ga-tp}(b/N)$. □

Proposition 2.15 (Transitivity). *Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2$ be such that M_1 and M_0 are (μ, σ)-limit over some $M' \prec_{\mathcal{K}} M_0 \prec_{\mathcal{K}} M_1$, where $\mathcal{M}_i$ witnesses that M_i is (μ, σ)-limit over M' and $\mathcal{M}_0 \subset \mathcal{M}_1$. Then $a \downarrow^{\mathcal{M}_0}_{M_0} M_2$ iff $a \downarrow^{\mathcal{M}_0}_{M_0} M_1$ and $a \downarrow^{\mathcal{M}_1}_{M_1} M_2$.*

Proof. $(\Rightarrow)$ By monotonicity.

$(\Leftarrow)$ Suppose $a \downarrow^{\mathcal{M}_0}_{M_0} M_1$ and $a \downarrow^{\mathcal{M}_1}_{M_1} M_2$. Notice that M_1 is universal over M_0. Therefore, by extension property (proposition 2.7), there exists $b \models \text{ga-tp}(a/M_1)$ such that $b \downarrow^{\mathcal{M}_0}_{M_0} M_2$. By monotonicity, we have that $b \downarrow^{\mathcal{M}_1}_{M_1} M_2$. Since $a, b \downarrow^{\mathcal{M}_1}_{M_1} M_2$, $\text{ga-tp}(a/M_1) = \text{ga-tp}(b/M_1)$ and in particular M_1 is a limit model over M_0, then by stationarity (proposition 2.14) we have that $\text{ga-tp}(a/M_2) = \text{ga-tp}(b/M_2)$. Since $b \downarrow^{\mathcal{M}_0}_{M_0} M_2$, then $a \downarrow^{\mathcal{M}_0}_{M_0} M_2$. □

References

[ArBe] C. Argoty and A. Berenstein, *Hilbert spaces with Unitary operators*, Math. Logic Quarterly, vol 55 (2009), no. 1, 37–50

[Ba] J. Baldwin, *Categoricity in Abstract Elementary Classes*, monograph, http://www.math.uic.edu/~jbaldwin/pub/AEClec.pdf

[Be1] I. Ben-Yaacov, *Positive Model Theory and compact abstract theories*, J. Math. Log., 3(1), pp. 85-118, 2003.

[Be2] I. Ben-Yaacov, *Uncountable dense categoricity in cats*, J. Symbolic Logic, 70(3), pp. 829-860, 2005.

[BeBeHeUs] I. Ben-Yaacov, A. Berenstein, C. W. Henson and A. Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis Vol. 2 (Eds. Z. Chatzidakis, D. Macpherson, A. Pillay and A. Wilkie) London Math Soc. Lecture Note Series Nr 350, Cambridge Univ Press 2008.

[Gr] R. Grossberg, *Classification theory for abstract elementary classes*, Logic and Algebra, ed. Yi Zhang, Contemporary Mathematics, Vol. 302, AMS, (2002), pp. 165-204.

[GrVa] R. Grossberg and M. VanDieren, *Galois-stability for Tame Abstract Elementary Classes*, J. Math. Logic. 6, No. 1 (2006), pp. 25-49.

[GrVaVi] R. Grossberg , M. VanDieren and A. Villaveces, *Uniqueness of Limit Models in Classes with Amalgamation*, submitted.

[HeIo] C.W. Henson and J. Iovino, *Ultraproducts in analysis*, in Analysis and Logic (Mons, 1997), vol. 262 of London Math. Soc. Lecture Note Ser., pp. 1-110, Cambridge University Press, 2002.

[Hi] Å. Hirvonen, *Categoricity in homogeneous complete metric spaces*, Ph.L thesis, University of Helsinki. Available at http://www.helsinki.fi/~asaekman/paperit/categoricityv2.pdf

[Po] P. Poitevin, *Model Theory of Nakano Spaces*, Ph.D. thesis, University of Illinois at Urbana-Champaigne. Available at http://w3.salemstate.edu/~lpoitevin/thesis.pdf

[PoZa] P. Poitevin, P. Zambrano, *Model Theory of Nakano Spaces: Independence in the Bounded case.* In process.

[Sh 394] S. Shelah, *Categoricity of abstract classes with amalgamation*, Annals of Pure and Applied Logic, 1999.

[ViZa2] A. Villaveces and P. Zambrano, *Limit Models in Metric Abstract Elementary Classes: Existence and Uniqueness.* Submitted to the pre-prints of the Mittag-Leffler Institute. 2010.

E-mail - A. Villaveces: *avillavecesn@unal.edu.co*, Departamento de Matemáticas Universidad Nacional de Colombia, AK 30 # 45-03, Bogotá - Colombia

E-mail - P. Zambrano: *phzambranor@unal.edu.co*, Departamento de Matemáticas Universidad Nacional de Colombia, AK 30 # 45-03, Bogotá - Colombia