THE ROYAL SWEDISH ACADEMY OF SCIENCES
INSTITUT MITTAG-LEFFLER
Auravägen 17, SE-182 60 Djursholm, Sweden
Tel. +46 8 622 05 60 Fax. +46 8 622 05 89
info@mittag-leffler.se www.mittag-leffler.se

A Hanf Number for Saturation and Omission

J. Baldwin and S. Shelah

REPORT No. 31, 2009/2010, fall
ISSN 1103-467X
ISRN IML-R- -31-09/10- -SE+fall

A Hanf number for saturation and omission

John T. Baldwin[*]
University of Illinois at Chicago

Saharon Shelah
Hebrew University of Jerusalem
Rutgers University

March 15, 2010

Abstract

Suppose $\boldsymbol{T} = (T, T_1, p)$ is a triple of two countable theories in vocabularies $\tau \subset \tau_1$ and a τ_1-type p over the empty set. We show the Hanf number for the property: There is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated is essentially equal to the Löwenheim number of second order logic.

Newelski [3] asked to calculate the Hanf number of the following property P_N.

Definition 0.1 *We say $M_1 \models \boldsymbol{T}$ where $\boldsymbol{T} = (T, T_1, p)$ is a triple of two countable theories in vocabularies $\tau \subset \tau_1$ and p is a τ_1-type over the empty set if M_1 is a model of T_1 which omits p, but $M_1 \restriction \tau$ is saturated. Let $\boldsymbol{K}_{\boldsymbol{T}}$ denote the class of models M_1 which satisfy $\boldsymbol{T}$.*

For $\boldsymbol{K} = \boldsymbol{K}_{\boldsymbol{T}}$ for some $\boldsymbol{T}$ in a vocabulary with cardinality κ, let $P^{\kappa}_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ hold if $|\tau_1| \leq \kappa$ and for some M_1 with $|M_1| = \lambda$, $M_1 \models \boldsymbol{T}$. If $\boldsymbol{T}$ is in a countable vocabulary, we write $P^{c}_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$. Finally, $P^{f}_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ is the same property restricted to triples where T_1 and T are finitely axiomatizable in finite vocabularies.

$\mathrm{spec}(\boldsymbol{T})$ *is the collection of cardinals λ such that there is an M_1 satisfying $\boldsymbol{T}$ with $|M_1| = \lambda$,*

Recall Hanf's observation [1] that for any such property $P(\boldsymbol{K}, \lambda)$, where $\boldsymbol{K}$ is ranges over a *set* of classes of models, there is a cardinal $\kappa = H(P)$ such that: if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. $H(P)$ is called the Hanf number of P. E.g. $P(\boldsymbol{K}, \lambda)$ might be the property that $\boldsymbol{K}$ has a model of power λ. Similarly the Löwenheim number $\ell(P)$ of a set P of classes is the least cardinal μ such that any class $\boldsymbol{K} \in P$ that has a model has one of cardinal $\leq \mu$.

Theorem 0.2 *Assume the collection of λ with $\lambda^{<\lambda} = \lambda$ is a proper class. $H(P^f_N) = \ell(L^{II})$ where L^{II} denotes the collection of sentences of second order logic.*

[*]We give special thanks to the Mittag-Leffler Institute where this research was conducted. This is paper F995 in Shelah's bibliography. Baldwin was partially supported by NSF-0500841. Shelah thanks the Binational Science Foundation for partial support of this research.

Since $H(P^c_N) \geq H(P^f_N)$, this shows that that the Hanf number in the abstract is at least $\ell(L^{II})$, as asserted.

Jouko Vaananen provided the following summary of the effect of this result by indicating the size of $\ell(L^{II})$. $\ell(L^{II})$ is bigger than the first (second, third, etc) fixed point of any normal function on cardinals that itself can be described in second order logic. For example it is bigger than the first κ such that $\kappa = \beth_\kappa$, bigger than the first κ such that there are κ cardinals λ below κ such that $\lambda = \beth_\lambda$, etc. It is easy to see that if there are measurable (inaccessible, Mahlo, weakly compact, Ramsey, huge) cardinals, then the Lowenheim number of second order logic exceeds the first of them (respectively, the first inaccessible, Mahlo, weakly compact, Ramsey, huge) (and second, third, etc). So even under $V = L$, the Löwenheim number is bigger than any 'large' cardinal that is second order definable and consistent with $V = L$. Such results are discussed in Vaananen's paper "Hanf numbers of unbounded logics"[4]. A result of Magidor [2] shows the Lowenheim number of second order logic is always below the first supercompact. Vaananen's paper "Abstract logic and set theory II: Large cardinals" gives lower bounds for the Lowenheim number of equicardinality quantifiers and thus *a fortiori* for second order logic [5]. In simple terms, if $E(\kappa)$ is the statement that $2^\kappa \geq \kappa^{++}$ then the first κ cardinals (if any) such that $E(\kappa)$ holds is less than the Lowenheim number of second order logic. This shows that by forcing we can push the Lowenheim number up at will.

We make the following assumption throughout.

Assumption 0.3 *Assume the collection of λ with $\lambda^{<\lambda} = \lambda$ is a proper class.*

This assumption follows from GCH, but if GCH fails badly the only such cardinals are strongly inaccessible. The key point for our use of the condition is that $\lambda^{<\lambda} = \lambda$ is a sufficient condition for the existence of a saturated model in λ. We will explore this issue for stable theories, in the absence of this condition, elsewhere. In Section 1 we review some properties of second order logic and show the equality of two 'Löwenheim numbers'; this equality demonstrates the assumption is harmless in our context. In Section 2, we state two technical results, prove one, and deduce Theorem 0.2 from them. In Section 3, we prove the more difficult technical result. Newelski's question arose in the study of the model theory of groups and the existence of groups of bounded order.

The authors acknowledge very fruitful discussions with Jouko Väänänen and Tapani Hyttinen concerning the material.

1 Some Second Order Logic

By (pure) second order logic, we mean the logic with individual variables and variables for relations of all arities. The atomic formulas are equalities between variables and expressions $X(\mathbf{x})$ where X is an n-ary relation and $\mathbf{x}$ is an n-tuple of variables. Note that a structure A for this logic is simply a set so is determined entirely by its cardinality. But we use the full semantics; the n-ary relation variables range over all n-ary relations on A.

We put our restriction to $\lambda = \lambda^{<\lambda}$ in a more general setting. In general for any class $\boldsymbol{K}$ of models write $\mathrm{spec}(\boldsymbol{K})$ for the collection of λ such that there is a model in $\boldsymbol{K}$ with cardinality λ. We describe some technical variants for the second order case that are relevant here.

Definition 1.1 *Let ψ be a sentence of second order logic.*

1. $\mathrm{spec}^1(\psi) = \{\lambda : \lambda \models \psi\}$.

2. $\mathrm{spec}^2(\psi) = \{\lambda : \lambda = \lambda^{<\lambda} \wedge \lambda \models \psi\}$

Note that there is a sentence χ in second order logic which has a model of size λ if and only if $\lambda^{<\lambda} = \lambda$. Namely, let χ assert there is an extensional relation R on sets such that each element denotes, via R, a set of smaller cardinality than the universe and each such set is coded by R. We will generally write $\lambda^{<\lambda} = \lambda$ to denote this sentence.

Definition 1.2 *Define H^2 and ℓ^2 to the be Hanf and Lowenheim numbers with respect to* spec^2.

We can show

Lemma 1.3 $H(L^{II}) = H^2(L^{II})$ *and* $\ell(L^{II}) = \ell^2(L^{II})$

Proof. One direction is easy. For every sentence ψ of second order logic, there is a sentence ψ^* such that:

$$\mathrm{spec}^2(\psi) = \mathrm{spec}^1(\psi^*).$$

ψ^* just expresses the conjunction of ψ with $\lambda^{<\lambda} = \lambda$. Recall that for either spectrum $\ell^i(L^{II}) = \sup\{\min\{\mathrm{spec}^i(\phi)\} : \phi \in (L^{II}$ has a model $\}$ and similarly $H^i(L^{II}) = \sup\{\sup\{\mathrm{spec}^i(\phi)\} : \phi \in (L^{II}$ is bounded $\}$. Since every 2-spectrum is a 1-spectrum $\ell^2(L^{II}) \leq \ell^1(L^{II})$ and $H^2(L^{II}) \leq H^1(L^{II})$.

But the opposite inequality also holds. Let ϕ be a sentence with a non-empty 2-spectrum. Let $f(\lambda)$ denote the least $\mu > \lambda$ with $\mu^{<\mu} = \mu$. It is easy to construct for each second order sentence ϕ a sentence ϕ^* such that

$$\mathrm{spec}(\phi^*) = \mathrm{spec}^2(\phi^*) = \{f(\lambda) : \lambda \in \mathrm{spec}(\phi)\}.$$

Clearly the map $\phi \mapsto \phi^*$ shows $\ell^2(L^{II}) \geq \ell^1(L^{II})$ and $H^2(L^{II}) \geq H^1(L^{II})$.

$\square_{1.3}$

2 The main result

We prove Theorem 2.2 in Section 3. Recall our notation from Definition 0.1.

Notation 2.1 *We will write $\boldsymbol{T}$ (possibly with subscripts) for a triple (T, T_1, p). The expression '$\boldsymbol{T}$ has a model in λ' means there is a model of T_1 with cardinality λ that omits p and whose reduct to $L(T) = \tau$ is saturated.*

We concentrate first on $P^f_N(\boldsymbol{K}_{\boldsymbol{T}}, \lambda)$ from Definition 0.1. We need some additional coding to handle and arbitrary theories.

Theorem 2.2 *For every second order sentence ϕ, there is a triple $\boldsymbol{T}_\phi$ in a finite vocabulary such that if $\lambda^{<\lambda} = \lambda$, then the following are equivalent:*

1. *$\boldsymbol{T}_\phi$ has a model in λ.*
2. *ϕ has a model in every cardinal strictly less than λ.*

Note that the following extends from finitely axiomatizable to 'arithmetic' by coding a model of arithmetic in the second order sentence. And it easy to see that the theory constructed in Theorem 2.2 is recursive. This observation is generalized in Theorem 2.11 to remove the restrictions on axiomatizability. For simplicity of exposition we deal first with the case of finite axiomatizabilty. Note however, that coding of syntax and satisfaction is used in the proof of Lemma 2.3.

Lemma 2.3 *For every $\boldsymbol{T}$, with finitely axiomatizable T_1, there is a second order $\phi_{\boldsymbol{T}}$, such that $\phi_{\boldsymbol{T}}$ has a model in λ if and only if $\boldsymbol{T}$ has a model in λ.*

Since T_1 is finitely axiomatizable, it is easy to write a second order sentence θ such that if $M \models \theta$, $M \models T_1$, M omits p and $M \restriction \tau$ is saturated. $\square_{2.3}$

We now deduce Theorem 0.2 from these two results.

Claim 2.4 $H(P^f_N) \leq \ell^2(L^{II})$ *where L^{II} denotes second order logic.*

Proof. Lemma 2.3 shows that for any $\boldsymbol{T}$, there is a $\phi_{\boldsymbol{T}}$ with $\mathrm{spec}(\boldsymbol{T}) = \mathrm{spec}(\phi_{\boldsymbol{T}})$. Suppose for contradiction that $H(P^f_N) > \ell^2(L^{II})$. Then there is a triple $\boldsymbol{T}$ with a bounded spectrum and the bound is greater than $\ell^2(L^{II})$. Trivially, $\mathrm{spec}^2(\phi_{\boldsymbol{T}})$ is an initial segment of the cardinals satisfying $\mu^{<\mu} = \mu$ as we can choose a saturated elementary submodel of a given member of $\boldsymbol{T}$ of the appropriate cardinality which omits p. But then, $\neg\phi_{\boldsymbol{T}}$ has a model and $\min \mathrm{spec}(\neg\phi_{\boldsymbol{T}}) > \ell^2(L^{II})$. This contradicts the definition of the Löwenheim number.

$\square_{2.4}$

Lemma 2.5 $H(P^f_N) \geq \ell^2(L^{II})$ *where L^{II} denotes second order logic.*

Proof. Suppose for contradiction that there is a second order sentence ψ such that $\lambda_0 = \min(\mathrm{spec}^2(\psi)) > H(P^f_N)$. Let λ_1 be the least cardinal satisfying $\lambda^\lambda = \lambda$ and $\geq \lambda_0$. Let $\hat{\psi}$ express $(\exists U)(\psi^U \wedge \lambda^{<\lambda} = \lambda)$. We apply Theorem 2.2 to $\neg(\hat{\psi})$ Note that $\hat{\psi}$ is true on all cardinals satisfying $\lambda^\lambda = \lambda$ and $\geq \lambda_0$ and false on all $\mu < \lambda_1$. By Theorem 2.2, $\lambda_1 \models \boldsymbol{T}_{\neg(\hat{\psi})}$ and $\lambda_1 \geq H(P^f_N)$. So $\boldsymbol{T}_{\neg(\hat{\psi})}$ and therefore $\neg(\hat{\psi})$ has arbitrarily large models. But $\neg(\hat{\psi})$ has no models larger than λ_1. This contradiction yields the theorem. $\square_{2.5}$

We could slightly more easily prove

$$H(P^f_N) \leq \ell^2(L^{II}) \leq H(P^c_N),$$

which gives our answer to Newelski's question but is not quite as sharp. That is, if we had just required $\boldsymbol{T}_\phi$ in Theorem 2.2 to be in a countable language rather than finitely axiomatizable, this would have no effect on the proof of Lemma 2.5 and it would have simplified the proof of Theorem 2.2 since we could have worked with countably many constants and omitted the function g. This inequality in these results is unsatisfying and with a little more effort we convert it to an equality. The key idea is to see that we can use the same ideas to code the syntax of infinitary second order logic.

Definition 2.6 *Let $L_{\theta^+,\kappa}(II)$ denote second order logic allowing strings of second order quantifiers of cardinality $< \kappa$ and conjunctions and disjunctions of cardinality $\leq \theta$.*

Remark 2.7 *Note that the Löwenheim number of $L_{\theta^+,\kappa}(II)$ is a limit cardinal of cofinality $> \theta$ and is an accumulation point of $\{\mu : \mu = \mu^{<\mu}\}$.*

We now show how to code the Löwenheim number of sentences of $L_{\theta^+,\kappa}(II)$ by pairs of a set A of ordinals and a sentence in L^{II}. We write L_θ for the θ stage in the construction of the inner model L.

Notation 2.8 *We denote by $L(II, \tau)$ the second order logic in the vocabulary τ consisting of unary predicates P and Q and a binary relation $<$.*

Lemma 2.9 *For every $\kappa \leq \theta$ and every sentence $\phi \in L_{\theta^+,\kappa}(II)$ we can find a pair (A_ϕ, ψ_ϕ) such that:*

1. *(a) $\psi_\phi \in L(II, \tau)$*
   (b) $\psi_\phi \vdash$ '$(P, <)$ is a well ordering, $Q \subseteq P$';
2. *For any cardinal $\lambda = \lambda^{<\lambda} > \theta$, the following are equivalent.*
   (a) ϕ has no model of card $< \lambda$.
   (b) There is a model M of ψ_ϕ with cardinality λ such that $(P^M, <^M)$ has order type λ and $A =$
   $\{\alpha < \kappa : \text{for some } a \in Q^M \subseteq P^M, \alpha = \text{otp}(\{b \in P^M : b <^M a\}, <^M)\}$.

Proof. By Lemma 1.3, we may assume $\lambda^{<\lambda} = \lambda$. Let $A_\phi \subseteq L_\theta$ be the set of ordinals of subformulas of ϕ in a standard coding of $L_{\theta^+,\kappa}(II)$ in L_θ.

Define ψ_ϕ so that $M \models \psi_\phi$ iff M satisfies the properties we now describe. First, for some $N \subseteq M$, with $|N| = |M|$, $(N, \epsilon) \approx (H(\mu), \epsilon)$ for some μ with $\mu^{<\mu} = \mu$. Then $P^N = P^M$ and $Q^N = Q^M$. Further, let ψ_ϕ assert Q is the set of ordinals (contained in the ordinal P^M) coding subformulas of ϕ under the standard inductive definition of $L_{\lambda,\kappa}(II)$. Further a function G^N defines truth of subformulas of our given formula ϕ on subsets b of N and by this coding, N satisfies '$b \models \neg\phi$' if N models $|b| < |N|$.

Then, if $|M| = \lambda$ and $(P^M, <^M)$ has order type θ, and Q^M is interpreted as the set of ordinals A in 2b), the coding in N will correctly represent truth of ϕ and ϕ will fail on all subsets of N with cardinality $< \lambda$. Thus 2b) implies 2a). Clearly if 2a) holds we can construct a model M satisfying 2b).

$\square_{2.9}$

Definition 2.10 *For ψ_ϕ defined as in Lemma 2.9,* $\mathrm{spec}(\psi_\phi, \theta, A)$ *is the set of the cardinalities of models M of ψ with $(P^M, <^M, Q^M) \approx (\theta, <, A)$.*

Theorem 2.11 *For any cardinal θ, the following four cardinals are equal.*

1. *λ_1 is the Hanf number of P_N^θ.*

2. *λ_2 is the Löwenheim number of $L_{\theta^+,\omega}(II)$.*

3. *λ_3 is the Löwenheim number of $L_{\theta^+,\theta^+}(II)$.*

4. *$\lambda_4 = \sup\{\sup \mathrm{spec}(\psi_\phi, \theta, A) : \psi_\phi \in L(II, \tau)$ and $A \subset \theta$ such that $\mathrm{spec}(\psi_\phi, \theta, A)$ is bounded$\}$.*

Proof. We chose the logic $L_{\theta^+,\omega}$ precisely so $\lambda_1 \leq \lambda_2$ (by a proof like that of Lemma 2.3 but now we have conjunctions of cardinality θ) and clearly $\lambda_2 \leq \lambda_3$. Lemma 2.9 yields:

$$\{\min(\mathrm{spec}^2(\phi)) : \phi \in L_{\theta^+,\theta^+}\} \subseteq \{\sup(\mathrm{spec}^2(\theta, \psi_\phi, A)) : \phi \in L_{\theta^+,\theta^+} \text{ is bounded}\}.$$

(We can replace ϕ by a ϕ^* whose only model is the model of ϕ with minimum cardinality to guarantee the containment.) Thus, $\lambda_3 \leq \lambda_4$.

The proof that $\lambda_4 \leq \lambda_1$ is obtained by modifying the proof of Theorem 2.3. Add to the vocabulary in the T_ϕ from the proof in section 3 of Theorem 2.3, symbols $P, Q, <$ and use the same coding ideas to guarantee that $Q \subseteq P$ and both are well-ordered by $<$. Thus for each ψ_ϕ, we can construct $\boldsymbol{T}_{\psi_\phi}$ with the two spectra related as in Theorem 2.3. This yields $\lambda_4 \leq \lambda_1$ by slightly modifying the argument for Lemma 2.5. $\square_{2.11}$

3 Essential Lemmas

Now we prove Theorem 2.2. For convenience, we list here the two vocabularies. We describe the axioms of T and T_1 below.

Notation 3.1

1. *τ contains unary predicates Q_1, Q_2, a binary relation R and partial binary functions F and F_2. It contains two constant symbols c_0, c_ω and a unary function symbol g.*

2. *τ_1 adds a unary predicate Q_0 and a binary relation $<_1$.*

Remark 3.2 (Proof Sketch) For each second order ϕ, we construct a triple $\boldsymbol{T}_\phi$. But most of the construction is independent of the particular ϕ and so we first construct a theory T_1 which does not depend on ϕ. The vocabulary τ will contain unary predicates Q_1, Q_2. The axioms will assert that Q_1, Q_2 partition the universe. Q_0 is in τ_1. Omission of the type p will guarantee that $Q_0 \subset Q_1$ is countable. Omission of the type in a model M of T_1 whose τ-reduct $\aleph_1$-saturated and some coding involving the partial order $<_0$ in τ will guarantee that $Q_1(M)$ is well-ordered by a relation symbol $<_1$ in τ_1. A relation symbol R in τ will code subsets of Q_1 by elements of Q_2. Thus first order quantification on Q_2 will encode second order quantification on Q_1. In particular, we can code a given second order sentence ϕ and thus extend T_1 to T_ϕ. But the encoding will be 'correct' only on subsets whose every subset is coded in Q_2. But if $\mu < \lambda$ and M is λ-saturated, μ is a $<_1$-initial segment Q_1. Since $\mu < \lambda$ each subset of μ is coded by a type of size μ so the encoded semantics is correct and μ is a model of ϕ.

Proof of Theorem 2.2. We gradually introduce the vocabulary and theory explaining the use of various predicates as they are introduced; we repeat a bit of the proof sketch. Below we say certain conditions hold to mean they hold in any model of T. We first describe τ and T. In particular, τ contains unary predicates Q_1, Q_2 that partition the universe.

There is a binary relation $<_0$, which is a partial order of Q_1. There is a partial function F mapping $Q_1 \times Q_1$ into Q_1. We write F_a for the partial function from Q_1 into Q_1 indexed by a. The partial order $<_0$ satisfies: $a \leq_0 b$ implies $F_a \subset F_b$.

We have two further properties of F. F_{c_0} is the empty function. For every $a \in Q_1$ and every $e \in Q_1$, if $e \notin \operatorname{dom} F_a$, then there are $b, d \in Q_1$ with $a <_0 b$ and $F_b = F_a \cup \{\langle e, d\rangle\}$.

Further there is a pairing function F_2 on Q_1 and an extensional relation R between Q_1 and Q_2 so that each element of Q_2 codes a subset of Q_1 via R. We write U_b for $\{a \colon R(b, a)\}$ (for $a \in Q_1$ and $b \in Q_2$).

T asserts that Q_1 is preserved by g, that g is a permutation, and $Q_1(c_0)$.

The set of $\{U_a : a \in Q_2\}$ is closed under Boolean operations and if U_b is such a set so is $F_a(U_b)$ for any $a \in Q_1$. For each $a \in Q_1$, there is $b \in Q_2$ such that $U_b = \{c \colon c <_1 a\}$.

Secondly, we turn to the description of τ_1 and T_1. In τ_1, there is a unary relation Q_0 such that $Q_0 \subset Q_1$ and T_1 asserts Q_0 is preserved by g and c_0, c_ω are in Q_0. Thus, each $g^i(c_0) \in Q_0$. Further, there is a binary τ_1-relation $<_1$, which is a linear order of Q_1 and such that on Q_1, $x <_1 g(x)$ and $x < c_\omega$ implies $g(x) < c_\omega$. Thus, $\langle g^i(c_0) \colon i < \omega\rangle \cup \{c_\omega\}$ name countably many elements of Q_1 which are $<_1$- ordered in order type $\omega + 1$. T_1 further asserts $(Q_1, <_1)$ is 'internally well-ordered' in the following sense. For every $a \in Q_2$, if U_a is non-empty, it has a $<_1$-least element.

The type p asserts $Q_0(x)$ and x is not a $g^i(c_0)$.

Claim 3.3 *If a model M of T_1 is such that its reduct to τ is an $\aleph_1$-saturated model of T but M omits p, $(Q_1, <_1)$ is a well-ordering in M.*

Proof. Suppose there is a countable $<_1$-descending chain $B = \{b_i : i < \omega\}$ in $(Q_1, <_1)$. Using the properties of F, we can define a $<_0$-increasing chain of a_n in Q_1 such that $F_{a_n} = \{\langle c_1, b_1\rangle, \ldots \langle g^n(c_0), b_n\rangle\}$, where the $g^i(c_0)$ are images of c_0 by iterating g. Since the model is $\aleph_1$-saturated there is an $a_\omega \in Q_1$ such that each $F_{a_n} \subset F_{a_\omega}$. But then $B = F_{a_\omega}(\{g^i(c_0) : i < \omega\})$. Note that while the choice of b_i involved the τ_1-symbol $<_1$, the existence of a_ω is by the consistency of a τ-type so the use of saturation is legitimate.

Since M omits p, $\{g^i(c_0) \colon i < \omega\} = \{a : a <_1 c_\omega\}$ and therefore is coded by an element of Q_2. By the closure properties of the coded sets, $B = U_d$ for some $d \in Q_2$. This contradicts the internal well-ordering of Q_1. $\square_{3.3}$

Now translate ϕ to the first order formula $\phi^*(v)$ by translating each second bound order variable X to a first order formula in x and v. Replace each occurrence of $X(z)$ by $R(z,v) \wedge R(z,x)$. This translation has the following consequence. (This is immediate for monadic second order but we included a pairing function F_2 on Q_1 so it extends to arbitrary sentences.)

Fact 3.4 *If $M \models T$, $a \in Q_2(M)$ and each subset of U_a is coded by an element of $Q_2(M)$, then $M \models \phi^*(a)$ if and only $U_a(M) \models \phi$.*

Add the following axiom to T_1 to obtain T_ϕ

$$(\forall u)(\forall w)[((\forall z)R(z,w) \leftrightarrow z <_1 u) \rightarrow \phi^*(w)].$$

Claim 3.5 *If $\mu < \lambda = \lambda^{<\lambda}$ and M is model of T_ϕ with cardinality λ that omits p but whose reduct to τ is saturated then $\mu \models \phi$.*

Conversely, if ϕ is true on all $\mu < \lambda = \lambda^{<\lambda}$, there is a model M_1 of T_ϕ with cardinality λ that omits p but whose reduct to τ is saturated.

Proof. Since $\mu < \lambda$, μ is an initial segment of Q_1 so $\mu = \{a \in Q_1 : R(y,d)\}$ for some $d \in Q_2$. But then each subset Y of μ gives rise to a type $q_Y(x)$:

$$\{R(y,d)\} \cup \{R(y,x) : y \in Y\} \cup \{\neg R(y,x) : y \notin Y\}.$$

For each Y the τ-type $q_Y(x)$ has cardinality less than λ and so is realized by saturation. We finish by Fact 3.4.

For the converse, well-order Q_1 by $<_1$ in order type λ. Add in Q_2 a code for each subset of cardinality $< \lambda$. Let the F_a list the partial functions of cardinality less than λ from Q_1 to Q_1 and let $<_0$ denote the natural partial ordering on Q_1 induced by inclusion of the named functions. Since ϕ is true below λ, each infinite initial segment in λ defines a model of ϕ and the definition of T_ϕ shows that we have a saturated model of T when we take the reduct to τ. Finally, let Q_0 include exactly the first ω elements of Q_1.

$\square_{3.5}$

Letting $\boldsymbol{T}_\phi$ be the triple (T, T_ϕ, p) we have a triple satisfying Theorem 2.2.

References

[1] William Hanf. Models of languages with infinitely long expressions. In *Abstracts of Contributed papers from the First Logic, Methodology and Philosopy of Science Congress, Vol.1*, page 24. Stanford University, 1960.

[2] M. Magidor. On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics*, 10:147–157, 1971.

[3] Ludomir Newelski. Bounded orbits and measures on a group. preprint.

[4] Jouko Vaananen. On the hanf numbers of unbounded logics. In B.Mayoh F.Jensen and K.Moller, editors, *Proceedings from 5th Scandinavian Logic Symposium*, pages 309–328. Aalborg University Press, 1979.

[5] Jouko Vaananen. Abstract logic and set theory ii: Large cardinals. *Journal of Symbolic Logic*, 47:335–346, 1982.