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Around Superstability in Metric Abstract Elementary Classes: Limit Models and r-Towers

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AROUND SUPERSTABILITY IN METRIC ABSTRACT ELEMENTARY CLASSES: LIMIT MODELS AND R-TOWERS.

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Abstract. We study versions of limit models adapted to the context of *metric abstract elementary classes*. Under superstability-like assumptions, we prove some generalizations of theorems from [GrVaVi]. We prove criteria for existence and uniqueness of limit models in the metric context.

1. Preliminaries - Why limit models and towers?

The Model Theory of metric structures can be approached in a fruitful way from the Abstract Elementary Class perspective — extending in some senses the framework of First Order Continuous Model Theory [BeBeHeUs] and in other senses benefitting from the enormous richness of the Stability Theory in Abstract Elementary Classes. Other authors (Hirvonen [Hi] in her thesis with Hyttinen, Usvyatsov and Shelah) have provided essentially two major frameworks for dealing with contexts outside "generalized first order".

Hirvonen and Hyttinen have developed a solid framework for categoricity transfer of metric AEC and for the study of $\aleph_0$-stable classes of metric structures (a good analysis of primary models, basic items in the definition itself, etc.).

Our focus here goes more towards an analysis of "superstability" in metric AEC. Of course this goal is long-winded, but we provide first steps in that direction in this paper. In particular, building mainly on ideas from the discrete AEC setting coming from [GrVaVi], and related more distantly to Shelah's ideas in [Sh600], we approach here the connection between two facets of (protean) superstability: limit models (existence and first steps towards uniqueness).

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The main constructions in our paper are versions of towers adapted to the metric context (r-towers and metric r-towers). Towers have been used extensively by Grossberg, Shelah, VanDieren, Villaveces and Yarden in their work on AEC before. Here we adapt them to the metric setting and use them to prove various lemmas useful to an approach of uniqueness of limit models in metric AEC (in a forthcoming paper). Towers can be regarded as a strong generalization of the concept of Galois type: a Galois type is (an equivalence class) of triples (M, N, a) where $M \prec N$ and $a \in N \setminus M$ — towers "refine" the way the element a is connected to M inside N and provides a very robust situation where a is replaced by a long sequence $(a_i)_i$, and the models M and N themselves are "sliced through". Extension properties of triples, and ultimately, independence and "forking" calculus-like properties of the triples may be lifted in a robust way to the towers. This has been explored by the authors mentioned above in the usual AEC setting - we begin the exploration here of the *metric* version.

In [GrVaVi] the authors prove the uniqueness of limit models under superstability-like assumptions for AEC. Here we study the behavior of r-towers under superstability-like assumptions for the metric setting.

Note: we base many of our results here in the constructions of [ViZa1], where we define and study the notions of ε-splitting and r-independence – both of them adapted to the metric case. We study there conditions for good behavior of stationarity, existence, extension, etc. We use freely these notions in this paper, and refer to their statement in [ViZa1] at crucial places here.

2. Existence of limit models in MAECs

The following lemma will be useful later: it provides relative saturation criteria by iterating ω-many times dense relative saturation.

Lemma 2.1. *Suppose that we have an increasing $\prec_{\mathcal{K}}$-chain of models $(N_n : n < \omega)$ such that N_{n+1} realizes a dense subset of* ga-$S(N_n)$. *Then, every type in* ga-$S(N_0)$ *is realized in* $N_\omega := \overline{\bigcup_{n<\omega} N_n}$.

Proof. See [ViZa1, Lemma 1.18]. □

Definition 2.2. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$. We say that N is μ-**d**-universal over N iff for every $N' \succ_{\mathcal{K}} N$ such that $dc(N) = \mu$ we have that there exists a $\prec_{\mathcal{K}}$-embedding $f : N' \to M$ which fixes pointwise M. We say that N is **d**-universal over M iff N is $dc(M)$-**d**-universal over M.

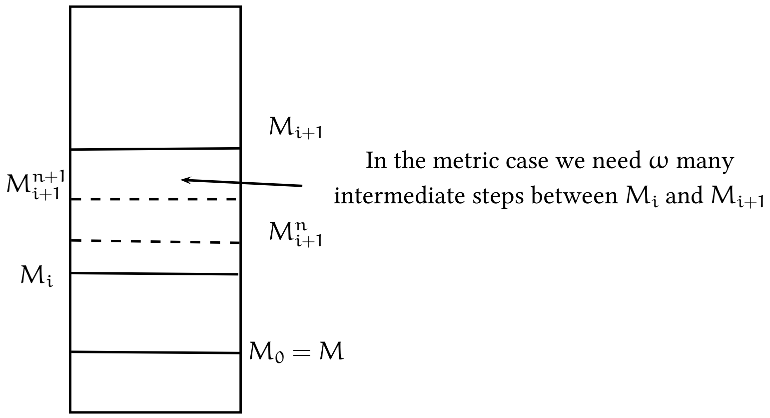
We drop the prefix **d** if it is clear that we are working in a metric setting.

Definition 2.3. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$, where $dc(M) = \mu$. We say that N is (μ, θ)-**d**-limit over N iff there exists an increasing and continuous $\prec_{\mathcal{K}}$-chain $(M_i : i < \theta)$ such that $\overline{\bigcup_{i<\theta} M_i} = N$, $dc(M_i) = \mu$ for every $i < \theta$ and also M_{i+1} is μ-**d**-universal over M_i.

Definition 2.4. We say that $\mathcal{K}$ is μ-**d**-stable iff for every $M \in \mathcal{K}$ such that $dc(M) \leq \mu$ we have that $dc(\text{ga-S}(M)) \leq \mu$

We now prove the existence of universal extensions in the setting of Metric Abstract Elementary Classes. We point out that this is an adaptation of the proof of the existence of universal extensions over a given model M in the setting of Abstract Elementary Classes (see [GrVa]). In that proof, under μ-stability, we can consider an increasing and continuous $\mathcal{K}$-chain $\langle M_i : i < \mu \rangle$ such that $M_0 := M$ and where M_{i+1} realizes every Galois-type in ga-S(M_i). So, $\bigcup_{i<\mu} M_i$ is universal over M. But in this setting, we cannot consider directly from μ-**d**-stability that M_{i+1} realizes every type in ga-S(M_i). But we use Lemma 2.1 in a suitable way for guaranteeing that requirement.

Proposition 2.5 (Existence of universal extensions). *Let $\mathcal{K}$ be a MAEC μ-**d**-stable with AP. Then for all $\mathcal{M} \in \mathcal{K}$ such that $dc(\mathcal{M}) = \mu$ there exists $\mathcal{M}^* \in \mathcal{K}$ universal over $\mathcal{M}$. such that $dc(\mathcal{M}^*) = \mu$*



Proof. The proof follows almost along the same lines as the proof of existence of universal models in usual AECs (see Claim 2.9 of [GrVa] and Claim 1.15.1 of [Sh600]); that is, by trying to capture realizations of types along the construction in a coherent way, and building the universal extension as a union of a chain (we do not repeat all the details of the proof, but point out the differences).

In our metric setting, we need to be careful with the way we realize the types along the construction: although this cannot be done in an immediate way in each successor stage as in [GrVa], lemma 2.1 provides the realizations we need of dense subsets of the typespace in ω many steps.

We construct an increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \mu \rangle$ such that $M_0 := M$, M_{i+1} is the completion of the union of a resolution $(M^n_{i+1} : n < \omega)$ where $M^0_{i+1} := M_i$ M^{n+1}_{i+1} realizes a dense subset of ga-S(M^n_{i+1}) and $\mathrm{dc}(M^n_{i+1}) = \mu$ for every $n < \omega$. This is possible by μ-**d**-stability of $\mathcal{K}$. Take $M^* := \overline{\bigcup_{i<\mu} M_i}$. M^* turns out to be universal over M — by the same argument as in Claim 2.9 of [GrVa].

□

Corollary 2.6. *Let $\mathcal{K}$ be a MAEC μ-**d**-stable with AP. Then for all $M \in \mathcal{K}$ such that $\mathrm{dc}(M) = \mu$ there exists $M^* \in \mathcal{K}$ limit over M such that $\mathrm{dc}(M^*) = \mu$*

Proof. Iterate the construction given in proposition 2.5.

□

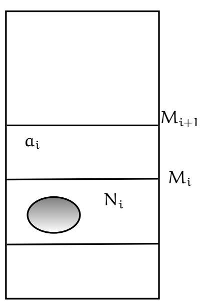
3. Limit Models and r-Towers

Throughout this section, we assume that all our models have density character μ, all orderings denoted by I, I', I_β, etc. have cardinality μ as well, and $\mathrm{cf}(I) = \mathrm{cf}(I') = \mathrm{cf}(I_\beta) > \omega$.

Assumption 3.1 (superstability). *For every a and every increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \sigma \rangle$ and $\mathcal{M}_j$ a resolution of M_j ($j < \sigma$):*

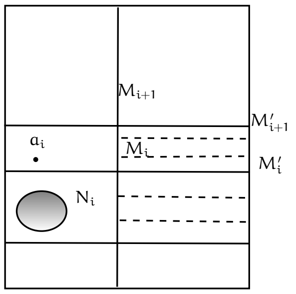
1. *If $p \upharpoonright M_i \perp^{\mathcal{M}_0}_{M_0} M_i$ for all $i < \sigma$, then $p \perp^{\mathcal{M}_0}_{M_0} \overline{\bigcup_{i<\sigma} M_i}$.*
2. *if $\mathrm{cf}(\sigma) > \omega$, there exists $j < \sigma$ such that $a \perp^{\mathcal{M}_j}_{M_j} \bigcup_{i<\sigma} M_i$.*
3. *if $\mathrm{cf}(\sigma) = \omega$, there exists $j < \sigma$ such that $a \perp^{\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.*

Definition 3.2 (r-Towers). Let I be a well-order, $\mathfrak{M} := (M_i : i \in I)$ be an $\prec_{\mathcal{K}}$-increasing chain, $\overline{a} := (a_i : i \in I)$, $\mathfrak{N} := (N_i : i < \sigma)$ a sequence of models in $\mathcal{K}$, $\mathcal{M} := (\mathcal{M}_j : j \in I)$ is a sequence of resolutions $\mathcal{M}_j$ of M_j ($j \in I$) and $\mathcal{N} := (\mathcal{N}_j : j \in I)$ is a sequence of resolutions $\mathcal{N}_j$ of N_j ($j \in I$). We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is an *r-tower* iff for every $i \in I$ we have that M_i is a σ-limit model over N_i, $a_i \in M_{i+1} \setminus M_i$ and $a_i \perp^{\mathcal{N}_i}_{N_i} M_i$.



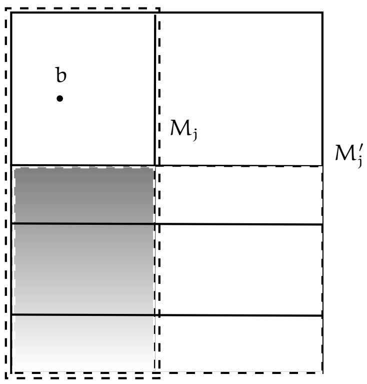
Definition 3.3 (Extension of r-towers). Let $I \le I'$ be well-orders, $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}_{\mu,I}$ and $(\mathfrak{M}', \overline{a}', \mathfrak{N}', \mathcal{M}', \mathcal{N}') \in \mathcal{K}_{\mu,I'}$. We say that $(\mathfrak{M}', \overline{a}', \mathfrak{N}', \mathcal{M}', \mathcal{N}')$ extends $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ (which we denote by $(\mathfrak{M}', \overline{a}', \mathfrak{N}', \mathcal{M}', \mathcal{N}') > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$) iff for every $i \in I$:

(1) M'_i is a proper universal model over M_i
(2) $\mathcal{M}_i \subset \mathcal{M}'_i$.
(3) $a_i = a'_i$
(4) $N_i = N'_i$
(5) $\mathcal{N}_i = \mathcal{N}'_i$



3.1. **Reduced r-Towers.**

Definition 3.4. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N})$ is a *reduced r-tower* iff for every extension $(\mathfrak{M}', \overline{a}', \mathfrak{N}', \mathcal{M}', \mathcal{N}') > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ and for every $j \in I$ we have that $\overline{\bigcup_{i\in I} M_i} \mathop{\downarrow}^{\mathcal{M}_j}_{M_j} M'_j$ (i.e.: for every $b \in \overline{\bigcup_{i\in I} M_i} = \bigcup_{i\in I} M_i$, $b \mathop{\downarrow}^{\mathcal{M}_j}_{M_j} M'_j$).



Proposition 3.5 (Density of Reduced r-Towers). *Every r-tower* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *has an extension which is a reduced r-tower.*

Proof. Suppose not. Then, we can construct an $<$-increasing sequence $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha : \alpha < \mu^+ \rangle$ of r-towers such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^0 := (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ and such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\alpha+1}$ witnesses that

$(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ is not reduced.

For a fixed $l \in I$, by assumption 3.1, for every c we have that there exists a minimal $\alpha < \mu^+$ such that $c \downarrow^{\mathcal{M}_l^\alpha}_{M_l^\alpha} \bigcup_{\alpha<\mu^+} M_l^\alpha$ (which we denote by α_c^l). Define $\alpha_c := \sup\{\alpha_c^l : l \in I\} < \mu^+$ (because $|I| = \mu$). Notice that for every $i \in I$, $M_i^{\alpha_c^i} \prec_{\mathcal{K}} M_i^{\alpha_c} \prec_{\mathcal{K}} \bigcup_{\alpha<\mu^+} M_i^\alpha$, so by monotonicity we have that $c \downarrow^{\mathcal{M}_i^{\alpha_c}}_{M_i^{\alpha_c}} \bigcup_{\alpha<\mu^+} M_i^\alpha$ for every $i \in I$ (since by definition of $<$ we have that $\mathcal{M}_i^{\alpha_c^i} \subseteq \mathcal{M}_i^{\alpha_c}$).

Take any $\gamma_0 < \mu^+$. We have that $\bigcup_{i\in I} M_i^{\gamma_0}$ has density character μ, so we can take $B_{\gamma_0} \subseteq \bigcup_{i\in I} M_i^{\gamma_0}$ of cardinality μ such that $\overline{B_{\gamma_0}} = \bigcup_{i\in I} M_i^{\gamma_0}$. Defining $f_0 : B_{\gamma_0} \to \mu^+$ as $c \mapsto \alpha_c$, we have that there exists $\gamma_0' < \mu^+$ such that $\alpha_c < \gamma_0'$ for every $c \in B_{\gamma_0}$. Take $\gamma_1 := \max\{\gamma_0, \gamma_0'\} + 1$.

Suppose that we have defined $\gamma_n < \mu^+$ $(n < \omega)$. Take $B_{\gamma_n} \subseteq \bigcup_{i\in I} M_i^{\gamma_n}$ of cardinality μ such that $\overline{B_{\gamma_n}} = \bigcup_{i\in I} M_i^{\gamma_n}$. Taking $f_n : B_{\gamma_n} \to \mu^+$ as $c \mapsto \alpha_c$, we have that there exists $\gamma_n' < \mu^+$ such that $\alpha_c < \gamma_n'$ for every $c \in B_{\gamma_n}$. Take $\gamma_{n+1} := \max\{\gamma_n, \gamma_n'\} + 1$.

We have that $\gamma := \sup\{\gamma_n : n < \omega\} < \mu^+$.

Take $b \in \bigcup_{i\in I} M_i^\gamma$ which witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\gamma$ is not reduced: i.e.: there exists $k \in I$ such that $b \not\downarrow^{\mathcal{M}_k^\gamma}_{M_k^\gamma} M_k^{\gamma+1}$. We know that there exists $j \in I$ such that $b \in M_j^\gamma$, so $j > k$.

Since $b \in M_j^\gamma := \overline{\bigcup_{\alpha<\gamma} M_j^\alpha}$, there exists a sequence (b_n) in $\bigcup_{\alpha<\gamma} M_j^\alpha$ such that $(b_n) \to b$. Since $b \not\downarrow^{\mathcal{M}_k^\gamma}_{M_k^\gamma} M_k^{\gamma+1}$, by [ViZa1, Lemma 2.13] (Continuity of Independence) there exists $N < \omega$ such that $b_N \not\downarrow^{\mathcal{M}_k^\gamma}_{M_k^\gamma} M_k^{\gamma+1}$. Notice that there exists $\alpha_N < \gamma$ such that $b_N \in M_j^{\alpha_N}$. By definition of γ, there exists $M < \omega$ such that $\alpha_N < \gamma_M$, therefore $b_N \in M_j^{\gamma_M} \subseteq \bigcup_{i\in I} M_i^{\gamma_M}$. Take a sequence $(c_n) \in B_{\gamma_M}$ such that $c_n \to b_N$. Since $b_N \not\downarrow^{\mathcal{M}_k^\gamma}_{M_k^\gamma} M_k^{\gamma+1}$, by [ViZa1, Lemma 2.13], we have that there exists $K < \omega$ such that $c_K \not\downarrow^{\mathcal{M}_k^\gamma}_{M_k^\gamma} M_k^{\gamma+1}$. But we have that $c_K \downarrow^{\mathcal{M}_k^\alpha}_{M_k^\alpha} \bigcup_{\alpha<\mu^+} M_k^\alpha$, where $\alpha := \alpha_{c_K}$. Also we have that $\alpha_{c_K} < \gamma_{M+1} < \gamma$, therefore $M_k^{\alpha_{c_K}} \prec_{\mathcal{K}}$

$M_k^\gamma \prec_{\mathcal{K}} M_k^{\gamma+1} \prec_{\mathcal{K}} \bigcup_{\alpha<\mu^+} M_k^\alpha$, so by monotonicity (see [ViZa1, Proposition 2.4]) we have that $c_K \downarrow_{M_k^\gamma}^{\mathcal{M}_k^\gamma} M_k^{\gamma+1}$ (contradiction). Therefore the proposition is true. □

Proposition 3.6 (completion of union of chains of reduced r-towers).
If $\langle ((\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\gamma : \gamma < \beta\rangle$ is an $<$-increasing chain of reduced r-towers (where $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\gamma \in \mathcal{K}_{\mu, I_\gamma}$ and $I_\alpha \subseteq I_\gamma$ if $\alpha < \gamma < \beta$), the completion of the union of this sequence is a reduced r-tower indexed by $I_\beta := \bigcup_{\gamma<\beta} I_\gamma$.

Proof. Suppose not. Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\beta$ be an r-tower which witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\beta$ is not a reduced r-tower. Let $b \in \overline{\bigcup_{i\in I_\beta} M_i^\beta} = \bigcup_{i\in I_\beta} M_i^\beta$ (because $cf(I_\beta) > \omega$) and $i \in I_\beta$ be such that $b \not\downarrow_{M_i^\beta}^{\mathcal{M}_i^\beta} M_i$. Therefore, there exists $j \in I_\beta$ such that $b \in M_j^\beta$. Notice that $j > i$.

Since $b \in M_j^\beta := \overline{\bigcup_{\gamma(j)\le\alpha<\beta} M_j^\alpha}$ (where $\gamma(j) := \min\{\gamma < \beta : j \in I_\gamma\}$), there exists a sequence (b_n) in $\bigcup_{\gamma(j)\le\alpha<\beta} M_j^\alpha$ such that $b_n \to b$. By [ViZa1, Lemma 2.13] (continuity of r-independence), there exists $N < \omega$ such that $b_N \not\downarrow_{M_i^\beta}^{\mathcal{M}_i^\beta} M_i$.

Defining $\gamma(i) := \min\{\gamma < \beta : i \in I_\gamma\}$, let $\max\{\gamma(i), \gamma(j)\} \le \gamma < \beta$ be such that $b_n \in M_j^\gamma \subseteq \bigcup_{i\in I_\gamma} M_i^\gamma$. Since $b_N \not\downarrow_{M_i^\beta}^{\mathcal{M}_i^\beta} M_i$, by monotonicity ([ViZa1, Proposition 2.4]) we have that $b_N \not\downarrow_{M_i^\gamma}^{\mathcal{M}_i^\gamma} M_i$. This contradicts the fact that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\gamma$ is a reduced r-tower. So, the proposition is true. □

3.2. **Full-relativeness of r-towers.**

Definition 3.7 (strong type). Let M be a σ-limit model

(1) $$\mathfrak{St}(M) := \left\{ (p, N) : \begin{array}{l} N \prec_{\mathcal{K}} M \\ N \text{ is a } \theta\text{-limit model} \\ M \text{ is universal over } N \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \downarrow_N^{\mathcal{N}} M \\ \text{for some resolution } \mathcal{N} \text{ of } N. \end{array} \right.$$

(2) Two strong types $(p_l, N_l) \in \mathfrak{St}(M_l)$ $(l \in \{1, 2\})$ are *parallel* (which we denote by $(p_1, N_1) \parallel (p_2, N_2)$) iff for every $M' \succ_{\mathcal{K}} M_1, M_2$ with density character μ, there exists $q \in \text{ga-S}(M')$ which extends

both p_1 and p_2 and $q \downfree^{\mathcal{N}_l}_{N_l} M'$ ($l \in \{1,2\}$) (where $\mathcal{N}_l$ is the resolution of N_l which satisfies $p_i \downfree^{\mathcal{N}_l}_{N_l} M_l$).

Assumption 3.8. *Through this subsection, assume that* I *is a well order which has a cofinal sequence* $(i_\alpha : \alpha < \theta)$*, where* $\mathrm{cf}(\theta) > \omega$.

Definition 3.9 (Metric r-Towers). An r-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is called a *metric r-tower* if the resolution witnessing that M_i is a (μ, σ)-limit model over N_i is spread-out. A spread-out resolution $\mathcal{M}$ of M is a resolution where for every γ, $M^{\gamma+1}$ is an ω_1-limit over M^γ.

Definition 3.10 (Full relative r-towers). Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ be a r-tower indexed by I. Let $(M_i^\gamma : \gamma < \sigma)$ be a sequence which witnesses that M_i is a (μ, σ)-limit model. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a relative r-tower with respect to $(M_i^\gamma)_{i \in I, \gamma < \sigma}$ iff for every $i_\alpha \leq i < i_{\alpha+1}$ and $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ there exists $i \leq j < i_{\alpha+1}$ such that $(p, M_i^\gamma) \parallel (\text{ga-tp}(a_j/M_j), N_j)$.

Proposition 3.11. *Suppose that for every* $\alpha < \theta$ *there are* $\mu \cdot \omega$ *many elements between* i_α *and* $i_{\alpha+1}$*. Let* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *be a full relative r-tower with respect to* $(M_i^\gamma)_{i \in I, \gamma < \sigma}$*. Then* $M := \overline{\bigcup_{i \in I} M_i}$ *is a limit model over* M_{i_0}.

Proof. It is enough to prove that $M_{i_{\alpha+1}}$ is universal over M_{i_α}. Let $p :=$ ga-tp$(a/M_{i_\alpha}) \in$ ga-S(M_{i_α}) and $\varepsilon > 0$. So, by assumption 3.1 there exists $\gamma := \gamma_\varepsilon < \sigma$ such that $a \downfree^{\varepsilon}_{M_{i_0}^{\gamma_\varepsilon}} M_{i_0}$.

By construction, $M_{i_\alpha}^{\gamma+1}$ is a (μ, ω_1)-limit model over $M_{i_\alpha}^\gamma$. Let $(M_i^* : i < \omega_1)$ be a resolution which witnesses that.

Consider $q := p \upharpoonright M_{i_\alpha}^{\gamma+1}$, so by assumption 3.1 there exists $i < \omega_1$ such that $q \downfree_{M_i^*} M_{i_\alpha}^{\gamma+1}$. By extension over universal models [ViZa1, 2.7] (notice that $M_{i_\alpha}^{\gamma+1}$ is universal over M_i^*), there exists $q^* \in$ ga-S(M_{i_α}) an extension of q such that $q^* \downfree_{M_i^*} M_{i_\alpha}$. So, $(q^*, M_i^*) \in \mathfrak{St}(M_{i_\alpha})$. By full relativeness of $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$, there exists $i_\alpha \leq j_1 < i_{\alpha+1}$ such that $(q^*, M_i^*) \parallel (\text{ga-tp}(a_j/M_j), N_j)$. Therefore, $q^* =$ ga-tp(a_j/M_{i_α}) and so q^* is realized in M_{j_1}.

By monotonicity of non-ε-splitting, we have that p does not ε-split over M_i^* (since p does not ε-split over $M_{i_\gamma}^\gamma$ and $M_{i_\alpha}^\gamma \prec_{\mathcal{K}} M_i^*$); i.e. $p \downfree^{\varepsilon}_{M_i^*} M_{i_\alpha}$. Since $q^* \downfree_{M_i^*} M_{i_\alpha}$, then $q^* \downfree^{\varepsilon}_{M_i^*} M_{i_\alpha}$ (by monotonicity of non-ε-splitting).

Also, since $q = p \upharpoonright M_{i_\alpha}^{\gamma+1}$ and $q^* \supset q$, then $q^* \upharpoonright M_{i_\alpha}^{\gamma+1} = p \upharpoonright M_{i_\alpha}^{\gamma+1}$. Notice that $M_{i_\alpha}^{\gamma+1}$ is universal over M_i^*.

Since $p, q^* \downarrow^{\varepsilon}_{M^*_i} M_{i_\alpha}$, by a weak version of stationarity [ViZa1, Lemma 2.6], we have that $\mathbf{d}(p, q^*) < 2\varepsilon$. Therefore, M_{j_1} realizes a dense subset of ga-S(M_{i_α}).

Doing a similar argument, we can construct an increasing sequence $(j_n : n < \omega)$ in I (where $j_0 := i_\alpha$) such that $i_\alpha \le j_n < i_{\alpha+1}$, where $M_{j_{n+1}}$ realizes a dense subset of ga-S(M_{j_n}).

Therefore, by [ViZa1, 1.18] we have that $M^* := \overline{\bigcup_{n<\omega} M_{j_n}} \prec_{\mathcal{K}} M_{i_{\alpha+1}}$ realizes every type over $M_{j_0} = M_{i_\alpha}$, so $M_{i_{\alpha+1}}$ does. □

The following fact is proved in a similar way like the discrete case (see [GrVaVi]). For the sake of completeness, we give a proof of this result.

Proposition 3.12. *If* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu,I}$*, there exists* $(\mathfrak{M}', a, \mathfrak{N}, \mathcal{M}', \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu,I}$ *such that for every limit* $i \in I$*,* M'_i *is a* (μ, μ)*-limit over* $\overline{\bigcup_{j<i} M_j}$

Proof. First, we construct by induction on $i \in I$ a model $M^+_i \succ_{\mathcal{K}} M_i$ and a directed system $(f_{i,j} : i < j \in I)$ of $\prec_{\mathcal{K}}$-embeddings (as in the discrete AEC case, one may prove that the "union axioms" for metric AEC also hold for directed systems) such that $f_{i,j} : M^+_i \to M^+_j$ and $f_{i,j} \restriction M_i = id_{M_i}$.

Suppose $(M^+_k : k \le i)$ and $(f_{k,l} : k < l \le i)$ are constructed. We give the construction of M^+_{i+1} and $f_{i,i+1}$. The construction of $f_{j,i+1}$ $(j < i)$ are given by definition of directed system. Let M^*_{i+1} be a limit model over M^+_i and M_{i+1}. Since $a_{i+1} \downarrow^{\mathcal{N}_{i+1}}_{N_{i+1}} M_{i+1}$ and M_{i+1} is universal over N_{i+1} (by definition of r-tower), by the extension property ([ViZa1, 2.7]) and invariance of r-independence there exists $f \in Aut(\mathbb{M}/M_{i+1})$ such that $a_{i+a} \downarrow^{\mathcal{N}_{i+1}}_{N_{i+1}} f[M^*_{i+1}]$. Define $M^+_{i+1} := f[M^*_{i+1}]$ and $f_{i,i+1} := f \restriction M^+_i$.

For limit $i \in I$, first take the directed limit of $(M^+_k : k \le i)$ and $(f_{k,l} : k < l \le i)$ and then consider M^+_i a limit model over this directed limit and (μ, μ)-limit over $\overline{\bigcup_{j<i} f_{j,i}[M^+_j]}$.

Fix $j \in I$. Let $f_{j,\sup(I)}$ and $M'_{j,\sup(I)}$ be the respective directed limit of this directed system. Notice that $f_{j,\sup(I)} \restriction M_j = id_{M_j}$. Define $M'_j := f_{j,\sup(I)}[M^+_j]$.

Notice that the r-tower $(\mathfrak{M}', \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ defined in this way satisfies the requirements of the proposition. □

Lemma 3.13 (Weak Full Relativeness). *Given* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu, I_n}$, *there exists* $(\mathfrak{M}', a, \mathfrak{N}, \mathcal{M}', \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu, I_{n+1}}$ *such that for every* $(p, N) \in \mathfrak{St}(M_i)$ *(where* $i \in I_n$ *and* $i_\alpha \leq i < i_{\alpha+1}$*) there exists* $i \leq j < i_{\alpha+1}$ *such that* $(\text{ga-tp}(a_j/M'_j), N_j) \parallel (p, N)$.

Proof. Let $M'_{i_{\alpha+1}}$ be a (μ, μ)-limit model over $\overline{\bigcup_{j<i_{\alpha+1}, j \in I_n} M_j}$ (by proposition 3.12). Let $\langle M'_i : l \in I_{n+i}, i_\alpha + \mu \cdot n < l < \alpha + 1 \rangle$ be an enumeration of a resolution which witnesses that $M'_{i_{\alpha+1}}$ is (μ, μ)-limit over $\overline{\bigcup_{j<i_{\alpha+1}, j \in I_n} M_j}$.

Let $\mathfrak{S} := \{(p, N)^l_\alpha : i_\alpha + \mu \cdot n < l < i_{\alpha+1}\}$ be an enumeration of a dense subset of $\bigcup \{\mathfrak{St}(M_i) : i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$ (by μ-stability). Therefore, given $(p, N)^l_\alpha \in \mathfrak{S}$ there exists $i \in I_n$ such that $i_\alpha \leq i < i_{\alpha+1}$ such that $(p, N)^l_\alpha \in \mathfrak{St}(M_i)$. So $p^l_\alpha \mathop{\smile\!\!\!\!\!\!|}\,^{\mathcal{N}^l_\alpha}_{N^l_\alpha} M_i$. Since by definition of strong type M_i is universal over N^l_α and $M_i \prec_{\mathcal{K}} M'_l$, by [ViZa1, 2.7] there exists $p^* \in \text{ga-S}(M'_l)$ which extends p^l_α and $p^* \mathop{\smile\!\!\!\!\!\!|}\,^{\mathcal{N}^l_\alpha}_{N^l_\alpha} M'_l$. Notice that $M'_{succ_{I_{n+1}}(l)}$ is universal over M'_l (by construction), then there exists $a_l \in M'_{succ_{I_{n+1}}(l)}$ such that $a_l \models p^*$. Consider $N_l := N^l_\alpha$. So, $a_l \mathop{\smile\!\!\!\!\!\!|}\,^{\mathcal{N}}_{N_l} M'_l$. The r-tower constructed in this way satisfies the requirements of the proposition. □

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