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1 Introduction

We shall investigate whether second order equivalence of two models, or equivalence in some stronger logic than second order logic implies isomorphism in certain cardinalities. We always assume that our vocabulary is finite.

1.1 Definition. We call the following condition *Ajtai Hypothesis*: Second order equivalence implies isomorphism for countable models in finite vocabularies. This means: in any finite vocabulary, if two countable models satisfy the same second order theory then they are isomorphic.

We call Ajtai Hypothesis restricted to ordinals *Fraïssé Hypothesis*. That is: All countable ordinals have different second order theories.

Ajtai [2] has proved that Ajtai Hypothesis is independent of ZFC. We are looking for related results in the cardinality $\aleph_0$ and similar results in bigger infinite cardinals. The name Ajtai Hypothesis is not commonly used but it is our own invention. It might be better to talk about Ajtai Property instead of Ajtai Hypothesis as Ajtai did not suggest Ajtai Hypothesis to be true. However, the name Ajtai Hypothesis is compatible with Fraïssé Hypothesis used by Wiktor Marek. Fraïssé Hypothesis has been studied by Fraïssé [3] and Marek [10], [11].

In the first chapter we will recall the proof of Ajtai and use his method to prove various results related to Ajtai Hypothesis in countable cardinality.

In the second chapter we will use Ajtai's method to prove that it is independent of ZFC whether $L^2_{\kappa,\omega}$-equivalence implies isomorphism for any

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models of regular cardinality κ in any finite vocabulary. We will also prove that these implications are independent of each other for different κ's and we will give an analogous result for singular cardinals. We consider the results in Chapter 3 as the main results of this paper.

In Chapter 4 we investigate the relation between Ajtai Hypothesis and various large cardinal axioms.

Notation

ZF-formulas mean formulas in language of set theory. ZF-equivalence of two structures means that they satisfy same sentences of language of set theory. Generally, if L is a logic, $\mathfrak{A} \equiv_L \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy same sentences of L. We use L^2 to refer to second order logic. In L^2 we can quantify over all relations over the universe of the model, thus our second order logic means the second order logic with full semantics. There are also other second order logics which do not use full semantics such as monadic second order logic where we can quantify over unary relations only, and second order logic with Henkin semantics [5]. Generally L^n refers to n:th order logic with full semantics. $H(\kappa)$ means the set of sets hereditarily smaller than κ i.e. $\{X :$ the transitive closure of X has cardinality less than $\kappa\}$. The symbol $\restriction$ means "restricted to". Depending on context this can mean reduct of a model to a smaller vocabulary or restriction of some operations to some set. The forcing name of a given set X is denoted by a dot on top of that set: $\dot{X}$. Interpretation of a set in a given model of ZFC is denoted by the set with the model of ZFC as superscript: for example ω_1^L means ω_1 of L. The reals mean the same as the powerset of ω.

Notation which is not explained is standard as used for example in Jech's book [6].

2 Ajtai's result, countable case

2.1 Lemma. *The structure $(\omega, <)$ is definable in second order logic in any infinite model.*

Proof. A model $(X, <_X)$ is isomorphic to $(\omega, <)$ if and only if X is an infinite set, $<_X$ is a linear order on X and every element of X has finitely many predecessors in the linear order. Infiniteness and finiteness are expressible in second order logic in any infinite model so the claim follows. □

2.2 Definition. In the model $(\omega, <)$ we can quantify over subsets of ω in second order logic. We say that a second order formula $\phi(U, V)$ is second order definable well-order of the reals if for any countable model the following hold in $(\omega, <)$:

1. For any subsets $X, Y, Z \subseteq \omega$ the linear order axioms hold i.e.

$$\neg\phi(X,X) \wedge (\phi(X,Y) \wedge \phi(Y,Z)) \rightarrow \phi(X,Z) \wedge (X = Y \vee \phi(X,Y) \vee \phi(Y,X))$$

2. There is no infinite descending sequence in the linear order i.e. for any sequence $(X_n)_{n \in \mathbb{N}}$ of subsets of ω there is an n such that $\phi(X_{n+1}, X_n)$ does not hold.

If ϕ is a second order definable well-order of the reals we can define another formula ϕ' in empty vocabulary which says that there is a model $(X, <_X)$ isomorphic to $(\omega, <)$ and ϕ holds in $(X, <_X)$ where $<$ is replaced by $<_X$ and the quantifiers are relativized to X. Thus if we have a second order definable well-order of the reals then we can in any infinite model (even in a model with empty vocabulary) define $(\omega, <)$ inside that model by second order quantifiers and have (up to isomorphism same) second order definable well-order ϕ' there. Often we don't bother to make distinction between ϕ and ϕ': If we are talking about ϕ in a model which is not $(\omega, <)$ we mean ϕ'.

Third order (or generally n:th order) definable well-order of the reals is defined in an analogous way. Also second order definable well-order of the powerset of κ for uncountable κ, which is used in Chapter 3, is defined in an analogous way.

We recall that Ajtai proved the independence of Ajtai Hypothesis from ZFC. We will now present the first part of the proof of Ajtai:

2.3 Theorem. *[2] If there is a second order definable well-order of the powerset of ω, then second order equivalence implies isomorphism for countable models in any finite vocabulary. If the well-order is Σ^1_n for $n \geq 2$, then Σ^1_{n+1}-equivalence implies isomorphism for countable models in any finite vocabulary.*

Proof. A model in a finite vocabulary can be coded into a k-ary relation R for some k. If there is a Σ^1_n-definable well-order of the reals for $n \geq 2$ then there is also a Σ^1_n-definable well-order of the powerset of ω^n. Let $<$ be such a well-order of the powerset of ω^n. In second order logic we can talk about the $<$-least subset R_0 of ω^n which is isomorphic to the model in question. For every k-tuple $(m_1, \ldots m_k)$ of natural numbers we can say in second order logic that it belongs to R_0:

$$\exists R_0 (R \cong R_0 \wedge \forall R_1 (R_1 \cong R \rightarrow (R_0 = R_1 \vee R_0 < R_1)) \wedge R_0(m_1, \ldots, m_k))$$

Similarly we can say in second order logic that some n-tuple does not belong to R_0. If two countable models are now second order equivalent, they have the same set R_0. Thus they have same isomorphism type and they are isomorphic.

We will next show that if there is a second order definable well-order of the reals, Σ^1_k-equivalence for certain k implies isomorphism. Let us assume our second order definable well-order of the reals is Δ^1_n for some $n \geq 2$. We make the assumption $n \geq 2$ to make complexity calculations simpler; in all our applications $n \geq 2$ so it does not do any harm. Note that if a well-order is Σ^1_n then it is Π^1_n because $x < y \Leftrightarrow x \neq y \wedge \neg y < x$. Similarly every Π^1_n well-order is Σ^1_n. Thus a well-order is Σ^1_n iff it is Π^1_n iff it is Δ^1_n. Also two models are Σ^1_n-equivalent iff they are Π^1_n-equivalent as we will show. Assume not: there are Σ^1_n-equivalent models $\mathfrak{A}$ and $\mathfrak{B}$ which are not Π^1_n-equivalent. Assume ϕ is such a Π^1_n formula that $\mathfrak{A} \models \phi$ and $\mathfrak{B} \not\models \phi$. Now $\neg\phi$ is such a Σ^1_n formula that $\mathfrak{A} \not\models \neg\phi$ and $\mathfrak{B} \models \neg\phi$, so the models are not Σ^1_n-equivalent, which is a contradiction. The proof that Π^1_n-equivalence implies Σ^1_n-equivalence is the same.

So let $<$ be a Δ^1_n well-order of the reals. Then the formula $R_1 \cong R \rightarrow R_0 \leq R_1$ is Π^1_n. The formula $R_1 \cong R$ does not make any difference because it is Σ^1_1 and the formula $R_0 \leq R_1$ is Π^1_n for some $n \geq 2$. Now the formula $R \cong R_0 \wedge \forall R_1 (R_1 \cong R \rightarrow R_0 \leq R_1) \wedge R_0(m_1, m_2, \ldots, m_n)$ is Π^1_n and the formula

$$\exists R_0 (R \cong R_0 \wedge \forall R_1 (R_1 \cong R \rightarrow R_0 \leq R_1) \wedge R_0(m_1, m_2, \ldots, m_n))$$

is Σ^1_{n+1}. Hence Σ^1_{n+1}-equivalence implies isomorphism. □

2.4 Corollary. *If $V = L$ then second order equivalence implies isomorphism for countable models in any finite vocabulary (i.e. Ajtai Hypothesis holds).*

Proof. In L there is a second order definable well-order of the powerset of ω.

In the following formula X and Y denote subsets of ω and $<_{L_\alpha}$ is the canonical well-order of α:th level of the constructible sets.

$$X < Y \Leftrightarrow \exists \alpha \exists L_\alpha \exists <_{L_\alpha} (X \in L_\alpha \wedge Y \in L_\alpha \wedge X <_{L_\alpha} Y)$$

□

2.5 Remark. In fact the well-order of the reals in L is Δ^1_2, thus if $V = L$ then Σ^1_3-equivalence implies isomorphism for countable models.

Generally if there is a Σ^1_n well-order of the reals, any two countable Σ_{n+1}-equivalent models are isomorphic. Hence they are second order equivalent and the full second order theory of a countable model is determined by its Σ_{n+1}-theory. However, it does not follow that every second order sentence is equivalent to a Σ^1_{n+1} sentence for countable models.

2.6 Corollary. *Ajtai Hypothesis is consistent with $V \neq L$.*

Proof. By result of Harrington [4] it is consistent with ZFC that the continuum is as big as desired but it has a Δ^1_3-definable well-order. □

If we have a second order definable well-order of the reals with a parameter r then any two countable models which satisfy same second order theory with parameter r are isomorphic. This can be seen by just adding a parameter to the proof of Theorem 2.3. However, in this article we do not give much attention to the case where we allow parameters: We are generally interested in possibility to determine isomorphism types of models by their theories in languages having sentences smaller than the cardinality of the model. Thus using a real parameter in a language to determine isomorphism type of a countable model (a real) is a bit disappointing. However we mention the following remark:

2.7 Remark. Harrington has proved [4] that it is consistent that Martin's Axiom holds, the continuum is as big as wanted and there is a second order definable well-order of the reals using a real parameter. It follows that the following are consistent with each other:

1. Martin's Axiom
2. For some real parameter second order equivalence with that real parameter implies isomorphism for countable models .

2.8 Open question. Is Martin's Axiom consistent with Ajtai Hypothesis.

2.9 Open question. If $V = L$, are there two countable non-isomorphic models which have same monadic second order theory?

A second order definable well-order of the reals is also consistent with measurable and Woodin cardinals, which cannot exist in L. We will return to these large cardinals in Chapter 4.

By Theorem 2.3 Ajtai Hypothesis is consistent. In all our examples where Ajtai Hypothesis holds this is based on a second order definable well-order of the reals.

2.10 Open question. Is it consistent that Ajtai Hypothesis holds, but there is no second order definable well-order of the reals?

If that is not consistent, then Ajtai Hypothesis is equivalent to the existence of a second order definable well-order of the reals. We have an idea how it might be possible to prove that these conditions are not equivalent.

Suppose there is a model of ZFC with the following properties (We do not know yet if such a model exists) :

1. There is no second order definable well-order of the reals.
2. There are second order definable sets $X_i \subset \mathbb{R} : i \in \omega$ such that each X_i has a second order definable well-order and $\mathbb{R} = \bigcup_{i\in\omega} X_i$.

Suppose now $\mathfrak{A}$ and $\mathfrak{B}$ are two second order equivalent countable models. Now $\mathfrak{A}$ is isomorphic to some real a and $\mathfrak{B}$ is isomorphic to some real b. Assume i and j are such indexes that $a \in X_i$ and $b \in X_j$. Let $X = X_i \cup X_j$. Now X is second order definable and there is a second order definable well-order of X. We assumed $\mathfrak{A}$ and $\mathfrak{B}$ are second order equivalent, so for all $n \in \omega$ n belongs to the the least real in X isomorphic to $\mathfrak{A}$ if and only if n belongs to the the least real in X isomorphic to $\mathfrak{B}$. Now $\mathfrak{A}$ and $\mathfrak{B}$ have same isomorphism type and they are isomorphic.

We have another idea, suggested by Shelah, how it might be possible to have Ajtai Hypothesis without second order definable well-order of the reals. Assume there is a second order definable set of reals which contains exactly one real of each isomorphism type. Then we can use the idea of Ajtai's proof to show that Ajtai Hypothesis holds. The problem is to find a model of ZFC in which there is a second order definable set of reals which contains exactly one real of each isomorphism type but there is no second order definable well-order of the reals. We are working to find such a model, using a construction suggested by Shelah.

We show next that it is not consistent that Σ^1_1-equivalence implies isomorphism for countable models.

2.11 Theorem. *For any infinite cardinal κ there are two non-isomorphic Σ^1_1-equivalent models of Peano Axioms of cardinality κ. In particular there are two Σ^1_1-equivalent countable models of Peano Axioms which are not isomorphic.*

Proof. We start by proving the claim for $\kappa = \omega$. This proof works equally well for all $\kappa < 2^{\aleph_0}$. For $\kappa \geq 2^{\aleph_0}$ the claim follows from a simple cardinality argument.

We construct an elementary chain of length ω_1 of countable models of Peano Axioms. We put $\mathfrak{A}_0$ to be the standard model of arithmetic. We recall that there are $2^{\aleph_0}$ different types in arithmetic. If $A \subseteq \omega$, then by compactness theorem $\Sigma_A = \{n\text{:th prime number divides } x : n \in A\} \cup \{\neg n\text{:th prime number divides } x : n \notin A\}$ is a consistent set of formulas. Thus if $A \subseteq \omega$ and $B \subseteq \omega$, Σ_A and Σ_B can be completed to types and these types are different. In any countable model only countably many types Σ_A are satisfied, so by Compactness Theorem there is always some countable elementary extension which realizes some new type Σ_A. It is thus easy to get an elementary chain of length ω_1 of countable non isomorphic models of Peano Axioms. However, we want some of the models in chain to be Σ_1^1-equivalent. In order to do that, we make sure that Σ_1^1 sentences true in models of the chain are increasing. Thus for each Σ_1^1 formula $\exists R\phi$ which is true in the standard model of arithmetic we put a new relation to the vocabulary of $\mathfrak{A}_0$ and interpret it in such a way that the formula ϕ is satisfied. If $\mathfrak{A}_{\alpha+1}$ satisfies some Σ_1^1 sentences (in the original vocabulary) which are not true in $\mathfrak{A}_\alpha$ then we add new relations to the model so that every Σ_1^1 sentence is satisfied by a relation in the model. We are making the vocabulary bigger and bigger, but it does not matter. If σ is the vocabulary of $\mathfrak{A}_\alpha$ and τ is the vocabulary of $\mathfrak{A}_\beta$, $\alpha < \beta$, then $\mathfrak{A}_\alpha \preccurlyeq \mathfrak{A}_\beta \restriction \sigma$. Since there are only countably many Σ_1^1 sentences, there is such an $\alpha < \omega_1$ that from α forward all models in the chain are Σ_1^1-equivalent. Thus from some α forward, all models in the chain are Σ_1^1-equivalent but not isomorphic.

The proof above works equally well for all $\aleph_\alpha < 2^{\aleph_0}$. In any cardinality κ there are 2^κ nonisomorphic models of arithmetic. Therefore in cardinalities $\kappa \geq 2^{\aleph_0}$ Σ_1^1-equivalence does not imply isomorphism and thus Σ_1^1-equivalence does not imply isomorphism in any infinite cardinality.

The theorem is formulated for Peano Axioms, but the proof works equally well for any theory which has $2^{\aleph_0}$ many types and more than continuum many non-isomorphic models in all cardinalities greater than or equal to the continuum.

$\square$

We showed above that Σ_1^1-equivalence does not imply isomorphism for countable models. We proved earlier that Σ_3^1-equivalence implies isomorphism for countable models in L. However we don't know yet what is the least n where it is possible that Σ_n^1-equivalence implies isomorphism for countable models.

2.12 Open question. Is it consistent that Σ_2^1-equivalence implies isomorphism for countable models?

We will now recall the second part of the independence proof of Ajtai. Note that in the proof we do not assume anything about the ground model. Consequently if we add a Cohen real to any model of ZFC, as is done in the proof, in the generic extension Ajtai Hypothesis fails.

2.13 Theorem. *[2] It is consistent with ZFC, that there are two countable non-isomorphic models which satisfy the same sentences of language of set theory. In particular the models are second order equivalent and equivalent in n:th order logic for all n.*

Proof. We add a Cohen-generic real to the set theoretic universe. Recall that the forcing conditions are functions from finite subsets of ω to $\{0,1\}$. A forcing condition p is stronger than a forcing condition q iff p extends q. If G is a subset of ω, we denote by F^G the set of all subsets of ω which differ from G only in finitely many points. Let now G be a generic real and $-G$ the complement of G. We are discussing the models $(F^G \cup \omega, <_\omega, P_G)$, where $<_\omega$ is the natural order of ω and P_G is the relation which tells which natural numbers n belong to which sets in F^G, and the corresponding model to $-G$. We denote these models M^G and M^{-G}. We claim that these two models satisfy the same sentences of language of set theory, but are not isomorphic. If some sentence of language of set theory is true in M^G, then it is forced by some forcing condition p. But p is finite and does not determine M^G at all. Assume $p \Vdash \phi(M^G) \wedge \neg\phi(M^{-G})$. So there is a generic filter G containing p such that $V^G \models \phi(M^G) \wedge \neg\phi(M^{-G})$. Now consider another generic filter G' which agrees with G on domain of p but is complement of G outside domain of p. Now $V^G = V^{G'}$, but the models M^G and M^{-G} swap places: $\dot{M^G}^{V^G} = \dot{M^{-G}}^{V^{G'}}$ and $\dot{M^G}^{V^{G'}} = \dot{M^{-G}}^{V^G}$. Thus the forcing condition p can not force any formula of language of set theory with parameters from the ground model to be true in M^G and false in M^{-G}.

But $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ are non-isomorphic. Since ω is a rigid structure, in an isomorphism every set in F^G should be mapped to exactly the same set in F^{-G}. But this is impossible because $G \notin F^{-G}$.

□

2.14 Remark. If two countable models are not isomorphic to each other then then they are separated by some $L_{\omega_1,\omega}$-sentence. The logic $L_{\omega_1,\omega}$ is related to Dynamic Ehrenfeucht-Fraïssé games, see for example [19] for definitions. For any non-$L_{\omega_1,\omega}$-equivalent countable models $\mathfrak{A}$ and $\mathfrak{B}$ there is an $\alpha < \omega_1$ such that I has a winning strategy in Dynamic Ehrenfeucht-Fraïssé game $EFD_\alpha(\mathfrak{A}, \mathfrak{B})$. The least such α is called Scott Watershed for $\mathfrak{A}$ and $\mathfrak{B}$. The bigger the Scott Watershed is, the harder the models are to distinguish by a $L_{\omega_1,\omega}$-sentence. The models M^G and M^{-G} satisfy same sentences of language of set theory, so they are in a way hard to distinguish from each other. However, the Scott Watershed of the pair (M^G , M^{-G}) is a very small ordinal: $\omega + 1$.

In the proof of Theorem 2.13 we added one generic real to the set theoretic universe and got two second order equivalent non-isomorphic models. But actually by a little modification of the proof, we can add many generic reals to the universe and get many countable second order equivalent non-isomorphic models:

2.15 Theorem. *Let κ^+ be an infinite cardinal. It is consistent with ZFC that there are κ^+ countable ZF-equivalent non-isomorphic models.*

Proof. We add κ^+ generic reals to L. Forcing conditions are finite functions from $\kappa^+ \times \omega$ to $\{0, 1\}$. A forcing condition p is stronger than another forcing condition q iff p extends q. If G is a generic set for this notion of forcing, for all $\alpha < \kappa^+$, $f_\alpha = \{n : G(\alpha, n) = 1\}$ is a generic real. Note that for all $\alpha < \beta < \kappa^+$ f_α and f_β differ in infinitely many points. Thus if we construct models around f_α and f_β as in Theorem 2.13, we get countable non-isomorphic models. We denote these models by M^{f_α} and M^{f_β}. We will show that the models are ZF-equivalent. Assume not: then there is a forcing condition p and a ZF-sentence ϕ with possibly parameters from the ground model such that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$. So there is a generic filter G containing p such that $V^G \models \phi(M^{f_\alpha}) \wedge \neg\phi(M^{f_\beta})$. But there is another generic filter G' which agrees with G in all ordinals different from α and β, agrees with G in α and β in the domain of p and chances digits of α to digits of β and vice versa outside the domain of p. Now $V^G = V^{G'}$, $p \in G'$ and the interpretations of $\dot{M}^{f_\alpha}$ and $\dot{M}^{f_\beta}$ swap places in the two generic extensions. Thus it is impossible that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$. □

2.16 Theorem. *Fraïssé Hypothesis implies that there is a third order definable well-order of a subset of the reals which has length ω_1.*

Proof. The ordinal ω_1 is definable in third order logic. In third order logic we can also define a truth definition for all countable ordinals i.e. a mapping from ω_1 to the second order theories of the ordinals in ω_1. For details see definition 4.9 and lemma 4.10. We fix some Gödel-numbering of second order sentences and consider second order theories as real numbers. From Fraïssé Hypothesis it follows that countable ordinals have different second order theories and thus our mapping maps them to different reals. Thus we have a third order definable injective mapping from ω_1 to the reals, so we have a third order definable well-order of length ω_1 of a subset of the reals. □

2.17 Theorem. *If there is a second order definable well-order of length ω_1 of a subset of the reals then Fraïssé Hypothesis holds.*

Proof. Let X be the subset of the reals in the assumption and let α be a countable ordinal. In the second order definable well-order of X there is the α:th real in the well-order of X. In second order logic we can talk about this real by sentences of the following form:

"There is an initial segment of the well-order of X which has the same order type with this model and the supremum of this initial segment contains (or does not contain) n."

If α and β are different countable ordinals then X has an α:th real a and a β:th real b and $a \neq b$. Thus there is some $n \in \omega$ where a and b disagree and for this n α and β disagree about a second order sentence of the form above.

□

2.18 Theorem. *Consider the following conditions:*

1. *There is a second order definable well-order of the reals*
2. *Ajtai Hypothesis*
3. *Fraïssé Hypothesis*
4. *There is a third order definable well-order of a subset of the reals which has length ω_1,*
5. *There is a second order definable well-order of length ω_1 of a subset of the reals.*

The following implications hold:
1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4.
5. $\Rightarrow$ 3.

Proof. 1. $\Rightarrow$ 2. Theorem 2.3. 2. $\Rightarrow$ 3. is trivial. 3. $\Rightarrow$ 4. Theorem 2.16. 5. $\Rightarrow$ 3. Theorem 2.17. □

Recall that the implication 2. $\Rightarrow$ 1. in the theorem above is an open question. Implication 2. $\Rightarrow$ 3. is proper, as after adding a Cohen real to L countable ordinals still satisfy different second order theories. This is because in V^G we can still talk about which second order sentences the ordinals satisfy in L. And this does not hold only for ordinals: any model which is countable in L is the unique countable model in V which satisfies its second order theory. It follows that the negation of Fraïssé Hypothesis implies $\omega_1^L \neq \omega_1^V$.

At this point we note that Ajtai Hypothesis and the Continuum Hypothesis do not decide each other in any way. We give the following examples:

1. Ajtai Hypothesis and the Continuum Hypothesis both hold in L.

2. If we add for example $\aleph_2$ Cohen generic reals to L, then Ajtai Hypothesis and the Continuum Hypothesis both fail.

3. Harrington[4] gives a model of ZFC in which the continuum is large but has a Δ^1_3 well-order. Consequently the Continuum Hypothesis fails but Ajtai Hypothesis holds.

4. If we add one Cohen generic real to L then Ajtai Hypothesis fails but the Continuum Hypothesis holds.

Also $\diamondsuit$ does not decide the truth of Ajtai Hypothesis. In L the $\diamondsuit$ principle holds and Ajtai Hypothesis holds. We will show that $\diamondsuit$ can be forced to be true by a small forcing which does not destroy large cardinals, and we can have a model with $\diamondsuit$ and a proper class of Woodin cardinals. From Theorem 4.6 it follows that $\diamondsuit$ is consistent with the negation of Ajtai Hypothesis. We will now introduce a forcing which makes $\diamondsuit$ true. We have found the forcing from Jech's book [6], exercise 15.23.

2.19 Lemma. *(Folklore) Let $Q = \{\langle S_\beta : \beta < \alpha\rangle, \alpha < \omega_1\}$, where $S_\beta \subseteq \beta$ for all $\beta < \alpha$. Let p be stronger than q if and only if p extends q. Let G be Q-generic. Then $V[G] \models \diamondsuit$.*

Proof. We will show that $\bigcup G$ is a $\diamondsuit$-sequence. Thus we need to show that for any forcing names $\dot{C}$ and $\dot{X}$, if $p \Vdash (\dot{C}$ is closed unbounded subset of ω_1 and $\dot{X} \subseteq \omega_1)$ then there is a q stronger than p such that $q = \langle S_\beta : \beta \leq \alpha\rangle$ and $q \Vdash (\alpha \in \dot{C}$ and $\dot{X} \cap \alpha = S_\alpha)$.

So assume $p \Vdash (\dot{C}$ is a closed unbounded subset of ω_1 and $\dot{X} \subseteq \omega_1)$. We will define inductively an ω-sequence of forcing conditions in such a way, that the upper limit of this sequence will do the job. We use len(p) to denote the length of the forcing condition p.

1. $p_0 = p$

2. p_1 is a forcing condition strictly stronger than p_0 such that $p_1 \Vdash \alpha_1 \in \dot{C}$ for some $\alpha_1 > \text{len}(p_0)$. This is possible because $\dot{C}$ is unbounded subset of ω_1.

3. p_2 is a forcing condition strictly stronger than p_1 such that it decides $\dot{X} \cap \alpha_1$, and $\text{len}(p_2) > \alpha_1$. This is possible because our forcing is $\aleph_0$-closed and it does not add any new subsets to countable sets. Thus $\dot{X} \cap \alpha_1$ is some set from the ground model and there is some forcing condition which decides which set from the ground model it is.

4. p_3 is a forcing condition strictly stronger than p_2 such that $p_3 \Vdash \alpha_2 \in \dot{C}$ for some $\alpha_2 > \text{len}(p_2)$.

5. p_4 is a forcing condition strictly stronger than p_3 such that it decides $\dot{X} \cap \alpha_2$, and $\text{len}(p_4) > \alpha_2$.

   $\vdots$

Let α be the supremum of the ordinals $\text{len}(p_n)$, $n \in \omega$. Since the sequence $\alpha_1, \alpha_2, \ldots$ converges to α and $\dot{C}$ is closed, $q \Vdash \alpha \in \dot{C}$ for any q which is stronger than all p_n:s. Also for any $\beta < \alpha$ there is some forcing condition p_n which decides whether $\beta \in \dot{X}$. Now we can define q to be as p_n:s for $\beta < \alpha$ and at α we can define it to be $\dot{X} \cap \alpha$.

□

Ajtai [2] has proved that it is consistent with ZFC that there are two different countable ordinals which satisfy same standard ZF-formulas. However, the model of ZFC in the proof is not necessarily transitive, so there might be some non-standard ZF or second order formulas which do not agree about those ordinals.

Marek [11] states without proof that in the Levy model, where all cardinals below the first inaccessible cardinal are collapsed to countable ordinals, Fraïssé Hypothesis fails. He also states a result of G. Sacks that if ω_1^L is collapsed to ω, then Fraïssé Hypothesis fails. We will next present a proof for this. Note that the failure of Fraïssé Hypothesis is consistent relative to consistency of ZFC.

2.20 Theorem. *It is consistent that Fraïssé Hypothesis fails.*

Proof. Let L be the ground model. We make a forcing which collapses ω_1 to ω. The forcing conditions are injective functions from finite subsets of ω to ω_1. A condition p is stronger than a condition q iff p extends q.

We make the following remark: If a is an element of the ground model, ϕ is a second order sentence and $p \Vdash \phi(a)$ then $1 \Vdash \phi(a)$. This is because in this forcing any forcing condition can not determine the generic extension in any way. If G is a generic filter for this forcing and p is a forcing condition then there is another generic filter G' containing p such that $V^G = V^{G'}$.

We claim that after the forcing there are two different ordinals smaller than ω_2^L which have same second order theory. Assume not. Then after the forcing all ordinals smaller than ω_2^L have different second order theories. For each ordinal $\alpha < \omega_2^L$, the relation $1 \Vdash \phi(\alpha)$ is definable in the ground model and the real $r_\alpha = \{n : n \text{ is a Gödel number of such a second order sentence } \phi \text{ that } 1 \Vdash \phi(\alpha)\}$ is definable in the ground model and belongs to ground model. Now the mapping $\alpha \mapsto r_\alpha$ is an injective mapping from ω_2 to the reals and it exists in L which is a contradiction.

□

We will give another proof for the consistency of the existence of two non-isomorphic second order equivalent countable linear orders. In the proof we construct two linear orders, which "look like" the two models in the proof of Theorem 2.13.

2.21 Theorem. *It is consistent with ZFC that there are two (or κ^+) countable non-isomorphic second order equivalent linear orders.*

Proof. Recall the models $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ from the proof of Theorem 2.13. We expand these models by adding linear orders ("lexicographic orders") to the sets F^G and F^{-G}. That is: in "lexicographic order" $X < Y$ iff there is $n \in \omega$ such that below n X and Y have the same elements, but $n \notin X$ and $n \in Y$. Note that these lexicographic orders are definable in second order logic, so the expanded models are second order equivalent.

Now we want to modify these lexicographic orders in such a way that they reflect the structure of the sets in F^G and F^{-G}. For each subset X of $\mathbb{N}$ we construct the following linear order denoted by $<_X$:

We denote by $<^1_X$ the following linear order: In the beginning there are four points. After the four starting points there is a $\mathbb{Q}$-component. Then if X has the first digit zero (respectively one) there are two (respectively three) points in the linear order. If $<^n_X$ has been defined, we denote by $<^{n+1}_X$ the linear order which has $<^n_X$ in the beginning, then a $\mathbb{Q}$-component and then two points (if $n+1$:th digit of X is 0) or three points (if $n+1$:th digit of X is 1). Finally we define $<_X = \bigcup_{n\in\mathbb{N}} <^n_X$.

The construction is definable in second order logic, so the mapping $X \mapsto <_X$ with domain F^G is definable in $(F^G \cup \omega, <_\omega, P_G)$. Similarly the mapping $X \mapsto <_X$ with domain F^{-G} is definable in $(F^{-G} \cup \omega, <_\omega, P_{-G})$.

Now we can define the linear order $<_G$ as follows:

$$dom <_G = \bigcup_{X \in F^G} dom <_X$$

where $dom <_X \cap dom <_Y = \emptyset$ for all different X and Y. If x and y are in $dom <_G$ then $x <_G y$ iff one of the following holds

1. There are X and Y such that $x \in <_X$ and $y \in <_Y$ and X is smaller than Y in the lexicographic order of F^G.

2. There is X such that $x \in <_X$ and $y \in <_X$ and $x <_X y$.

The construction of $<_G$ is second order definable in $(F^G \cup \omega, <_\omega)$. In similar way we can define another linear order $<_{-G}$ in $(F^{-G} \cup \omega, <_\omega, P_{-G})$. Because $(F^G \cup \omega, <_\omega)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$ are second order equivalent, also the linear orders $<_G$ and $<_{-G}$ are second order equivalent.

But the models are not isomorphic because the model constructed from $-G$ does not have an interval which starts with four points, then has ω Q-components and between the Q-components 2 points when the corresponding digit of G is 0 and 3 points when the corresponding digit of G is 1.

If we add κ^+ generic reals as in Theorem 2.15 then we get κ^+ non-isomorphic second order equivalent linear orders. □

2.22 Theorem. *It is consistent with ZFC that there are two countable second order equivalent non-isomorphic models of arithmetic.*

Proof. Let α and β be second order equivalent countable non-isomorphic ordinals, which consistently exist by Theorem 2.20. Let σ be a minimal type [7]. We extend the prime model of arithmetic by taking α-canonical and β-canonical extensions over the type σ. That is: we take the Ehrenfeucht-Mostowski models which are generated by the sequences of elements of the minimal type σ, and we let the generating sequences have order types α and β. The models are second order equivalent, but they are not isomorphic as there is no order preserving mapping of the generators of the first model to the generators of the second model. It is also impossible to have an isomorphism from one model to the other which would map the set of generators to a set other than generators in the other, because both structures are rigid [7] (p.70). □

2.23 Definition. A second order sentence ϕ is a *complete second order sentence*, if all such models $\mathfrak{A}$ and $\mathfrak{B}$ that $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \phi$ are second order equivalent.

The following is an unpublished result of Solovay [17].

2.24 Theorem. *It is independent of ZFC whether all models that satisfy the same complete second order sentence are isomorphic. Models which satisfy the same complete second order sentence have the same cardinality.*

Proof. Let $V = L$ and let ϕ be a complete second order sentence. If there were more than one non-isomorphic models of ϕ then there would be some model $\mathfrak{A}$ which is the $<_L$-least model of ϕ and some model $\mathfrak{B}$ of ϕ which is not isomorphic to the $<_L$-least model of ϕ. But now ϕ can not be complete because $\mathfrak{A}$ satisfies second order sentence "is isomorphic to the $<_L$-least model of ϕ" and $\mathfrak{B}$ does not.

We have proved earlier that if we add a Cohen-generic real G to L, we get second order equivalent non-isomorphic models $(F^G \cup \omega, <_\omega, P_G)$ and $(F^{-G} \cup \omega, <_\omega, P_{-G})$. In fact the models satisfy the same complete second order sentence. This sentence says: The universe of the model is $\omega \cup \{X \subseteq \omega : |X \cap -G| < \aleph_0\}$ where G is some L-generic subset of ω and there is also the natural order of ω and a relation which tells which elements of ω belong to which subsets of ω.

We will now show that models which satisfy same complete second order sentence have same cardinality. Assume not. Then there are models of different cardinalities which satisfy a complete second order sentence ϕ. Some of these models is of smallest cardinality where there is a model of ϕ and some others are not. Assume $\mathfrak{A}$ is a model of ϕ of the least possible cardinality and $\mathfrak{B}$ is a model of ϕ of some bigger cardinality. Now in $\mathfrak{B}$ the second order sentence "there is a model of ϕ which has cardinality less than cardinality of this model" is true and in $\mathfrak{A}$ it is false. Thus ϕ is not a complete second order sentence.

□

2.25 Definition. $\mathfrak{A} \preceq_{L^2} \mathfrak{B}$ means $\mathfrak{A}$ is second order elementary submodel of $\mathfrak{B}$. This means: $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ and for any second order formula $\phi(X_1, \dots X_n, x_1, \dots, x_m)$ and relations $A_1, \dots A_n \in A$ and elements $a_1, \dots, a_m \in A$,

if $\mathfrak{A} \models \phi(A_1, \dots A_n, a_1, \dots, a_m)$ then $\mathfrak{B} \models \phi(A_1, \dots A_n, a_1, \dots, a_m)$.

We are interested in the following question: is it possible to have two such models $\mathfrak{A}$ and $\mathfrak{B}$ that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \ncong \mathfrak{B}$ and $\mathfrak{A} \equiv_{L^2} \mathfrak{B}$. Clearly it is impossible to have $\mathfrak{A} \preceq_{L^2} \mathfrak{B}$ as the universe of $\mathfrak{A}$ satisfies the formula saying that every element belongs to it in $\mathfrak{A}$ but that is not the case in $\mathfrak{B}$. However, we will give such $\mathfrak{A}$ and $\mathfrak{B}$ that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \ncong \mathfrak{B}$ and $(\mathfrak{A}, a_1, \dots, a_n)$ and $(\mathfrak{B}, a_1, \dots, a_n)$ satisfy the same formulas of language of set theory for all first order parameters $a_1, \dots, a_n \in A$. This result is easy to get if one thinks models of empty vocabulary in different cardinalities, but we give an example were both models $\mathfrak{A}$ and $\mathfrak{B}$ have the same cardinality $\aleph_0$.

2.26 Theorem. *It is consistent with ZFC that there exist two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_0$ satisfying the following: $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \ncong \mathfrak{B}$ and $(\mathfrak{A}, a_1, \dots, a_n) \equiv_{ZF} (\mathfrak{B}, a_1, \dots, a_n)$ for all elements $a_1, \dots a_n \in A$.*

Proof. We force ω generic reals to the set theoretic universe. Forcing conditions are finite functions $f : \omega \times \omega \to \{0, 1\}$, and a forcing condition

p is stronger than a forcing condition q iff p extends q. Define $dom\mathfrak{A} = \bigcup_{i\in 6\mathbb{N}} dom\mathfrak{A}_i$, where $\mathfrak{A}_i$ is the Ajtai model constructed from the i:th generic real. Define $<^{\mathfrak{A}}=$ the natural order of ω. Define $P^{\mathfrak{A}} = \bigcup_{i\in 6\mathbb{N}} P^{\mathfrak{A}_i}$. Define similar way $dom\mathfrak{B} = \bigcup_{i\in 2\mathbb{N}} dom\mathfrak{A}_i$, $<^{\mathfrak{B}}=$ the natural order of ω and $P^{\mathfrak{B}} = \bigcup_{i\in 2\mathbb{N}} P^{\mathfrak{A}_i}$.

The models are not isomorphic because $\mathfrak{B}$ contains some subsets of ω which $\mathfrak{A}$ does not contain, and in an isomorphism every subset of ω is mapped to itself.

We claim that $(\mathfrak{A}, a_1, \ldots, a_n) \equiv_{ZF} (\mathfrak{B}, a_1, \ldots, a_n)$ for arbitrary elements $a_1, \ldots a_n \in \mathfrak{A}$. Suppose not: there is a forcing condition p and a formula ϕ such that $p \Vdash \phi((\dot{\mathfrak{A}}, \dot{a}_1, \ldots, \dot{a}_n)) \wedge \neg\phi((\dot{\mathfrak{B}}, \dot{a}_1, \ldots, \dot{a}_n))$. Let G be a generic filter which contains p. It is possible to construct another generic filter G' such that $V^G = V^{G'}$, $\dot{a}_1^{V^G} = \dot{a}_1^{V^{G'}}, \ldots, \dot{a}_n^{V^G} = \dot{a}_n^{V^{G'}}$ and $\dot{\mathfrak{B}}^{V^G} = \dot{\mathfrak{A}}^{V^{G'}}$. This is possible because the forcing condition p is finite. For those i which determine the elements $a_1, \ldots, a_n$ we let G and G' agree about everything. In the domain of p we let G and G' agree about everything. Otherwise we let G' produce in the indexes $6\mathbb{N}$ those generic reals which G produces in the indexes $2\mathbb{N}$, and in the indexes $\mathbb{N} \setminus 6\mathbb{N}$ those generic reals which G produces in the indexes $2\mathbb{N}+1$. Because of p it may be impossible to produce exactly the same generic reals, but it is possible to produce reals which are the same except in finitely many digits. However, finitely many digits do not make any difference to the model $\dot{\mathfrak{A}}^{V^{G'}}$ and we get $\dot{\mathfrak{B}}^{V^G} = \dot{\mathfrak{A}}^{V^{G'}}$. But now it can not be so that $p \Vdash \phi((\dot{\mathfrak{A}}, \dot{a}_1, \ldots, \dot{a}_n)) \wedge \neg\phi((\dot{\mathfrak{B}}, \dot{a}_1, \ldots, \dot{a}_n))$.

□

3 Generalizing Ajtai's result to uncountable cardinals

We have shown in Theorem 2.3 that it is consistent that second order equivalence implies isomorphism for models of cardinality $\aleph_0$. But is it consistent that second order equivalence implies isomorphism for models of cardinality $\aleph_1$? It is easy to show by a simple cardinality argument that second order equivalence does not necessarily imply isomorphism for models of cardinality $\aleph_1$:

There are $2^{\aleph_0}$ L^2-theories. There are $2^{\aleph_1}$ models which are not isomorphic to each other. It is clear that if $2^{\aleph_0} < 2^{\aleph_1}$, then there are second order equivalent non-isomorphic models of cardinality $\aleph_1$. However, if $2^{\aleph_0} = 2^{\aleph_1}$ we don't know what happens:

3.1 Open question. Is it consistent that $2^{\aleph_0} = 2^{\aleph_1}$ and second order equivalence implies isomorphism for models of cardinality $\aleph_1$?

In the first chapter we saw the result of Ajtai that it is independent of ZFC whether all countable models in any finite vocabulary can be characterized up to isomorphism by their second order theories. By some coding sentences of second order logic are natural numbers and second order theories are real numbers. Countable models are also real numbers, so the question whether all two different reals of latter type correspond to two different reals of former type is natural. We note that first order theories also correspond to real numbers but all countable models can not be characterized up to isomorphism by their first order theories.

All models of cardinality κ can be characterized up to isomorphism by a L_{κ^+,κ^+} sentence, "Scott sentence". However, these Scott sentences have the same cardinality as the model in question. In this article we are interested in the possibility of characterizing models up to isomorphism by theories, where the sentences have cardinality smaller than the model.

We mention the following things about possibility to characterize models up to isomorphism by infinitary languages. In cardinality ω of the models $L_{\omega_1,\omega}$-equivalence implies isomorphism. Generally $L_{\infty,\omega}$ equivalence is equivalent to the existence of a back-and-forth set. Back-and-forth-equivalence implies isomorphism only in cardinality ω so $L_{\infty,\omega}$ is not good in characterizing uncountable models up to isomorphism. Nadel and Stavi [15] have investigated logics $L_{\infty,\lambda}$ and showed that these are not successfull in characterizing all models up to isomorphism in cardinality λ, where λ is an uncountable successor cardinal.

Thus infinitary languages are not good in characterizing all models up to isomorphism in uncountable cardinal λ, if we don't allow the infinitary language to have sentences of cardinality λ. Higher order languages are also not succesfull. As they have only continuum many sentences they cannot characterize all models up to isomorphism in a cardinality which has more than continuum many models.

We will introduce certain infinitary second order languages $L^2_{\kappa,\omega}$ for all regular cardinal κ and prove that it is independent of ZFC whether all models of cardinality κ in any finite vocabulary can be characterized up to isomorphism by $L^2_{\kappa,\omega}$ theories. Sentences of $L^2_{\kappa,\omega}$ correspond to subsets of cardinals $\lambda < \kappa$ so this logic is not "too strong". We decided to formulate our theorems for $L^2_{\kappa,\omega}$ because it is a well-known logic. However, most of our results hold equally well for a fragment of $L^2_{\kappa,\omega}$ which contains atomic formulas, in which ordinals smaller than κ are definable and which is closed under second order quantifiers first order quantifiers and finite connectives.

3.2 Definition. Let $n \in \omega$ and let κ be a regular cardinal. The logic $L^2_{\kappa,\omega}$ is the smallest logic which

1. Contains all atomic formulas,
2. Is closed under negation, conjunctions of size less than κ, disjunctions of size less than κ, first order existential and universal quantifiers and second order existential and universal quantifiers.

3.3 Definition. We call the following *Ajtai Hypothesis* for a regular cardinal κ: In any finite vocabulary, if two models of cardinality κ satisfy the same $L^2_{\kappa,\omega}$-theory then they are isomorphic.

3.4 Lemma. *Let κ be a regular cardinal. In the logic $L_{\kappa,\omega}$ all ordinals smaller than κ are definable.*

Proof. This is done by induction on the ordinal $\alpha < \kappa$. Assume the Induction Hypothesis holds for all $\beta < \alpha$ i.e. there are formulas $\theta_\beta(y)$ which define ordinals $\beta < \alpha$. Now the formula which defines α is

$$\bigwedge_{\beta<\alpha} \exists y(y < x \wedge \theta_\beta(y)) \wedge \forall y(y < x \rightarrow \bigvee_{\beta<\alpha} \theta_\beta(y))$$

□

3.5 Remark. Many cardinality quantifiers are expressible in second order logic. In the logics $L^2_{\kappa^+,\omega}$ even more cardinality quantifiers are expressible. If an ordinal α is definable in second order logic (or in infinitary second order logic) then the quantifiers $\exists^{\geq\aleph_\alpha}$ and $\exists^{\aleph_\alpha}$ are definable in second order logic (or infinitary second order logic). Consequently if κ is a regular cardinal and $\alpha < \kappa$ is an ordinal, the quantifiers $\exists^{\geq\aleph_\alpha}$ and $\exists^{\aleph_\alpha}$ are definable in the logic $L^2_{\kappa,\omega}$.

Assume now α is definable in second order logic (or in infinitary second order logic). We will introduce a sentence which defines the quantifier $\exists^{\aleph_\alpha}$:

The sentence says that there are U, V, c and $<$ such that the following hold:

1. The relation $<$ defines a well-order in the unary predicate U
2. Unary predicate V contains those elements x satisfying the following:
   - $x \in U$
   - x has infinitely many predecessors
   - For all $y < x$: $|\{z : z < y\}| < |\{z : z < x\}|$.

3. c is the greatest element of V and $(V \setminus \{c\}, <\restriction V \setminus \{c\}) \cong \alpha$

4. $\exists \pi$ (π is a bijection from $\{y : y < c\}$ to $\{y : \phi(y)\}$).

When an ordinal α is given, 1.-3. define the cardinal $\aleph_\alpha$. Finally 4. says that there is a bijection from this cardinal to those elements which satisfy the formula ϕ. Thus this sentence is equivalent to $\exists^{\aleph_\alpha}\phi$. By replacing bijection by injection in 4. we get a sentence equivalent to $\exists^{\geq\aleph_\alpha}\phi$.

3.6 Theorem. *If κ is a regular cardinal and there is a second order definable well-order of the powerset of κ, then Ajtai Hypothesis holds at κ. In particular Ajtai Hypothesis holds at κ if $V = L$.*

Proof. As before in theorem 2.3, a model can be coded into a n-ary relation $R \subseteq \kappa^n$. For all n-tuples of ordinals smaller than κ we can say whether the tuple belongs to or does not belong to the least subset of κ^n which is isomorphic with the model in the well-order. The canonical well-order of L up to sets of cardinality κ is second order definable in any cardinality κ.

□

Thus second order definable well-order of the powerset of a regular cardinal κ implies Ajtai Hypothesis holds at κ. Whether the implication holds also to the other direction we don't know:

3.7 Open question. Are the following conditions equivalent?

1. There is a second order definable well-order of the powerset of κ.

2. Ajtai Hypothesis holds at cardinality κ.

In Theorem 3.6 we saw that in L Ajtai Hypothesis holds at any regular cardinal as there is a second order definable well-order of the powerset of κ. In fact we will get a better result:

3.8 Theorem. *Let κ be a regular cardinal and let $H(\kappa^+) \subset L[X]$ for some set X with $X \subseteq \lambda < \kappa$. Then Ajtai Hypothesis holds at cardinality κ.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of cardinality κ. By assumption $\mathfrak{A}$ and $\mathfrak{B}$ belong to $L[X]$ and hence are isomorphic to some sets in $L[X]$. In the infinitary second order language $L^2_{\kappa,\omega}$ we can talk about the least subset of κ^n in the canonical well-order of $L[X]$ which is isomorphic to $\mathfrak{A}$. We will now describe how this is done.

In the logic $L^2_{\kappa,\omega}$ all ordinals $\alpha < \kappa$ are definable by certain formulas θ_α. Now the set X is definable by the formula

$$X \subseteq \lambda \wedge \bigwedge_{\alpha \in X} \exists x(\theta_\alpha(x) \wedge x \in X) \wedge \bigwedge_{\alpha \notin X} \exists x(\theta_\alpha(x) \wedge x \notin X).$$

In the formula above we define the ordinal λ in the logic $L^2_{\kappa,\omega}$ as we say for all $\alpha < \lambda$ whether it belongs to X or not. We denote this formula which defines X by ϕ_X.

If the set X and an ordinal $\alpha < \kappa^+$ are given, the α:th level of the sets constructible from X is second order definable from these parameters. Also the canonical well-order of $L_\alpha[X]$ is second order definable from X and α. Let $\phi_{L_\alpha[X]}(Y, X, \alpha)$ be a second order formula which says that Y is the α:th level of the sets constructible from X and let $\phi_{<_{L_\alpha[X]}}(Z, X, \alpha)$ be a second order formula which says that Z is the canonical well-order of the α:th level of the sets constructible from X.

As usual, we assume that the model in question has been coded into a n-ary relation R. We are interested in sentences of the following form:

There are X, α, $L_\alpha[X]$, $<_{L_\alpha[X]}$ and R_0 such that the following hold:

1. $\phi_X(X)$
2. α is an ordinal
3. $\phi_{L_\alpha[X]}(L_\alpha[X], X, \alpha)$
4. $\phi_{<_{L_\alpha[X]}}(<_{L_\alpha[X]}, X, \alpha)$
5. $R_0 \in L_\alpha[X] \wedge R_0 \cong R \wedge \forall R_1((R_1 \in L_\alpha[X] \wedge R_1 \cong R) \rightarrow R_0 \leq_{L_\alpha[X]} R_1)$
6. $(\alpha_1, \ldots, \alpha_n) \in R_0$

The first four formulas say that α is an ordinal and the sets X, $L_\alpha[X]$ and $<_{L_\alpha[X]}$ are what they are supposed to be. The fifth formula says that R_0 belongs to $L_\alpha[X]$ and it is the least model in the canonical well-order of $L_\alpha[X]$ which is isomorphic to the model in question. The sixth formula says that a tuple $(\alpha_1, \ldots, \alpha_n)$ belongs to R_0. Similarly we can say that a tuple does not belong to R_0.

If two models of cardinality κ are now $L^2_{\kappa,\omega}$-equivalent, then they satisfy all same sentences of the form above. Thus they have same set R_0 and consequently they are isomorphic.

□

3.9 Corollary. *It is consistent that there is a measurable cardinal κ and Ajtai Hypothesis holds everywhere above the cardinality of the powerset of κ.*

Proof. There is a model of ZFC [16] such that there is a measurable cardinal κ and every set is constructible from a certain subset of the powerset of κ. $\square$

We will now state without a proof a lemma which is needed to show the indenpendence of Ajtai Hypothesis at a regular cardinal κ. In definition 4.7 we will give exact definition of $L^2_{\kappa,\omega}$-formulas as set theoretic objects and prove the lemma.

3.10 Lemma. *Let $n \in \omega$. Every formula of $L^2_{\kappa^+,\omega}$ can be defined by a ZF-formula using a subset of κ as a parameter .*

If κ is an inaccessible cardinal, every formula of $L^2_{\kappa,\omega}$ can be defined by a ZF-formula using a subset of some $\lambda < \kappa$ as a parameter .

3.11 Theorem. *Let κ be a regular cardinal. It is consistent with ZFC that there are two ZF-equivalent non-isomorphic models of cardinality κ. The models are also $L^2_{\kappa,\omega}$-equivalent for all n.*

Proof. We add a Cohen-generic subset G to κ. The forcing conditions are mappings of cardinality smaller than κ from κ to $\{0,1\}$. We define the model $(F^G \cup \kappa, <_\kappa, R_G)$. Here F^G is the set of all subsets of κ which agree with G everywhere except in a set of cardinality smaller than κ, $<_\kappa$ is the natural order of κ and R_G is a relation which tells which elements of κ belong to which sets in F^G. The model $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ is defined in an analogous way.

We note that this forcing is $< \kappa$-closed so it does not add any new subsets to cardinals smaller than κ. If κ is inaccessible, all cardinals below κ are preserved and κ remains inaccessible.

We also note that no forcing condition can determine the model $(F^G \cup \kappa, <_\kappa, R_G)$ in any way, as a forcing condition defines the value of G only in a subset of κ which has cardinality less than κ. This means: for any forcing condition p there are two generic filters G and G' containing p such that

$$V^G = V^{G'}, \quad \dot{(}F^G \cup \kappa, <_\kappa, R_G)^{V^G} = \dot{(}F^{-G} \cup \kappa, <_\kappa, R_{-G})^{V^{-G}}$$

and

$$\dot{(}F^G \cup \kappa, <_\kappa, R_G)^{V^{-G}} = \dot{(}F^{-G} \cup \kappa, <_\kappa, R_{-G})^{V^G}.$$

Thus the models $(F^G \cup \kappa, <_\kappa, R_G)$ and $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ are ZF-equivalent with parameters from the ground model. As the forcing does not add any new subsets to any cardinals smaller than κ, the models are by previous lemma $L^2_{\kappa,\omega}$-equivalent. But they are not isomorphic: well-ordered structure of κ is rigid, so every subset of κ would be mapped in an isomorphism to itself. However $G \in (F^G \cup \kappa, <_\kappa, R_G)$ and $G \notin (F^{-G} \cup \kappa, <_\kappa, R_{-G})$, so there is no isomorphism. $\square$

3.12 Theorem. *Let κ be a regular cardinal. It is consistent with ZFC that in all cardinalities $\lambda \geq \kappa$ there are two ZF-equivalent non-isomorphic models of cardinality λ.*

Proof. We have proved that adding a Cohen subset to a regular cardinal κ produces two models of cardinality κ which are non-isomorphic but satisfy same formulas of set theory with parameters from ground model. In fact, Cohen subsets produce such models in all cardinalities $\lambda \geq \kappa$. This is because we can extend the universes of the models defined in the Theorem 3.11 by adding λ new elements and putting them to some new unary relation. These new models can be constructed from the models introduced in Theorem 3.11 and the term λ, and thus they are ZF-equivalent. □

We have proved that it is independent of ZFC whether Ajtai Hypothesis holds at a regular cardinal κ or not. It happens that these are also relatively independent of each other, as the following theorem demonstrates.

3.13 Theorem. *Let J be a finite set of regular cardinals. It is consistent that Ajtai Hypothesis fails in J and holds at every regular cardinal outside J.*

Proof. We start from L and use iterated forcing to add Cohen subsets to all cardinals in J, adding a Cohen subset first to the smallest cardinal in J and proceeding this way from down to up. We note that GCH holds in L and adding a single Cohen subset preserves GCH so GCH is preserved all the way through our forcing. Also cardinals are preserved. Let κ be a cardinal in J. It follows from Factor Lemma (see for example [6], Lemma 21.8) that the iterated forcing can be decomposed into $P_{<\kappa} * P_\kappa * P_{>\kappa}$ as follows. $P_{<\kappa}$ preserves GCH and cardinals and P_κ adds a Cohen subset to κ. Thus after $P_{<\kappa} * P_\kappa$ we have GCH, cardinals are preserved and Ajtai Hypothesis fails at κ. $P_{>\kappa}$ is κ closed and thus does not add any subsets to cardinals smaller than or equal to κ. Consequently, $P_{>\kappa}$ does not chance the truth of Ajtai Hypothesis, which is false at κ after the forcing.

Let now $\kappa \notin J$. Our forcing can be decomposed to $P_{<\kappa} * P_{>\kappa}$. $P_{<\kappa}$ adds some Cohen subsets below κ and $P_{>\kappa}$ adds subsets only to cardinals greater than κ. Thus after the forcing $H(\kappa^+) \subseteq L[X]$ for some $X \subseteq \lambda < \kappa$ and from Theorem 3.8 it follows that Ajtai Hypothesis holds at κ. □

3.14 Theorem. *Let J be a set which contains some successor cardinals and possibly ω. It is consistent that Ajtai Hypothesis fails at all cardinals which belong to J, and holds at all successor cardinals outside J and at all inaccessible cardinals which do not have a cofinal subset in J.*

Proof. Let L be the ground model. We make an iterated forcing with full support in all limit stages, which proceeds from down up and adds Cohen subsets to all cardinals in J. Menas has written an article about forcing of this type and calls it *backward Easton forcing* [13].

The forcing conditions are as follows:

1. If $\omega \in J$, then P_0 is the set of finite partial functions from ω to $\{0,1\}$. A forcing condition p is stronger than forcing condition q if and only if p extends q. If $\omega \notin J$, then P_0 is the trivial forcing.

2. Assume $\alpha = \beta^+$ and P_γ has been defined for all $\gamma \leq \beta$.

   If $\aleph_\alpha \in J$, we define P_α to be the set of sequences $p_\gamma, \gamma \leq \alpha$ where the γ:th coordinate belongs to P_γ for each $\gamma < \alpha$ and the α:th coordinate is a forcing name $\dot{X}$ such that $p \restriction \alpha \Vdash \dot{X}$ is a partial function from $\aleph_\alpha$ to $\{0,1\}$ and $|\dot{X}| < \aleph_\alpha$. If p and q are two iterated forcing notions of length α then p is stronger than q if and only if $p \restriction \alpha$ is stronger than $q \restriction \alpha$ and $p \restriction \alpha \Vdash p(\alpha)$ and $q(\alpha)$ are partial functions from $\aleph_\alpha$ to $\{0,1\}$ which have cardinality smaller than $\aleph_\alpha$ and $p(\alpha) \supseteq q(\alpha)$.

   If $\aleph_\alpha \notin J$ then P_α is the trivial forcing.

3. If α is a limit ordinal, forcing conditions in P_α are tuples p of length α such that for each $\beta < \alpha \quad p \restriction \beta \Vdash p(\beta) \in P_\beta$. This forcing has full support in all limit stages, which means that in limit stages all coordinates of a forcing condition may be non zero. A forcing condition p is stronger than q if and only if $p \restriction \beta$ is stronger than $q \restriction \beta$ for each $\beta < \alpha$.

We will inductively show that for all cardinals κ the following conditions will hold after the forcing:

1. κ remains a cardinal.

2. If κ is ω or a successor cardinal, Ajtai Hypothesis fails at κ iff $\kappa \in J$. If κ is inaccessible cardinal and there is no cofinal subset of κ in J then Ajtai Hypothesis holds at κ.

3. Generalized Continuum Hypothesis holds up to cardinal κ.

Let us assume Induction Hypothesis holds for all cardinals below κ. By Factor Lemma [6] the forcing can be decomposed into parts:

$$P_{<\kappa} * P_\kappa * P_{>\kappa}$$

in such a way that after the forcing $P_{<\kappa}$ Induction Hypothesis holds below κ and if $\kappa \in J$, then P_κ adds a Cohen subset to κ and if $\kappa \notin J$ P_κ is the trivial forcing. $P_{>\kappa}$ is κ^+-closed, so it does not make any chance to Induction Hypothesis in cardinals less or equal to κ.

If $\kappa \in J$ then the Cohen forcing makes Ajtai Hypothesis false at κ and adding a single Cohen subset does not make GCH false at κ.

If $\kappa \notin J$, empty forcing does not make GCH false at κ. Also $H(\kappa^+) \subseteq L(X)$ for $X \subseteq \lambda < \kappa$ which codes all the previously added generic subsets, so from Theorem 3.8 it follows that Ajtai Hypothesis holds at κ.

We still need to show that GCH is preserved at limit cardinals.

1. λ is a singular limit cardinal. From Induction Hypothesis we know that GCH holds below λ. Because our ground model was L and the failure of singular cardinal hypothesis implies $0^\sharp$ exists, so after our forcing it can't be that $\neg SCH(\lambda)$. Thus SCH (λ). Now λ is a strong limit cardinal so $2^\lambda = \lambda^{cf(\lambda)} =_{SCH} \lambda^+$.

2. Let κ be an inaccessible cardinal. All subsets of κ in V^G are constructible from a single set of cardinality κ which codes all the generic sets added below κ. Thus the powerset of κ has cardinality κ^+.

□

3.15 Remark. If we allow J to be a proper class in the assumption of Theorem 3.14, the Theorem seems still to be valid. Then we just need to use a proper class of forcing conditions and the length of the iteration is a proper class.

In this chapter we have already given generalization of Ajtai's result to regular cardinals. Next we will turn our attention to the case of singular cardinals. For the case of regular cardinals the languages $L^2_{\kappa,\omega}$ had important role. For the singular cardinals κ we introduce a language which has same role as the languages $L^2_{\kappa,\omega}$ had for regular cardinals κ:

3.16 Definition. Let κ be a singular cardinal. Let $L^2_\kappa = \bigcup_{\lambda<\kappa} L^2_{\lambda^+,\omega}$.

3.17 Definition. We call the following *Ajtai Hypothesis* for a singular cardinal κ: In any finite vocabulary, if two models of cardinality κ satisfy the same L^2_κ-theory then they are isomorphic.

Note that the set of L_κ formulas is closed under normal finitary first order connectives and quantifiers, but not under conjunctions or disjunctions of length $cf(\kappa)$.

Two important facts about the languages L_κ are the following:

1. Every ordinal $\alpha < \kappa$ is definable in L_κ.

2. Every formula of L_κ can be expressed as formula of language of set theory using a subset of some $\lambda < \kappa$ as a parameter.

3.18 Theorem. *If $V = L$ then L^2_κ-equivalence implies isomorphism in any singular cardinal κ.*

Proof. We have showed before in Theorem 3.6 that if $V = L$ then all $L^2_{\kappa,\omega}$-equivalent models of cardinality κ are isomorphic for any regular cardinal κ. Because all ordinals less than κ are definable in L^2_κ, the proof we used there works without any changes for L^2_κ. □

3.19 Theorem. *Let $\kappa = \aleph_\alpha$ be a singular cardinal. It is consistent that Ajtai Hypothesis fails at κ.*

Proof. As in Theorem 3.14, we use the full support iterated Cohen forcing. This time we add generic subsets to all regular cardinals smaller than κ.

Recall that for each regular $\aleph_\beta < \kappa$ our forcing creates two models M^G_β and M^{-G}_β of cardinality $\aleph_\beta$ which are $L^2_{\aleph_\beta,\omega}$-equivalent and non-isomorphic. We define the models M^G_κ and M^{-G}_κ as follows:

M^G_κ contains the α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the β:th coordinate is either M^G_β or M^{-G}_β,

2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the β:th coordinate is $\emptyset$,

3. The set of indexes β where the β:th coordinate is M^{-G}_β is not cofinal in α.

Similarly we define M^{-G}_κ to contain those α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the β:th coordinate is either M^G_β or M^{-G}_β,

2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the β:th coordinate is $\emptyset$,

3. The set of indexes β where the β:th coordinate is M^G_β is not cofinal in α.

Clearly the models are non-isomorphic as there is no sequence in M^{-G}_κ which could be mapped to the sequence in M^G_κ which contains only the models M^G_β.

We will now prove that the models are L^2_κ-equivalent. Assume not. Then there is a forcing condition p such that $p \Vdash \dot\phi \in L^2_\kappa \wedge \dot\phi(\dot{M^G}_\kappa) \wedge \neg\dot\phi(\dot{M^{-G}}_\kappa)$

for some forcing name $\dot{\phi}$. Thus there is some generic filter G such that $p \in G$ and $V^G \models \phi(M^G) \wedge \neg\phi(M^{-G})$. The sentence ϕ is a sentence in the language of set theory with a subset of some $\aleph_{\gamma^+} < \kappa$ as a parameter.

We will now construct another generic filter G' which contains p and $\dot{\phi}^{V^G} = \dot{\phi}^{V^{G'}}$. The elements of G' are made from elements of G by the following modification:

1. Up to stage γ^+ (where the formula ϕ appears) no modification is done.
2. In the the domain of p no modification is done.
3. Above stage γ^+ outside the domain of p the forcing condition is chanced to its mirror image, i.e. the domain remains the same but zeros and ones chance places.

Clearly $p \in G'$. Also up to stage γ^+ G' and G agree about everything, so $\dot{\phi}^{V^G} = \dot{\phi}^{V^{G'}}$. After stage γ^+ G' adds essentially complement of those sets G adds to all regular cardinals. There is difference only in the domain of p which is always of a smaller cardinality. In particular $M_\beta^G = M_\beta^{-G'}$ and $M_\beta^{-G} = M_\beta^{G'}$ for all $\gamma^+ < \beta < \alpha$. Also $V^G = V^{G'}$. Now $\dot{M^G}^{V^{G'}} = M^{-G}$ and $\dot{M^{-G}}^{V^{G'}} = M^G$ i.e. the models chance places in the generic extensions. However, the formula ϕ is the same and $V^G = V^{G'}$ so ϕ can not be true in one model and false in the other. □

3.20 Corollary. *Assuming the consistency of an inaccessible cardinal, there is a model of ZFC in which Ajtai Hypothesis fails at all cardinals.*

Proof. We start from a model of ZFC which satisfies $V = L$ and there is an inaccessible cardinal. Let κ be the least inaccessible cardinal in that model. We proceed from down to up and add by iterated Cohen forcing generic subsets to all regular cardinals smaller than κ. At limit stages we take full support. After the forcing Ajtai Hypothesis fails at all cardinals smaller than κ and κ remains inaccessible. Thus $V_\kappa^{V^G}$ satisfies ZFC and Ajtai Hypothesis fails at every cardinal.

□

Ajtai's original proof did not only show independence of Ajtai Hypothesis, but it showed independence of whether n:th order equivalence implies isomorphism for countable models for arbitrary $n \geq 2$. This is also true for the generalization of Ajtai's result to arbitrary regular cardinals, which we presented earlier in this chapter. When we use iterated forcing and add

Cohen subsets first to smaller cardinals and then to bigger cardinals, adding Cohen subsets to bigger cardinals does not change (infinitary) second order equivalence of models at smaller cardinals. However, it might change (infinitary) higher order equivalence of models for some stronger higher order logics. The following question is an example about the problem:

3.21 Open question. Let P be an iterated forcing which adds first a Cohen subset to $\aleph_0$ and then a Cohen subset to $\aleph_1$. Let M_0^G and M_0^{-G} be the usual models constructed from the generic set and its complement in cardinality $\aleph_0$. Are the models M_0^G and M_0^{-G} third order equivalent after the forcing?

4 Ajtai Hypothesis and large cardinal axioms

In this chapter we will discuss how some large cardinal axioms are related to Ajtai Hypothesis. First we will discuss consistency of some large cardinal axioms with second order definable well-orders of the reals. Then we will give an exact definition of $L^2_{\kappa,\omega}$-formulas as set theoretic objects and show that the sentence "There are two $L^2_{\kappa,\omega}$-equivalent non-isomorphic models of cardinality κ" is Σ^2_1. We will also show that if we have enough large cardinals then Ajtai Hypothesis is false. In the end we will discuss third order definable well-orders of the reals.

From the proof of Theorem 2.3 and some facts about the consistency of well-orders of the reals with large cardinals we get the following results:

4.1 Theorem. *It is consistent that there is a measurable cardinal and Σ^1_4-equivalence implies isomorphism for countable models. It is consistent that there are n Woodin cardinals and Σ^1_{n+3}-equivalence implies isomorphism for countable models. The above results are of course relative to consistency of the relevant large cardinal axioms.*

Proof. The existence of a measurable cardinal with a Δ^1_3 well-order of the reals is consistent [16]. Also for each natural number n it is consistent to have n Woodin cardinals and Σ^1_{n+2} well-order of the reals [12]. From Theorem 2.3 it follows that it is consistent that there are n Woodin cardinals and Σ^1_{n+3}-equivalence implies isomorphism for countable models. Also it is consistent that there is a measurable cardinal and Σ^1_4-equivalence implies isomorphism for countable models. □

4.2 Lemma. *It is possible to code all finite vocabularies as natural numbers by some Gödel numbering.*

Proof. Divide the set of prime numbers to infinitely many infinite parts P_n in some second order definable way. Then take a countably infinite set of constants and a countably infinite set of relation and function symbols of each arity, and assign in some second order definable way a different prime number code for any symbol. Now a finite vocabulary can be coded as the number which we get if we multiply all codes of the symbols in the vocabulary with each other. □

4.3 Lemma. *Given a finite vocabulary σ, the set of L^2 terms in vocabulary σ are second order definable. Also the set of free variables in a L^2-term is second order definable. Given a model $\mathfrak{A}$ in vocabulary σ, a L^2-term t and an assignment of L^2-variables s which contains the free variables of t in its domain, the interpretation of term $t_s^{\mathfrak{A}}$ is second order definable.*

Proof. We define a rank for $L^2(\sigma)$-terms.

1. Constants and variables have rank 0.

2. If rank of terms $t_1, \ldots, t_n$ have been defined and F is a n-ary fuction symbol in σ or n-ary second order function variable then rank $F(t_1, \ldots t_n)$ is sup{rank (t_i) +1: $1 \leq i \leq n$}.

A set t is a $L^2(\sigma)$ term iff there is a set X such that $t \in X$ and every set in X is either a $L^2(\sigma)$-term of rank 0 or is a result of applying a function in σ or a second order function variable to sets in X. This can be said in second order logic.

For a $L^2(\sigma)$-term t define X' to be the smallest set which satisfies the definition. X' is the set of subterms of t and it is definable in second order logic. Once we have X' defined, we can define the rank for terms in X' and by induction on rank define the free variables of all subterms and interpretations of subterms with a given assignment. □

4.4 Lemma. *Ajtai Hypothesis at cardinality ω is true in V if and only if it is true in $L[\mathbb{R}]$.*

Proof. We define inductively rank for $L^2_{\kappa,\omega}$ formulas ϕ,

1. rank(ϕ)=0 for atomic ϕ.

2. rank($\bigwedge \Psi$)= rank($\bigvee \Psi$) = sup { rank (ϕ)+1 : $\phi \in \Psi$}

3. If $\phi = \neg\psi, \phi = \exists x_n\psi, \phi = \forall x_n\psi, \phi = \exists X_i^n\psi, \phi = \forall X_i^n\psi, \phi = \exists F_i^n\psi$ or $\phi = \forall F_i^n\psi$ then rank(ϕ)=rank(ψ) +1.

In case of second order formulas, conjunctions and disjunctions are of length 2 and rank is always finite. Generally $L^2_{\kappa,\omega}$ formulas have rank less than κ.

Given a finite vocabulary τ, second order formulas in vocabulary τ are inductively definable in similar way as terms in Lemma 4.3. Also for any $L^2(\tau)$-formula the set of its subformulas is definable. In this set we can define rank for all subformulas, and by induction on rank the set of free variables in a given subformula. An interpretation for finitely many first order and second order variables in a countable model is a real number. Consequently every interpretation which exists in V exists in $L[\mathbb{R}]$. The truth predicate for a countable model $\mathfrak{A}$ i.e. the set of ordered tuples $\langle \phi, s \rangle$ such that $\mathfrak{A} \models_s \phi$ is inductively definable. This means that the truth predicate for formulas of rank 0 is definable and if the truth predicate for formulas of rank $< n$ is defined then it is definable also for formulas of rank n. Finally the truth predicate is definable as union of these "partial truth predicates". The truth predicate is definable in $L[\mathbb{R}]$ because it is an inductive definition and its existence is provable from ZF. Axiom of Choice may be false in $L[\mathbb{R}]$ but it is not needed. Also the truth predicate of V equals truth predicate of $L[\mathbb{R}]$ because they are determined by the reals and V and $L[\mathbb{R}]$ have the same reals.

Let us now look at the sentence which says that countable Ajtai Hypothesis is false: $\exists \tau \exists \mathfrak{A} \exists \mathfrak{B} \exists \Pi_{\mathfrak{A}} \exists \Pi_{\mathfrak{B}}(\mathfrak{A}$ and $\mathfrak{B}$ have vocabulary τ and $|\mathfrak{A}| = |\mathfrak{B}| = \omega$ and $\neg \mathfrak{A} \cong \mathfrak{B}$ and $\Pi_{\mathfrak{A}}$ is a truth predicate of second order formulas for $\mathfrak{A}$ and $\Pi_{\mathfrak{B}}$ is a truth predicate of second order formulas for $\mathfrak{B}$ and $\Pi_{\mathfrak{A}}$ and $\Pi_{\mathfrak{B}}$ contain exactly the same sentences).

If the sentence is true in one of V and $L[\mathbb{R}]$ then all the sets witnessing the truth of the sentence exist also in the other. Thus the sentence is also true in the other and the claim follows. □

The proof of the next lemma can be found from Larson's book [9].

4.5 Lemma. *If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ, then no forcing construction in V_δ can chance the theory of $L[\mathbb{R}]$.*

4.6 Theorem. *If there is a measurable cardinal above a limit of Woodin cardinals then Ajtai Hypothesis is false.*

Proof. Assume there is a measurable cardinal above a limit of Woodin cardinals. We add a Cohen generic real G to V as in Theorem 2.13. Now Ajtai Hypothesis is false in $V[G]$. By Lemma 4.4 Ajtai Hypothesis is false in $L[\mathbb{R}]^{V[G]}$. By assumption and Lemma 4.5 Ajtai Hypothesis is false in $L[\mathbb{R}]^V$ and by Lemma 4.4 Ajtai Hypothesis is false in V.

□

We will now give exact definition of $L^2_{\kappa,\omega}$-formulas as set theoretic objects and prove Lemma 3.10.

4.7 Definition. We will introduce certain coding where all $L^2_{\kappa,\omega}$ formulas are coded by subsets of κ, or in fact by subsets of ordinals smaller than κ. First the atomic formulas:

1. A symbol in the vocabulary of the model which has been assigned prime number code n as described in Lemma 4.2 is $\langle 1, n\rangle$
2. $x_\alpha = \langle 2, \alpha\rangle$
3. $c_\alpha = \langle 3, \alpha\rangle$
4. $R^n_\alpha = \langle 4, n, \alpha\rangle$ These are the codes for relation variables
5. $F^n_\alpha = \langle 5, n, \alpha\rangle$ These are the codes for function variables
6. $F^n_i(t_1, \ldots, t_n) = \langle 6, F^n_i, t_1, \ldots, t_n\rangle$.
7. $t_i \equiv t_j = \langle 7, t_i, t_j\rangle$
8. $R^n_i(t_1, \ldots, t_n) = \langle 8, R^n_i, t_1, \ldots, t_n\rangle$.

We describe now how to code objects of the form above by subsets of κ in a systematic way. There is a second order definable bijection from κ to $\kappa \times \kappa$. The objects are coded in such a way that the n:th κ codes the n:th co-ordinate in the tuple. For example $c_{\omega+1}$ has in the beginning of the first κ three ones and the rest are zeros. In the beginning of the second κ it has $\omega+1$ ones and the rest are zeros, and all the other κ:s have just zeros. $F^1_1(c_{\omega+1})$ has 6 ones in the beginning of the first κ, code of F^1_1 in the second κ and in the third κ the subset of κ coding $c_{\omega+1}$ we just described.

By this coding the predicates "X is a $L^2_{\kappa,\omega}(\tau)$ term" and "X is a $L^2_{\kappa,\omega}(\tau)$ atomic formula" are definable in second order logic.

The non-atomic formulas are coded as follows:

1. $\neg\phi = \langle 9, \{\phi\}\rangle$
2. $\bigwedge X = \langle 10, X\rangle$
3. $\bigvee X = \langle 11, X\rangle$
4. $\exists x_\alpha \phi = \langle 12, x_\alpha, \{\phi\}\rangle$
5. $\forall x_\alpha \phi = \langle 13, x_\alpha, \{\phi\}\rangle$

6. $\exists R^n_\alpha \phi = \langle 14, R^n_\alpha, \{\phi\}\rangle$

7. $\forall R^n_\alpha \phi = \langle 15, R^n_\alpha, \{\phi\}\rangle$

8. $\exists F^n_\alpha \phi = \langle 16, F^n_\alpha, \{\phi\}\rangle$

9. $\forall F^n_\alpha \phi = \langle 17, F^n_\alpha, \{\phi\}\rangle$

These are coded by subsets of κ similar way as atomic formulas except that there are also sets of formulas. For example objects of type 2 are coded by 10 ones in the first κ and in the second κ a code for the set X. X is a set of formulas of size less than κ and we use again the second order definable bijection from κ to $\kappa \times \kappa$. The first κ codes some formula in X, the second κ codes some other formula in X, if there is any, and so on until after some α many κ:s there are no more formulas in X and the rest are just zeros.

We have defined $L^2_{\kappa,\omega}$-formulas as subsets of κ but in fact in all subsets coding a $L^2_{\kappa,\omega}$-formula the set of ones is not cofinal in κ. Thus we can as well think them as subsets of some $\alpha < \kappa$ when we have cut away the zeros from the end. This proves Lemma 3.10.

We note that proof of Lemma 4.3 works also for $L^2_{\kappa,\omega}(\sigma)$-terms in cardinality κ.

4.8 Lemma. *Given a finite vocabulary σ the relation "X is a $L^2_{\kappa,\omega}(\sigma)$-sentence" is second order definable in a model of cardinality κ.*

Proof. The second order sentence says that there is a set $Y = Y_1 \cup Y_2$ containing X such that every set in Y_1 is either $L^2_{\kappa,\omega}(\sigma)$-term of rank 0 or is a result of applying functions in σ or second order function variables to elements in Y_1. Also every element in Y_2 is either $L^2_{\kappa,\omega}$ atomic formula or is formed from other sets in Y by operations described in Definition 4.7 and Y is the smallest set satisfying this definition. By this definition Y is the set of subformulas and subterms of X. The sentence says further that there is a function F which maps all elements of Y to the set of their free variables and F maps X to $\emptyset$.

□

4.9 Definition. Let $\mathfrak{A}$ be a model and τ be a finite vocabulary. The truth predicate T for the logic $L^2_{\kappa,\omega}(\tau)$ in the model $\mathfrak{A}$ is a binary relation. As elements it has ordered pairs of $L^2_{\kappa,\omega}(\tau)$-formulas and interpretations of less than κ many variables of $L_{\kappa,\omega}(\tau)$ in the model $\mathfrak{A}$ satisfying the following conditions:

1. If t_i and t_j are $L^2_{\kappa,\omega}(\tau)$ terms and variables of t_i and t_j belong to the domain of an interpretation s then $\langle t_i = t_j, s\rangle \in T$ if and only if ${t_i}^{\mathfrak{A}}_s = {t_j}^{\mathfrak{A}}_s$.

2. If R is a n-ary relation symbol in τ and $t_1, \ldots t_n$ are $L^2_{\kappa,\omega}(\tau)$-terms such that their variables belong to the domain of s then $\langle R(t_1, \ldots t_n), s\rangle \in T$ if and only if $\langle {t_1}^{\mathfrak{A}}_s, \ldots {t_n}^{\mathfrak{A}}_s\rangle \in R^{\mathfrak{A}}$.

3. If X is an n-ary relation variable and $t_1, \ldots t_n$ are $L^2_{\kappa,\omega}$-terms such that their variables belong to the domain of s $\langle X(t_1, \ldots t_n), s\rangle \in T$ if and only if $\langle {t_1}^{\mathfrak{A}}_s, \ldots {t_n}^{\mathfrak{A}}_s\rangle \in X^{\mathfrak{A}}_s$.

4. $\langle \neg\phi, s\rangle \in T$ if and only if $\langle \phi, s\rangle \notin T$.

5. If Ψ is a set of $L^2_{\kappa,\omega}(\tau)$-formulas and for all $\phi \in \Psi$ it is defined whether $\langle \phi, s\rangle \in T$ or not, then $\langle \bigwedge \Psi, s\rangle \in T$ if and only if $\langle \phi, s\rangle \in T$ for all $\phi \in \Psi$.

6. If Ψ is a set of $L^2_{\kappa,\omega}(\tau)$-formulas and for all $\phi \in \Psi$ it is defined whether $\langle \phi, s\rangle \in T$ or not, then $\langle \bigvee \Psi, s\rangle \in T$ if and only if $\langle \phi, s\rangle \in T$ for some $\phi \in \Psi$.

7. $\langle \exists x_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for some interpretation s' such that s and s' are same except possibly in x_α.

8. $\langle \forall x_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for all interpretations s' such that s and s' are same except possibly in x_α.

9. $\langle \exists X_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for some interpretation s' such that s and s' are same except possibly in X_α.

10. $\langle \forall X_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for all interpretations s' such that s and s' are same except possibly in X_α.

11. $\langle \exists F_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for some interpretation s' such that s and s' are same except possibly in F_α.

12. $\langle \forall F_\alpha \phi, s\rangle \in T$ if and only if $\langle \phi, s'\rangle \in T$ for all interpretations s' such that s and s' are same except possibly in F_α.

4.10 Lemma. *If Π is a set of ordered pairs of $L^2_{\kappa,\omega}(\tau)$-sentences and assignments for less than κ variables in a model $\mathfrak{A}$ then there is a second order sentence with a third order parameter Π which is true if and only if Π is the truth predicate of $\mathfrak{A}$.*

Proof. This is just formalizing Definition 4.9 in second order logic. This is possible because given a model $\mathfrak{A}$ of cardinality κ in a vocabulary τ , the set of $L^2_{\kappa,\omega}$-terms, formulas, free variables in formulas, assignments for less than κ variables and interpretations of terms with given assignments including the free variables of the term are second order definable. From these it follows that the case of atomic formulas is definable in second order logic. The other cases are definable as well, because we need to quantify only over sets of cardinality κ in the truth definition. Note that we cannot quantify over the truth predicate in second order logic because it is too big and we need a third order quantifier for that. But given a model and a predicate, checking whether the predicate is the truth predicate for the model is possible in second order logic.

□

In the following lemma we denote by $truth(X, \mathfrak{A})$ the second order sentence with a third order parameter X which says that X is a truth definition for $L^2_{\kappa,\omega}(\sigma)$ formulas in model $\mathfrak{A}$ where σ is the vocabulary of $\mathfrak{A}$ and κ is the cardinality of $\mathfrak{A}$.

4.11 Lemma. *"There are two $L^2_{\kappa,\omega}$-equivalent non-isomorphic models of cardinality κ" is a Δ^2_1 property in cardinality κ.*

Proof. The Σ^2_1-sentence:

$\exists\Pi_1\exists\Pi_2\exists\sigma\exists\mathfrak{A}\exists\mathfrak{B}(\mathfrak{A}$ and $\mathfrak{B}$ have vocabulary $\sigma \wedge |\mathfrak{A}| = |\mathfrak{B}| = \kappa \wedge \neg\mathfrak{A} \cong \mathfrak{B} \wedge truth(\Pi_1, \mathfrak{A}) \wedge truth(\Pi_2, \mathfrak{B}) \wedge \forall X(X$ is a $L^2_{\kappa,\omega}(\sigma)$-sentence $\rightarrow ((X, \emptyset) \in \Pi_1 \leftrightarrow (X, \emptyset) \in \Pi_2)))$.

The Π^2_1-sentence:

$\exists\sigma\exists\mathfrak{A}\exists\mathfrak{B}\forall\Pi_1\forall\Pi_2(\mathfrak{A}$ and $\mathfrak{B}$ have vocabulary $\sigma \wedge |\mathfrak{A}| = |\mathfrak{B}| = \kappa \wedge \neg\mathfrak{A} \cong \mathfrak{B}\wedge(truth(\Pi_1, \mathfrak{A})\wedge truth(\Pi_2, \mathfrak{B})) \rightarrow \forall X(X$ is a $L^2_{\kappa,\omega}(\sigma)$-sentence $\rightarrow ((X, \emptyset) \in \Pi_1 \leftrightarrow (X, \emptyset) \in \Pi_2)))$.

□

Some large cardinal axioms imply that there is no second order definable well order of the reals. In particular this holds for large cardinal axioms that imply Projective Determinacy. These axioms possibly imply that there are two second order equivalent countable models which are not isomorphic. If that is the case, we can ask the question: does third order equivalence imply isomorphism for countable models? From a result of Abraham and Shelah [1] it follows that it is consistent, as it is always possible to force a third order definable well-order of the reals with a small forcing which preserves large cardinals.

4.12 Lemma. *If there is a second order definable well-order of the reals, then there is a second order definable non-measurable set of reals. If Projective Determinacy holds, then all second order definable sets of reals are measurable. Consequently if PD holds, there is no second order definable well-order of the reals.*

Proof. Recall the construction of a non-measurable set of reals by Vitali. We define an equivalence relation in the interval $[0, 1]$: $x \sim y \Leftrightarrow \quad x - y$ is a rational number. By the Axiom of Choice there is a set which contains exactly one member from each equivalence class. Such a set turns out as is well-known to be non-measurable. If $<$ is a second order definable well-order of the reals then there is a second order definable non-measurable set of reals. We can define this set to contain the $<$-least element from each equivalence class.

By a result of Mycielski and Steinhaus, every second order definable set of reals is measurable assuming Projective Determinacy [14].

□

Next we will note that if the Proper Forcing Axiom holds, then there is no second order definable well order of the reals. Consequently, one cannot use Ajtai's proof to show the consistency of Ajtai Hypothesis with Proper Forcing Axiom. If Ajtai Hypothesis is consistent with the Proper Forcing Axiom, then Ajtai Hypothesis can hold without second order definable well-order of the reals.

4.13 Lemma. *Proper Forcing Axiom implies that there is no second order definable well-order of the reals.*

Proof. Proper Forcing Axiom implies Axiom of Determinacy holds in $L[\mathbb{R}]$ which implies Projective Determinacy [18].

□

4.14 Open question. Is the Proper Forcing Axiom consistent with Ajtai Hypothesis?

4.15 Theorem. *It is consistent that Martin's Maximum holds and third order equivalence implies isomorphism for countable models.*

Proof. By Larson's result [8] Martin's Maximum is consistent with the existence of a well-order of the reals definable in $H(\aleph_2)$ without parameters. The relation $X \in H(\aleph_2)$ is definable in third order logic. Recall that $X \in H(\aleph_2)$ iff the cardinality of the transitive closure of X is at most $\aleph_1$. This can be said in third order logic:

$\exists\alpha < \omega_2 \exists f$ (f is a function $\wedge$ dom$(f) = \alpha^+ \wedge f(0) = \emptyset \wedge f(\alpha) = X \wedge \forall\beta \leq \alpha(f(\beta) \subseteq \mathcal{P}(\bigcup_{\gamma<\beta} f(\gamma)) \wedge |f(\beta)| \leq \aleph_1)$).

We can quantify over elements of $H(\aleph_2)$ in third order logic thus Martin's Maximum is consistent with a third order definable well-order of the reals. Consequently it is consistent that Martin's Maximum holds and third order equivalence implies isomorphism for countable models. □

4.16 Theorem. *If Ajtai Hypothesis holds below a measurable cardinal then it holds at the measurable cardinal.*

Proof. Assume that is not the case. Then Ajtai Hypothesis holds below a measurable cardinal κ, but there are two models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ which are equivalent in infinitary second order logic but not isomorphic. Let j be an elementary embedding from V into transitive class M with critical point κ. Since j is elementary embedding, $j(\kappa)$ is the least cardinal where Ajtai Hypothesis fails in M. We will show that Ajtai Hypothesis fails in κ in M, which will be a contradiction. We assume that $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of κ^n. Then $j(\mathfrak{A})$ and $j(\mathfrak{B})$ are subsets of $j(\kappa)^n$. $\mathfrak{A} = j(\mathfrak{A}) \cap \kappa^n$ and $\mathfrak{B} = j(\mathfrak{B}) \cap \kappa^n$, thus the models $\mathfrak{A}$ and $\mathfrak{B}$ belong to M. Similarly any subset of κ in V belongs to M. Thus the models $\mathfrak{A}$ and $\mathfrak{B}$ are infinitary second order equivalent but not isomorphic in M.

In fact some weaker assumption than measurability is sufficient for the result above. We introduce now the concept of Σ^m_n indescribable cardinal. □

4.17 Definition. A cardinal κ is Σ^m_n indescribable if for all $U \subseteq V_\kappa$ and for all Σ^m_n sentences ϕ if $(V_\kappa, \epsilon, U) \models \phi$ then there is an $\alpha < \kappa$ such that $(V_\alpha, \epsilon, U \cap V_\alpha) \models \phi$.

4.18 Theorem. *If Ajtai Hypothesis holds below a Σ^2_1 indescribable cardinal κ then it holds at κ.*

Proof. Assume towards contradiction that Ajtai Hypothesis fails at κ. Recall that by Lemma 4.11 the failure of Ajtai Hypothesis at κ is Σ^2_1 in κ. Then by Σ^2_1 indescribability there is an $\alpha < \kappa$ such that $(V_\alpha, \epsilon) \models \phi$, where ϕ expresses the negation of Ajtai Hypothesis at the cardinality of the model in question. But then Ajtai Hypothesis fails at the cardinality of V_α, so κ is not the first cardinal where Ajtai Hypothesis fails, contradiction. In fact we need here only a weaker version of Σ^2_1 indescribability: we don't need to use any subset of V_κ. □

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