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ON THE MAXIMAL RESOLVABILITY OF MONOTONICALLY NORMAL SPACES

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ABSTRACT. We continue the work of [5] and show that all Monotonically Normal spaces are maximally resolvable if and only if every ultrafilter is maximally decomposable. Hence the existence of Monotonically Normal space which is not maximally resolvable is equi-consistent with the existence of a measurable cardinal.We also show that it is consistent (modulo the consistency of a measurable cardinal) that there is a Monotonically Normal space X with $\Delta(X) = \aleph_\omega$ which is not ω_1 resolvable. As follows from the results of [5] this is the best possible.

1. INTRODUCTION

A topological space X is called λ resolvable (λ a cardinal) if X contains λ many mutually disjoint dense subsets. A natural bound on the resolvability of X is $\Delta(X) = \min\{|G| |G \text{open}, G \neq \emptyset\}$. We say that X is maximally resolvable if X is $\Delta(X)$ resolvable. The expectation is that "nice" spaces should be maximally resolvable, an expectation verified by the well known fact that metric and linearly ordered spaces are maximally resolvable. It is also well known that there is a regular space X with no isolated points such that X is not even 2-resolvable.

The following useful fact was observed by El̀kin in [4]:

Lemma 1.1. *A space X is λ-resolvable iff every nonempty open set in X contains a nonempty subset which is λ resolvable in the subspace topology.*

For a space X $\Delta(X)$ is a very important parameter for studying resolvability properties of X. We shall be interested in spaces where the cardinality of every nonempty open subset is the same, namely spaces where $\Delta(X) = |X|$.It is immediate that every non-empty subset of a space X has an non empty open subset G such that $\Delta(G) = |G|$. One way in which we shall use Lemma 1.1 is when we would shall want to prove the maximal resolvability of a space X and we shall assume, without loss of generality that $\Delta(X) = |X|$.

As made clear by [5] a natural class of spaces for studying the problem of resolvability is given by the following definition:.

Definition 1.2. *A space X is called* monotonically normal *,MN, if it is T_1 and there is an operation $H(x,U)$ defined on all pairs (x,U) such that U is open in X and $x \in U$, such that the following holds for every* $(x,U) \in \mathrm{dom}(H)$

(1) $x \in H(x,U) \subset U$ and $H(x,U)$ is open
(2) If $H(x,U) \cap H(y,V) \neq \emptyset$ then $x \in V$ or $y \in U$.

It is easily seen that metric spaces and linearly ordered spaces are monotonically normal. Recall that a set D in a space X is called *strongly discrete* if the points of D can be separated by pairwise disjoint neighborhoods. The following is essentially proved in [3]:

Theorem 1.3. *If X is monotonically normal and if $x \in X$ is in the closure of $Y \subseteq X$ then there is a strongly discrete $Y^* \subseteq Y$ such that x is in the closure of Y^*.*

We say that a space satisfying the conclusion of Theorem 1.3 is a *tightly strongly discrete space* (TSD). The purpose of this paper is to continue the study of resolvability of MN spaces started by [5], but most of the results actually work for the larger class of TSD spaces. Of course when we get counterexamples to resolvability it will be more interesting to get an example which is a MN space.

In [5] it is shown that every TSD space with no isolated points (also called *crowded* space) is ω resolvable. If one assumes the existence of a measurable cardinal κ then one can get an example of a MN space X with $\Delta(X) = \kappa$ which does not have any subspace which is ω_1 resolvable. In the same paper it is shown how to get assuming that there is an ω_1 decendingly complete ultrafilter on $\aleph_\omega$ (see the definition below) a MN space X with $\Delta(X) = \aleph_\omega$ which is not ω_2 resolvable. Also it is shown that $\aleph_\omega$ is the smallest cardinal for which we can find such space. These results left several natural open questions like: can you find in ZFC an MN space which is not maximally resolvable? Is it consistent to have a MN space X with $\Delta(X)$ less than the first measurable which is not ω_1 resolvable?

In this paper we completely settle all these problems by showing that the connection between the the existence of MN spaces which are not maximally resolvable and decendingly complete ultrafilters indicated by [5] is actually even tighter . In fact we shall prove Theorem 1.7 in which the notion of decendingly complete ultrafilter plays a central role.

Definition 1.4. *An ultrafilter $\mathcal{F}$ on a set I is called λ-decendingly complete (λ a cardinal) if for every sequence of sets $\langle A_\alpha | \alpha < \lambda \rangle$, such that for every $\alpha < \beta < \lambda$ $A_\alpha \in \mathcal{F}$ and $A_\beta \subseteq A_\alpha$ we have $\bigcap_{\alpha<\lambda} A_\alpha \in \mathcal{F}$.*

If $\mathcal{F}$ is not λ-decendingly (namely λ-decendingly *incomplete* .) we use an equivalent terminology:

Definition 1.5. *An ultrafilter $\mathcal{F}$ on a set I is called λ decomposable if there is a partition of I into λ disjoint subsets $\langle A_\alpha | \alpha < \lambda \rangle$ such that for every $S \subseteq \lambda$ of cardinality $< \lambda$ $\bigcup_{\alpha \in S} A_\alpha \notin \mathcal{F}$.*

It is easily seen that $\mathcal{F}$ is λ-decendingly incomplete iff it is λ- decomposable. Recall that an ultrafilter $\mathcal{F}$ on a set I is uniform if for every $A \in \mathcal{F}$ $|A| = |I|$.

Definition 1.6. *An uniform ultrafilter $\mathcal{F}$ on a set I is called* completely decomposable *if it is λ decomposable for every $\lambda \leq |I|$.*

The main theorem of this paper is:

Theorem 1.7. *The following are equivalent :*
(M1) Every TSD space X is maximally resolvable.
(M2) Every MN space is maximally resolvable.
(M3) Every ultrafilter is maximally decomposable.

Theorem 1.7 will follow from a ‘local’ version of the theorem dealing with a space X such that $\Delta(X) = |X| = \lambda$ for a fixed cardinal λ.We shall break the ”local” theorem into two theorems dealing respectively with λ regular and λ singular.

Theorem 1.8. *Let λ be a regular cardinal cardinal. The following are equivalent*
(R1) Every TSD space X with $\Delta(X) = |X| = \lambda$ is λ (hence maximally) resolvable.
(R2) Every MN space X with $\Delta(X) = |X| = \lambda$ is λ (hence maximally) resolvable.
(R3) Every uniform ultrafilter on λ is completely decomposable.

Theorem 1.9. *Let λ be singular. The following are equivalent*
(S1) Every TSD space X with $\Delta(X) = |X| = \lambda$ is λ (hence maximally) resolvable.
(S2) Every MN space X with $\Delta(X) = |X| = \lambda$ is λ (hence maximally) resolvable.
(S3) For every $\mu < \lambda$ there is $\nu < \lambda$ such that for every $\nu < \zeta \leq \lambda$ and every uniform ultrafilter $\mathcal{F}$ on ζ, $\mathcal{F}$ is μ decomposable.

Theorem 1.7 follows easily from Theorems 1.8 and 1.9. The only point which not completely trivial is that if every TSD space X for which $\Delta(X) = |X|$ being maximally resolvable is equivalent by Lemma 1.1 to having every TSD space being maximally resolvable.

So the maximal resolvability of MN spaces is tightly connected with every ultrafilter being maximally decomposable. What about the existence of ultrafilters which are not maximally decomposable? An obvious example is a non principle κ complete ultrafilter on a measurable cardinal κ. This ultrafilter is not γ decomposable for every $\gamma < \kappa$. (So as proved in [5] assuming the existence of a measurable cardinal one get an MN space X with $\Delta(X) = |X| = \kappa$ which is not even ω_1 resolvable.)

Assuming the consistency of measurable cardinal is necessary for getting a model in which there is an ultrafilter which is not maximally decomposable. This follows from the results of Donder ([2]) that if there is no inner model with measurable cardinal then every ultrafilter is maximally decomposable.Actually the property of ultrafilters studied in [2] is regularity of ultrafilters and the the main result is that if there is no inner model with measurable cardinal then every uniform ultrafilter $\mathcal{F}$ on a cardinal λ is (ω, κ)-regular for every $\kappa < \lambda$ (For singular λ he gets that the ultrafilter is (ω, λ) regular). But it easily follows from the definition of (γ, δ)-regularity that if an ultrafilter is (ω, κ) regular then it is κ-decomposable. Since every uniform ultrafilter on λ is λ decomposable it follows that if there is no inner model with measurable cardinal then every ultrafilter is maximally decomposable.

Can we get decendingly complete ultrafilters on cardinals below the first measurable? Prikry in [8] got a model in which a singular cardinal carries a uniform ultrafilter which is ω_1 decendingly complete. This cardinal is still rather large (cardinal fixed point etc.). In [1] (using the model of [7] which assumes the existence of a cardinal κ which is κ^+ supercompact) we get a model in which there is a uniform ultrafilter on $\aleph_{\omega+1}$ which is ω_1 decendingly complete. Though it was not stated in the paper it is rather easy to see that in the same model there is a uniform ultrafilter on $\aleph_\omega$ which is ω_1-decendingly complete. This was used in [5] to get a model in which there is a MN space X with $\Delta(X) = \aleph_\omega$ which is not ω_2 resolvable. In Section 4 we shall analyze the model of [7] and show that the space constructed in [5] is not even ω_1 resolvable, which by their results, is the best possible. Sometime around the same time in which we proved the consistency (from supercompact) of a ω_1 decendingly complete ultrafilter on $\aleph_\omega$ and $\aleph_{\omega+1}$ Woodin proved, assuming the consistency of measurable cardinal, that it is consistent to have an ω_1 -decendingly complete ultrafilter on $\aleph_\omega$ ([11]). We believe that the same analysis we describe in Section 4 for the model of [1] applies to the model constructed by Woodin, so it will give in his model an MN space with $\Delta(X) = |X| = \aleph_\omega$ which is not ω_1 resolvable. (If one wants to get similar results for cardinals larger than $\aleph_\omega$ like $\aleph_{\omega+1}$ one must start with a much stronger consistency assumptions as we shall show in section 4.

So it follows from Theorem 1.7 ,[2] and [11]:

Theorem 1.10. *The following are equiconsistent:*

(1) The existence of a measurable cardinal
(2) The existence of a MN space which is not maximally resolvable (or equivalently the existence of a TSD space which is not maximally resolvable)
(3) The existence of a MN space X with $\Delta(X) = \aleph_\omega$ which is not ω_1 resolvable

The structure of the paper is as follows: In Section 2 we shall recall the definition of filtration F on an infinity branching tree T and the associated space $X(T, F)$ given in [5] . In [5] the authors were mainly concerned with the case that the tree T was the tree of all finite sequences of λ. (We shall call this a λ tree. We shall have to generalize this notion to the Tree (with the filtration) being *essentially* λ tree .We shall generalize the results of [5] about resolvability (and unresolvability) of the space $X(T, F)$. These spaces we play a major role in the proof of the main theorem.In section 3 we shall prove the main theorem. In section 4 we shall argue that in the model of [7] there is an ω_1-decendingly complete ultrafilter $\mathcal{F}$ on $\aleph_\omega$ and the corresponding space $X = X(T, \mathcal{F})$ is an example of a MN space which not ω_1 resolvable, where T is the tree $\aleph_\omega^{<\omega}$. (We have $\Delta(X) = |X| = \aleph_\omega$.)

2. Filtered Trees and The Associated Spaces

For us a tree T will always be a tree of finite sequences where T is closed under taking initial segments and the tree order is $s < t$ iff s is an initial segment of t . We shall assume (unless otherwise stated) that our trees are infinitely branching everywhere, namely if $t \in T$ then the set of all immediate successors of t in T,$S_T(t)$, is infinite.A subtree is of T is a subset closed under initial segments. The subtree of T, R is a *complete* subtree of T if for every $t \in R$ either $S_R(t) = S_T(t)$ or $S_R(t) = \emptyset$.(We are not necessarily assuming that R is infinitely splitting every where.).

Definition 2.1. *Given a tree T a filtration of T is a map F assigning to every $t \in T$ a filter $F(t)$ such that $F(t)$ is a uniform filter on $D_t =$*

$\{x|t^\frown x \in S_T(t)\}$. The pair (T,F) is called a filtered tree. If for every $t \in T$ $F(t)$ is an ultrafilter we say that F is an ultra-filtration of T and the pair (T,F) is an ultra-filtered tree. If for every $t \in T$ D_t is a fixed set D and $F(t)$ is a fixed filter $\mathcal{F}$ on D, we denote (T,F) also by $(T,\mathcal{F})$. For a filtered tree (T,F) and for $t \in T$ the restriction of (T,F) to t ($(T,F)_t$) is the filtered tree (T_t, F_t) where $T_t = \{v|t^\frown v \in T\}$ and $F_t(v) = F(t^\frown v)$. When F is understood from the context we shall also denote $(T,F)_t$ by T_t. The filtered tree (T,F) is called a λ-filtered tree if the tree T is the set of all finite sequences of the cardinal λ i.e. T is $\lambda^{<\omega}$.The filtered tree (T,F) is called essentially *λ-filtered tree if for every $t \in T$ there is a cardinal $\mu_t \le \lambda$ such that $S_T(t) = \{t^\frown \beta|\beta < \mu_t\}$ and such that for every $\nu < \lambda$ and $t \in T$*

$$\{\beta < \mu_t|\mu_{t^\frown\beta} > \nu\} \in F(t) \tag{1}$$

.

The definition of essentially λ-filtered tree means that while we allow a node to have less that λ immediate successors , these successors has splittings which are arbitrarily large in λ and this is still true if we restrict the successors of t to a set in $F(t)$.This definition will be mainly used in the case that λ is singular, though it makers sense also in the case λ is regular. Note that if λ is regular and (T,F) is an essentially λ-filtered tree then if $t \in T$ and $\mu_t < \lambda$ then $\{\beta < \mu|\mu_{t^\frown\beta} = \lambda\} \in F(t)$. (Otherwise we shall not be able to satisfy (1).) Note also that if (T,F) is a (an essentially) λ filtered tree and $t \in T$ then $(T,F)_t$ is a (an essentially) λ - filtered tree.

Given a filtered tree there is a natural topology defined on T ([5]): $U \subseteq T$ is open if whenever $t \in U$ the set $\{x \in D_t|t^\frown x \in U\} \in F(t)$.The resulting space is denoted by $X(T,F)$. It is easily seen that if (T,F) is an essentially-λ filtered tree then $\Delta(X(T,F)) = |X(T < F) = \lambda$ .Also it follows that if U is open in $X(T,F)$ and $t \in U$ there is an open subtree of $(T,F)_t$, W such that $\{t^\frown v|v \in W\} \subseteq U$.

The following three statements are proved in [5]:

Theorem 2.2. *(1) For every filtered tree $X(T,F)$ $X(T,F)$ is a MN space. (Theorem 3.2 in* [5]*).*

(2) Let (T,F) be ultra-filtered tree and μ be a cardinal such that for all $t \in T$ $F(t)$ is a μ-decendingly complete ultrafilter then $X(T,F)$ is not μ^+-resolvable. (Theorem 3.5 of [5]*)*

(3) Let (T,F) be a λ-ultra-filtered tree such that for every $t \in T$ $F(t)$ is completely decomposable. Then $X(T,F)$ is λ resolvable. Hence it is maximally resolvable. (Theorem 3.13 in [5]*)*

The main theorem of this section

Theorem 2.3. *Let (T, F) be an essentially λ-ultra-filtered tree . Assume that for every $\eta < \lambda$ and for every $t \in T$*

$$\{\beta < \mu_t | F(t^\frown\beta) \quad \text{is } \eta\text{-decomposable}\} \in F(t) \tag{2}$$

Then the space $X(F, T)$ is λ resolvable . (hence it is maximally resolvable.)

The proof of Theorem 2.3 is similar to the proof of Theorem 3.13. of [5] but we need some generalizations and we use slightly different terminology. (We use ranking functions on well founded trees rather than the λ- good functions used in [5].) Since we are considering also well founded trees we shall drop the assumption that a tree is automatically infinitely branching every where. Recall that a tree is called well founded if it has no infinite branch. For any tree T let T^* be the set of maximal elements of T. It is well known that a well founded tree T has a ranking function $\text{rank}_T : T \to \alpha$ onto some ordinal α defined by $\text{rank}_T(t) = \sup\{\text{rank}(s) + 1 | s \in S_t\}$. α is always a successor ordinal $\alpha = \text{rank}_T(\emptyset) + 1$. The ordinal $\text{rank}_T(\emptyset)$ is called the rank of the well founded tree T and denoted by $\text{rank}(T)$. Note that the ranking function is always *onto* α. We need an operation of gluing together well founded trees

Definition 2.4. *Let T be tree. Suppose that for every $t \in T^*$ we are a given another tree T_t. The glued tree of T and $\{T_t | t \in T^*\}$, denoted by $T \bigoplus \{T_t | t \in T^*\}$ is the tree $T \cup \{t^\frown v | t \in T^*, v \in T_t\}$*

Recall that for us a tree is always a tree of finite sequences.So that the glued tree is a tree . Also if T is well founded and for every $t \in T^*$ T_t is a well founded tree then the glued tree, $T \bigoplus \{T_t | t \in T^*\}$, is well founded. Also it easily seen in this case that if R is the glued tree $T \bigoplus \{T_t | t \in T^*\}$ then for $t \in T^*$, $v \in T_t$ we have $\text{rank}_R(t^\frown v) = \text{rank}_{T_t}(v)$ A typical case in which we use this operation is when we shall start from a given tree S, we shall take T to be a (complete) subtree of R and for $t \in T^*$ T_t is a (complete) subtree of $R_t = \{v | t^\frown v \in R\}$ then the glued tree $T \bigoplus \{T_t | t \in T^*\}$ is a (complete) subtree of R.

Definition 2.5. *Let (T, F) be a filtered tree. Let R be a well founded complete subtree of T. R is said to be homogenous subtree of (T, F) if for every subtree of R, U which is open in R in the subspace topology induced by $X(T, F)$ we have that for every $t \in U$ $\text{rank}_U(t) = \text{rank}_R(t)$.*

The main lemma for the proof of the Theorem 2.3 is the following:

Lemma 2.6. *Let (T, F) and λ satisfy the assumptions of Theorem 2.3 then*

(1) *If λ is singular or λ is regular and $\mu_\emptyset < \lambda$ then there is an homogenous well founded complete subtree of T ,R such that* $\text{rank}(R) = \lambda + 1$.
(2) *If λ is regular and $\mu_\emptyset = \lambda$ then there is an homogenous well founded complete subtree of T, R such that* $\text{rank}(R) = \lambda$.

Note that the two conclusions in the two cases of Lemma 2.6 are not necessarily exclusive.

Lemma 2.6 will follow from the following lemma:

Lemma 2.7. *Let (T, F) and λ satisfy the assumptions of Theorem 2.3 then for for every successor $\alpha < \lambda$ there is a complete homogenous well founded subtree R such that* $\text{rank}(R) = \alpha$.

We shall prove Lemma 2.7 by induction on α when the induction assumption will be that the lemma is true for successor $\beta < \alpha$ for *every* filtered tree (T, F) satisfying the assumptions of Theorem 2.3 for λ. Note that if (T, F) satisfies the assumptions of theorem 2.3 then for every $t \in T$ $(T, F)_t$ satisfies the same requirements for λ.

The case $\alpha = 1$ is trivial by taking R to be a tree which contains only the root and the immediate successors of the root $R = \{\emptyset\} \cup S_\emptyset$. If $\alpha = \beta + 1$ where β is successor we use the induction assumption for β and for each of the essentially λ-ultra-filtered trees T_t where t is an immediate successor of the root of T to get a complete homogenous well founded subtree of T_t, R_t such that $\text{rank}(R) = \beta$. Let W be the subtree of T made up of the root and the immediate successors of the root. The glued tree $R = W \bigoplus \{R_t | t \in W^*\}$ is easily verified to be a well founded tree . For each immediate successor of $\emptyset$, t, $\text{rank}_R(t) = \text{rank}_{R_t}(\emptyset) = \beta$. Hence $\text{rank}_R(\emptyset) = \beta + 1 = \alpha$.It is also easily verified that R is an homogenous and complete subtree of T using the fact that for every $t \in W^*$ R_t was a homogenous complete subtree of T_t.

Now we deal the case $\alpha = \beta + 1$ where β is a limit ordinal. Let $\eta = \text{cof}(\beta)$. (Of course $\eta < \lambda$.) Let $\langle \delta_\gamma | \gamma < \eta \rangle$ be an increasing sequence of *successor* ordinals cofinal in β. Here we use the assumption that the ultra-filtered tree (T, F) satisfy the assumptions of Theorem 2.3 So we know that

$$D = \{\zeta < \mu_\emptyset | F(\langle \zeta \rangle) \quad \text{is} \quad \eta \quad \text{decomposable}\} \in F(\emptyset) \tag{3}$$

. We first define a well founded complete subtree of T , W. W will be made up of three levels W_0 contains only the root of T, $\emptyset$. W_1 is made up of all the immediate successors in T of $\emptyset$ which is $S_\emptyset$. Let $\widehat{W_1}$ be the set $\{\langle \zeta \rangle | \zeta \in D\}$. W_2 is the set of the immediate successors of the members of $\widehat{W_1}$. $W = W_0 \cup W_1 \cup W_2$. Namely we take the first two levels of T and we include a node on the third level only if it is the successor of a node on the second level whose associated ultrafilter is η decomposable.

The set of maximal members of W, W^* is clearly $(W_1-\widehat{W_1})\cup W_2$.Fix $s\in\widehat{W_1}$. By definition and the assumption $F(s)$ is η decomposable. Hence we can find a partition $\{A^s_\rho|\rho<\eta\}$ of μ_s such that for every $E\subseteq\eta$ of cardinality $<\eta$ $\bigcup\limits_{\rho\in E}A^s_\rho\notin F(s)$. For every $t\in W_2$ let $s(t)$ be the unique $s\in W_1$ such that t is the immediate successor of s and let $\rho(t)$ be the unique $\rho<\eta$ such that $t=s(t)^\frown\zeta$ for some $\zeta\in A^{s(t)}_\rho$. For each $t\in W_2$ we use the induction assumption for the ultra-filtered tree T_t and the successor ordinal $\delta_{\rho(t)}$ and get a complete well founded homogenous subtree of T_t ,R_t such that $\operatorname{rank}(R_t)=\delta_{\rho(t)}$. For $t\in W_1-\widehat{W_1}$ let R_t be the trivial tree $\{\emptyset\}$. We have defined R_t for every $t\in W^*$, so we can form the glued tree $R=W\bigoplus\{R_t|t\in W^*\}$.$R$ is clearly a complete well founded subtree of T. We have to argue that R is homogenous and that $\operatorname{rank}(R)=\alpha$.We shall give a combined argument for these two facts.

Let U be a subtree of R which is open in the relative topology of $X(T,F)$. If $t\in W_2\cap U$ consider $U_t=\{v|t^\frown v\in U\}$. It is easily verified to be a subtree of R_t which is open in the subspace topology of R_t induced by $X((T,F)_t)$.Hence by R_t being homogenous we get for every $v\in U_t$

$$\operatorname{rank}_U(t^\frown v)=\operatorname{rank}_{U_t}(v)=\operatorname{rank}_{R_t}(v)=\operatorname{rank}_R(t^\frown v) \tag{4}$$

In particular for $t\in W_2\cap U$ $\operatorname{rank}_U(t)=\operatorname{rank}_{U_t}(\emptyset)=\operatorname{rank}_{R_t}(\emptyset)=\delta_{\rho(t)}=\operatorname{rank}_R(t)$. So we verified the needed fact about U for every node which is above the first level of tree. For $s\in W_1-\widehat{W_1}$ we have $\operatorname{rank}_U(s)=\operatorname{rank}_R(s)=0$ since s is a maximal member of both the tree R and the tree U. What about $s\in\widehat{W_1}$? Let $Z_s=\{\zeta|s^\frown\zeta\in U\}$. By U being open we have $Z_s\in F(s)$. $\{A^s_\rho|\rho<\eta\}$ witnessed the η decomposability of $F(s)$, hence a set in the ultrafilter $F(s)$ must intersect η members of the partition $\{A^s_\rho|\rho<\eta\}$.Since the set of immediate successors of s in U is $S^U_s=\{s^\frown\zeta|\zeta\in Z_s\}$ we get that $\{\rho(t)|t\in S^U_s\}$ is unbounded in η so

$$\beta=\sup\{\delta_{\rho(t)}|t\in S^U_s\} \tag{5}$$

Since β is a limit ordinal we also get

$$\beta=\sup\{\delta_{\rho(t)}+1|t\in S^U_s\} \tag{6}$$

But now

$$\text{rank}_U(s) = \sup\{\text{rank}_U(t) + 1 | t \in S_s^U\} = \sup\{\rho(t) | t \in S_s^U\} = \beta \tag{7}$$

. The value of $\text{rank}_U(s)$ turned out to be independent of U so in particular if we take $U = R$ we get $\text{rank}_R(s) = \beta$ which implies that for every U we have $\text{rank}_U(s) = \text{rank}_R(s) = \beta$.

We are left with determining the value of $\text{rank}_R(\emptyset)$ and showing that $\text{rank}_U(\emptyset) = \text{rank}_R(\emptyset)$. We have shown that for every appropriate U , if s is in U an immediate successor of $\emptyset$ then either $\text{rank}_U(s) = \beta$ (if $s \in U \cap \widehat{W_1}$) and $\text{rank}_U(s) = 0$ otherwise. By definition of the topology, U open and $\emptyset \in U$ implies that $\{\zeta < \mu_\emptyset | \langle\zeta\rangle \in U\} \in F(\emptyset)$. Also $D \in F(\emptyset)$. so we have for some $\zeta < \mu_\emptyset \quad \langle\zeta\rangle \in U \cap W_1$. So $U \cap \widehat{W_1} \neq \emptyset$.

$$\text{rank}_U(\emptyset) = \sup\{\text{rank}_U(s) + 1 | s \in U \cap W_1\} \tag{8}$$

. We argued that the sup in equation (8) is over a set that contains at most the two value 1 and $\beta + 1 = \alpha$. Since we proved that the value $\beta + 1$ does appear in the set over which we take the sup we proved

$$\text{rank}_U(\emptyset) = \alpha$$

. By taking $U = R$ we get $\text{rank}(R) = \text{rank}_R(\emptyset) = \alpha$. □Lemma 2.6.

In the proof of Lemma 2.6 when we dealt with the case $\alpha = \beta + 1$ where β is a limit ordinal, we assumed about β that the induction assumption for successor ordinals less than β holds and that $\text{cof}(\beta) = \eta < \lambda$. It means that the same proof also work in the case $\beta = \lambda$ provided λ is singular, hence we have a proof of

Lemma 2.8. *Let (T, F) and λ satisfy the requirements of Theorem 2.3 where λ is singular then there is a complete well founded homogenous subtree of T , R such that* $\text{rank}(R) = \lambda + 1$.

Proof of Lemma 2.6 The case λ singular follows immediately from Lemma 2.8. So assume λ regular and the first subcase we consider is when $\mu = \mu_\emptyset = \lambda$. Let W be the subtree of T composed of the first two levels . i.e. $W = \{\emptyset\} \cup S_\emptyset$.In our case $S_\emptyset = \{\langle\beta\rangle | \beta \in \lambda\}$. For $t = \langle\beta\rangle$ use Lemma 2.7 for $(T, F)_t$ and get a well founded homogenous complete subtree of T_t , R_t, such that $\text{rank}(R_t) = \beta + 1$. The glued tree $R = W \bigoplus \{R_t | t \in S_\emptyset\}$ is clearly a complete homogenous well founded subtree of T whose rank

$$\begin{aligned} \text{rank}(R) = \text{rank}_R(\emptyset) &= \sup\{\text{rank}_R(t) + 1 | t \in S_\emptyset\} \\ &= \sup\{\text{rank}(R_t) + 1 | t \in S_\emptyset\} \\ &= \sup\{\beta + 2 | \beta \in \lambda\} = \lambda \end{aligned} \tag{9}$$

So we are left with the subcase λ regular but $\mu = \mu_\emptyset < \lambda$. As we noted after the definition of essentially λ-filtered tree in this case $D = \{\beta < \mu | \mu_{\langle\beta\rangle} = \lambda\} \in F(\emptyset)$. For $t \in D$ use the previous subcase for the essentially λ-filtered tree $(T, F)_t$ to get a complete well founded homogenous subtree of T_t , R_t such that $\text{rank}(R_t) = \lambda$. Let $\widehat{D} = \{\langle\beta\rangle | \beta \in D\}$. For $t \in S_\emptyset - widehat D$ let R_t be the trivial tree containing only the root $\emptyset$. (Of course in this case $\text{rank}(R_t) = 0$. As in the previous subcase we form $W = \{\emptyset\} \cup S_\emptyset$ and the glued tree $R = W \bigoplus \{R_t | t \in S_\emptyset\}$. As above R is a well founded complete homogenous subtree of T and

$$\text{rank}(R) = \sup\{\text{rank}(R_t) + 1 | t \in S_\emptyset\} \tag{10}$$

. The set on which the sup is evaluated in equation (10) contains only two values $\lambda + 1$ and 1 so we get $\text{rank}(R) = \lambda + 1$. □ (Lemma 2.6).

Proof of Theorem 2.3. We are given a λ ultra-filtered tree (T, F) satisfying the assumptions of Theorem 2.3. We shall define by induction an increasing sequence of well founded complete subtrees of T, $\langle T_n | n \in \omega\rangle$ such that $T = \bigcup_{n\in\omega} T_n$. T_0 is the trivial tree $\{\emptyset\}$. Suppose that T_n is defined. For every $t \in T_n^*$ let R_t be well founded complete homogenous subtree of T_t such that $\text{rank}(R_t) \geq \lambda$. Such a subtree exists by Lemma 2.6. (Actually we have $\text{rank}(R_t) \in \{\lambda, \lambda + 1\}$ but what we need is only that $\text{rank}(R_t) \geq \lambda$.) Let $T_{n+1} = T_n \bigoplus \{R_t | t \in T_n^*\}$. Note that by the completeness of R_t in T_t and the fact that $\text{rank}(R_t) > 0$ (for $t \in T_n^*$) we have $S_t \subseteq T_{n+1}$. It is therefore easily verified by induction that the n-th level of T is included in T_n, so $T = \bigcup_{n\in\omega} T_n$.Also for every $n \in \omega$ T_n is a well founded complete subtree of T.

For $t \in T$ let $h(t)$ be the minimal n such that $t \in T_n$.If $h(n) > 0$ we have a unique $s(t) \in T^*_{h(t)-1}$ such that $s(t) < t$ in the tree order. Hence we have a unique $v(t) \in R_{s(t)}$ such that $t = s(t)^\frown v(t)$. $R_{s(t)}$ is a well founded tree of rank $\leq \lambda + 1$. Let $\alpha(t) = \text{rank}_{R_{s(t)}}(v(t)$. $\alpha(t) \leq \lambda + 1$.AS a special case we define $\alpha(\emptyset) = 0$.

Theorem 2.3 will follow from

Lemma 2.9. *For every $\beta \in \Lambda$ the set $D_\beta = \{t | \alpha(t) = \beta\}$ is dense in the space $X(T, F)$.*

The sets $\{D_\beta | \beta \in \lambda\}$ witness the λ -resolvability of $X(T, F)$. So suppose that U is a non empty open subset of $X(T, F)$ and $\beta < \lambda$ Without loss of generality we can assume that U is a subtree of T_t. Let $t \in U$. $t = s(t)^\frown v(t)$. By definition of our topology and the fact that $T_{h(t)}$ is well founded we can find $t' \in U \cap T^*_{h(t)}$ and $t < t'$ in the tree order. Since $t' \in T^*_{h(t)}$, $R_{t'}$ is defined.The set $W = \{v \in R_{t'} | (t')^\frown v \in U\}$ is a non empty open subtree of $R_{t'}$. By the homogeneity of $R_{t'}$ for every $v \in W$ $\mathrm{rank}_W(v) = \mathrm{rank}_{R_{t'}}(v)$ So we have

$$\text{(11)} \qquad \mathrm{rank}(W) = \mathrm{rank}_W(\emptyset) = \mathrm{rank}_{R_{t'}}(\emptyset) \geq \lambda$$

Since $\beta < \lambda$ we must have some $v \in W$ such that $\mathrm{rank}_W(v) = \beta$. Again by homogeneity of W $\mathrm{rank}_{R_{t'}}(v) = \mathrm{rank}_W(v) = \beta$. Note that $v \neq \emptyset$. Let $u = (t')^\frown v$. We get $u \in U$, $h(u) = h(t)+1$, $s(u) = t'$ and $v(u) = v$. So $\alpha(u) = \mathrm{rank}_{R_{t'}}(v(u)) = \beta$. So we found a point of D_β in U and we verified the denseness of D_β. □ (Lemma 2.9) □ (Theorem 2.3).

For the future proofs of Theorems 1.8 and 1.9 we need the following corollary of Theorem 2.3

Corollary 2.10. *Let λ be a singular cardinal for which condition (S3) of Theorem 1.9 holds, then for every essentially λ-ultra-filtered tree (T, F) $X(T, F)$ is maximally resolvable.*

Corollary 2.10 follows easily from Theorem 2.3 because the condition (S3) together with the definition of essentially λ-filtered tree imply (2.

We also need a converse of Corollary 2.10

Lemma 2.11. *Let λ be singular. Suppose that condition (S3) fails for λ, i.e. there is $\mu < \lambda$ such that either there is an uniform ultrafilter on λ which is μ decendingly complete or*

$$\text{(12)} \qquad \{\eta < \lambda | \textit{there is a uniform } \mu \textit{ decendingly complete ultrafilter on } \eta\}$$

is cofinal in λ. Then there is an essentially λ-ultra-filtered tree (T, F) such that $X(T, F)$ is not λ resolvable.

Proof of Lemma 2.11: If there is a uniform ultrafilter $\mathcal{F}$ on λ which is η decendingly incomplete for some $\eta < \lambda$ then the filtered tree $X(T, \mathcal{F})$ is not η^+ resolvable by Theorem 2.2 (2). So assume that every uniform ultrafilter on λ is μ decomposable for every $\mu < \lambda$. We claim that we can not have unboundedly many ὴs in λ carrying a uniform $\nu = \mathrm{cof}(\lambda)$ decendingly complete ultrafilter. Assume otherwise. So the set $D = \{\eta < \lambda | \text{there is a } \nu \text{ decendingly complete uniform ultrafilter on} \eta\}$ is cofinal in λ. Fix an increasing sequence $\langle \rho_\delta | \delta < \nu \rangle$ cofinal in D. For $\delta < \nu$ let $\mathcal{F}_\delta$ be a ν decendingly complete uniform ultrafilter on ρ_δ. Since ν is regular by a theorem in [6] each $\mathcal{F}_\delta$ is also ν^+ decendingly complete. Fix a uniform ultrafliter $\mathcal{G}$ on ν and define an ultrafilter $\mathcal{H}$ on λ by

$$\text{(13)} \qquad A \in \mathcal{H} \quad \text{iff} \quad A \subseteq \lambda \quad \text{and} \quad \{\delta < \nu | A \cap \rho_\delta \in \mathcal{F}_\delta\} \in \mathcal{G}$$

$\mathcal{H}$ is easily verified to be uniform ultrafilter on λ (It is uniform because $\mathcal{G}$ is uniform on ν so for every $B \in \mathcal{G}$ $\{\rho_\delta | \delta \in B\}$ is cofinal in λ.)

Claim 2.12. *$\mathcal{H}$ is ν^+ decendingly complete*

Claim 2.12 contradicts our assumption that every uniform ultrafilter on λ is η decomposable for every $\eta < \lambda$.

Proof of claim 2.12: Let $\langle A_\alpha | \alpha < \nu^+ \rangle$ be a $\subseteq$ decreasing sequence of subsets of λ which are in $\mathcal{H}$. For $\alpha < \nu^+$ let $B_\alpha = \{\delta < \nu | A_\alpha \cap \rho_\delta \in \mathcal{F}_\delta\}$. By our assumptions and the definition of $\mathcal{H}$ $B_\alpha \in \mathcal{G}$. The sequence $\langle B_\alpha | \alpha < \nu^+ \rangle$ is a decreasing sequence of subsets of ν of length ν^+ . Hence this sequence has to be eventually constant. Let B^* be the eventual value of the B_α's. Of course $B^* \in \mathcal{G}$. For $\delta \in B^*$ the sequence $\langle A_\alpha \cap \rho_\delta | \alpha < \nu^+ \rangle$ is a decreasing sequence of subsets of ρ_δ that are all in $\mathcal{F}_\delta$. $\mathcal{F}_\delta$ is a ν^+ decendingly complete ultrafilter, therefore $\bigcap_{\alpha<\nu^+} (A_\alpha \cap \rho_\delta) \in \mathcal{F}_\delta$. Hence

$$B^* \subseteq \{\delta < \nu | (\bigcap_{\alpha<\nu^+} A_\alpha) \cap \rho_\delta \in \mathcal{F}_\delta\} \in \mathcal{G} \tag{14}$$

By definition of $\mathcal{H}$ it means $\bigcap_{\alpha<\nu^+} A_\alpha \in \mathcal{H}$. $\square$(Claim 2.12).

It follows from our assumptions that for some $\eta < \lambda$ we can define an increasing sequence $\langle \rho_\delta \rangle | \delta < \nu$ cofinal in λ such that for $\delta < \nu$ ρ_δ carries a uniform η-decendingly complete ultrafilter $\mathcal{F}_\delta$. By claim 2.12 we can assume that for every $\delta < \nu$ $\mathcal{F}_\delta$ is ν-decomposable.

We shall define an essentially λ-ultra-filtered tree (T, F) such that all $t \in T$ $F(t)$ is η decendingly complete. By Theorem 2.2 (2) the associated space $X(T, F)$ is not η^+ resolvable. The definition of (T, F) will be by induction on the levels of T where for every $t \in T$ For some $\delta < \nu$ $\mu_t = \rho_\delta$ and $F(t) = \mathcal{F}_\delta$.For $t = \emptyset$ pick any $\delta < \nu$ and define $\mu_t = \rho_\delta$ and $F(t) = \mathcal{F}_\delta$. Given t that was already was put in T and for whom μ_T and $F(t)$ were defined. These definitions determine the set of immediate successors of t in T ,namely $S_t = \{t^\frown\beta | \beta < \mu_t\}$. We have to define for $s \in S_t$ μ_s and $F(s)$. By our inductive assumption for some $\delta < \nu$ $\mu_t = \rho_\delta$ and $F(t) = \mathcal{F}_\delta$. We know that $\mathcal{F}_\delta = F(t)$ is ν-decomposable. Hence we can represent $\rho_\delta = \mu_t$ as a disjoint union of a family of sets of size ν , $\{A_\delta | \delta < \nu\}$ such that every subset of ν,B, of cardinality less than ν $\bigcup_{\delta\in B} A_\delta \notin F(t)$. For $\beta < \mu_t$ let $\delta(\beta)$ be the unique member of ν such that $\beta \in A_{\delta(\beta)}$. Define for $\beta < \mu_t$ $\mu_{t^\frown\beta} = \rho_{\delta(\beta)}$ and $F(t^\frown\beta) = \mathcal{F}_{\delta(\beta)}$. It is easily verified that this definition yields an essentially λ-ultra-filtered tree and since the ultrafilters $\mathcal{F}_\delta$ are all η decendingly complete we get that $X(T, F)$ is not η^+ resolvable. $\square$(Lemma 2.11).

3. Proofs of the Main Theorems

In this section we shall prove Theorems 1.8 and 1.9, from which our main theorem (1.7) follows.

Definition 3.1. *Let X be a topological space.*

1. *For $x \in X$ define $T(x)$ to be the set of cardinals μ such that there exists a set $Y \subseteq (X - \{x\})$, $|Y| = \mu$,x is in the closure of Y but for any $Z \subseteq Y, |Z| < \mu$ x is not in the closure of Z.*
2. *For $x \in X$ let $\rho(x) = \sup T(x)$*
3. *Suppose that $|X| = \lambda$. A point $x \in X$ is called bounded if either λ is regular and $\lambda \notin T(x)$ or λ is singular and $\rho(x) < \lambda$. (In particular in the case λ singular we also have $\lambda \notin T(x)$).*

The following theorem generalizes Lemma 3.16 of [5].

Lemma 3.2. *Let X be a space such that $\Delta(X) = |X| = \lambda$ and such that the set of bounded points is dense in X, then X is λ-resolvable.*

Proof: We shall construct a family $\langle D_\alpha | \alpha < \lambda \rangle$ of mutually disjoint dense subsets of X by induction of length λ. Let $\langle (x_\gamma, \alpha_\gamma) | \gamma < \lambda \rangle$ be an enumeration of all the pairs (x, α) such that x is a bounded point of X and α is an ordinal in λ,possibly with repetitions . In the case λ is singular we shall assume that $\rho(x_\gamma) \leq \gamma$. (We may have a slight problem for $\gamma < \min\{\rho(x) | x \text{ is bounded}\}$ but that for these γ's we define $(x_\gamma, \alpha_\gamma)$ to be some default value that does not require us to do any thing in our inductive construction.) We define our approximation to $\langle D_\alpha | \alpha < \lambda \rangle$, $\langle D^\delta_\alpha | \alpha < \lambda \rangle$ by induction on $\delta < \lambda$ where for fixed $\alpha < \lambda$ the sequence $\langle D^\delta_\alpha | \delta < \lambda \rangle$ is increasing , $D^0_\alpha = \emptyset$, and for limit δ $D^\delta_\alpha = \bigcup_{\gamma < \delta} D^\gamma_\alpha$. We carry the inductive assumption that $\langle D^\delta_\alpha | \alpha < \lambda \rangle$ are mutually disjoint and that for every $\gamma < \delta$ x_γ is in the closure of $D^\delta_{\alpha_\gamma}$. We also assume that

$$|W_\delta = \bigcup_{\alpha < \lambda} D^\delta_\alpha| < \lambda$$

. It implies that in every stage δ only $< \lambda$ many D^δ_α are non empty. In the case λ is singular we shall strengthen the assumption about the cardinality of W_δ by requiring $|W_\delta| \leq \delta$.This inductive assumption is easily verified to be kept by the limit stages in the construction both in λ regular case and in the λ singular case.

So we have to specify the construction only in the case $\delta = \gamma + 1$. We make $D^\delta_\alpha = D^\gamma_\alpha$ except for the case $\alpha = \alpha_\gamma$ .The set W_γ is of cardinality $< \lambda$. We assumed $\Delta(X) = |X| = \lambda$, so every point is the the closure of the complement of a set of cardinality $< \lambda$. In particular x_γ is in the closure of $X - W_\gamma - \{x_\gamma\}$. So there is $Y \subseteq (X - W_\gamma - \{x_\gamma\})$ such that $|Y| \in T(x_\gamma)$.x_γ is a bounded point so in the case λ regular we get that $|Y| < \lambda$ and in the case λ singular $|Y| \leq \gamma$. (Recall that we assumed that our enumeration in this singular case is such that $\rho(x_\gamma) \leq \gamma$.) For $\alpha = \alpha_\gamma$ we define $D^\delta_\alpha = D^\gamma_\alpha \cup Y$. Our construction of Y guarantees that $\langle D^\delta_\alpha | \alpha < \lambda\rangle$ is a family of mutually disjoint sets and that x_γ is in the closure of $D^\delta_{\alpha_\gamma}$. At the end we define for $\alpha < \lambda \quad D_\alpha = \bigcup_{\delta<\lambda} D^\delta_\alpha$. By construction these sets are mutually disjoint and for every bounded point $x \in X$ and for $\alpha < \lambda \quad x$ is in the closure of D_α. (The pair $(x, \alpha) = (x_\gamma, \alpha_\gamma)$ for some $\gamma < \lambda$.) Since we assumed that the set of bounded points is dense in X we get that each $\alpha < \lambda \quad D_\alpha$ is dense in X. $\Box$(Lemma 3.2).

The next Theorem motivates our interest in essentially λ-ultra-filtered trees, because they become useful when the set of bounded points is not dense in our space.This theorem generalizes the proof of Theorem 3.17 in [5].

Theorem 3.3. *Let X be a TSD space such that $\Delta(X) = \lambda$ is a singular cardinal. Let $\eta = \mathrm{cof}(\lambda)$.Assume that the set of bounded points is not dense in X and that there is $\zeta < \lambda$ such that for every cardinal μ, $\zeta \leq \mu < \lambda$ every uniform ultrafilter on μ is η decomposable. Then there exists an essentially-λ-ultra-filtered tree (T, F) and a continuous one to one map, g ,from $X(T, F)$ into X.*

Proof of Theorem 3.3: We shall construct the tree T together with the assignments of μ_t and $F(t)$ to every $t \in T$. In addition we shall define by the same induction for every $t \in T$ the value in X of the map g on t and an open neighborhood of $g(t)$, U_t. U_t will be defined such that if $s, t \in T$ and $s < t$ (in the tree order,) then $U_t \subseteq U_s, g(s) \notin U_t$ and if s and t are incomparable in the tree order $U_s \cap U_t = \emptyset$. Fix $\zeta < \lambda$ as in the statement of the theorem and fix a sequence $\langle \gamma_\delta | \delta < \eta\rangle$ cofinal in λ.

The induction will be done in the following order: At the n-th stage we shall have a tree T_n of height n. Its n levels are as usual the 0 level, the 1 level... $n-1$-level. T_n is intended be the subtree of made up first n levels of the final T.) At the end of the n-th stage we shall determine for every $t \in T_n \quad g(t)$, U_t and for $s \in T_n - T^*_n$ (i.e. s is at level $j < n-1$) μ_s and $F(s)$.For $t \in T^*_n$ (i.e. for t at the $n-1$-level of T_n.) we shall have defined at this stage a cardinal μ^-_t such that $\zeta \leq \mu^-_t < \lambda$. Note that in this stage μ_t and $F(t)$ are not defined yet. They will be defined at the next stage. μ^-_t is a commitment that μ_t ,when it is defined will be at least μ^-_t.In order to guarantee that the final (F,T) will be an essentially λ-ultra-filtered tree we shall have an inductive assumption that for every $s \in T_n$ which is at the $n-2$-level and for every $\xi < \lambda$

$$\{\beta < \mu_s | \xi \leq \mu^-_{s^\frown\beta}\} \in F(s) \tag{15}$$

. Since, when we define μ_t for $t \in T^*_n$ at the next stage we honor our commitments and make $\mu_t \geq \mu^-_t$, it easily verified by induction that for $\xi < \lambda$

$$\{\beta < \mu_s | \xi \leq \mu_{s^\frown\beta}\} \in F(s) \tag{16}$$

.

We assumed that the set of bounded points is not dense in X so we start the induction by picking an non empty open subset of X such that U contains no bounded point. We pick any point $x \in U$ and we define $T_0 = \{\emptyset\}$,$U_\emptyset = U$,$g(\emptyset) = x$ and $\mu^-_\emptyset = \zeta$. This concludes the 0-stage of the induction.

Suppose that T_n with all the additional information we specified above is already defined. Fix $t \in T^*_n$.$g(t) \in U_\emptyset$,hence it is not bounded. So $\sup(T(g(t)) \geq \lambda$. Note that we can have $\lambda \in T(g(t))$ but this does change the argument bellow because we can have $\mu_t = \lambda$. So we can find $Y \subseteq X$ such that $|Y| \geq \mu^-_t$, $g(t) \in \bar{Y}$ and for every $Z \subseteq Y$, $|Z| < |Y|$ $\quad g(t) \notin \bar{Z}$. Without loss of generality we can assume that $Y \subseteq U_t$. Also, since X is TSD space we can assume that Y is strongly discrete. So let $\{W_y | y \in Y\}$ be a family of mutually disjoint open sets such that $y \in W_y$. Without loss of generality we can assume $W_y \subseteq U_t$. We are ready for the next step of the induction.

We define $\mu_t = |Y|$.By the choice of Y we honored our commitment that $\mu^-_t \leq \mu_t$. Fix an enumeration of Y $\quad \langle y_\beta | \beta < \mu_t\rangle$. The successors of t in T_{n+1} are ,as expected, $S_t = \{t^\frown\beta | \beta < \mu_t\}$. For $s = t^\frown\beta \in S_t$ we define $g(s) = y_\beta$, $U_s = W_\beta$. We still have to define $F(t)$ and for $s \in S_t$ $\quad \mu^-_s$. We first define a uniform filter on μ_t $\quad F^-(t)$. For $A \subseteq \mu_t$ we put: $A \in F^*(t)$ iff there is an open neighborhood W of $g(t)$ such that $\{\beta < \mu_t | y_\beta \in W\} \subseteq A$.

It is easily verified that $F^*(t)$ is a filter. $F^*(t)$ is a uniform filter on μ_t because if it is not uniform some open neighborhood of $g(t)$, W will contain less than μ_t members of Y but this will imply that $g(t)$ is in the closure of a subset of Y of cardinality $< |Y|$, which is a contradiction.So $F^*(t)$ can be extended to a uniform ultrafilter on μ_t. Define $F(t)$ to be such an ultrafilter. Since $\zeta \leq \mu_t$ the ultrafilter $F(t)$ is η decomposable. Let $\langle A_\delta | \delta < \eta\rangle$ be a partition of μ_t such that for every $B \subseteq \eta$, $|B| < \eta$ we have $\bigcup\limits_{\delta \in B} A_\delta \notin F(t)$.

For $\beta < \mu_t$ define $\mu^-_{t^\frown\beta}$ to be γ_δ where $\delta < \eta$ is the unique ordinal such that $\beta \in A_\delta$. The assumption about the partition $\langle A_\delta | \delta < \eta \rangle$ assures us that equation 15 is also true for t.This concludes the definition of the $n+1$ stage of the induction.

Our final tree T is the union of the different T_n's. The values of μ_t and $F(t)$ for $t \in T$ were defined along the way. Clearly (T, F) is an essentially λ-ultra-filtered tree. g defines a one to one map from $X(T, F)$ into X. So we only have to verify that g is continuous. Let U be open in X. Assume that $g^{-1}(U) \neq \emptyset$. We show that $g^{-1}(U)$ is open in $X(T, F)$. Let $t \in g^{-1}(U)$, i.e. $g(t) \in U$. By definition of $F^-(t)$ and of $g(s)$ for $s \in S_t$ we get $\{\beta < \mu_t | g(t^\frown\beta) \in U\}$ is in $F^-(t) \subseteq F(t)$. So $\{\beta < \mu_t | t^\frown\beta \in g^{-1}(U)\} \in F(t)$. Since it holds for every $t \in g^{-1}(U)$ we proved that $g^{-1}(U)$ is open and that g is continuous. $\square$ (Theorem 3.3).

The following result is essentially the proof of Theorem 3.17 of [5]. But note also that a small modification of the proof of Theorem 3.3 will prove it. (If you assume that λ is regular and all the points in X of the form $g(t)$ are not bounded then in the induction step defining the successors of $t \in T_n$, we can take $Y \subseteq U_t$ such that $|Y| = \lambda$ and for every $Z \subseteq Y \quad g(t) \notin \bar{Z}$. So we can assume that always $\mu_t = \lambda$ and we do not need the assumption about the decomposability of ultrafilters. Hence (T, F) is a λ-ultra-filtered tree.)

Theorem 3.4. *Let X be a TSD space with $\Delta(X) = |X| = \lambda$ where λ is a regular cardinal. Assume that the set of bounded points in X is not dense in X. Then there is a λ-ultra-filtered tree (T, F) and a one to one continuous map from $X(T, F)$ into X.*

We have gathered all the ingredients needed for the proof of Theorems 1.8 and 1.9. We shall try to run the proofs of these two theorems in parallel. In both case the implication from the first clause ((R1) and (S1)) to the second ((R2) and (S2)) is trivial as any MN space is TSD. For the implication from the second clause to the third clause assume that we are given a cardinal λ such that if λ is regular then (R2) holds and if λ is singular then (S2) holds.

We claim that in both cases every uniform ultrafilter on λ is completely decomposable. Otherwise if $\mathcal{F}$ is such an ultrafilter which not μ decomposable for some $\mu < \lambda$ (Note that every uniform ultrafilter on λ is automatically λ decomposable.) Consider the λ ultra-filtered tree $(T, \mathcal{F})$ where $T = \lambda^{<\omega}$. $X(T, \mathcal{F})$ is an MN space which by Theorem 2.2 is not μ^+ resolvable. Hence we got a MN space X with $\Delta(X) = |X| = \lambda$ which is not λ resolvable. So (R2) and (S2) fail respectively. For the case λ regular we verified (R3)

In the case λ singular Lemma 2.11 gives us , assuming that (S3) fails, an essentially λ-ultra-filtered tree (T, F) such that $X(T, F)$ is not λ resolvable, so we got a MN space X with $\Delta(X) = |X| = \lambda$ which is not λ resolvable, contradiction to (S2).

Now assume that (R3) and (S3) for λ regular and λ singular respectively. Let X be a TSD space with $\Delta(X) = |X| = \lambda$. We shall prove the λ resolvability of X by referring to Lemma 1.1. Let G be a non empty open subset of X. If the set of bounded points is dense in G then Lemma 3.2, applied to G as a subspace of X, shows that G is λ resolvable. If the set of bounded points is not dense in G we invoke Theorem 3.3 in case λ is singular and Theorem 3.4 when λ is regular. In the case λ regular we get λ- ultra-filtered tree (T, F) and a one to one continuous map $g : X(T, F) \to G$. By our assumption of (R3) every uniform ultrafilter on λ is maximally decomposable, so in particular it is true for every ultrafilter in the filtration F. By Theorem 2.2(3) the space $X(T, F)$ is λ resolvable.

In the case λ singular, the condition (S3) implies that the assumptions of Theorem 3.3 are satisfied . So we get an essentially λ ultra-filtered tree (T, F) and a continuous one to one map $g : X(T, F) \to G$. Since we assume (S3) Corollary 2.10 implies that $X(T, F)$ is λ resolvable.

In both the regular and the singular case we found a space Y which is λ resolvable and a one to one continuous map $g : Y \to G$. Let $Z \subseteq G$ be the range of g. If $\{D_\alpha | \alpha < \lambda\}$ is a family of mutually disjoint dense subsets of Y the family $\{g'' D_\alpha | \alpha < \lambda\}$ is a family of mutually disjoint dense subsets of Z. So we found in G a subspace which is λ resolvable. By Lemma 1.1 we proved the λ resolvability of X. $\Box$(Theorems 1.8,1.9,1.7).

4. A Model with ω_1 irresolvable space

In this section we shall present a model of Set Theory in which there is a MN space X such that $\Delta(X) = |X| = \aleph_\omega$ which is not ω_1 resolvable. Both parameters of this statement are optimal because of a result of [5] every NM space with no isolated points is ω resolvable, so ω_1 in the above statement is optimal. In the same paper it is shown that very MN space X with $\Delta(X) = \aleph_n$ for some $n \in \omega$ is maximally resolvable. (It actually follows also from our Theorem 1.8 because by [6] every ultrafilter on $\aleph_n, n \in \omega$ is maximally decomposable.

The more general frame that allows us to get MN spaces that are not ω_1 resolvable is:

Theorem 4.1. *Let $V_1 \subseteq V_2$ be two transitive models of ZFC such that*

(1) $\kappa > \omega_1^{V_1} = \omega_1^{V_2}$ *is a cardinal in* V_1.
(2) There is a countable subset of κ $A = \{\alpha_n | n \in \omega\} \in V_2 - V_1$ *such that no* $B \in V_1$ *of cardinality* $< \kappa$ *covers* A. .
(3) Let $\mathcal{G}$ *be in* V_2 *the filter on* κ *defined in* $B \in \mathcal{G}$ *iff* $A - B$ *is finite. Then* $\mathcal{G} \cap V_1 \in V_1$.
(4) Let $\mathcal{F} \in V$ *be a uniform ultrafilter on* κ *extending* $\mathcal{G} \cap V$.

Then in V $\mathcal{F}$ *is uniform* ω_1 *decendingly complete ultrafilter on* κ. *Also if we let* T *be the tree* $\kappa^{<\omega}$ *then the space* $X = X(T, \mathcal{F})$ *is a MN space with* $\Delta(X) = |X| = \kappa$ *which is not* ω_1 *resolvable.*

Note that $\mathcal{F}$ as above always exits because by assumption (3) of Theorem 4.1 , if $B \in \mathcal{G} \cap V_1$ $|B| = \kappa$.

Proof of Theorem 4.1: $\mathcal{F}$ is ω_1 decendingly complete because given a descending sequence of sets (in V_1) in $\mathcal{F}$ $\langle B_\zeta | \zeta < \omega_1 \rangle$, we get that $\langle B_\zeta \cap A | \zeta < \omega_1 \rangle$ is in V_2 a decreasing sequence of subsets of the countable set A of length ω_1. (Here we use the assumption $\omega_1^{V_1} = \omega_1^{V_2}$.) so this sequence is constant for $\eta < \zeta < \omega_1$ for some $\eta < \omega_1$. Let $A^* \subseteq A$ be this constant. Consider $B = \bigcap_{\zeta < \omega_1} B_\zeta$. Clearly we have $A^* \subseteq B$. Consider the symmetric difference $C = B \Delta B_\eta$. C has empty intersection with A, so $\kappa - C \in \mathcal{G} \subset \mathcal{F}$, since $\mathcal{F}$ is an ultrafilter and $B_\eta \in \mathcal{F}$ we get $B \in \mathcal{F}$.

We know by Theorem 2.2 that the space X is MN with $\Delta(X) = \kappa$. (We also know by the same theorem that X is not ω_2 resolvable.). The point of the present theorem is to show that in our particular case X is not even ω_1 resolvable.

Lemma 4.2. *X has no subspace which is ω_1 resolvable.*

In [5] there is a description of a general procedure for finding the closure of a set $D \subseteq X(T, F))$ for any ultra-filtered tree (X, T) as follows: We define by induction on ordinals an increasing sequence of subsets of $X(T, F)$: $D_0 = D$, $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$ for α limit and

$$D_{\beta+1} = \{t | t \in T, \{x | t^\frown x \in D_\beta\} \in F(t)\}$$

. The closure of D, $\bar{D}$ is D_β where β is the minimal ordinal such that $D_{\beta+1} = D_\beta$. Also $\bar{D}$ can be described as the union of all the D_β's.

If $t \in \bar{D}$ then we define $\beta(t)$ to be either 0 if $t \in D_{\beta+1}$ or the minimal β such that $t \in D$.In our particular case of the given $X = X(T, \mathcal{F})$ we get

Lemma 4.3. *Let* $D \subseteq X$ *and let* $t \in \bar{D}$ *then there is a finite sequence of members of* A, s *such that* $t^\frown s \in D$.

Proof of Lemma 4.3: We prove the lemma by induction on $\beta(t)$. If $\beta(t) = 0$ then $t \in D$ and we can take s to be the empty sequence. If $\beta(t) = \beta > 0$ it means that $t \in D_{\beta+1}$. By definition $D_{\beta+1}$

$$B = \{\alpha < \kappa | t^\frown\alpha \in \bar{D}, \beta(t^\frown\alpha) < \beta\} \in \mathcal{F}$$

. By the definition of $\mathcal{F}$ $B \cap A$ is infinite. Pick $\alpha \in B \cap A$. So $t^\frown\alpha \in \bar{D}$ and $\beta(t^\frown\alpha) < \beta$. By the induction assumption there is a finite sequence of elements of A, s such that $t^\frown\alpha^\frown s \in D$ but then $\hat{s} = \alpha^\frown s$ is a finite sequence of members of A such that $t^\frown\hat{s} \in D$. $\square$(Lemma 4.3).

Note that even though $t, D \in V_1$ we have to go to V_2 to find the right s such that $t^\frown s \in D$ because $A \in V_2 - V_1$. Suppose now that $Y \subseteq X$ is ω_1 resolvable. Let $\{D_\gamma | \gamma < \omega_1\}$ be a family of mutually disjoint dense subsets of Y. Let $t \in Y$. Since for every $\gamma < \omega_1$ $t \in \bar{D}_\gamma$ we use Lemma 4.3 for t and D_γ and find a finite sequence of members of A , s_γ such that $t^\frown s_\gamma \in D_\gamma$. The sequence $\langle s_\gamma | \gamma < \omega_1\rangle$ is defined in V_2 and it has length ω_1 there. (We use again that $\omega_1^{V_1} = \omega_1^{V_2}$.) A is countable set so there are $\gamma \neq \delta$ in ω_1 such that $s = s_\gamma = s_\delta$. But then $t^\frown s \in D_\gamma$ and $t^\frown s \in D_\delta$ which contradicts the assumption that the family $\{D_\gamma | \gamma < \omega_1\}$ is a family of mutually disjoint sets. $\square$(Lemma 4.2,Theorem 4.1).

Our goal now to describe how one can get a situation described by the assumptions of Theorem 4.1, a situation that we shall name "the two models situation", for the smallest possible κ , which is $\kappa = \aleph_\omega$. We shall refer to the construction of [7]. There we start from a cardinal κ which is κ^+ supercompact and get a model in which κ becomes $\aleph_\omega$, it is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$. The way the model is defined is that we start from the ground model V in which κ is κ^+ supercompact and $2^\kappa > \kappa^+$ and force with forcing notion P which introduces an increasing sequence of elements of $P_\kappa(\kappa^+)$ $\langle P_n | n \in \omega\rangle$ such that $\bigcup_{n\in\omega} P_n = \kappa^+$. ($P_\kappa(\lambda)$ is the set of all subsets of λ of cardinality less than κ. In the following passages when we use the notation $P_\kappa(\lambda)$ we shall mean it in the sense of V.) We also introduce collapsing functions that makes $\kappa = \aleph_\omega$. In the resulting universe (which we shall denote by V_2) κ^+ is collapsed because it becomes of cofinality ω. We then define a submodel of V_2 V_1 such that $V \subseteq V_1 \subseteq V_2$ such that V_1 has all the bounded subsets of κ which are in V_2 . Hence V_1 and V_2 agree about cardinals $\leq \kappa$ and $V_1 \models \kappa = \aleph_\omega$ but in V_1 $(\kappa^+)^V$ is still a cardinal. So that in particular the sequence $\langle P_n | n \in \omega\rangle$ is not in V_1. More than that no set in V_1 of cardinality less than κ^+ covers $\{P_n | n \in \omega\}$.

In [1] there is a proof that if we define in V_2 the filter $\mathcal{G}$ on $P_\kappa(\kappa^+)$ by $A \in \mathcal{G}$ iff $A - \{P_n | n \in \omega\}$ is finite then we have $\mathcal{G} \cap V_1 \in V_1$. Since the cardinality in V of $P_\kappa(\kappa^+)$ is κ^+ we can copy the sequence $\langle P_n | n \in \omega \rangle$ to κ^+ together with the filter $\mathcal{G}$ and get the two models situation to hold for $V_1 \subseteq V_2$ and κ^+ which is $\aleph_{\omega+1}$ of V_1. In particular we get in that in V_1 there is a NM space X with $\Delta(X) = \aleph_{\omega+1}$ and X is not ω_1 resolvable. We shall argue that for the same pair of universes V_1, V_2 we also have the two models situation for $\aleph_\omega$. We do it by converting the sequence $\langle P_n | n \in \omega \rangle$ into an ω sequence in κ while maintaining the information about the original sequence. Note that in the construction of [7] we were interested in cardinal arithmetic and hence we made the assumption $2^\kappa = \kappa^{++}$ but we do not have to make any specific assumption on cardinal arithmetic in the present case , so we can have in our model GCH or its negation.

In V let H be the set of all order preserving functions from an ordinal in κ into an ordinal in κ. In V H has cardinality κ so let g be a one to one function from H onto κ. If C is set of ordinals, let f_C be the unique order preserving map of $\mathrm{ot}(C)$ onto C. ($\mathrm{ot}(C)$is the order type of C.) If $C \subseteq D$ are sets of ordinals then clearly $e(C, D) = f_D^{-1} \circ f_C \in H$. Since the sequence $\langle P_n | n \in \omega \rangle$ is increasing, i.e For $n \in \omega$ $P_n \subseteq P_{n+1}$ we can consider the set $A = \{e(P_n, P_{n+1}) | n \in \omega\}$. This set is a countable subset of subset of κ which belongs to V_2. This set can not be in V_1 because from it we can reconstruct $\langle P_n | n \in \omega \rangle$. This reconstruction is done as follows:From A we can generate the commutative sequence of embeddings $\{e(P_n, P_m) | n \leq m \in \omega\}$. This system has a direct limit which is an ordered set isomorphic to $\bigcup_{n \in \omega} P_n$. But this union is $(\kappa^+)^V$, so from the direct limit we can recover for each $n \in \omega$ $e(P_n, \kappa^+)$. But the last embedding has a range which is exactly P_n.

From the last fact follows that A can not be covered by a set in V_1 of cardinality $< \kappa$. Assume otherwise. Let $B \in V_1$ such that $A \subseteq B$ and $|B| < \kappa$. (It does not matter whether we compute the cardinality of B in V_1 or in V_2 because they have the same cardinals $\leq \kappa$.) If $h \in V_1$ is a one to one map of $|B|$ onto B , $h^{-1}(A) \in V_1$ because it is a bounded subset of κ and V_1 and V_2 agree about such sets. But then $A \in V_1$ which is a contradiction.

As before we shall define a filter $\mathcal{K}$ on κ by $B \subseteq \kappa$ is in $\mathcal{K}$ iff $A - B$ is finite.

Claim 4.4. $\mathcal{K} \cap V_1 \in V_1$

Proof of claim 4.4: We refer to some facts from [7] and [1] about the particular forcing notion $\mathcal{P}$ used to create V_2 over V and the particular way we defined V_1 in V_2. Note that as usual V_2 is generated by forcing over V_1 by a forcing notion which is a subset of $\mathcal{P}$. Denote it by $\mathcal{Q}$. $\mathcal{Q}$ is defined in V_1 and is typically not in V. [7] introduces a certain group of automorphisms of the forcing notion $\mathcal{P}$, Φ. As usual every $\phi \in \Phi$ has a natural operation on all the $\mathcal{P}$ names. We denote this operation also by ϕ.

In [7] it is shown that Φ has the following properties:

(F1) Every member of V_1 is the realization by the generic filter of a name which is invariant under all the automorphisms of Φ.If $B \in V_1$ then in V_1 we can determine such name for B.
(F2) In particular $\mathcal{Q}$ is invariant under all $\phi \in \Phi$, namely if in V_1 $p \in \mathcal{Q}$ then $\phi(p) \in \mathcal{Q}$.
(F3) The sequence $\langle P_n | n \in \omega \rangle$ has a canonical name which is almost invariant under all $\phi \in \Phi$. It means that if $\mathring{S}$ is the canonical name for this sequence ,$G \subseteq \mathcal{P}$ a generic filter then the realization of $\mathring{S}$ by $\phi''G$ differs from the realization of $\mathring{S}$ by G only by finitely many values.
(F4) In V_1 the operation of Φ on $\mathcal{Q}$ is transitive. Namely if $q_0, q_1 \in \mathcal{Q}$ then there is $\phi \in \Phi$ such that $\phi(q_0)$ is compatible with q_1.

Note that A was defined from $\langle P_n | n \in \omega \rangle$ in such a way that a finite change in this sequence yields a finite change in the resulting A. $\mathcal{K}$ was defined from A in such a way that finite change of A does not change the resulting $\mathcal{K}$. So condition (F3) above implies that if $\mathring{\mathcal{K}}$ is the canonical name for $\mathcal{K}$ then for every $\phi \in \Phi \quad \phi$ does not move $\mathring{\mathcal{K}}$. So suppose that $B \subseteq \kappa$,$B \in V_1$.By (F1) above pick in V_1 a name for B which invariant under all the operations of Φ. Let $\mathring{B}$ be this name.

Subclaim 4.5. *$B \in \mathcal{K}$ iff there is $q \in \mathcal{Q}$ such that $q \Vdash \mathring{B} \in \mathring{\mathcal{K}}$*

If $B \in \mathcal{K}$ and $G \subseteq \mathcal{P}$ is the generic filter over V then there is $q \in G$ such that $q \Vdash \mathring{B} \in \mathring{\mathcal{K}}$.But G is also a generic filter in $\mathcal{Q}$ over V_1. So the left side of subclaim 4.5 implies the right side. For the other direction suppose that there is $q \in \mathcal{Q}$ such that $q \Vdash \mathring{B} \in \mathring{\mathcal{K}}$. But $B \notin \mathcal{K}$. So there is $p \in G$ such that $p \Vdash \mathring{B} \notin \mathring{\mathcal{K}}$. Again $p \in \mathcal{Q}$. By condition (F4) above there is $\phi \in \Phi$ such that $\phi(q)$ is compatible with p. But the names $\mathring{B}$ and $\mathring{\mathcal{K}}$ are invariant under ϕ so $\phi(q) \Vdash \mathring{B} \in \mathring{\mathcal{K}}$. This is a contradiction to p and $\phi(q)$ being compatible. $\square$(Subclaim 4.5)

Subclaim 4.5 gives a definition of $B \in \mathcal{K}$ in V_1 which implies that $\mathcal{K} \cap V_1 \in V_1$.$\square$(Claim 4.5).

We have described a pair of models V_1, V_2 which satisfy the two models situation simultaneously for $\aleph_\omega$ and $\aleph_{\omega+1}$ (In the sense of V_1). Hence we proved

Theorem 4.6. *Assume the consistency of κ which is κ^+) supercompact, then it is consistent that there are NM spaces X, X_1 such that they are not ω_1 resolvable , $\Delta(X) = |X| = \aleph_\omega$ and $\Delta(X_1) = |X_1| = \aleph_{\omega+1}$.*

The generalization of of the forcing of [7] gives:

Theorem 4.7. *Let κ be a supercompact cardinal and let α be a countable ordinal. Then there is an appropriate forcing extension the following holds: For every cardinal λ, $\aleph_\omega \leq \lambda \leq \aleph_\alpha$ there is a MN space X with $\Delta(X) = |X| = \lambda$ and X is not ω_1 resolvable.*

It is not known what is the exact consistency strength needed for the proofs of Theorems 4.6 and 4.7. But it is quite large and for instance a typical result is the following:

Theorem 4.8. *Let κ be a successor cardinal such that $\kappa = \lambda^{+n}$ for some $n \in \omega$ and λ a singular cardinal. (This happens for instance when κ is below the first inaccessible.). Assume also that for $\mu < \lambda$ $\mu^{\aleph_0} < \lambda$. Assume that there is a TSD space X with $\Delta(X) = \lambda$ and such that X is not maximally resolvable. Then for every $n \in \omega$ there is an inner model with n Woodin cardinals.*

Proof of Theorem 4.8: By Theorem 1.8 there is an uniform ultrafilter on κ , $\mathcal{F}$ such that $\mathcal{F}$ is not completely decomposable.So it is μ decendingly complete for some $\mu < \kappa$.We can assume without loss of generality that μ is regular.(Decendingly completeness for μ is equivalent for decendingly completeness for $\mathrm{cof}(\mu)$.) The ultrafilter is obviously κ decomposable. The main theorem of [6] is that an ultrafilter which is η^+ decomposable for regular η is also η decomposable. $\kappa = \lambda^{+n}$, so by applying [6] $n-1$ times we get that $\mathcal{F}$ is λ^+ decomposable.(In particular we get $\mu < \lambda$.)

We convert $\mathcal{F}$ to a uniform ultrafilter on λ^+ by the usual Rudin Keisler reduction. Namely we fixed a partition of κ into λ^+ sets such that the union of a subfamily of size $\leq \lambda$ is not in $\mathcal{F}$. $\langle A_\alpha | \alpha < \lambda^+ \rangle$ be this partition and let g be the function form κ into λ^+ defined by $g(\gamma)$ for $\gamma < \kappa$ is the unique $\alpha < \lambda^+$ such that $\gamma \in A_\alpha$. Define the ultrafilter $\mathcal{G}$ on λ^+ by $B \subseteq \lambda^+ \in \mathcal{G}$ iff $g^{-1}(B) \in \mathcal{F}$. It is easily seen that $\mathcal{G}$ is a uniform ultrafilter on λ^+ which is μ-decendingly complete.

By [9] if for a singular cardinal λ and regular $\mu < \lambda$ we have a a uniform μ-decendingly complete ultrafilter on λ^+ then the combinatorial principle $\square_\lambda$ fails. (See also [1]) By Corollary 5 of [10] Projective

Determinacy holds (PD). PD is well known to imply that for every $n \in \omega$ there is an inner model with n Woodin cardinals. $\square$(Theorem 4.8).

If we re just interested in getting a model with an MN space X with $\Delta(X) = \aleph_\omega$ which is not ω_1 resolvable then the right consistency assumption is the existence of a measurable cardinal. Woodin (**??**) , starting form one measurable cardinal, constructed a model which $\aleph_\omega$ carries a uniform ω_1 decendingly complete ultrafilter. In that proof the model is the smaller model of a pair of models satisfying the two models situation.So in his model there is a MN space X with $\Delta(X) = \aleph_\omega$ such that X is not ω_1 resolvable. By the results of [2] we need to assume the consistency of a measurable cardinal, so we proved Theorem 1.10.

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