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1. Introduction

One way to formulate the Baire Category Theorem is that no compact space can be covered by countably many nowhere dense sets. Soon after Cohen's discovery of forcing, it was realized that it was natural to consider strengthenings of this statement in which one replaces *countably many* with $\aleph_1$*-many.* Even taking the compact space to be the unit interval, this already implies the failure of the Continuum Hypothesis and therefore is a statement not provable in ZFC. Additionally, there are ZFC examples of compact spaces which can be covered by $\aleph_1$ many nowhere dense sets and hence some restriction must be placed on the class of compact spaces in order to obtain even a consistent statement.

Still, there are natural classes of compact spaces for which the corresponding statement about Baire Category — commonly known as a *forcing axiom* — is consistent. The first and best known example is *Martin's Axiom for* $\aleph_1$ *dense sets (*$\mathrm{MA}_{\aleph_1}$*)* whose consistency was isolated from solution of Souslin's problem [16]. This is the forcing axiom for compact spaces which do not contain uncountable families of pairwise disjoint open sets. For broader classes of spaces, it is much more natural to formulate the class and state the corresponding forcing axiom in terms of the equivalent language of forcing notions.

Foreman, Magidor, and Shelah have isolated to broadest class of forcings for which a forcing axiom is relatively consistent — those forcings which preserve stationary subsets of ω_1 [8]. The corresponding

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forcing axiom is known as *Martin's Maximum (*MM*)* and has a vast wealth of consequences which are still being developed (many are in fact consequences of the weaker *Proper Forcing Axiom* (PFA)).

Many consequences of MM (and in fact $\mathrm{MA}_{\aleph_1}$ itself) are examples of Π_2 sentences concerning the structure $H(\omega_2) = (H(\omega_2), \in, \omega_1, \mathrm{NS}_{\omega_1})$. Woodin has produced a canonical forcing extension of $L(\mathbb{R})$, under an appropriate large cardinal assumption, which is provably optimal in terms of the Π_2 sentences which its $H(\omega_2)$ satisfies [17]. Not surprisingly, the theory of the $H(\omega_2)$ of this model largely coincides with the consequences of MM which concern $H(\omega_2)$.

What will concern us in the present paper is the extent to which there is a corresponding strongest forcing axiom which is consistent with the Continuum Hypothesis (CH). More specifically, Woodin has posed the following problem.

Problem 1.1. [17] *Are there two Π_2-sentences ψ_1 and ψ_2 in the language of $(H(\omega_2), \in, \omega_1, \mathrm{NS}_{\omega_1})$ such that ψ_1 and ψ_2 are each individually Ω-consistent with* CH *but such that $\psi_1 \wedge \psi_2$ Ω-implies* ¬CH*?*

Here "Ω-consistent" means something weaker than "provably forceable from large cardinals" and "Ω-implies" means something weaker than just "implies."

Even though CH implies that $[0, 1]$ can be covered by $\aleph_1$-many nowhere dense sets, some forcing axioms are in fact compatible with CH. Early on, Jensen established that Souslin's Hypothesis was consistent with CH (see [4]). Shelah then developed a general framework for establishing consistency results with CH by iterated forcing [15]. The result was a largely successful but *ad hoc* method which Shelah and others used to prove that many consequences of MM are consistent with CH. Moreover, with a few exceptions, it was known that starting from a ground model with a supercompact cardinal, these consequences of MM could all be made to hold in a single forcing extension which satisfies CH.

The purpose of the present paper is to prove that Problem 1.1 has a positive answer, at least if it is consistent that there is an inaccessible limit of measurable cardinals (usually this question is discussed in the context of much stronger large cardinal hypotheses). In fact the two sentences ψ_1 and ψ_2 can be chosen so that their conjunction implies $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and neither requires the use of the non-stationary ideal on ω_1 as a predicate.

The relative consistency of these sentences with CH is obtained by adapting Eisworth's preservation theorems for not adding reals [6] (which are closely based on Shelah's framework noted above) in two different — and necessarily incompatible — ways. Traditionally, the two ingredients in any preservation theorem of this sort are *completeness* and some form of $(<\omega_1)$-*properness*. In one preservation theorem (which is essentially proved in [6]), the *completeness* condition is weakened while maintaining the other requirement. In the other preservation theorem the *completeness* condition is strengthened slightly from the condition in [6], but $(<\omega_1)$-*properness* is replaced by the weaker combination of *properness and* $(<\omega_1)$-*semiproperness*.

The paper is organized as follows. In Section 2 we formulate the two Π_2 sentences and outline the tasks which must be completed to prove the main theorem. Section 3 contains a discussion of the preservation theorems which will be needed for the main result, including the proof of a new preservation theorem for not adding reals. Section 4 is devoted to the analysis of the single step forcings associated to one of the Π_2-sentences. The relationship between α-semiproperness and the preservation of α-club guessing will be discussed in Section 5. In Section 6 we re-examine the forcing construction from Section 4 which requires the existence of measurable cardinals and argue that the construction can be carried out using the $\aleph_n$'s if one has an appropriate form of reflection. Finally, Section 7 will contain some concluding remarks.

The reader is assumed to have familiarity with proper forcing and with countable support iterated forcing constructions. While we aim to keep the present paper relatively self contained, the reader will benefit from familiarizing themselves with the arguments of [2] and [6]. We will also deal with revised countable support and will use [12] as a reference. The notation is mostly standard for set theory and we will generally follow the conventions of [9] and [10]. We will now take the time to fix some notational conventions which are not entirely standard. If θ is a regular cardinal, then $H(\theta)$ will denote the collection of all sets of hereditary cardinality less than θ. Unless explicitly stated otherwise, θ will always denote an uncountable regular cardinal. If X is an uncountable set, we will let $[X]^{\aleph_0}$ denote the collection of all countable subsets of X. If X has cardinality ω_1, then an ω_1-*club* in $[X]^{\aleph_0}$ is a cofinal subset which is closed under taking countable unions and is well ordered in type ω_1 by containment. At certain points we will need to code hereditarily countable sets as elements of 2^ω. If $r \in 2^\omega$ and A is in $H(\omega_1)$, then we say that r *codes* A if $(\mathrm{tc}(A), \in, A)$ is isomorphic to (ω, R_1, R_2) where $R_1 \subseteq \omega^2$ and $R_2 \subseteq \omega$ are defined by

$$((i,j) \in R_1) \leftrightarrow (r(2^{i+1}(2j+1)) = 1)$$

$$i \in R_2 \leftrightarrow r(2i+1) = 1$$

(Here tc denotes the transitive closure operation.) While not every r in 2^ω codes an element of $H(\omega_1)$, every element of $H(\omega_1)$ has a code in 2^ω. Also, if f is a function from a set of ordinals of ordertype ω into $2^{<\omega}$, then we will say that f *codes* $A \in H(\omega_1)$ if, for some cofinite subset X of the domain of f, $\bigcup f[X]$ is a single infinite length sequence which codes A. Finally, if r and s are elements of $2^{\leq\omega}$, we will let $\Delta(r,s)$ denote the least i such that $r(i) \neq s(i)$ (if no such i exists, we define $\Delta(r,s) = \min(|r|,|s|)$).

2. Two Π_2-sentences

In this section we will present the two Π_2-sentences which are used to resolve Problem 1.1 and will prove that their conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$. This will be done by appealing to the following theorem of Devlin and Shelah.

Theorem 2.1. [3] *The equality* $2^{\aleph_0} = 2^{\aleph_1}$ *is equivalent to the following statement: There is an* $F : H(\omega_1) \to 2$ *such that for every* $g : \omega_1 \to 2$*, there is an* $X \in H(\omega_2)$ *such that whenever* M *is a countable elementary submodel of* $(H(\omega_2), \in, X)$*,*

$$F(\overline{X}) = g(\delta).$$

where $\overline{X}$ *and* δ *are the images of* X *and* ω_1 *respectively under the transitive collapse of* M*.*

The first sentence is essentially the same as one used by Caicedo and Veličković in [2] in order to prove that BPFA implies there is a Δ_1-well ordering of the $H(\omega_2)$, definable from a parameter in $H(\omega_2)$. We will now take some time to recall the definitions associated to this coding. Given $x \subseteq \omega$, let $\sim_x$ be the equivalence relation on $\omega \setminus x$ defined by letting $m \sim_x n$ iff $[m,n] \cap x = \emptyset$. Given two further subsets y and z of ω, let $(I_k)_{k<t}$ (for some $t \leq \omega$) be the increasing enumeration of the set of $\sim_x$–equivalence classes intersecting both y and z, and let the oscillation of x, y and z be the function $o(x,y,z) : t \to 2$ defined by

$$o(x,y,z) = 0 \text{ if and only if } min(I_k \cap y) \leq min(I_k \cap z)$$

Let $\vec{C} = \langle C_\delta : \delta \in Lim(\omega_1)\rangle$ be a *ladder system* on ω_1 (so that each C_δ is a cofinal subset of δ ordertype ω), and let $\alpha < \beta < \gamma$ be limit ordinals greater than ω_1. Let $N \subseteq M$ be countable subsets of γ with $\{\omega_1, \alpha, \beta\} \subseteq N$ such that, for all $\xi \in \{\omega_1, \alpha, \beta, \gamma\}$, $\sup(N \cap \xi) < \sup(M \cap \xi)$ and $\sup(M \cap \xi)$ is a limit ordinal. We are going to specify a way of decoding a finite binary sequence from $\vec{C}$, N, M, α and β. This decoding will be a very minor variation of the one defined in [2].

Let $\overline{M}$ be the transitive collapse of M, and let $\pi : M \longrightarrow \overline{M}$ be the corresponding collapsing function. Let $\omega_1^{\bar{N}}$ and $\omega_1^{\bar{M}}$ denote the respective ordertypes of $N \cap \omega_1$ and $M \cap \omega_1$. Let $\alpha_M = \pi(\alpha)$, $\beta_M = \pi(\beta)$ and $\gamma_M = ot(M)$. *The height of N in M with respect to $\vec{C}$* is defined as $n(N, M) = |\omega_1^{\bar{N}} \cap C_{\omega_1^{\bar{M}}}|$. Set

$$x = \{|\pi(\xi) \cap C_{\alpha_M}| \,:\, \xi \in \alpha \cap N\},$$

$$y = \{|\pi(\xi) \cap C_{\beta_M}| \,:\, \xi \in \beta \cap N\},$$

$$z = \{|\pi(\xi) \cap C_{\gamma_M}| \,:\, \xi \in N\}.$$

If the length of $o(x, y, z)$ is at least $n(N, M)$, then we define

$$s(N, M) = s^{\vec{C}}_{\alpha,\beta}(N, M) = o(x, y, z)$$

Otherwise we leave $s(N, M)$ undefined. If s is a finite length binary sequence, we define $\bar{s}$ to be the sequence of the same length l with its digits reversed: $\bar{s}(i) = s(l - i)$.

If $\alpha < \beta < \gamma$ are ordinals in the interval (ω_1, ω_2) of uncountable cofinality and $f : \omega_1 \to 2^\omega$, then we say that (α, β, γ) *codes f (relative to $\vec{C}$)* if there is an ω_1-club $\langle N_\xi : \xi < \omega_1\rangle$ in $[\gamma]^{\aleph_0}$ such that

- $\{\omega_1, \alpha, \beta\} \subseteq N_0$;
- for all $\nu < \omega_1$ and all $\xi \in \{\omega_1, \alpha, \beta, \gamma\}$, $\sup(N_\nu \cap \xi)$ is a limit ordinal;
- for all $\nu_0 < \nu_1 < \omega_1$ and all $\xi \in \{\omega_1, \alpha, \beta, \gamma\}$, $\sup(N_{\nu_0} \cap \xi) < \sup(N_{\nu_1} \cap \xi)$;
- for every limit $\nu < \omega_1$, there is a $\nu_0 < \nu$ such that if $\nu_0 < \xi < \nu$, then

  $$\Delta(\bar{s}(N_\xi, N_\nu), f(N_\nu \cap \omega_1)) \geq n(N_\xi, N_\nu),$$

  where the functions s and n are computed using the parameters $\vec{C}$, α, β, and γ.

It is not difficult to show that if (α, β, γ) codes both f and g with respect to some $\vec{C}$, then there is a closed unbounded set of δ such that $f(\delta) = g(\delta)$.

Let us pause for a moment to note that the assertion *for some $\vec{C}$, every f is coded by some triple* (α, β, γ) implies that $2^{\aleph_0} = 2^{\aleph_1}$. To see this, define $F(\overline{\mathcal{N}}, \bar{\alpha}, \bar{\beta}) = 1$ whenever $\mathcal{N}$ is an ω_1-club in $[\gamma]^{\aleph_0}$, $\alpha < \beta < \gamma < \omega_2$ are as above, M is a countable elementary submodel of $H(\omega_2)$ containing $\{\mathcal{N}, \alpha, \beta\}$, $s(N_\xi, N_\nu)(0) = 1$ for a cobounded set of $\xi < \nu = M \cap \omega_1$, and $(\bar{M}, \in, \overline{\mathcal{N}}, \bar{\alpha}, \bar{\beta})$ is the collapse of $(M, \in, \mathcal{N}, \alpha, \beta)$. Now let $g\colon \omega_1 \to 2$ be given and define $f(\delta)$ to be the sequence which takes the constant value $g(\delta)$. If $\mathcal{N}$ witnesses that (α, β, γ) codes f, and

M is a countable elementary submodel of $H(\omega_2)$ containing $\{\mathcal{N}, \alpha, \beta\}$, then $F(\overline{\mathcal{N}}, \bar{\alpha}, \bar{\beta}) = g(\delta)$. By Theorem 2.1, this implies $2^{\aleph_0} = 2^{\aleph_1}$.

Define ψ_1 to be the assertion that for every $A : \omega_1 \to 2$ and for every ladder system $\vec{C}$, there is a triple (α, β, γ) and a function $f : \omega_1 \to 2^\omega$ such that (α, β, γ) codes f relative to $\vec{C}$ and for each $\delta < \omega_1$, $f(\delta)$ is a code for $A \restriction \delta$. We will prove in Section 4 that the conjunction of ψ_1 and CH can be forced over any model in which there is an inaccessible limit of measurable cardinals.

Now we will turn to the task of defining a Π_2-sentence ψ_2 which, together with ψ_1, provides a solution to Problem 1.1. Suppose for a moment that $\langle N_\xi : \xi < \omega_1 \rangle$ witnesses that (α, β, γ) codes $A : \omega_1 \to 2$ relative to $\vec{C}$. If X_i $(i < \omega)$ is an infinite increasing sequence in $\{N_\xi : \xi < \omega_1\}$, define the *height* of $\{X_i\}_{i<\omega}$ to be $\delta = \omega_1 \cap \bigcup_{i<\omega} X_i$. Observe that, together with α, β and $\vec{C}$, $\{X_i\}_{i<\omega}$ uniquely determines $A \restriction \delta$. Moreover, $A \restriction \delta$ can be recovered just from the isomorphism type of the structure

$$(N, \in, \omega_1, \alpha, \beta\, ;\, X_i : i < \omega),$$

where $N = \bigcup_{i<\omega} X_i$. We will refer to a structure arising in this way as a ψ_1-structure and say that this structure *codes* $A \restriction \delta$. The statement ψ_2 is the assertion that for every ladder system $\vec{C}$, every triple $\alpha < \beta < \gamma$ of ordinals strictly between ω_1 and ω_2 and every ω_1-club $\mathcal{N}$ in $[\gamma]^{\aleph_0}$, there is a function $f : \omega_1 \to 2^{<\omega}$ such that for every limit $\delta < \omega_1$, $f \restriction C_\delta$ codes (in the sense discussed at the end of the introduction) the transitive collapse of a structure

$$(N, \in, \omega_1, \alpha, \beta\, ;\, X_i\, :\, i < \omega),$$

where $\{X_i\}_{i<\omega}$ is an increasing sequence in $\mathcal{N}$ of height greater than δ and $N = \bigcup_{i<\omega} X_i$. In Section 3, we will prove that ψ_2 is relatively consistent with CH.

We now have the following proposition.

Proposition 2.2. *$\psi_1 \wedge \psi_2$ implies $2^{\aleph_0} = 2^{\aleph_1}$. In fact, $2^{\aleph_0} = 2^{\aleph_1}$ follows from the existence of a ladder system $\vec{C}$ for which the conjunction of ψ_1 and ψ_2, both relative to $\vec{C}$, holds.*

Proof. Fix a ladder system $\vec{C}$ and suppose that ψ_1 and ψ_2 are true. If $t : \delta \to 2^{<\omega}$ for some countable limit ordinal δ and if $t \restriction C_\delta$ codes a ψ_1-structure which in turn codes $A \restriction \delta^*$ for some $\delta^* > \delta$ and $A : \omega_1 \to 2$, then define $F(t) = A(\delta)$. Now if (α, β, γ) codes $g : \omega_1 \to 2$ relative to $\vec{C}$ as witnessed by $\mathcal{N}$ and $f : \omega_1 \to 2^{<\omega}$ witnesses the corresponding instance of ψ_2, then $F(f \restriction \delta) = g(\delta)$ for every limit ordinal δ. By Theorem 2.1, $2^{\aleph_0} = 2^{\aleph_1}$. □

We will finish this section by mentioning that both ψ_1 and ψ_2 imply that $\diamondsuit$ fails. Let us say that an ω_1-club of $[\gamma]^\omega$ (for some $\gamma < \omega_2$ of uncountable cofinality) is *typical* in case for all $\nu_0 < \nu_1 < \omega_1$, $N_{\nu_0} \cap \omega_1$ and $\sup(N_{\nu_0})$ are limit ordinals, $(N_{\nu_0} \cap \omega_1) + \omega \leq N_{\nu_1} \cap \omega_1$, and $\sup(N_{\nu_0}) < \sup(N_{\nu_1})$. The following fact shows that our methods do not extend to show nonexistence of a Π_2-maximal model for $\diamondsuit$.

Fact 2.3. *$\diamondsuit$ implies the failure of ψ_1. In fact, $\diamondsuit$ implies that there is a ladder system $\vec{C}$ with the property that for every ordinal γ in ω_2 of uncountable cofinality and every typical ω_1-club $\langle N_\nu : \nu < \omega_1\rangle$ of $[\gamma]^\omega$ there are stationarily many $\nu < \omega_1$ such that for every $\xi < \nu$, $|C_{N_\nu \cap \omega_1} \cap N_\xi| > |C_{ot(N_\nu)} \cap \sup(\pi``N_\xi)|$, where π is the collapsing function of N_ν.*

Proof. It is easy to fix a natural notion of coding in such a way that for every $\gamma < \omega_2$ and every ω_1-club $\langle N_\nu : \nu < \omega_1\rangle$ of $[\gamma]^\omega$ there is a set $X \subseteq \omega_1$ and the are a closed unbounded set of $\delta < \omega_1$ such that $X \cap \delta$ codes a directed system $\mathcal{S} = \langle \delta_\nu, i_{\nu,\nu'} : \nu \leq \nu' < \delta\rangle$ where, for all $\nu \leq \nu' < \delta$, $\delta_\nu = ot(N_\nu)$ and $i_{\nu,\nu'} = \pi_{N_{\nu'}} \circ \pi^{-1}_{N_\nu}$ (where π_{N_ν} denotes the collapsing function of N_ν). Let us fix such a notion of coding. Let $\vec{X} = (X_\nu)_{\nu<\omega_1}$ be a $\diamondsuit$-sequence. We define from $\vec{X}$ a ladder system $\vec{C} = \langle C_\delta : \delta \in Lim(\omega_1)\rangle$ in the following way.

Let $\delta \in Lim(\omega_1)$ and suppose X_δ codes a directed system $\mathcal{S} = \langle \delta_\nu, i_{\nu,\nu'} : \nu \leq \nu' < \delta\rangle$ with well-founded direct limit, where the δ_ν's are countable limit ordinals above β, and each $i_{\nu,\nu'}$ is an order-preserving map from δ_ν to $\delta_{\nu'}$, $i_{\nu,\nu'} \neq id$. Suppose that for all $\nu < \delta$ it is true that $crit(i_{\nu,\nu+1})$ is a limit ordinal and $crit(i_{\nu,\nu+1}) + \omega \leq crit(i_{\nu+1,\nu+2})$, where $crit(i_{\nu,\nu'})$ is the least ordinal moved by $i_{\nu,\nu'}$, and that $\sup(range(i_{\nu,\nu+1})) < \delta_{\nu+1}$. Let η be the direct limit of $\mathcal{S}$ and let $i_{\nu,\delta} : \delta_\nu \to \eta$ be the corresponding limit map for each $\nu < \delta$. We assume η is an ordinal. Suppose, in addition, that none of C_δ, C_η have been defined yet. Then we pick C_δ and C_η in such way that for every $\nu < \delta$, $|C_\delta \cap crit(i_{\nu,\delta})|$ is bigger than $|C_\eta \cap \sup(range(i_{\nu,\delta}))|$. Now, using the fact that $\vec{X}$ is a $\diamondsuit$-sequence it is not difficult to check that $\vec{C}$ is a ladder system as required. □

On the other hand it is easy to see that $\diamondsuit$ – and in fact $\clubsuit$ – implies the failure of ψ_2. To see this, let $\langle C_\delta : \delta \in Lim(\omega_1)\rangle$ be a $\clubsuit$–sequence. Suppose that $f : \omega_1 \to 2^{<\omega}$ is such that for all limit $\delta < \omega_1$ there is a co-finite set $X \subseteq C_\delta$ such that $\bigcup f[X]$ is a member of 2^ω. There is then some $n < \omega$ such that $S = \{\nu \in \omega_1 : |f(\nu)| = n\}$ is unbounded in ω_1. But if δ is such that $C_\delta \subseteq S$, then $\bigcup f[C_\delta]$ is finite, which is a contradiction.

3. Iteration theorems

In this section we will review and adapt Eisworth's general framework for verifying that an iteration of forcings does not add new reals. We will need two preservation results, one which is essentially established in [6] and one which is an adaptation of the result in [6] to iterations of totally proper α-semiproper forcings. In the course of the section, we will also establish that ψ_2 is relatively consistent with CH.

Before we begin we will review some of the definitions which we will need in this section. A *forcing* $\mathbb{Q}$ is a partial order with a greatest element $1_{\mathbb{Q}}$. A cardinal θ is *sufficiently large* for a forcing $\mathbb{Q}$ if $\mathcal{P}(\mathcal{P}(\mathbb{Q}))$ is an element of $H(\theta)$. We will say that M *is a suitable model for* $\mathbb{Q}$ if $\mathbb{Q}$ is in M and M is a countable elementary submodel of $H(\theta)$ for some θ which is sufficiently large for $\mathbb{Q}$. If M is a suitable model for $\mathbb{Q}$ and q is in $\mathbb{Q}$, then we will say that q is $(M, \mathbb{Q})$*-generic* if whenever $r \leq q$ and $D \in M$ is a dense subset of $\mathbb{Q}$, r is compatible with an element of $D \cap M$. If, moreover, $\{p \in \mathbb{Q} \cap M : q \leq p\}$ is an $(M, \mathbb{Q})$-generic filter, then we say that q is *totally* $(M, \mathbb{Q})$*-generic.* $\mathbb{Q}$ is *(totally) proper* if whenever M is a suitable model for $\mathbb{Q}$ and q is in $\mathbb{Q} \cap M$, q has a (totally) $(M, \mathbb{Q})$-generic extension. It is easily verified that a forcing is totally proper if and only if it is proper and does not add any new reals.

Remark 3.1. It is important to note that if $\mathbb{Q}$ is totally proper and M is suitable for $\mathbb{Q}$, it need not be the case that every $(M, \mathbb{Q})$-generic condition is totally $(M, \mathbb{Q})$-generic. It is true that every $(M, \mathbb{Q})$-generic condition can be *extended* to a totally $(M, \mathbb{Q})$-generic condition. This distinction is very important in the discussion of when an iteration of forcings adds new reals.

A *suitable tower* (in $H(\theta)$) for $\mathbb{Q}$ is a set $\mathcal{N} = \{N_\xi : \xi < \eta\}$ (for some ordinal η) such that for some θ which is sufficiently large for $\mathbb{Q}$:

- each N_ξ is a countable elementary submodel of $H(\theta)$ having $\mathbb{Q}$ as a member;
- if $\nu < \eta$ is a limit ordinal, then $N_\nu = \bigcup_{\xi<\nu} N_\xi$;
- if $\nu < \eta$ is a successor ordinal, then $\{N_\xi : \xi < \nu\}$ is in N_ν.

Since a tower is naturally ordered by $\in$, we notationally identify it with the corresponding sequence. A condition q is $(\mathcal{N}, \mathbb{Q})$-generic if it is $(N, \mathbb{Q})$-generic for each N in $\mathcal{N}$. A partial order $\mathbb{Q}$ is η*-proper* if whenever $\mathcal{N} = \langle N_\xi : \xi < \eta \rangle$ is a suitable tower for $\mathbb{Q}$ and q is in N_0, then q has a $(\mathcal{N}, \mathbb{Q})$-generic extension. If a forcing is η-proper for every $\eta < \omega_1$, we will say that it is $(<\omega_1)$*-proper.*

Now we will return to our discussion of iterated totally proper forcing.

Definition 3.2. Suppose that η is a countable ordinal and $\mathbb{P} * \dot{\mathbb{Q}}$ is an iteration of forcings such that $\mathbb{P}$ is η-proper. The iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is *η-complete* if *whenever*

(1) $\langle N_\xi : \xi < 1 + \eta \rangle$ is a suitable tower of models for $\mathbb{P} * \dot{\mathbb{Q}}$,
(2) $G \subseteq \mathbb{P} \cap N_0$ is $(N_0, \mathbb{P})$-generic, and
(3) $(p, \dot{q})$ is in $\mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ with p in G,

then there is a $G^* \subseteq \mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ extending G with $(p, \dot{q}) \in G^*$ such that whenever r is a lower bound for G which is $(\langle N_\xi : \xi < 1+\eta \rangle, \mathbb{P})$-generic, then r forces G^*/G has a lower bound in $\dot{\mathbb{Q}}$.

Notice that if $\eta < \zeta$ and $\mathbb{P}*\dot{\mathbb{Q}}$ is η-complete, then $\mathbb{P}*\dot{\mathbb{Q}}$ is ζ-complete. By routine adaptations to the proof of Theorem 4 of [6], we obtain the following iteration theorem.

Theorem 3.3. *Let η, γ be ordinals, with $\eta < \omega_1$, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \,:\, \alpha < \gamma \rangle$$

be a countable support iteration with countable support limit $\mathbb{P}_\gamma$. Suppose that for all $\alpha < \gamma$,

- $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ *is $(<\omega_1)$–proper,*
- *the iteration $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ is η–complete.*

Then $\mathbb{P}_\gamma$ is totally proper.

We will now argue that this theorem is sufficient to prove that the conjunction of ψ_2 and CH can be forced over any model ZFC. By performing a preliminary forcing if necessary, we may assume that our ground model satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Suppose that $\vec{C}$, α, β, γ, and $\mathcal{N}$ represent an instance of ψ_2, i.e., $\vec{C}$ is a ladder system on ω_1, $\omega_1 < \alpha < \beta < \gamma < \omega_2$ and $\mathcal{N}$ is an ω_1-club contained in $[\gamma]^{\aleph_0}$. Define $\mathbb{Q} = \mathbb{Q}_{\vec{C},\alpha,\beta,\mathcal{N}}$ to be the collection of all q such that the domain of q is η for some countable limit ordinal η, q maps into $2^{<\omega}$, and q satisfies the conclusion of ψ_2 for $\delta \leq \eta$. Note that $\mathbb{Q}$ has cardinality $2^{\aleph_0} = \aleph_1$. A slightly involved but completely standard argument shows that $\mathbb{Q}$ is totally $(<\omega_1)$-proper. This is in fact essentially the same argument as the proof that the forcing to uniformize a ladder system coloring by countable approximations is $(<\omega_1)$-proper; see [15]. Since length-ω_2 countable support iterations of proper forcings of size $\aleph_1$ are $\aleph_2$-c.c. (see for instance Theorem 2.10 of [1]), standard book-keeping arguments reduce our task to verifying that an iteration of forcings of the form $\mathbb{Q}_{\vec{C},\alpha,\beta,\mathcal{N}}$ is ω-complete.

Lemma 3.4. *Suppose that*

- $\mathbb{P}$ *is a totally proper, ω-proper forcing;*
- *for each $\delta \in Lim(\omega_1)$, $\dot{C}_\delta$ is a $\mathbb{P}$-name for a cofinal subset of δ of ordertype ω;*
- *$\dot{\vec{C}}$ is a $\mathbb{P}$-name for the ladder system on ω_1 induced by the names $\dot{C}_\delta$ ($\delta \in Lim(\omega_1)$);*
- *$\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ are $\mathbb{P}$-names for an increasing sequence of ordinals between ω_1 and ω_2;*
- *$\dot{\mathcal{N}}$ is a $\mathbb{P}$-name for an ω_1-club contained in $[\dot{\gamma}]^{\aleph_0}$;*
- *$\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for the partial order $\mathbb{Q}_{\dot{\vec{C}},\dot{\alpha},\dot{\beta},\dot{\mathcal{N}}}$.*

Then $\mathbb{P} * \dot{\mathbb{Q}}$ is an ω-complete iteration.

Proof. Let $\langle N_k : k < \omega \rangle$ be a tower of models with $\mathbb{P} * \dot{\mathbb{Q}}$ in N_0, $G \subseteq \mathbb{P} \cap N_0$ be an $(N_0, \mathbb{P})$-generic filter, and $(p, \dot{q})$ be in $\mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ such that p is in G. Notice that some condition in G decides $\dot{q}$, $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ to be some q, α, β and γ, respectively. Let r be a real which codes the transitive collapse of

$$(\bigcup_{k<\omega} N_k \cap \gamma, \in, \alpha, \beta; N_k \cap \gamma : k < \omega).$$

The key point is that if $\bar{p}$ is $(\langle N_k : k < \omega \rangle, \mathbb{P})$-generic, then $\bar{p}$ forces $N_k \cap \gamma$ is in $\dot{\mathcal{N}}$ for all $k < \omega$.

Set $\delta = N_0 \cap \omega_1$. Notice that there is a ladder $\hat{C}_\delta$ on δ such that whenever C' is a ladder on δ which is in N_1, then $C' \setminus \hat{C}_\delta$ is finite and $\hat{C}_\delta$ consists only of ordinals not in the domain of $\dot{q}$ as decided by G. In particular if $\bar{p}$ is $(N_1, \mathbb{P})$-generic, then $\bar{p}$ forces that $\dot{C}_\delta$ is contained in $\hat{C}_\delta$ except for a finite set. Let f_δ be a bijection between $\hat{C}_\delta$ and $\{r \restriction n : n < \omega\}$. Standard arguments (similar to those referred to before the statement of the lemma) now allow us to build a $G^* \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ such that $(p, \dot{q})$ is in G^* and if

$$g = \cup\{s \in H(\omega)^{<\delta} : \exists p \in G(p, \check{s}) \in G^*\}$$

then f_δ is a restriction of g. It follows that whenever r is a lower bound for G which is $(\langle N_k : k < \omega \rangle, \mathbb{P})$-generic, then r forces G^*/G has a lower bound. □

Unlike ψ_2, it is generally not possible to force an instance of ψ_1 with an ω-proper forcing. Fortunately, assuming the existence of 3 measurable cardinals, there is a forcing to force an instance of ψ_1 which is $(< \omega_1)$-semiproper. In the remainder of this section, we formulate and prove a version of Theorem 4 of [6] which applies to iterations of totally proper $(<\omega_1)$-semiproper iterands. This seems to provide the first example of a forcing which is proper and $(<\omega_1)$-semiproper, but not $(<\omega_1)$-proper.

In order to state this definition we will borrow the following pieces of notation from [6] (originating in [15]): Given a set N and a forcing notion $\mathbb{P} \in N$, $\mathrm{Gen}(N, \mathbb{P})$ denotes the set of all $(N, \mathbb{P})$-generic filters $G \subseteq N \cap \mathbb{P}$. Furthermore, if $p \in N \cap \mathbb{P}$, $\mathrm{Gen}(N, \mathbb{P}, p) = \{G \in \mathrm{Gen}(N, \mathbb{P}) : p \in G\}$ and $\mathrm{Gen}^+(N, \mathbb{P}, p)$ denotes the set of all $G \in \mathrm{Gen}(N, \mathbb{P}, p)$ such that G has a lower bound in $\mathbb{P}$.

Given a partial order $\mathbb{P}$, a regular cardinal θ which is sufficiently large for $\mathbb{P}$, and a countable $N \prec H(\theta)$ with $\mathbb{P} \in N$, we say that a condition $p \in \mathbb{P}$ is $(\mathbb{P}, N)$-*semi-generic* if $p \Vdash \tau \in (\check{\omega}_1 \cap \check{N})$ for all $\mathbb{P}$-names τ for countable ordinals. Given a countable ordinal η, $\mathbb{P}$ is said to be η-*semi-proper* if for every suitable tower $\langle N_\xi : \xi < \eta \rangle$ with $\mathbb{P} \in N_0$, and every $p \in \mathbb{P} \cap N_0$, there is a condition $q \leq p$ in $\mathbb{P}$ which is $(\langle N_\xi : \xi < \eta \rangle, \mathbb{P})$-semi-generic, i.e., which is $(N_\xi, \mathbb{P})$-semi-generic for all $\xi < \eta$.

Given a countable elementary substructure N of $H(\theta)$ with $\mathbb{P} \in N$, and given $G \in Gen(N, \mathbb{P})$, we let $N[G]$ denote the set of G-interpretations of $\mathbb{P}$-names which are in N (see section 3 from [6] for details). From now on, given such N and G we will usually identify members x of $N[G]$ with one of its $\mathbb{P}$–names $\dot{x} \in N$. This allows us to give meaning to such notation as $\dot{x} \in \dot{y}$. The particular places where we make such identifications should be clear from the context.

In the following definition, we have replaced the condition that r be $(\langle N_\xi : \xi < 1 + \eta \rangle, \mathbb{P})$-generic from Definition 3.2 with the condition that it be merely $(\langle N_\xi : \xi < 1 + \eta \rangle, \mathbb{P})$-semi-generic. We call the corresponding notion η-semi-completeness, and note that it is a stronger condition than η-completeness.

Definition 3.5. Suppose that η is a countable ordinal and $\mathbb{P} * \dot{\mathbb{Q}}$ be an iteration of forcings such that $\mathbb{P}$ is η-proper. The iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is η-*semi-complete* if *whenever*

(1) $\langle N_\xi : \xi < 1 + \eta \rangle$ is a suitable tower of models for $\mathbb{P} * \dot{\mathbb{Q}}$,
(2) $G \subseteq \mathbb{P} \cap N_0$ is $(N_0, \mathbb{P})$-generic, and
(3) $(p, \dot{q})$ is in $\mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ with p in G,

then there is a $G^* \subseteq \mathbb{P} * \dot{\mathbb{Q}} \cap N_0$ extending G with $(p, \dot{q}) \in G^*$ such that whenever r is a lower bound for G which is $(\langle N_\xi : \xi < 1 + \eta \rangle, \mathbb{P})$-semi-generic, then r forces G^*/G has a lower bound in $\dot{\mathbb{Q}}$.

Even though we will be working exclusively with iterations of proper forcings in this paper, we will use the terminology of *revised countable support* iterations in order to prove the analogue of Theorem 3.3 for η-semi-complete iterations. By *revised countable support* (RCS) we mean either the original presentation of RCS due to Shelah [15], or the later reformulation due to Miyamoto [12]. Theorems 3.6 and 3.7 below are used in [11]. They in turn rely on facts that are proved in [12] but not explicitly stated in [15]. In [11] it is claimed, erroneously, that these facts apply to the presentation of RCS due to Donder and Fuchs [5]. Under the Donder-Fuchs presentation of RCS, an RCS iteration of proper forcings it identical to the corresponding countable support iteration, for which Theorem 3.6 fails. For the Shelah version and the Miyamoto versions, an RCS limit of proper forcings and the corresponding countable support limit are merely isomorphic on a dense set. It follows, in the end, that Theorem 3.8 is true when one uses countable support in place of revised countable support, though again our proof of this fact requires RCS. A similar situation holds in [11].

To facilitate the statements below, we let "$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}} : \alpha < \gamma \rangle$ has RCS limit $\mathbb{P}_\gamma$" include the case that $\gamma = \beta + 1$ and $\mathbb{P}_\gamma = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ (and similarly for countable support).

Theorem 3.6. *Suppose that γ is an ordinal and that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is an RCS iteration with RCS limit $\mathbb{P}_\gamma$. Fix $p \in \mathbb{P}_\gamma$ and $\beta < \gamma$. Suppose that A is a maximal antichain in $\mathbb{P}_\beta$ below $p \restriction \beta$, and $f \colon A \to P$ is a function such that for each $a \in A$, $f(a) \leq p$ and $f(a) \restriction \beta = a$. Then there is a condition $p' \leq p$ in $\mathbb{P}_\gamma$ such that $p \restriction \beta = p' \restriction \beta$ and each $a \in A$ forces that $p' \restriction [\beta, \gamma) = f(a) \restriction [\beta, \gamma)$.*

Theorem 3.7. *Suppose that γ is an ordinal and that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ is an RCS iteration with RCS limit $\mathbb{P}_\gamma$. Then for each $p \in \mathbb{P}_\gamma$ and each $q \leq p$ in $\mathbb{P}_\gamma$ there exist $\beta < \gamma$ and $r \in \mathbb{P}_\beta$ such that $r \leq q \restriction \beta$ and either $r \Vdash cof(\check{\gamma}) = \omega$ or $r \Vdash \forall \alpha \in (\check{\beta}, \check{\gamma})\, p(\alpha) = 1_{\dot{\mathbb{Q}}_\alpha}$.*

The following is our extension of Theorem 3.3 to η-semi-complete iterations. We will introduce two more useful facts before we start the proof.

Theorem 3.8. *Let η, γ be ordinals, with $\eta < \omega_1$, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \, : \, \alpha < \gamma \rangle$$

be a countable support iteration with countable support limit $\mathbb{P}_\gamma$. Suppose that for all $\alpha < \gamma$,

- $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ *is $(<\omega_1)$–semi-proper,*

- *the iteration* $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ *is* η*–semi-complete,*
- $\Vdash_{\alpha+1} |\mathbb{P}_\alpha| \leq \aleph_1$.

Then $\mathbb{P}_\gamma$ *is totally proper.*

A proof of the following fact appears in [11].

Fact 3.9. *Let* η *be a countable ordinal, let* γ *be an ordinal, and let*

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \,:\, \alpha < \gamma \rangle$$

be a revised countable support iteration with RCS limit $\mathbb{P}_\gamma$*. Suppose that for all* $\alpha < \gamma$*,*

- $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ *is* η*–semiproper, and*
- $\Vdash_{\alpha+1} |\mathbb{P}_\alpha| \leq \aleph_1$.

Then $\mathbb{P}_\gamma$ *is* η*–semiproper.*

The proof of Theorem 3.8 uses the following lemma, a simplified (and ostensibly weaker) version of Lemma 4.10 from the proof of (our) Fact 3.9 in [11].

Lemma 3.10. *Let* γ *be an ordinal, and let* η *be a countable ordinal. Suppose that* $\mathbb{P}_\gamma$ *is the RCS limit of an RCS iteration* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$ *such that for each* $\alpha < \gamma$*,*

- $1_{\mathbb{P}_\alpha}$ *forces* $\dot{\mathbb{Q}}_\alpha$ *to be* η*-semi-proper, and*
- $1_{\mathbb{P}_{\alpha+1}}$ *forces* $\mathbb{P}_\alpha$ *to have cardinality* $\aleph_1$*.*

Let θ *be sufficiently large for* $\mathbb{P}_\gamma$*. Fix* $\alpha \leq \beta \leq \gamma$*, and fix a suitable tower* $\langle N_\xi : \xi < \eta \rangle$ *for* $\mathbb{P}_\gamma$ *with* $\alpha, \beta \in N_0$*. Let* $s \in \mathbb{P}_\gamma$ *and* $r \in \mathbb{P}_\alpha$ *be such that*

- r *is* $(N_\xi, \mathbb{P}_\alpha)$*-semi-generic for all* $\xi < \eta$*,*
- $s \restriction \alpha \geq r$*,*
- r *forces that for some* $t \in \mathbb{P}_\gamma \cap N_0$*,* $s \restriction [\alpha, \gamma) = t \restriction [\alpha, \gamma)$*.*

Then there exists $r^\dagger \in \mathbb{P}_\beta$ *such that*

- $r^\dagger$ *is* $(N_\xi, \mathbb{P}_\beta)$*-semi-generic for all* $\xi < \eta$*,*
- $r^\dagger \leq s \restriction \beta$*,*
- $r^\dagger \restriction \alpha = r$*,*

To prove Theorem 3.8, let θ be a regular cardinal which is sufficiently large for $\mathbb{P}_\gamma$. Let $N \prec H(\theta)$ be countable with $\mathbb{P}$ and η in N, and let $p \in \mathbb{P}_\gamma \cap N$ be an arbitrary condition. We must produce a totally $(N, \mathbb{P}_\gamma)$-generic condition $q \leq p$. For each $\alpha \in N \cap (\gamma+1)$, let α^* denote the ordertype of $N \cap \alpha$. Fix a suitable tower $\overline{N} = \langle N_\xi : \xi \leq \eta\gamma^* \rangle$ with $N_0 = N$. The following claim is a variation of Claim 6.2 of [6]. In order to facilitate the statement of the claim, we let $N_{\eta\gamma^*+1}$ stand for V.

Claim 3.11. *Given $\alpha < \beta$ in $N_0 \cap (\gamma + 1)$, $p \in \mathbb{P}_\beta$ and*

$$G \in Gen^+(N_0, \mathbb{P}_\alpha, p \restriction \alpha) \cap N_{\eta\alpha^*+1},$$

there is a $G^\dagger \in Gen(N_0, \mathbb{P}_\beta, p) \cap N_{\eta\beta^*+1}$ such that whenever $r \in \mathbb{P}_\alpha$ is a lower bound for G that is $(N_\xi, \mathbb{P}_\alpha)$-semi-generic for all $\xi \in (\eta\alpha^*, \eta\gamma^*]$, there is an $r^\dagger \in \mathbb{P}_\beta$ such that

1. *$r^\dagger$ is a lower bound for $G^\dagger$,*
2. *$r^\dagger \restriction \alpha = r$,*
3. *$r^\dagger$ is $(N_\xi, \mathbb{P}_\alpha)$-semi-generic for every $\xi \in (\eta\beta^*, \eta\gamma^*]$.*

Theorem 3.8 follows from taking $\alpha = 0$ and $\beta = \gamma$ in Claim 3.11. Inducting primarily on γ, we assume that the claim holds for all $\gamma' < \gamma$ in place of γ, for this fixed sequence of N_ξ's. This will be useful in the limit case below.

Remark 3.12. Item (1) above implies that $\{q \restriction \mathbb{P}_\alpha \mid q \in G^\dagger\} = G$, since otherwise, these two generic filters could not have the same lower bound r in common.

Since $\mathbb{P}_0$ is the trivial forcing, the case $\alpha = 0$, $\beta = 1$ follows from the assumption that $\dot{\mathbb{Q}}_0$ is totally proper (and $(< \omega_1)$-semi-proper).

Now consider the case where $\beta = \beta_0 + 1$. We are given a $G \in Gen^+(N_0, \mathbb{P}, p \restriction \alpha) \cap N_{\eta\alpha^*+1}$, and, applying the induction hypothesis we may fix a $G_0^\dagger \in Gen(N_0, \mathbb{P}_{\beta_0}, p \restriction \beta_0) \cap N_{\eta\beta_0^*+1}$ satisfying the claim with β_0 in the role of β. Since $\mathbb{P}_\alpha$ is $(<\omega_1)$-semi-proper, the conclusion of the claim implies that $G_0^\dagger \in Gen^+(N_0, \mathbb{P}_{\beta_0}, p \restriction \beta_0)$. We apply the definition of "$\dot{\mathbb{Q}}_{\beta_0}$ is η-semi-complete for $\mathbb{P}_{\beta_0}$" in $N_{\eta\beta^*+1}$ with

$$\{N_0\} \cup \{N_\xi : \eta\beta_0^* + 1 \leq \xi \leq \eta\beta^*\},$$

$G_0^\dagger$ and $p(\beta_0)$ in place of $\langle N_\xi : \xi < 1 + \eta\rangle$, $\bar{G}$ and $\dot{q}$ there. This gives us an $G^\dagger \in Gen(N_0, \mathbb{P}_\beta, p)$ extending $G_0^\dagger$ such that whenever $r_0^\dagger$ is a lower bound for $G_0^\dagger$ which is $(\{N_0\} \cup \{N_\xi : \eta\beta_0^* + 1 \leq \xi \leq \eta\beta^*\}, \mathbb{P})$-generic, then $r_0^\dagger$ forces that $G^\dagger/G_0^\dagger$ has a lower bound in $\mathbb{Q}_{\beta_0}$. By elementarity, we may choose such a $G^\dagger$ in $N_{\eta\beta^*+1}$.

Now, whenever $r \in \mathbb{P}_\alpha$ is a lower bound for G that is $(N_\xi, \mathbb{P}_\alpha)$-semi-generic for all $\xi \in (\eta\alpha^*, \eta\gamma^*]$, there is by the choice of $G_0^\dagger$ a condition $r_0^\dagger \in \mathbb{P}_{\beta_0}$ such that

- $r_0^\dagger$ is a lower bound for $G_0^\dagger$,
- $r_0^\dagger \restriction \alpha = r$,
- $r_0^\dagger$ is $(N_\xi, \mathbb{P}_\alpha)$-semi-generic for every $\xi \in (\eta\beta_0^*, \eta\gamma^*]$.

There is a condition $s \in \mathbb{P}_\beta \cap N_{\eta\beta^*+1}$ such that $s \restriction \beta_0$ is $1_{\mathbb{P}_{\beta_0}}$ and $1_{\mathbb{P}_{\beta_0}}$ forces that $s(\beta_0)$ is a lower bound for $G^\dagger/G_0^\dagger$ if such a lower bound exists. By Lemma 3.10, then, there is an $r^\dagger$ as desired, with $r^\dagger \restriction \beta_0 = r_0^\dagger$ and $s \geq r^\dagger$. This takes care of the case where β is a successor ordinal.

Finally, suppose that β is a limit ordinal. Fix a strictly increasing sequence $\langle \alpha_n : n \in \omega \rangle \in N_{\eta\beta^*+1}$ which is cofinal in $N_0 \cap \beta$, with $\alpha_0 = 0$, and let $\langle D_n : n \in \omega \rangle \in N_{\eta\beta^*+1}$ be a listing of the dense open subsets of $\mathbb{P}_\beta$ in N_0.

Claim 3.13. *There exist sequences $\langle p_n : n \in \omega \rangle$, $\langle G_n : n \in \omega \rangle$ in $N_{\eta\beta^*+1}$ such that $p_0 = p$, $G_0 = G$ and, for all $n \in \omega$,*

- *$p_{n+1} \in N_0 \cap D_n$;*
- *$p_{n+1} \leq p_n$;*
- *$G_n \in Gen(N_0, \mathbb{P}_{\alpha_n}, p \restriction \alpha_n) \cap N_{\eta\alpha_n^*+1}$;*
- *whenever $r \in \mathbb{P}_{\alpha_n}$ is a lower bound for G_n that is $(N_\xi, \mathbb{P}_{\alpha_n})$-semi-generic for all $\xi \in (2\alpha_n^*, 2\gamma^*]$, there is an $r^+ \in \mathbb{P}_{\alpha_n+1}$ such that*
  - *r^+ is a lower bound for G_{n+1};*
  - *$r^+ \restriction \alpha_n = r$;*
  - *r^+ is $(N_\xi, \mathbb{P}_{\alpha_n+1})$-semi-generic for any ξ with $2\alpha_{n+1}^* < \xi \leq 2\gamma^*$.*

Given $n \in \omega$, $r \in \mathbb{P}_{\alpha_n}$ and $\delta \in (\alpha_n, \gamma] \cap N_0$, let $A(r, n, \delta^*)$ denote the statement that r is a lower bound for G_n and r is $(N_\xi, \mathbb{P}_{\alpha_n})$-semi-generic for all $\xi \in (\eta\alpha_n^*, \eta\delta^*]$. Then the last item of the claim says that for all $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \alpha_n^*, \gamma^*)$, there exists an $r^+ \in \mathbb{P}_{\alpha_{n+1}}$ such that

- $r^+ \restriction \alpha_n = r$;
- r^+ satisfies $A(r^+, n+1, \alpha_{n+1}^*, \gamma^*)$.

To verify the claim, suppose that p_n and G_n are given. We will verify that p_{n+1} and G_{n+1} exist as described in the claim. First note that $E = \{t \restriction \alpha_n \mid t \in D_n, t \leq p_n\}$ is dense in $\mathbb{P}_{\alpha_n}$ below $p_n \restriction \alpha_n$, and that $E \in N_0$. Since $p_n \restriction \alpha_n \in G_n$, there exists a $t \in E \cap G_n$. Applying the definition of E inside N_0, we get a $p_{n+1} \in N_0 \cap D_n$ with $p_{n+1} \leq p_n$ and $p_{n+1} \restriction \alpha_n \in G_n$, as desired.

Applying the induction hypothesis inside of $N_{\eta\beta^*+1}$, with α_n, α_{n+1} and β in place of α, β and γ, we can find a filter

$$G_{n+1} \in Gen(N_0, \mathbb{P}_{\alpha_{n+1}}, p_{n+1} \restriction \alpha_{n+1})$$

such that for any condition $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \beta^*)$ there is an $r' \in \mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r', n+1, \beta^*)$ such that $r' \restriction \alpha_n = r$. We need to see that for this G_{n+1}, for any condition $r \in \mathbb{P}_{\alpha_n}$ satisfying $A(r, n, \gamma^*)$ there is an $r^+ \in \mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r', n+1, \gamma^*)$ such that $r^+ \restriction \alpha_n = r$.

Fix such an r. Since r satisfies $A(r, n, \gamma^*)$, it satisfies $A(r, n, \beta^*)$. Fix $r' \in \mathbb{P}_{\alpha_{n+1}}$ such that $r' \restriction \alpha_n = r$ and r' satisfies $A(r, n+1, \beta^*)$. In order to apply Lemma 3.10, we want to see that there is an $r'' \in \mathbb{P}_{\alpha_{n+1}}$ satisfying $A(r'', n+1, \beta^*)$ such that $r'' \restriction \alpha_n = r$ and such that $r'' \Vdash_{\mathbb{P}_{\alpha_n}} r''/G_{\alpha_n} \in N_{2\beta^*+1}[G_{\alpha_n}]$, that is, that r forces (in $\mathbb{P}_{\alpha_n}$) that there is a $\mathbb{P}_{\alpha_n}$-name t in $N_{2\beta^*+1}$ such that $r''/G_{\alpha_n} = t_{G_{\alpha_n}}$.

If we force with $\mathbb{P}_{\alpha_n}$ below $r \restriction \alpha_n$, in $V[G_{\alpha_n}]$, $r'/G_{\alpha_n} \in \mathbb{P}_{\alpha_{n+1}}/G_{\alpha_n}$ satisfies condition d, i.e.,

- is a lower bound for $\{s/G_{\alpha_n} : s \in G_{n+1}\}$;
- is semi-generic for $(N_\xi[G_{\alpha_n}], \mathbb{P}_{\alpha_{n+1}}/G_{\alpha_n})$, for all $\xi \in (2\alpha^*_{n+1}, 2\beta^*]$.

So there exists a condition satisfying d in $N_{2\beta^*+1}[G_{\alpha_n}]$.

Let τ be a $\mathbb{P}_{\alpha_n} \restriction r$-name for an element of $\mathbb{P}_{\alpha_{n+1}}/G_n$ in $N_{2\beta^*+1}[G_{\alpha_n}]$ satisfying d. Viewing $\mathbb{P}_{\alpha_{n+1}}$ as $\mathbb{P}_{\alpha_n} * \dot{Q}_{\alpha_n,\alpha_{n+1}}$, let $r'' = (r, \tau)$.

Now apply Lemma 3.10. We have that

- r is $(N_\xi, \mathbb{P}_{\alpha_n})$-semi-generic for all $\xi \in (2\beta^*, 2\gamma^*]$;
- $r'' \restriction \alpha_n = r$;
- r forces that there is a $t \in \mathbb{P}_{\alpha_{n+1}} \cap N_{2\beta^*+1}$ such that $r'' \restriction [\alpha_n, \alpha_{n+1}) = t \restriction [\alpha_n, \alpha_{n+1})$.

Then by the lemma, there exists an r^+ which is $(N_\xi, \mathbb{P}_{\alpha_{n+1}})$-semi-generic for all $\xi \in (2\beta^*, 2\gamma^*]$, such that $r^+ \leq r''$ and $r^+ \restriction \alpha_n = r$. This verifies the claim.

Let $G^\dagger = \{t \in N_0 \cap \mathbb{P}_\beta \mid \exists n\, t \geq p_n\}$. Then

$$G^\dagger \in Gen(N_0, \mathbb{P}_\beta, p) \cap N_{\eta\beta^*+1}.$$

Subclaim 3.1. *$G^\dagger$ has a lower bound.*

To see this, let r be a lower bound for G that is $(N_\xi, \mathbb{P}_\alpha)$-semi-generic for all $\xi \in (\eta\alpha^*, \eta\gamma^*]$. The properties of the sequence $\langle G_n : n \in \omega\rangle$ allow us to build a sequence $\langle r_n : n \in \omega\rangle$ satisfying:

- $r_0 = r$;
- r_n is a lower bound for G_n in $\mathbb{P}_{\alpha^*}$;
- r_n is $(N_\xi, \mathbb{P}_{\alpha_n})$-semi-generic for all $\xi \in (\eta\alpha^*_n, \eta\gamma^*]$;
- $r_{n+1} \restriction \alpha_n = r_n$.

Finally let $r^+ = \bigcup_{n\in\omega} r_n \in \mathbb{P}_\beta$. Let us check that r^+ is a lower bound for $G^\dagger$. First note that by the argument presented in Remark 3.12, $\{q \restriction \alpha_n \mid q \in G_m\} = G_n$, whenever $n \leq m$. When $m \leq n$, $p_m \geq p_n$, so $p_m \restriction \mathbb{P}_{\alpha_n} \geq p_n \restriction \mathbb{P}_{\alpha_n}$. Since for each $n \in \omega$ we have $p_n \restriction \alpha_n \in G_n$, we get that for each such n, $\{p_m \restriction \alpha_n : m \in \omega\} \subseteq G_n$. Since $r \restriction \mathbb{P}_{\alpha_n}$ is a lower bound for G_n, we have that $r \leq p_n$ for all $n \in \omega$, and thus that r is a lower bound for $G^\dagger$. This proves the subclaim, and thereby the limit case and thereby Theorem 3.8.

4. The single step forcing for ψ_1

In this section we examine the single step forcings associated with ψ_1. Before proceeding, we will recall some terminology from [13]. Let X be an uncountable set and let θ be a regular cardinal with $\mathcal{P}([X]^{\aleph_0})$ in $H(\theta)$. $[X]^{\aleph_0}$ is topologized by declaring sets of the form

$$[a, M] = \{N \in [X]^{\aleph_0} : a \subseteq N \subseteq M\}$$

to be open whenever M is in $[X]^{\aleph_0}$ and a is a finite subset of M. If M is a countable elementary submodel of $H(\theta)$ with X in M, then $\Sigma \subseteq [X]^{\aleph_0}$ is *M-stationary* if $M \cap E \cap \Sigma$ is non empty whenever $E \subseteq [X]^{\aleph_0}$ is a club in M. If Σ is a function whose domain is a club of countable elementary submodels of $H(\theta)$, then we say that Σ is an *open stationary set mapping* if $\Sigma(M)$ is open and M-stationary whenever M is in the domain of Σ. If $\mathcal{N} = \langle N_\xi : \xi < \omega_1\rangle$ is a continuous $\subseteq$-chain where $\langle N_\xi : \xi \leq \nu\rangle$ is in $N_{\nu+1}$ for each ν, then we say that $\mathcal{N}$ is a *reflecting sequence* for Σ if whenever $\nu < \omega_1$ is a limit ordinal, there is a $\nu_0 < \nu$ such that

$$N_\xi \cap X \in \Sigma(N_\nu)$$

whenever $\nu_0 < \xi < \nu$. If $\mathcal{N} = \langle N_\xi : \xi \leq \delta\rangle$ is a sequence of countable successor length which has the above properties for all limit $\nu \leq \delta$, then we will say that $\mathcal{N}$ is a *partial reflecting sequence* for Σ. In [13] it is shown that PFA implies all open stationary set mappings admit reflecting sequences and that the forcing $\mathbb{P}_\Sigma$ of all countable partial reflecting sequences for an open stationary set mapping Σ is always totally proper.

Except for trivial cases $\mathbb{P}_\Sigma$ is not ω-proper. Moreover it will be ($<\omega_1$)-semiproper only under rather special circumstances. The following lemma gives a useful sufficient condition for when we can build generic conditions in $\mathbb{P}_\Sigma$ for a given suitable tower of models.

Lemma 4.1. *Let Σ be an open stationary set mapping and let θ be sufficiently large for $\mathbb{P}_\Sigma$. Suppose that $\mathcal{M} = \langle M_\delta : \delta \leq \alpha\rangle$ is a tower of elementary submodels of $H(\theta)$ which is suitable for $\mathbb{P}_\Sigma$ and which is a partial reflecting sequence for Σ. Then every condition in M_0 can be extended to a totally $(\mathcal{M}, \mathbb{P}_\Sigma)$-generic condition.*

Proof. This follows from the properness of $\mathbb{P}_\Sigma$ when $\alpha = 0$, and by the induction hypothesis, elementarity and the properness of $\mathbb{P}_\Sigma$ when α is a successor ordinal. When α is a limit ordinal, choose an increasing sequence $\langle \beta_i : i < \omega\rangle$ tending to α, such that for all δ in the interval $[\beta_0, \alpha)$, $M_\delta \cap X \in \Sigma(M_\alpha \cap H(\theta))$. Note that any condition in $\mathbb{P}_\Sigma$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for all $\delta < \alpha$ will be $(M_\alpha, \mathbb{P}_\Sigma)$-generic. The difficulty in what follows is in ensuring that a tail of the generic sequence we build falls inside of $\Sigma(M_\alpha \cap H(\theta))$. We will ensure that this happens for all members of the sequence containing M_{β_0}. We have that for each δ in the interval $[\beta_0, \alpha)$ there is a finite set $a_\delta \subset M_\delta \cap X$ such that $[a_\delta, M_\delta \cap X] \subset \Sigma(M_\alpha \cap H(\theta))$.

By elementarity and the induction hypothesis, we may assume first that s_0 is a condition in M_{β_0+1} which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for all $\delta \leq \beta_0$, and which extends any given condition $s \in M_0$. We may assume that the last member of s_0 is $M_{\beta_0} \cap X$, and we have then that a tail of s_0 is contained in $\Sigma(M_\alpha \cap H(\theta))$. Suppose now that $i \in \omega$, that s_i is a condition in M_{β_i+1} which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for all $\delta \leq \beta_i$, which extends s_0, whose last member is $M_{\beta_i} \cap X$, and such that every member of s_i containing $M_{\beta_0} \cap X$ is in $\Sigma(M_\alpha \cap H(\theta))$. We show how to choose s_{i+1} satisfying these conditions for $i+1$. If $a_{\beta_{i+1}} \subseteq M_{\beta_i+1}$, then we let s'_i be a condition in M_{β_0+1} extending s_i by one set which contains $a_{\beta_{i+1}}$, and, applying the induction hypothesis and elementarity we let s_{i+1} be a condition in $M_{\beta_{i+1}+1}$ as desired, extending s'_i.

If $a_{\beta_{i+1}}$ is not in M_{β_i+1}, we need to work harder to extend s_i while staying inside $\Sigma(M_\alpha \cap H(\theta))$. In this case, let $a(i,0) = a_{\beta_{i+1}}$ and let $\gamma(i,0)$ be the largest δ in (β_i, β_{i+1}) such that $a(i,0)$ is not contained in M_δ. Let $a(i,1)$ be a finite subset of $M_{\gamma(i,0)} \cap X$ such that

$$[a(i,1), M_{\gamma(i,0)} \cap X] \subseteq \Sigma(M_\alpha \cap H(\theta)).$$

Continue in this way, letting $\gamma(i, j+1)$ be the largest δ in $[\beta_i, \gamma(i,j))$ such that $\delta = \beta_i$ or $a(i, j+1)$ is not contained in M_δ, and, if $\gamma(i, j+1) > \beta_i$, letting $a(i, j+2)$ be a finite subset of $M_{\gamma(i,j+1)} \cap X$ such that

$$[a(i, j+2), M_{\gamma(i,j+1)} \cap X] \subseteq \Sigma(M_\alpha \cap H(\theta)).$$

As the $\gamma(i,j)$'s are decreasing, this sequence must stop at a point where $a(i,j) \subseteq M_{\beta_i+1}$ and $\gamma(i,j) = \beta_i$. Let k be this j. As $\langle a(i,j) : j \leq k \rangle$ is in $M_{\beta_{i+1}+1}$, we can argue in $M_{\beta_{i+1}+1}$, as follows.

Let $t(i,k)$ be a condition in M_{β_i+1} extending s_i such that every member of $t(i,k) \setminus s_i$ contains $a(i,k)$. Applying the induction hypothesis and elementarity, let $s(i,k)$ be a condition in $M_{\gamma(i,k-1)+1}$ extending $t(i,k)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for every $\delta \leq \gamma(i, k-1)$, and whose last member is $M_{\gamma(i,k-1)} \cap X$. For each positive $j < k$, let $t(i,j)$ be a condition in $M_{\gamma(i,j)+1}$ extending $s(i, j+1)$ such that every member of $t(i,j) \setminus s(i, j+1)$ contains $a(i,j)$, and let $s(i,j)$ be a condition in $M_{\gamma(i,j-1)+1}$ extending $t(i,j)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for every $\delta \leq \gamma(i, j-1)$, and whose last element is $M_{\gamma(i,j-1)} \cap X$. Finally, let $t(i,0)$ be a condition in $M_{\gamma(i,0)+1}$ extending $s(i,1)$ such that every member of $t(i,0) \setminus s(i,1)$ contains

$a(i,0)$, and let s_{i+1} be a condition in $M_{\beta_{i+1}+1}$ extending $t(i,0)$ which is $(M_\delta, \mathbb{P}_\Sigma)$-generic for all $\delta \leq \beta_{i+1}$ and whose last member is $M_{\beta_{i+1}} \cap X$.

Then every member of $s_{i+1} \backslash s_i$ is in $\Sigma(M_\alpha \cap H(\theta))$, as desired. Continuing in this way, the union of the s_i's will be the desired condition. $\square$

Now we return to our discussion of ψ_1. Let $\vec{C}$ be a ladder system and let κ_i $(i < 3)$ be an increasing sequence of cardinals greater than ω_2. For a fixed $A : \omega_1 \to 2$ we will define a totally proper forcing $\mathbb{Q}_{A,\vec{\kappa},\vec{C}}$ which collapses κ_2 to have cardinality ω_1 and adds a function $f : \omega_1 \to 2^\omega$ such that $f(\delta)$ is a code for $A \restriction \delta$ for each $\delta < \omega_1$, together with a witness $\mathcal{N}$ to the statement that $(\kappa_0, \kappa_1, \kappa_2)$ codes f with respect to $\vec{C}$. In order to improve readability, we will suppress terms from subscripts which are either clear from the context or which do not influence the truth of a given statement.

The forcing $\mathbb{Q}_{A,\vec{\kappa},\vec{C}}$ is the collection of all q such that:

1. q is a function from some countable successor ordinal $\delta + 1$ into $[\kappa_2]^{\aleph_0}$;
2. q is continuous and strictly $\subseteq$-increasing;
3. if $\nu \leq \delta$ is a limit ordinal, then there is a $\nu_0 < \nu$ and an $r \in 2^\omega$ such that r codes $A \restriction \nu$ and for all $\nu_0 < \xi < \nu$,

$$\Delta(\bar{s}_{\vec{\kappa}}(N_\xi, N_\nu), r) \geq n(N_\xi, N_\nu).$$

This forcing can be viewed as a two step iteration in which we first add, by countable approximations, a function $f : \omega_1 \to 2^\omega$ with the property that $f(\delta)$ codes $A \restriction \delta$ for each δ. Then we force to add a reflecting sequence (using the partial order described above) for the set mapping Σ_f, where $\Sigma_f(N)$ is the set of all M in $[\kappa_2]^{\aleph_0}$ such that $M \subseteq N$, $M \cap \kappa$ is bounded in $N \cap \kappa$ for κ in $\{\omega_1, \kappa_0, \kappa_1, \kappa_2\}$ and

$$\Delta(\bar{s}_{\vec{\kappa}}(M, N), f(N \cap \omega_1)) \geq n(M, N).$$

It is not difficult to verify that this is an open set mapping and it will follow from arguments below that it is in fact an open stationary set mapping. Hence $\mathbb{Q}_A$ can be regarded as a two step iteration of a σ-closed forcing followed by a forcing of the form $\mathbb{P}_\Sigma$.

Our goal in this section is to prove the following two lemmas. It follows from Theorem 3.8 and standard book-keeping and chain condition arguments that if there is a inaccessible cardinal which is a limit of measurable cardinals, then there is a proper forcing extension with the same reals which satisfies ψ_1.

Lemma 4.2. *If κ_i $(i < 3)$ is an increasing sequence of measurable cardinals, then $\mathbb{Q}_{A,\vec{\kappa}}$ is $(<\omega_1)$-semiproper.*

Lemma 4.3. *If $\mathbb{P}$ is a totally proper forcing and $\vec{\kappa}$, $\vec{C}$, and $\dot{A}$ are $\mathbb{P}$-names for objects as described above, then $\mathbb{P} * \dot{\mathbb{Q}}_{\dot{A}}$ is 1-semicomplete. In particular $\mathbb{Q}_{\dot{A}}$ is totally proper.*

Remark 4.4. The reader may be puzzled as to why we constructed $\mathbb{Q}_A$ by first forcing to produce the function f, since there are certainly functions f in V such that $f(\delta)$ codes $A \restriction \delta$. The problem arises in proving Lemma 4.3 — the argument below does not go through unless we force the function f as we are building the corresponding reflecting sequence.

Remark 4.5. It is interesting to note that it is much easier to obtain the consistency of $\psi_1[\vec{C}]$ with CH for *some* ladder system $\vec{C}$. Suppose that $\vec{C}$ is a ladder system on ω_1, $\mathbb{P}$ is a totally proper forcing, and $\dot{A}$ is a $\mathbb{P}$-name for an element of 2^{ω_1}. If M is a suitable model for $\mathbb{P} * \dot{\mathbb{Q}}_{\dot{A},\vec{C}}$ and p is totally $(M, \mathbb{P})$-generic, and $\dot{q}$ is forced by p to be an element of $M[G] \cap \dot{\mathbb{Q}}_{\dot{A},\vec{C}}$, then there is a $\dot{r}$ such that $(p, \dot{r})$ is a totally $(M, \mathbb{P} * \dot{\mathbb{Q}}_{\dot{A},\vec{C}})$-generic extension of $(p, \dot{q})$ (those familiar with preservation theorems for not adding reals with proper forcing should notice that this almost never works). This allows one to easily prove that if $\vec{C}$ is a fixed ladder system, then we can iterate forcings of the form $\dot{\mathbb{Q}}_{\dot{A},\vec{C}}$ without adding reals (and without the complex iteration machinery which we are about to employ). This shows that if we allow a fixed ladder system as a parameter, we can force $\psi_1[\vec{C}] \wedge$ CH over any model of ZFC (recall that $2^{\aleph_0} = \aleph_1$ follows from the existence of a ladder system $\vec{C}$ such that both $\psi_1[\vec{C}]$ and $\psi_2[\vec{C}]$ hold). The difficulty arises when we want to quantify out the parameter $\vec{C}$ in order to obtain a Π_2-sentence. The final section of [14] contains an example of a pair $\psi_1'[\vec{C}]$ and ψ_2' of Π_2 sentences having these same properties except that $\forall \vec{C} \psi_1'[\vec{C}]$ implies $2^{\aleph_0} = 2^{\aleph_1}$.

In [2], the proof of Lemma 5 actually yields the following lemma (stated in our notation).

Lemma 4.6. *Suppose that κ_i $(i < 3)$ is an increasing sequence of regular cardinals above ω_1 and $\vec{C}^i$ $(i < l)$ (for some $l \in \omega$) is a sequence of ladder systems on ω_1. If M is a countable elementary submodel of $H(\theta)$ for θ sufficiently large and $E \subseteq [\kappa_2]^{\aleph_0}$ is a club in M, then there is an n such that for any σ in $2^{<\omega}$ there is an N in $E \cap M$ such that*

$$o(x^i \setminus n, y^i \setminus n, z^i \setminus n) = \sigma$$

$$n^i(N, M) \leq n$$

for all $i < l$, where x^i, y^i, y^i, and n^i are computed from M and N as in the computation of $s_{\vec{\kappa}}^{\vec{C}^i}(N, M)$.

We will now prove Lemmas 4.2 and 4.3.

Proof of 4.3. Let $\mathbb{P}$ be proper and force that:

- $\dot{\kappa}_i$ $(i < 3)$ is an increasing sequence of regular cardinals above ω_1,
- $\langle \dot{C}_\xi : \xi \in Lim(\omega_1) \rangle$ is a ladder system on ω_1, and
- $\dot{A}$ is a function from ω_1 to 2.

Let $\dot{\mathbb{Q}}$ denote $\dot{\mathbb{Q}}_{\dot{A},\vec{\kappa},\vec{C}}$, $N_0 \in N_1$ be suitable models for $\mathbb{P} * \dot{\mathbb{Q}}$, G be an $(N_0, \mathbb{P})$-generic filter, and q be an element of $\mathbb{Q}^{N_0[G]}$. Observe that there is a condition in G deciding $\dot{\kappa}_i$ to be some κ_i for each $i < 3$. Furthermore, if $\delta = N_0 \cap \omega_1$, then there is an $A : \delta \to 2$ such that for every $\alpha < \delta$, there is a condition in G forcing $\dot{A} \restriction \check{\alpha} = \check{A} \restriction \check{\alpha}$. Fix an r in 2^ω such that r codes $A \restriction \delta$.

Notice that, by CH, if p is $(N_1, \mathbb{P})$–semi-generic and a lower bound for G, then p forces that the value of $\dot{C}_\delta$ is some element of N_1, where $\delta = N_0 \cap \omega_1$ (although it need not decide which is this value). Let C_δ^i $(i < \omega)$ enumerate all cofinal subsets of δ of ordertype ω which are elements of N_1 and let D_i $(i < \omega)$ enumerate all dense open subsets of $\mathbb{Q}$ which are in $N_0[G]$.

We will now build a sequence q_i $(i < \omega)$ of conditions in $\mathbb{Q}^{N_0[G]}$ such that:

- $q_{i+1} \leq q_i$ and q_{i+1} is in $N_0[G] \cap D_i$;
- if ξ is in $\mathrm{dom}(q_{i+1}) \setminus \mathrm{dom}(q_i)$, then $M = q_{i+1}(\xi)$ satisfies
  $$\Delta(\bar{s}^j(M, N_0 \cap \kappa_2), r) \geq n^j(M, N_0 \cap \kappa_2) \text{ for all } j \leq i,$$
  where s^j and n^j are computed using C_δ^j and $\vec{\kappa}$.

If this can be done, then any condition $\bar{p}$ which is an $(N_1, \mathbb{P})$–semi-generic lower bound for G will force that there is some $i_0 < \omega$ such that $\dot{C}_\delta = \check{C}_\delta^{i_0}$, and therefore that q_i $(i < \omega)$ will have a lower bound (namely the union of this sequence).

Suppose that we have constructed q_i and we wish to construct q_{i+1}. Following [13, 3.1] (or Lemma 4.1), it is sufficient to demonstrate that there is an elementary submodel M of $H((2^{\kappa_2})^+)$ such that D_i and q_i are in M and

$$\Delta(\bar{s}^j(M \cap \kappa_2, N_0 \cap \kappa_2), r) \geq n^j(M, N_0 \cap \kappa_2)$$

holds for all $j \leq i$. Let E be the collection of all sets of the form $M \cap \kappa_2$ such that M is a countable elementary submodel of $H((2^{\kappa_2})^+)$

such that q_i and D_i are in M. Let n be given as in Lemma 4.6 and let $\sigma = r \restriction (n+1)$. Find an M in E such that

$$o(x^j \setminus n, y^j \setminus n, z^j \setminus n) = \bar{\sigma},$$

$$n^j(M, N_0 \cap \kappa_2) \leq n$$

for all $j \leq i$. Then $\bar{s}^j(M, N_0 \cap \kappa_2)$ contains $r \restriction n$ as an initial part and therefore

$$\Delta(\bar{s}^j(M, N_0 \cap \kappa_2), r) \geq n^j(M, N_0 \cap \kappa_2).$$

This finishes the proof. □

Now we are ready to turn to the proof of Lemma 4.2. Since $\mathbb{Q}_{A,\vec{\kappa}}$ decomposes as an iteration of a σ-closed partial order followed by a forcing of the form $\mathbb{P}_\Sigma$, it is sufficient to verify the $(<\omega_1)$-semiproperness of the second factor. In fact we will show that if $\vec{\kappa}$ consists of measurable cardinals, $f : \omega_1 \to 2^\omega$ is any function, and $\Sigma_{f,\vec{\kappa}}$ is the open set mapping associated to f as above, then $\mathbb{P}_{\Sigma_{f,\vec{\kappa}}}$ is $(<\omega_1)$-semiproper.

For the rest of this section, let $\vec{\kappa} = \langle \kappa_0, \kappa_1, \kappa_2 \rangle$ be a fixed increasing sequence of 3 measurable cardinals, and fix a normal ultrafilter U_i on each κ_i. Let f be any fixed function from ω_1 to 2^ω and let $\vec{C}$ be a fixed ladder system on ω_1 We will denote $\mathbb{P}_{\Sigma_{f,\vec{\kappa}}}$ by $\mathbb{P}$.

Fix an $\alpha < \omega_1$ for which we wish to prove $\mathbb{P}$ is α-semiproper. Let θ be sufficiently large for $\mathbb{P}$ and let $\lhd$ be a well ordering of $H(\theta)$. For each $\xi < \omega_1$, fix a function $F_\xi : H(\theta)^{<\omega} \to H(\theta)$ closed under composition such that:

- if $M \subseteq H(\theta)$ is nonempty and closed under F_ξ, then the structure $(M, \in, \lhd, F_\eta \restriction M^{<\omega})_{\eta<\xi}$ is an elementary submodel of the structure $(H(\theta), \in, \lhd, F_\eta)_{\eta<\xi}$ (in particular F_ξ-closed sets are F_η-closed for all $\eta < \xi$);
- if M is nonempty and closed under F_0, then α, f, $\vec{\kappa}$, $\vec{U}$, $\vec{C}$, and $\mathbb{P}$ are in M.

Actually only the F_ξ's for non-limit ξ will be relevant.

If X is a subset of $H(\theta)$ and $\mathcal{F}$ is a family of functions $F : H(\theta)^{<\omega} \to H(\theta)$ then we will use $\mathrm{cl}_{\mathcal{F}}(X)$ to denote the closure of X under all $F \in \mathcal{F}$. Also, given $\xi < \omega_1$ we will write $\mathrm{cl}_\xi(X)$ to denote the closure of X under F_ξ. We will use the following well-known facts repeatedly in our argument.

Fact 4.7. *Let U be a normal measure on a measurable cardinal κ, let $\chi > 2^\kappa$ be a cardinal, and let M be an elementary submodel of $H(\chi)$ of size less than κ such that $U \in M$. Let also $\mathcal{F}$ be a family of functions $F : H(\chi)^{<\omega} \to H(\chi)$ closed under composition such that*

$F \restriction X \in M$ for every $F \in \mathcal{F}$ and every $X \in M$. If $\eta \in \bigcap(M \cap U)$, then $\mathrm{cl}_{\mathcal{F}}(M \cup \{\eta\}) \cap \kappa$ is an end-extension of $M \cap \kappa$.

Fact 4.8. *Let χ be an infinite cardinal and let M be an elementary submodel of $H(\chi)$. Let $\lambda < \mu$ be two cardinals in M, μ regular. If $F : H(\chi)^{<\omega} \to H(\chi)$ is such that $F \restriction X \in M$ for every $X \in M$, then for every $\xi \in \lambda$,*

$$\sup(\mathrm{cl}_F(M \cup \{\xi\}) \cap \mu) = \sup(M \cap \mu).$$

Also, given U, κ, χ, M and $\mathcal{F}$ as in Fact 4.7 and such that $|\mathcal{F}| < \kappa$, we will say that $(M_\xi)_{\xi<\kappa}$ is *the iteration of M relative to $\mathcal{F}$ and U* in case $(M_\xi)_{\xi<\kappa}$ is the unique $\subseteq$–continuous sequence such that $M_0 = M$ and such that, for all $\xi < \kappa$, $M_{\xi+1} = \mathrm{cl}_{\mathcal{F}}(M_\xi \cup \{\eta_\xi\})$, where $\eta_\xi = \min(\bigcap(U \cap N_\xi))$. We will also call $(\eta_\xi)_{\xi<\kappa}$ *the critical sequence of M relative to $\mathcal{F}$ and U.*

Lemma 4.1 reduces our task in proving Lemma 4.2 to proving the following lemma.

Lemma 4.9. *Let $\alpha < \omega_1$ be a limit ordinal and let $\langle N_\xi : \xi \leq \alpha\rangle$ be a suitable tower in $H(\theta)$ for $\mathbb{P}$ with N_ξ closed under F_ξ whenever $\xi \leq \alpha$ is a successor ordinal. Then there is a suitable tower $\langle N^*_\xi : \xi \leq \alpha\rangle$ in $H(\theta)$ such that:*

- *For every successor $\xi \leq \alpha$, $N_\xi \subseteq N^*_\xi$, $N_\xi \cap \omega_1 = N^*_\xi \cap \omega_1$, and N^*_ξ is closed under F_ξ.*
- *If $\nu \leq \alpha$ is a limit ordinal, then there is a $\nu_0 < \nu$ such that*

$$\Delta(\bar{s}(N^*_\xi \cap \kappa_2, N^*_\nu \cap \kappa_2), f(N^*_\nu \cap \omega_1)) \geq n(N^*_\xi \cap \kappa_2, N^*_\nu \cap \kappa_2)$$

whenever $\nu_0 < \xi < \nu$, where s denotes $s^{\vec{C}}_{\kappa_0,\kappa_1}$.

Proof. We start by proving the lemma for $\alpha = \omega$. Let $N = \bigcup_{j<\omega} N_j$ and let $\mathcal{F} = \{F_j : j \in \omega\}$. Let $(N'_j)_{j<\omega}$ and $(\eta^0_j)_{j<\omega}$ be the initial segment of length ω of, respectively, the iteration of N and the critical sequence of N, both relative to $\mathcal{F}$ and U_0. Let $N' = \bigcup_j N'_j$ and let $(N''_j)_{j<\omega}$ and $(\eta^1_j)_{j<\omega}$ be the initial segment of length ω of, respectively, the iteration of N' and the critical sequence of N', both relative this time to $\mathcal{F}$ and U_1. Let $N'' = \bigcup_j N''_j$. Finally, let $(N'''_j)_{j<\omega}$ and $(\eta^2_j)_{j<\omega}$ be the initial segment of length ω of, respectively, the iteration of N'' and the critical sequence of N'', both relative to $\mathcal{F}$ and U_2.

Each model N^*_i will be of the form $\mathrm{cl}_i(N_i \cup \bigcup_{r<3}\{\eta^r_k : k \in I^r_i\})$ for suitable finite subsets I^r_i of ω (for $r < 3$). It will follow in particular that $N_i \cap \omega_1 = N^*_i \cap \omega_1$. Furthermore, we will choose the sets I^r_i so that $i \subseteq I^r_i \subseteq I^r_{i+1}$ for all $r < 3$ and all $i < \omega$. This will ensure that each N^*_i is a member of N^*_{i+1} and also that we already know at the beginning of the construction exactly which set $\bigcup_j N^*_j$ is going to be. Indeed, $\bigcup_j N^*_j$ will be $N^* := \bigcup_{i<\omega} \mathrm{cl}_i(N_i \cup \bigcup_{r<3}\{\eta^r_j : j < \omega\}) = \bigcup_{j<\omega} N'''_j$.

Let $\delta = N \cap \omega_1$. Let π be the collapsing function of N^*, and let $C = \pi^{-1}``C_{\pi(\kappa_0)}$, $D = \pi^{-1}``C_{\pi(\kappa_1)}$ and $E = \pi^{-1}``C_{\pi(\kappa_2)}$.

Let $i < \omega$ be given and suppose $I^r_{i'}$ defined for $r < 3$ and $i' < i$. Let $n = n(N_i, N)$. I^0_i will be of the form $i \cup (\bigcup_{i'<i} I^r_{i'}) \cup \{k_j : j < n\}$ for a suitable increasing sequence $(k_j)_{j<n}$ of integers above $\bigcup_{i'<i} I^0_{i'}$ to be defined as follows, and similarly for I^1_i and I^2_i, with l_j's replacing the k_j's for I^1_i and with m_j's replacing the k_j's for I^2_i.

Let $M_0 = \mathrm{cl}_i(N_i \cup \mathcal{Y})$ for $\mathcal{Y} = \bigcup_{r<3}\{\eta^r_j : j \in i \cup (\bigcup_{i'<i} I^r_{i'})\}$). If $n = 0$ we can let $N^*_i = M_0$. Otherwise, let $f(\delta) \restriction n = \langle p_0, \ldots, p_{n-1}\rangle$. By the choice of $(\eta^r_j)_{j<\omega}$ (for $r < 3$) together with Fact 4.7, each of $\{\eta^0_j\}_{j<\omega}$, $\{\eta^1_j\}_{j<\omega}$ and $\{\eta^2_j\}_{j<\omega}$ is cofinal in $\kappa_0 \cap N^*$, $\kappa_1 \cap N^*$ and $\kappa_2 \cap N^*$, respectively. Choose integers k_q, l_q and m_q $(0 \le q \le n-1)$ and models M_t $(1 \le t \le 2n)$, satisfying the following conditions.

- $i \cup (\bigcup_{i'<i} I^0_{i'}) < k_0 < \ldots < k_{n-1}$;
- $i \cup (\bigcup_{i'<i} I^1_{i'}) < l_0 < \ldots < l_{n-1}$;
- $i \cup (\bigcup_{i'<i} I^2_{i'}) < m_0 < \ldots < m_{n-1}$;
- for all $q \in \{0, \ldots, n-1\}$,
  - $C \cap \eta^0_{k_q}$ has size strictly bigger than both $|D \cap \sup(M_{2q} \cap \kappa_1)|$ and $|E \cap \sup(M_{2q} \cap \kappa_2)|$;
  - $M_{2q+1} = cl_i(M_{2q} \cup \{\eta^0_{k_q}\})$;
  - if $p_{n-1-q} = 0$, then
  $$|C \cap \sup(M_{2q+1} \cap \kappa_0)| < |D \cap \eta^1_{l_q}| < |E \cap \eta^2_{m_q}|;$$
  - if $p_{n-1-q} = 1$, then
  $$|C \cap \sup(M_{2q+1} \cap \kappa_0)| < |D \cap \eta^2_{m_q}| < |E \cap \eta^1_{l_q}|;$$
  - $M_{2q+2} = cl_i(M_{2q+1} \cup \{\eta^1_{l_q}, \eta^2_{m_q}\})$.

Note the following consequences of these choices (and Facts 4.7 and 4.8), for all $q \in \{0, \ldots, n-1\}$.

- $\sup(M_{2q} \cap \kappa_0) < \eta^0_{k_q}$;
- $M_{2q+1} \cap [\sup(M_{2q} \cap \kappa_0),\, \eta^0_{k_{2q}}) = \emptyset$;
- $M_{2q+2} \cap \kappa_0 = M_{2q+1} \cap \kappa_0$;
- $\sup(M_{2q} \cap \kappa_1) = \sup(M_{2q+1} \cap \kappa_1) < \eta^1_{l_{2q}}$;
- $M_{2q+2} \cap [\sup(M_1 \cap \kappa_1),\, \eta^1_{l_0}) = \emptyset$;
- $\sup(M_{2q} \cap \kappa_2) = \sup(M_{2q+1} \cap \kappa_2) < \eta^2_{m_{2q}}$;
- $M_{2q+2} \cap [\sup(M_{2q+1} \cap \kappa_2),\, \eta^2_{m_{2q}}) = \emptyset$.

Now it is not hard to check that the string $\langle p_{n-1}, \dots, p_0\rangle$ is a final segment of $s(M_{2n} \cap \kappa_2, N^* \cap \kappa_2)$, which is what we wanted. This completes the proof for $\alpha = \omega$.

Now we can prove the lemma for general $\alpha < \omega_1$ by induction on α, α a limit ordinal. Let $(N_\nu)_{\nu\leq\alpha}$ be a tower as in the hypothesis of the lemma for α, assume the result proved for all $\beta < \alpha$, and let $(\alpha_i)_{i<\omega}$ be an increasing sequence converging to α.

We build a tower $(N^*_{\alpha_j+1})_{j<\omega}$ of countable models such that, letting $N^* = \bigcup_{j<\omega} N^*_{\alpha_j+1}$, for all $i < \omega$,

(a) $N_{\alpha_i+1} \subseteq N^*_{\alpha_i+1}$, $N_{\alpha_i+1} \cap \omega_1 = N^*_{\alpha_i+1} \cap \omega_1$, and $N^*_{\alpha_i+1}$ is closed under F_{α_i+1}, and
(b) $\Delta(\bar{s}(N^*_{\alpha_i+1} \cap \kappa_2, N^* \cap \kappa_2), f(N^* \cap \omega_1)) \geq n(N^*_{\alpha_i+1} \cap \kappa_2, N^* \cap \kappa_2)$.

We build this tower from $(N_{\alpha_i+1})_{i<\omega}$ as in the proof of the lemma for the case ω. Now we build by recursion towers $(N^*_\nu)_{\alpha_{i-1}+1<\nu\leq\alpha_i} \in N^*_{\alpha_i+1}$, for $i < \omega$, in the following way (where $\alpha_{-1} = -1$).

Let us fix $i < \omega$ and suppose $(N^*_\nu)_{\alpha_{i'-1}+1<\nu\leq\alpha_{i'}}$ has been built for all $i' < i$. Let $n = n(N^*_{\alpha_i+1} \cap \kappa_2, N^* \cap \kappa_2)$, and let $(M_\rho)_{\rho<2n+1}$ be the increasing sequence of models from the proof of the lemma for ω such that $N^*_{\alpha_i+1} = M_{2n}$. Let $\zeta_0 \in (\alpha_{i-1} + 1, \alpha_i]$ be maximal such that $n(N_\nu \cap \kappa_2, N^* \cap \kappa_2) = 0$ for all $\nu \in (\alpha_{i-1} + 1, \zeta_0)$. From the way $N^*_{\alpha_{i-1}+1}$ and M_1 have been built, one can see that in M_1 there is a tower $(N^\dagger_\nu)_{\alpha_{i-1}+1<\nu<\zeta_0}$ of models such that $N^*_{\alpha_{i-1}+1} \in N^\dagger_{\alpha_{i-1}+2}$ if $\zeta_0 > \alpha_{i-1}+2$, and such that

$(i)_0$ for every non-limit $\nu \in (\alpha_{i-1} + 1, \zeta_0)$, $N_\nu \subseteq N^\dagger_\nu$, $N^\dagger_\nu \cap \omega_1 = N_\nu \cap \omega_1$, and $N^\dagger_\nu$ is a countable subset of $H(\theta)$ closed under F_ν.

In particular we have that

$(ii)_0$ for every ν in $(\alpha_{i-1} + 1, \zeta_0)$, $n(N^\dagger_\nu \cap \kappa_2, N^* \cap \kappa_2) = 0$. Hence, for every countable $Q \subseteq \kappa_2$, $Q \in M_1$, if $Q \cap \omega_1 = N^\dagger_\nu \cap \omega_1$ and $N^\dagger_\nu \subseteq Q$, then $\Delta(\bar{s}(Q, N^* \cap \kappa_2), f(N^* \cap \omega_1)) \geq n(Q, N^* \cap \kappa_2)$.

(It suffices to set $N^\dagger_\nu = \mathrm{cl}_\nu(N_\nu \cup \bigcup_{r<3}\{\eta^r_k : k \in \bar{I}^r_i\})$ for all $\nu \in (\alpha_{i+1} + 1, \zeta_0)$, where $\bar{I}^0_i = i \cup (\bigcup_{i'<i} I^0_{i'}) \cup \{k_0\}$, $\bar{I}^1_i = i \cup (\bigcup_{i'<i} I^1_{i'})$ and $\bar{I}^2_i = i \cup (\bigcup_{i'<i} I^2_{i'})$, and where k_0 and each $I^r_{i'}$ is as in the proof of the lemma for ω applied to $(N_{\alpha_j+1})_{j<\omega}$.)

Now, by induction hypothesis we may find a tower $(N^*_\nu)_{\alpha_{i-1}+1<\nu<\zeta_0} \in M_1$ such that

$(A)_0$ for all non-limit $\nu \in (\alpha_{i-1}+1, \zeta_0)$, $N^\dagger_\nu \subseteq N^*_\nu$, $N^\dagger_\nu \cap \omega_1 = N^*_\nu \cap \omega_1$, and N^*_ν is a countable subset of $H(\theta)$ closed under F_ν, and

$(B)_0$ for all $\delta \in (\alpha_{i-1}+1,\, \zeta_0)$ there is some $\epsilon < \delta$ such that, for all $\xi \in [\epsilon,\, \delta)$, $\alpha_{i-1}+1 < \xi$, $\Delta(\bar{s}(N^*_\xi \cap \kappa_2, N^*_\delta \cap \kappa_2), f(N^*_\delta \cap \omega_1)) \geq n(N^*_\xi \cap \kappa_2, N^*_\delta \cap \kappa_2)$.

We take $(N^*_\nu)_{\alpha_{i-1}+1<\nu<\zeta_0}$ to be the $\lhd$–first such sequence in $H(\theta)$. The fact that $(N^\dagger_\nu)_{\alpha_{i-1}+1<\nu<\zeta_0}$ is in M_1 and that $(M_1, \in, \lhd, F_\nu)_{\nu<\zeta_0}$ is an elementary submodel of $(H(\theta), \in, \lhd, F_\nu)_{\nu<\zeta_0}$ implies that $(N^*_\nu)_{\alpha_{i-1}+1<\nu<\zeta_0}$ is in M_1 too.

Similarly, if $\zeta_1 \in (\alpha_{i-1}+1,\, \alpha_i]$ is maximal such that $n(N_\nu \cap \kappa_2, N^* \cap \kappa_2) = 1$ for all $\nu \in [\zeta_0,\, \zeta_1)$, in M_2 there is a tower $(N^\dagger_\nu)_{\zeta_0\leq\nu<\zeta_1}$ such that $(N^*_\nu)_{\nu<\zeta_0} \in N^\dagger_{\zeta_0}$ if $\zeta_1 > \zeta_0$, and such that

$(i)_1$ for all non-limit $\nu \in [\zeta_0,\, \zeta_1)$, $N_\nu \subseteq N^\dagger_\nu$, $N^\dagger_\nu \cap \omega_1 = N_\nu \cap \omega_1$, and $N^\dagger_\nu$ is a countable subset of $H(\theta)$ closed under F_ν, and

$(ii)_1$ for every ν in $[\zeta_0,\, \zeta_1)$ and every countable $Q \subseteq \kappa_2$ in M_2, if $Q \cap \omega_1 = N^\dagger_\nu \cap \omega_1$ and $N^\dagger_\nu \subseteq Q$, then $\Delta(\bar{s}(Q, N^* \cap \kappa_2), f(N^* \cap \omega_1)) \geq 1$.

(This time it suffices to set $N^\dagger_\nu = \mathrm{cl}_\nu(N_\nu \cup \bigcup_{r<3}\{\eta^r_k \,:\, k \in \bar{I}^r_i\})$ for all $\nu \in [\zeta_0,\, \zeta_1)$, where $\bar{I}^0_i = i \cup (\bigcup_{i'<i} I^0_{i'}) \cup \{k_0\}$, $\bar{I}^1_i = i \cup (\bigcup_{i'<i} I^1_{i'}) \cup \{l_0\}$ and $\bar{I}^2_i = i \cup (\bigcup_{i'<i} I^2_{i'}) \cup \{m_0\}$, and where k_0, l_0, m_0, and each $I^r_{i'}$ is again as in the proof of the lemma for ω applied to $(N_{\alpha_j+1})_{j<\omega}$.)

Again by induction hypothesis we find a tower $(N^*_\nu)_{\zeta_0\leq\nu<\zeta_1} \in M_2$ such that

$(A)_1$ for all non-limit $\nu \in [\zeta_0,\, \zeta_1)$, $N^\dagger_\nu \subseteq N^*_\nu$, $N^\dagger_\nu \cap \omega_1 = N^*_\nu \cap \omega_1$, and N^*_ν is a countable subset of $H(\theta)$ closed under F_ν, and

$(B)_1$ for all $\delta \in [\zeta_0,\, \zeta_1)$ there is some $\epsilon < \delta$ such that, for all $\xi \in [\epsilon,\, \delta)$, $\zeta_0 \leq \xi$, $\Delta(\bar{s}(N^*_\xi \cap \kappa_2, N^*_\delta \cap \kappa_2), f(N^*_\delta \cap \omega_1)) \geq n(N^*_\xi \cap \kappa_2, N^*_\delta \cap \kappa_2)$.

$(N^*_\nu)_{\zeta_0\leq\nu<\zeta_1}$ is now the $\lhd$–first such sequence in $H(\theta)$. The fact that $(N^\dagger_\nu)_{\zeta_0\leq\nu<\zeta_1}$ is in M_2 and that $(M_2, \in, \lhd, F_\nu)_{\nu<\zeta_1}$ is an elementary submodel of $(H(\theta), \in, \lhd, F_\nu)_{\nu<\zeta_1}$ implies that $(N^*_\nu)_{\zeta_0\leq\nu<\zeta_1}$ is in M_2 too.

Now if $\zeta_2 \in (\alpha_{i-1}+1,\, \alpha_i]$ is maximal such that $n(N_\nu \cap \kappa_2, N^* \cap \kappa_2) = 2$ for all $\nu \in [\zeta_1,\, \zeta_2)$ we proceed as in the above two cases, working inside of M_4 (this time with $\bar{I}^0_i = i \cup (\bigcup_{i'<i} I^0_{i'}) \cup \{k_0, k_1\}$, $\bar{I}^1_i = i \cup (\bigcup_{i'<i} I^1_{i'}) \cup \{l_0, l_1\}$ and $\bar{I}^2_i = i \cup (\bigcup_{i'<i} I^2_{i'}) \cup \{m_0, m_1\}$). In general, the tower corresponding to ζ_h, for $h \geq 2$, will be built inside M_{2h} as above with $\bar{I}^0_i = i \cup (\bigcup_{i'<i} I^0_{i'}) \cup \{k_0, \ldots k_{h-1}\}$, $\bar{I}^1_i = i \cup (\bigcup_{i'<i} I^1_{i'}) \cup \{l_0, \ldots l_{h-1}\}$ and $\bar{I}^2_i = i \cup (\bigcup_{i'<i} I^2_{i'}) \cup \{m_0, \ldots m_{h-1}\}$. This construction will of course have to stop after at most n steps.

Finally, it is not difficult to check that $(N^*_\nu)_{\nu\leq\alpha}$ is a tower witnessing the conclusion of the lemma for $(N_\nu)_{\nu\leq\alpha}$. □

5. α-SEMIPROPERNESS AND α-REFLECTION

One of the key observations in the proof that MM is consistent relative to a supercompact cardinal was that *stationary reflection* (which follows from SPFA and $\mathrm{FA}^+(\sigma\text{–closed})$) implies that preservation of NS_{ω_1} is *equivalent* to semiproperness. In this section we will examine an analogue of this phenomenon for α-semiproperness.

Let θ be an uncountable regular cardinal and let $\mathcal{E}_\theta^{[\alpha]}$ denote the collection of towers of countable elementary submodels of $H(\theta)$ of length α. Let $\mathcal{I}_\theta^{[\alpha]}$ denote the collection of all subsets of S of $\mathcal{E}_\theta^{[\alpha]}$ such that for some club $E \subseteq [H(\theta)]^{\aleph_0}$, no element of S has its range contained in E.

We will assume CH throughout this section unless otherwise noted. This will be done so that the club filter on $[H(\omega_1)]^{\aleph_0}$ is isomorphic to the club filter on $[\omega_1]^{\aleph_0}$.

Let $\mathrm{RP}_\theta^{[\alpha]}$ denote the assertion that whenever $\mathcal{S} \subseteq \mathcal{E}_\theta^{[\alpha]}$ is not in $\mathcal{I}_\theta^{[\alpha]}$, there is an continuous $\in$-chain $\mathcal{N}$ in $[H(\theta)]^{\aleph_0}$ such that the set

$$\{\langle \overline{N}_\xi : \xi < \alpha\rangle : (\langle N_\xi : \xi < \alpha\rangle \in \mathcal{S}) \wedge \forall \xi < \alpha (N_\xi \in \mathcal{N})\}$$

is not in $\mathcal{I}_{\omega_1}^{[\alpha]}$ (here $\overline{N}_\xi$ denotes the transitive collapse of N_ξ). We will refer to such a sequence $\mathcal{N}$ as a *reflecting sequence* for $\mathcal{S}$.

Observe that if we collapse the cardinality of $H(\theta)$ to ω_1 by a partial order which is σ-closed, then after the forcing if $\mathcal{N}$ is any ω_1-club which is a subset of $[H(\theta)]^{\aleph_0} \cap V$, then $\mathcal{N}$ satisfies the conclusion of $\mathrm{RP}_\theta^{[\alpha]}$. Furthermore, this property of $\mathcal{N}$ is preserved by going into any further forcing extension by an α-semiproper forcing notion. This show that if $\mathfrak{C}$ is a class of semiproper forcings including which can be iterated without adding new reals and we perform the usual iterated forcing construction of a model of $\mathrm{FA}(\mathfrak{C})$ by iterating forcings from $\mathfrak{C}$ in length a supercompact cardinal and performing book-keeping using the Laver function, then $\mathrm{RP}_\theta^{[\alpha]}$ holds in the resulting generic extension for every regular cardinal θ.

Proposition 5.1. *Let $\alpha < \omega_1$ and assume $\mathrm{RP}_\theta^{[\alpha]}$ holds for all θ. If $\mathbb{Q}$ is a forcing which preserves positive sets in $\mathcal{I}_{\omega_1}^{[\alpha]}$ and does not add new ω-sequences of elements of V, then $\mathbb{Q}$ is α-semiproper.*

Proof. Suppose that $\mathbb{Q}$ is a forcing which is not α-semiproper and set $\lambda = |2^{\mathbb{Q}}|^+$. Let $\mathcal{N}$ be a suitable tower of height α of countable elementary submodels of $H(2^{\lambda^+})$ such that for some q in N_0, there is no $(\mathcal{N}, \mathbb{Q})$-semigeneric extension. By elementarity, there is an $\mathcal{I}_\lambda^{[\alpha]}$ positive set $\mathcal{S}$ such that every element of every tower in $\mathcal{S}$ contains q and there is no extension of q which is semigeneric for any element of $\mathcal{S}$.

By our assumption, there a reflecting sequence $\mathcal{N}$ for $\mathcal{S}$; let $\mathcal{S}_0$ denote the $\mathcal{I}_{\omega_1}^{[\alpha]}$-positive set

$$\mathcal{S}_0 = \{\langle \overline{N}_\xi : \xi < \alpha \rangle : (\langle N_\xi : \xi < \alpha \rangle \in \mathcal{S}) \wedge \forall \xi < \alpha (N_\xi \in \mathcal{N})\}.$$

We now claim that q forces that $\check{\mathcal{S}}_0$ is in $\dot{\mathcal{I}}_{\omega_1}^{[\alpha]}$. Suppose that this is not the case and let $G \subseteq \mathbb{Q}$ be a V-generic filter containing q. Let $\langle M_\xi : \xi < \alpha \rangle$ be a tower of countable elementary submodels of $H(\lambda)^{V[G]}$ such that G and $\mathcal{N}$ are in M_0. Since $V^\omega \cap V[G] = V^\omega \cap V$, $\langle M_\xi \cap V : \xi < \alpha \rangle$ is in V. Since G is generic for each model in this tower, there is a condition $\bar{q}$ which forces this. But now $M_\xi \cap \bigcup \mathcal{N}$ is in $\mathcal{N}$ for each $\xi < \alpha$ and therefore $\bar{q}$ is semigeneric for some tower in $\mathcal{S}_0$, a contradiction. Therefore it must be that forcing with $\mathbb{Q}$ does not preserve that $\mathcal{S}_0$ is an $\mathcal{I}_{\omega_1}^{[\alpha]}$-positive set. □

6. The single step forcing for ψ_1 revised

We will now re-examine the single step forcing for the sentence ψ_1 and argue that, if $2^{2^{\aleph_0}} < \kappa_0 < \kappa_1 < \kappa_2$, then the forcing $\mathbb{Q}_{A,\vec{\kappa}}$ preserves $\mathcal{I}_{\omega_1}^{[\alpha]}$ for each α. In particular, if $\mathrm{RP}_\theta^{[\alpha]}$ holds for $\theta > 2^{2^{\kappa_2}}$, then $\mathbb{Q}_{A,\vec{\kappa}}$ is α-semiproper.

For the duration of this section, let $\vec{C}$ be a fixed ladder system on ω_1 and $\alpha < \omega_1$ be a fixed ordinal. In order to analyze the partial order $\mathbb{Q}_A$, we will need to examine a family of games related to those in [2]. We will formulate the games in a somewhat greater generality than we need. For this purpose we will assume that $\vec{\kappa}$ is an increasing sequence of m many regular cardinals, each greater that $2^{\aleph_1}$ where $0 < m < \omega$. Suppose that $\langle E_\xi : \xi < \alpha \rangle$ and $\langle F_\xi : \xi < \alpha \rangle$ are sequences such that for each $\xi < \alpha$, $E_\xi \subseteq \omega_1$ is a club and $F_\xi : \kappa_m^{<\omega} \to \kappa_m$. Define a game $\mathcal{G}_{\vec{E}}^{\vec{F}}$ as follows. In round i of the game, Player I plays ϱ_i and Player II responds by playing $\varsigma_i < \vartheta_i$:

I	ϱ_0		ϱ_1		ϱ_2	$\cdots$
II		$\varsigma_0 < \vartheta_0$		$\varsigma_1 < \vartheta_1$		$\cdots$

The rules of the game are that the players must play so that $\varrho_i < \varsigma_i < \vartheta_i < \varrho_{i+1}$ and ϱ_i, ς_i, and ϑ_i are all in κ_r if $i \equiv r \mod m$. The first player to break these rules loses. Now suppose that the game has been played and the rules have been followed. For $\nu < \omega_1$, let X_ν^ξ be the F_ξ-closure of $\nu \cup \{\varsigma_i : i < \omega\}$. Player II wins if for every $\xi < \alpha$ and every ν in E_ξ, $X_\nu^\xi \cap \omega_1 = \nu$ and $X_\nu^\xi \cap [\varrho_i, \varrho_{i+m}) \subseteq [\varrho_i, \vartheta_i)$ for all $i < \omega$. Notice that if Player II loses, then this is witnessed at a finite stage. Hence the game is determined.

Lemma 6.1. *Let $\langle \kappa_i : i < m \rangle$ be an increasing sequence of cardinals such that $2^{\aleph_1} < \kappa_0$. If $\vec{E} = \langle E_\xi : \xi < \alpha \rangle$ and $\vec{F} = \langle F_\xi : \xi < \alpha \rangle$ are such that Player II has a winning strategy in $\mathcal{G}^{\vec{F}}_{\vec{E}}$ and $F_\alpha : \omega_{m+3}^{<\omega} \to \omega_{m+3}$, then there is a club $E_\alpha \subseteq \omega_1$ such that Player II has a winning strategy in $\mathcal{G}^{\vec{F}'}_{\vec{E}'}$ where $\vec{E}' = \langle E_\xi : \xi \leq \alpha \rangle$ and $\vec{F}' = \langle F_\xi : \xi \leq \alpha \rangle$.*

Proof. This is similar to [2, Lemma 3]. Suppose that $\vec{F} = \langle F_\xi : \xi < \gamma \rangle$, F_γ, and $\vec{E} = \langle E_\xi : \xi < \gamma \rangle$ are given as in the statement of the lemma and let σ be a winning strategy for Player II in $\mathcal{G} = \mathcal{G}^{\vec{F}}_{\vec{E}}$. Suppose for contradiction that there is no club E_γ such that Player II has a winning strategy in $\mathcal{G}^{\vec{F}'}_{\vec{E}'}$. Since these games are determined, there is, for each candidate club E for E_γ, a winning strategy σ_E for Player I in the corresponding game $\mathcal{G}^{\vec{F}'}_{\vec{E}^\frown E}$. Since $2^{\omega_1} < \kappa_i$ for all $i < m$,

$$\sigma_*(p) = \sup\nolimits_E \ \sigma_E(p)$$

defines a strategy for Player I which is a winning strategy in $\mathcal{G}^{\vec{F}'}_{\vec{E}^\frown E}$ for any choice of E. Moreover, we may arrange that in round $md + r$, σ_* depends only on the moves played before round md. We will refer to this property of σ_* as *stability*.

Next observe that if p is a play in $\mathcal{G}$ such that, in each round, Player II plays ς as determined by σ and some ordinal ϑ' which is at least as large as the ordinal ϑ suggested by σ, then Player II wins the play p. If this happens (even for a partial play) we say that p *dominates* σ. Similarly, if p is a play of $\mathcal{G}$ such that in each round Player I plays an ordinal which is at least as large as what is suggested by σ_*, then Player I wins the play p. If this happens (even for a partial play) we say that p *dominates* σ and/or σ_*.

Fix a sufficiently large regular cardinal θ. Let P_ξ ($\xi \leq \omega_2 \cdot \omega$) be a continuous $\in$-chain of elementary submodels of $H(\theta)$, each of cardinality ω_2 and each containing ω_2 as a subset and $\vec{F}$, $\vec{E}$, σ, and σ_* as elements. Set $N_d = P_{\omega_2 \cdot d}$ for $d \leq \omega$. Let M_ξ ($\xi < \omega_1$) be a continuous $\subseteq$-chain of countable elementary submodels of N_ω such that $\vec{F}$, $\vec{E}$, σ, and N_d are in M_0 for each $d < \omega$ and $M_\xi \cap \omega_1$ is in $M_{\xi+1}$ for all $\xi < \omega_1$. By continuity of the sequence,

$$E_\gamma = \{M_\xi \cap \omega_1 : \xi < \omega_1\}$$

is a club. Set $\vec{E}' = \langle E_\xi : \xi \leq \gamma \rangle$ and let $\mathcal{G}'$ denote $\mathcal{G}^{\vec{F}'}_{\vec{E}'}$.

In order to derive a contradiction, it is sufficient to demonstrate how to defeat σ_* in the game $\mathcal{G}'$. We will describe how to inductively construct the play. Suppose that p_{md-1} has been played (setting $p_{-1} =$

$\langle\rangle$). We will assume that we have arranged p_{md-1} is in $N_d \cap M_0$ and that p_{md-1} dominates both σ and σ_*. We will now describe how to play m successive rounds so that the resulting play p_{md+m-1} is in $N_{d+1} \cap M_0$ and dominates σ and σ_*. Let $0 \leq r < m$ and suppose that p_{md+r-1} has been determined. Player I plays ϱ_{md+r} to be the least element of M_0 which is greater than $\sigma^*(p_{md+r-1})$. By our assumptions and the stability of σ, this ordinal must be in N_d. Let $(\varsigma_{md+r}, \vartheta^0)$ be the response by σ to this play and set ϑ_{md+r} to be equal to the least element of $N_{d+1} \cap M_0 \cap \kappa_r$ which is an upper bound for

$$\bigcup_{\xi < \omega_1} M_\xi \cap N_d \cap \kappa_r$$

This concludes the inductive construction and it should be clear that both p_{md+m-1} is in N_{d+1} and that it dominates both σ and σ_*.

By our assumptions, the play $p = \langle (\varrho_i, \varsigma_i, \vartheta_i) : i < \omega \rangle$ is a winning play for Player II in $\mathcal{G}$ but a winning play for Player I in $\mathcal{G}'$. This implies that there is a ν in E_γ such that for some m and r,

$$X \cap [\varrho_{md+r}, \varrho_{md+m+r}) \neq X \cap [\varrho_{md+r}, \vartheta_{md+r}).$$

where X is the F_γ-closure of $\nu \cup \{\varsigma_i : i < \omega\}$. Let ξ be such that $\nu = M_\xi \cap \omega_1$ and observe that $X \subseteq M_\xi$ since F_γ is in $M_0 \subseteq M_\xi$ and $\{\varsigma_i : i < \omega\} \subseteq M_0 \subseteq M_\xi$. Notice however that ϱ_{md+m+r} is in N_{d+1} and ϑ_{md+r} was chosen to be an upper bound for $M_\xi \cap N_{d+1} \cap \kappa_r$. It follows that

$$X \cap [\varrho_{md+r}, \varrho_{md+m+r}) = X \cap [\varrho_{md+r}, \vartheta_{md+r}),$$

which is a contradiction. It must therefore be that the conclusion of the lemma is true. □

We will now turn our task to showing that $\mathbb{Q}_{A,\vec{\kappa}}$ preserves $\mathcal{I}_{\omega_1}^{[\alpha]}$ provided that $\vec{\kappa}$ is an increasing sequence of 3 regular cardinals above $2^{2^{\aleph_0}}$. First we claim that we may assume that GCH holds below $\aleph_\omega$ and that $\kappa_i = \omega_{i+3}$. This is because this hypothesis can be forced by a σ-closed forcing and the hypotheses on κ_i ($i < 3$) are still satisfied in the generic extension. Furthermore, $\mathbb{Q}_{A,\vec{\kappa}}$ is the same forcing when computed in the generic extension and clearly if $\mathbb{Q}_{A,\vec{\kappa}}$ preserves $\mathcal{I}_{\omega_1}^{[\alpha]}$ in $V[G]$, the same must be true in V. Let $\theta < \aleph_\omega$ be sufficiently large for $\mathbb{Q}_{A,\vec{\kappa}}$ and set $\kappa_3 = \theta^+$. The purpose of arranging this cardinal arithmetic is that if M is an elementary submodel of $H(\theta)$ of cardinality κ_3, then M must be $H(\theta)$.

From this point on, let h be a bijection between $H(\omega_1)$ and ω_1. A sequence $\langle \nu_\xi : \xi \leq \alpha \rangle$ in ω_1 is said to be *suitable* if for every $\gamma < \alpha$,

$h(\langle \nu_\xi : \xi \leq \gamma \rangle) < \nu_{\gamma+1}$. By Lemma 4.1, it is sufficient to prove the following lemma.

Lemma 6.2. *For every sequence $\langle r_\xi : \xi < \omega_1 \rangle$ of elements of 2^ω and $\alpha < \omega_1$ there is a club $E \subseteq \omega_1$ such that if $\langle \nu_\xi : \xi \leq \alpha \rangle$ is a suitable sequence of elements of E and $F : H(\theta)^{<\omega} \to H(\theta)$, then there is a continuous $\in$-chain $\langle N_\xi : \xi \leq \alpha \rangle$ of countable F-closed subsets of $H(\theta)$ such that:*

- *$\langle r_\xi : \xi < \omega_1 \rangle$ is in N_0;*
- *if $\gamma \leq \alpha$ is a limit ordinal, there is a $\gamma_0 < \gamma$ such that if $\gamma_0 < \xi < \gamma$, then $N_\xi \cap \kappa_2$ is in $\Sigma_{r_{\nu_\gamma}}(N_\gamma \cap \kappa_2)$;*
- *$N_\xi \cap \omega_1 = \nu_\xi$.*

This will be proved in a sequence of lemmas. From this point on, let $\langle r_\xi : \xi < \omega_1 \rangle$ be given and let α be fixed. Let $\lhd$ be a well ordering of $H(\theta)$ in type κ_3 and fix a sequence $\langle F_\xi : \xi < \omega_1 \rangle$ of functions $F_\xi : H(\theta)^{<\omega} \to H(\theta)$ such that:

- if M is closed under F_ξ, then M is an elementary submodel of $(H(\theta), \in, \lhd)$ and if X is in M, then so is $\langle F_\eta \restriction X : \eta < \xi \rangle$;
- if M is closed under F_0, then α and $\langle r_\xi : \xi < \omega_1 \rangle$ are in M.

We will be interested in playing the games described above with $r = 4$. Applying Lemma 6.1 for games played with the cardinals $\langle \kappa_i : i < 4 \rangle$, it is possible to build a $\subseteq$-descending sequence $\langle E_\xi : \xi < \omega_1 \rangle$ of clubs such that for each $\gamma < \omega_1$, Player II has a winning strategy σ_γ in $\mathcal{G}_\gamma = \mathcal{G}_{\vec{E} \restriction \gamma}^{\vec{F} \restriction \gamma}$ and E_γ is in any F_γ-closed subset of $H(\theta)$ (here we are abusing notation and writing $\vec{F} \restriction \gamma$ when we really mean $\langle F_\xi \restriction [\omega_5]^{<\omega} : \xi < \gamma \rangle$). Let $\vec{\nu} = \langle \nu_\xi : \xi \leq \alpha \rangle$ be a suitable continuous sequence of elements of E_α.

Define Γ_ξ to be the set of all finite sequences $\vec{\varsigma} = \langle \varsigma_0 < \ldots < \varsigma_{k-1} \rangle$ such that there are ϱ_i $(i < k)$ and ϑ_i $(i < k)$ with $\langle \varrho_i, (\varsigma_i, \vartheta_i) : i < k \rangle$ being a winning position for Player II in $\mathcal{G}_\xi$. Such a sequence $\langle \varrho_i, (\varsigma_i, \vartheta_i) : i < k \rangle$ is said to be a *completion* of $\vec{\varsigma}$. A simple but important observation is that if $\xi < \eta$, then $\Gamma_\eta \subseteq \Gamma_\xi$. (This is the source of our need to work with games involving sequences of clubs E_ξ rather than a single club E.) Note that Γ_ξ is in any F_ξ-closed subset of $H(\theta)$. Observe that if τ is an element of Γ_ξ, then only the trivial permutation of the coordinates of τ results in another element of Γ_ξ and hence elements of Γ_ξ are uniquely determined by their range. As a result we will often identify an element of Γ_ξ with its range. Note, however, that *τ is an initial part of τ'* is always interpreted by viewing τ and τ' as sequences.

Lemma 6.3. *Suppose that $\gamma \leq \alpha$ and M is F_γ-closed and $M \cap \omega_1 = \nu_\gamma$. If $\bar{\gamma} < \gamma$ and τ is in $M \cap \Gamma_\gamma$, then there is a τ' in Γ_γ such that:*

- *τ is an initial part of τ';*
- *τ' is in M;*
- *$X = \mathrm{cl}_{F_\gamma}(\tau') \cap \kappa_2$ satisfies that $X \cap \omega_1 = \nu_{\bar\gamma}$ and*

$$[\tau, X] \subseteq \Sigma_{r_{\nu_\gamma}}(M \cap \kappa_2).$$

Proof. This is the same as in Lemma 5 of [2] with two exceptions. First, κ_i ($i < 3$) in the current discussion plays the same role as ω_{i+2} ($i < 3$) in their discussion. In the rounds involving the cardinal κ_3, Player I plays arbitrarily. Second, there is no need to make preliminary plays as in [2]. Instead, Player I arranges that the resulting sequence $o(x, y, z)$ has length $l \geq n = C_{M\cap\omega_1} \cap X$ and satisfies $o(x, y, z)(l - i) = r_{\nu_\gamma}(i)$ for all $i < n$. □

Lemma 6.4. *If $\gamma \leq \alpha$ is a limit ordinal, M is F_γ-closed, $M \cap \omega_1 = \nu_\gamma$, and $\tau \in M \cap \Gamma_\gamma$, then there is a continuous $\in$-chain $\langle M_\xi : \xi \leq \gamma\rangle$ with union M such that:*

- *M_ξ is F_η-closed where η is the greatest limit ordinal less than or equal to ξ;*
- *$M_\xi \cap \omega_1 = \nu_\xi$;*
- *for all limit $\eta \leq \gamma$ there is a $\eta_0 < \eta$ such that if $\eta_0 < \xi < \eta$, then $M_\xi \cap \kappa_2 \in \Sigma_{r_{\nu_\eta}}(M_\eta \cap \kappa_2)$;*
- *τ is in M_0.*

Proof. This will be proved by induction on γ. There are two cases, depending on whether γ is a limit of limit ordinals. If $\gamma = \beta + \omega$ and β is either 0 or a limit ordinal, then we will construct a sequence τ_i ($i < \omega$) using Lemma 6.3 such that:

- $\tau_0 = \tau$ and τ_{i+1} extends τ_i;
- τ_i is in $\Gamma_{\beta+1} \cap M$.
- If $M_{\beta+i} = \mathrm{cl}_{F_{\beta+1}}(\tau_{i+1} \cup \nu_{\beta+i})$, then

$$M_{\beta+i} \cap \omega_1 = \nu_{\beta+i}$$

$$[\tau_{i+1}, M_{\beta+i} \cap \kappa_2] \subseteq \Sigma_{r_{\nu_\gamma}}(M \cap \kappa_2);$$

- the set $\cup\{\tau_i : i < \omega\} \cap \kappa_3$ is cofinal in $M_\gamma \cap \kappa_3$.

This last condition implies that $\bigcup_{i=0}^{\infty} M_{\beta+i} = M_\gamma$.

If $\beta = 0$, we are done. Otherwise, working within $M_{\beta+1}$, we can now apply the induction hypothesis to M_β and τ to construct a sequence $\langle M_\xi : \xi < \beta\rangle$ in $M_{\beta+1}$ with union M_β such that:

- for all $\xi < \beta$, M_ξ is $F_{\eta+1}$-closed where η is the greatest limit ordinal less than or equal to ξ;
- for all $\xi < \beta$, $M_\xi \cap \omega_1 = \nu_\xi$;

- for all limit $\eta \leq \beta$ there is a $\eta_0 < \eta$ such that if $\eta_0 < \xi < \eta$, then $M_\xi \cap \omega_5 \in \Sigma_{r_{\nu_\eta}}(M_\eta \cap \omega_5)$;
- τ is in M_0.

It follows that the sequence $\langle M_\xi : \xi < \gamma\rangle$ has the desired properties.

Now consider the case where γ is a limit of a strictly increasing sequence γ_i $(i < \omega)$ such that $\gamma_0 = 0$ and if $0 < i$, then γ_i is a limit ordinal. M_γ contains the club E_γ and the set Γ_γ since M_γ is assumed to be closed under $F_{\gamma+1}$. Applying Lemma 6.3, it is possible to build a sequence τ_i $(i < \omega)$ in $\Gamma_\gamma \cap M$ such that:

- $\tau_0 = \tau$ and τ_{i+1} extends τ_i;
- If M_{γ_i} is the F_γ-closure of $\tau_{i+1} \cup \nu_{\gamma_i}$, then $M_{\gamma_i} \cap \omega_1 = \nu_{\gamma_i}$ and
$$[\tau_i, M_{\gamma_i} \cap \omega_5] \subseteq \Sigma_{r_{\gamma_i}}(M_\gamma \cap \omega_5);$$
- the set $\cup\{\tau_i : i < \omega\} \cap \omega_i$ is cofinal in $M_\gamma \cap \kappa_i$ for all $i < 4$.

The induction hypothesis now easily allows us to build $\langle M_\xi : \xi \leq \gamma_i\rangle$ for each $i < \omega$ such that:

- if $\xi \leq \gamma_i$, then M_ξ is $F_{\eta+1}$-closed where η is the greatest limit ordinal less than or equal to ξ;
- if $\xi \leq \gamma_i$, $M_\xi \cap \omega_1 = \nu_\xi$;
- for all limit $\eta \leq \gamma_i$ there is a $\eta_0 < \eta$ such that if $\eta_0 < \xi < \eta$, then $M_\xi \cap \kappa_2 \in \Sigma_{r_{\nu_\eta}}(M_\eta \cap \kappa_2)$;
- τ is in M_0 and τ_{j+2} is contained in M_{γ_j+1} for all $j < i$.

This completes the proof. □

7. Concluding remarks

The Π_2 sentences which we employed to resolve Problem 1.1 are quite *ad hoc* in nature and it is natural to ask whether there are simpler examples. In particular, it is unclear whether there are Π_2-sentences which have already been studied in the literature which solve Problem 1.1.

Until the present article, the study of preservation theorems for not adding reals largely centered on the degree to which $(< \omega_1)$-properness can be dispensed with in theorems like Theorem VIII.4.5 of [15] (which is the precursor to [6] and Theorems 3.3 and 3.8 above). For instance it is an open problem whether the hypothesis of $(< \omega_1)$-semiproperness can be removed as a hypothesis to Theorem 3.8 (an example in [15, XVIII] shows that some large cardinal assumption is necessary[1]; a different presentation of this theorem can be found in [7]).

[1] Shelah has indicated in private communication that there is a hole in his argument in [15, XVIII] where he claims that his iteration counterexample exists in any model of CH.

The relevance of this to the present discussion is that Shelah has shown that a different hypothesis which is unrelated to $(< \omega_1)$-properness can be substituted in order to obtain a preservation theorem for not adding reals [15,]. This iteration theorem allows one to establish, for instance, that the following Π_2-sentence is relatively consistent with CH: *For every ladder system $\langle C_\alpha : \alpha < \omega_1 \rangle$ on ω_1, then there is a club $E \subseteq \omega_1$ such that $E \cap C_\alpha$ is finite for all $\alpha < \omega_1$.* This sentence is a special case of the following stronger statement: *For every sequence $\langle D_\alpha : \alpha < \omega_1 \rangle$ in which $D_\alpha \subseteq \alpha$ is closed for each $\alpha < \omega_1$, there is a club $E \subseteq \omega_1$ such that if $\delta < \omega_1$, there is a $\delta_0 < \delta$ with $E \cap (\delta_0, \delta)$ either contained in or disjoint from D_δ.* While it is unknown whether this Π_2-sentence is consistent with CH, it is known that there is a canonical class of single step forcings which are totally proper and whose iterations are 1-semicomplete.

The present article underscores that the notion of *completeness* is not as robust as one might hope. The results in this paper show that there is an important distinction between 1-semicomplete iterations and ω-complete iterations. In [6], the apparent added flexibility of 2-complete over 1-complete iterations was important to the argument. While this was largely dismissed as a technical detail at the time, it may now warrant further investigation.

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