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Abstract

We consider limit sets of some conformal iterated function systems, and introduce classes of subsets of these limit sets, with the property that the classes are closed under countable intersections and that all sets in the classes have large Hausdorff dimension. We show some applications in ergodic theory, Diophantine approximation and complex dynamics.

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1 Introduction

In [1], Falconer defined classes $\mathcal{G}^s$ of G_δ-subsets of $\mathbb{R}^n$, for $0 < s \leq n$. Falconer proved that these classes are invariant under countable intersections, and that any set in $\mathcal{G}^s$ has at least Hausdorff dimension s. Moreover these classes are invariant under inverse images of bi-Lipschitz maps.

In this paper we will study classes of subsets of limit sets Λ of conformal graph directed Markov systems. Simple examples of such sets Λ are the interval, the middle third Cantor set, and Julia sets of polynomials of degree larger than 1. We are interested in such classes which have the intersection and dimension properties of $\mathcal{G}^s$.

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As is easily seen, the complement of a set in $\mathcal{G}^s$ cannot contain an open non-empty set. Since the complement of Λ may contain open non-empty sets, the classes of Falconer can not be directly used in our case. We will therefore introduce a modification of Falconer's classes, consisting of subsets of Λ with the desired properties. This will be done in Section 3. Before this is done we need some notations and results, presented in Section 2.

In Section 4, we consider some applications of these classes. We show that sets of points for which the ergodic averages of a Hölder continuous function g do not converge, are in our classes of sets, and we will apply this to piecewise expanding interval maps and Julia sets of polynomials. We will also study subsets of the middle third Cantor set, with some Diophantine properties, and show that these sets are in our classes.

2 Graph directed Markov systems

Let Σ_A be a transitive subshift of finite type over the alphabet $\{1, \ldots, q\}$ governed by a transition matrix A. We endow Σ_A with the product topology.

Elements in Σ_A will be denoted by $\boldsymbol{i} = i_1 i_2 \ldots$ The topology of Σ_A is generated by the metric d_Σ defined by $d_\Sigma(\boldsymbol{i}, \boldsymbol{j}) = 2^{-n}$, where n is the smallest number such that $i_n \neq j_n$.

We define a cylinder to be a set of the form

$$C_{i_1 \ldots i_k} = \{ \boldsymbol{j} \in \Sigma_A : j_1 \ldots j_k = i_1 \ldots i_k \}$$

and we refer to k as the generation of the cylinder $C_{i_1 \ldots i_k}$. On Σ_A we have the shift map σ defined by $\sigma \colon i_1 i_2 \ldots \mapsto i_2 i_3 \ldots$

We will now define a *graph directed Markov system*. This can be done in a much more general way, see for instance [7]. However in this paper we will only consider the following simplification. Let X be a compact and non-empty subset of $\mathbb{R}^n$, endowed with the usual metric. We assume that there are contractions on X denoted by $(f_i)_{i=1}^q$ and $(f_{i,j})_{i,j}$, where ij is a word in Σ_A. We will consider compositions of the contractions of the form $f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_{n-1},i_n}$ where $i_1 i_2 \ldots$ is an element in Σ_A. We will call such a system of contractions together with Σ_A, a graph directed Markov system.

For any element $i_1 i_2 \ldots$ in Σ_A, there is a point $x \in X$ such that

$$\{x\} = \bigcap_{n=1}^{\infty} f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_{n-1},i_n}(X).$$

This defines a mapping $\pi \colon \Sigma_A \to X$. Since all f_i and $f_{i,j}$ are contractions, π is continuous. The limit set or attractor of the graph directed Markov system is the set $\pi(\Sigma_A)$ and will be denoted by Λ_A. It is compact since Σ_A is compact and π is continuous. We endow Λ_A with the subset topology that comes from the topology of X.

We will now state some assumptions on the graph directed Markov system, that will be used in this paper.

First of all, we will only consider the case when all the contractions of the graph directed Markov system are conformal maps. Such a system is called a *conformal graph directed Markov system*.

The first two assumptions are

> The compact set X is the closure of its interior, and the images of the interior of X under the conformal contractions are disjoint. (1)

and

> There are numbers $0 < \lambda_1 < \lambda_2 < 1$ such that we have $\lambda_1 < \|\mathrm{d}_x f_{i,j}\| < \lambda_2$ and $\lambda_1 < \|\mathrm{d}_x f_i\| < \lambda_2$ for all i, j and $x \in X$. (2)

We will also need the following bounded distortion property.

> There is a constant κ such that for any sequence $\boldsymbol{i} \in \Sigma_A$, any integer $n > 0$ and any $x, y \in X$ it holds that
> $$\kappa^{-1} \leq \frac{\|\mathrm{d}_x(f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_{n-1},i_n})\|}{\|\mathrm{d}_y(f_{i_1} \circ f_{i_1,i_2} \circ \cdots \circ f_{i_{n-1},i_n})\|} \leq \kappa. \tag{3}$$

Assumption (3) is satisfied if for example there is an $\alpha > 0$ such that

$$\big|\|\mathrm{d}_x f_{i,j}\| - \|\mathrm{d}_y f_{i,j}\|\big| \leq \sup_{z \in X} \|(\mathrm{d}_z f_{i,j})^{-1}\|^{-1} |x - y|^\alpha$$

holds for all i, j and x, y, and similarly for the maps f_i, see [7]. It is shown in [7] that, under assumptions (1) and (2), such an α exists as long as the dimension n of the space is larger than one. So, assumption (3) is in fact only needed when $n = 1$.

3 A class with large intersection properties

We define a semi-metric d on Σ_A by $d(\boldsymbol{i}, \boldsymbol{j}) = |\pi(\boldsymbol{i}) - \pi(\boldsymbol{j})|$, so that

$$d(C_{i_1 \ldots i_n}) = |\pi(C_{i_1 \ldots i_n})|.$$

Note however that this semi-metric does not necessarily generate the topology of Σ_A. In fact, this need not be a metric, since there can be different $\boldsymbol{i}$ and $\boldsymbol{j}$ such that $\pi(\boldsymbol{i}) = \pi(\boldsymbol{j})$.

We define outer measures M_∞^t on Σ_A by

$$M_\infty^t(A) = \inf \Big\{ \sum_i d(C_i)^t : A \subset \bigcup_i C_i \Big\},$$

where each C_i is a cylinder of some generation.

We will work mainly with M^t_∞, but for some future use we also introduce the outer measures M^t_δ and M^t on Σ_A, defined by

$$M^t_\delta(A) = \inf\Big\{ \sum_i d(C_i)^t : A \subset \bigcup_i C_i,\ d(C_i) \leq \delta \Big\},$$

where each C_i is a cylinder of some generation, and $M^t(A) = \lim_{\delta\to 0} M^t_\delta(A)$. The t-dimensional Hausdorff measure is

$$H^t(A) = \lim_{\delta\to 0} \inf\Big\{ \sum_i d(U_i)^t : A \subset \bigcup_i U_i,\ d(U_i) \leq \delta \Big\}.$$

We note that $M^t(A) \geq M^t_\infty(A)$ for all sets A, and that M^t is equivalent to the t-dimensional Hausdorff measure H^t.

The outer measures M^t_∞, M^t_δ, M^t and H^t are defined on subsets of Σ_A. Using the projection $\pi\colon \Sigma_A \to X$, we project these outer measures to outer measures on X, and denote them by $\mathfrak{M}^t_\infty$, $\mathfrak{M}^t_\delta$, $\mathfrak{M}^t$ and $\mathfrak{H}^t$. Equivalently we may define

$$\mathfrak{M}^t_\infty(A) = \inf\Big\{ \sum_i |\pi(C_i)|^t : A \subset \bigcup_i \pi(C_i) \Big\},$$

where each C_i is a cylinder, and similarly for the measures $\mathfrak{M}^t_\delta$, $\mathfrak{M}^t$ and $\mathfrak{H}^t$.

We have now defined the Hausdorff measures H^t and $\mathfrak{H}^t$. The Hausdorff dimension of a set $E \subset \Sigma_A$ is defined to be the number s such that $H^t(E) = 0$ if $t > s$ and $H^t(E) = \infty$ if $t < s$. The Hausdorff dimension of a set $E \subset \Lambda_A$ is defined similarly using the measures $\mathfrak{H}^t$.

We will use the following classes of sets, which are modifications of Falconer's intersection classes from [1].

Definition 1. Let $\mathcal{G}^t(\Lambda_A)$, $0 < t \leq \dim_{\mathrm{H}}(\Lambda_A)$ be the class of G_δ-sets $F \subset \Lambda_A$ such that

$$\mathfrak{M}^t_\infty(F \cap \pi(C)) = \mathfrak{M}^t_\infty(\pi(C))$$

holds for all cylinders C.

We are mainly interested in the classes $\mathcal{G}^t(\Lambda_A)$, but to develop the theory, we prefer to work with classes of subsets of Σ_A. Therefore we also introduce the following classes.

Definition 2. Let $\mathcal{G}^t(\Sigma_A)$, $0 < t \leq \dim_{\mathrm{H}}(\Lambda_A)$ be the class of G_δ-sets $F \subset \Sigma_A$ such that

$$M^t_\infty(F \cap C) = M^t_\infty(C)$$

holds for all cylinders C.

Our main result on these classes is the following theorem.

Theorem 1. *The classes $\mathcal{G}^t(\Lambda_A)$ and $\mathcal{G}^t(\Sigma_A)$ are closed under countable intersections and the Hausdorff dimension of any set in one of these classes is at least t.*

The proof of Theorem 1 is in Section 5.

4 Applications

4.1 Non-typical points in ergodic theory

In this section we will state some results that are generalisations of the results in [2] and [10].

For functions $g\colon \Sigma_A \to \mathbb{R}$, consider the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1} g(\sigma^k \boldsymbol{i}), \tag{4}$$

as $n \to \infty$. For $x \in \Lambda_A$ such that $\pi(\boldsymbol{i}) = x$, let $A_g(x)$ be the set of accumulation points for the expression (4).

Let Σ be a shift space equipped with the metric d_Σ. We will say that a function $g\colon \Sigma \to \mathbb{R}$ is Hölder continuous if there are constants K and $\alpha > 0$ such that $|g(\boldsymbol{i}) - g(\boldsymbol{j})| < K d_\Sigma(\boldsymbol{i}, \boldsymbol{j})^\alpha$ holds for all $\boldsymbol{i}, \boldsymbol{j} \in \Sigma$. Denote by $C_\alpha^K(\Sigma)$ the family

$$C_\alpha^K(\Sigma) = \{\, g\colon \Sigma \to \mathbb{R} : |g(\boldsymbol{i}) - g(\boldsymbol{j})| < K d_\Sigma(\boldsymbol{i}, \boldsymbol{j})^\alpha,\ \forall \boldsymbol{i}, \boldsymbol{j} \in \Sigma \,\},$$

and by $C_\alpha^K(X)$ the family

$$C_\alpha^K(X) = \{\, \phi\colon X \to \mathbb{R} : |\phi(x) - \phi(y)| < K|x-y|^\alpha,\ \forall x, y \in X \,\}.$$

We note that if $\phi\colon X \to \mathbb{R}$ is in $C_\alpha^K(X)$, then $\phi \circ \pi$ is in $C_{\alpha_0}^{K_0}(\Sigma)$, for some constants α_0 and K_0 that do not depend on ϕ, i.e. the projection π transfers functions in $C_\alpha^K(X)$ to functions in $C_{\alpha_0}^{K_0}(\Sigma)$.

We will prove the following.

Theorem 2. *Consider a conformal graph directed Markov system, satisfying the assumptions* (1), (2) *and* (3). *For any sequence $(g_i)_{i=1}^\infty$ of real valued continuous functions on Σ_A, and each sequence $(x_i)_{i=1}^\infty$ of points in $\mathbb{R}$, it holds that*

$$\dim_{\mathrm{H}} \left(\cap_{i=1}^\infty \{\, y \in \Lambda_A : x_i \in A_{g_i}(y) \,\}\right) = \inf_i \dim_{\mathrm{H}} \left(\{\, y \in \Lambda_A : x_i \in A_{g_i}(y) \,\}\right).$$

We will combine Theorem 2 with Theorem 9 of Barreira's and Saussol's article [9], to get the following corollary.

Corollary 1. *Consider a conformal graph directed Markov system, satisfying the assumptions* (1), (2) *and* (3). *Let* $(g_i)_{i=1}^{\infty}$ *be a sequence of Hölder continuous maps on* Σ_A *such that no* g_i *is cohomologous to a constant. Then the set of points for which the ergodic averages do not converge for any of the maps* g_i *has the same Hausdorff dimension as* Λ_A.

Theorem 2 and Corollary 1 are generalisations of results in [2], where the case $\Lambda_A = [0, 1]$ is considered, i.e. where there are no holes. Moreover, in [2], each function g_i is the characteristic functions of a cylinder $C_{j_1 \cdots j_k}$, so that the ergodic average (4) is the frequency of occurrence of the word $j_1 \ldots j_k$ in a sequence $\boldsymbol{i} \in \Sigma_A$.

Note that to prove Theorem 2 it is sufficient to prove that each of the sets $\{ y \in \Lambda_A : x_i \in A_{g_i}(y) \}$ is in $\mathcal{G}^t(\Lambda_A)$ for any t less than $\dim_H(\{ y \in \Lambda_A : x_i \in A_{g_i}(y) \})$. The proofs of Theorem 2 and Corollary 1 are in Section 6.

Corollary 2. *Let* $0 = a_0 < \cdots < a_q = 1$, *and let* $f \colon [0, 1] \to [0, 1]$ *be monotone and* C^2 *on each of the intervals* (a_k, a_{k+1}), *with* $0 < \lambda_1 \leq |f'| \leq \lambda_2 < \infty$. *The system* $(f, [0, 1])$ *generates a subshift* Σ_f *on an alphabet of* q *symbols. Assume* Σ_f *is transitive and that* $(0, 1) \setminus \cup_{k=0}^{q-1} f((a_k, a_{k+1}))$ *does not contain any isolated points. Let* $(g_i)_{i=1}^{m}$ *be a family of Hölder continuous maps on* $[0, 1]$ *such that no* g_i *is cohomologous to a constant. Then the set of points in* $[0, 1]$, *such that no ergodic average of any* g_i *converges, has Hausdorff dimension* 1.

In [10] another method is used to prove theorems similar to Corollary 2. Note also that Corollary 3, is similar to Corollary 2.

Corollary 3 in [3] considers a special case of Corollary 2, namely maps of the form $f_\beta \colon x \mapsto \beta x \mod 1$, and with g_i being characteristic functions. For this special case Corollary 3 in [3] is stronger by stating that the set of points for which the ergodic averages do not converge contains a set from a class similar to $\mathcal{G}^1([0, 1])$, which is independent of the parameter β. This makes it possible to handle countably many maps g_i simultaneously and to take intersections between sets defined using different maps f_β.

The proof of Corollary 2 is in Section 7.

4.2 Diophantine approximation and Cantor sets

Let K be the middle third Cantor set. With $n = 3$, $\Sigma_A = \{0, 2\}^{\mathbb{N}}$, and an appropriate definition of the iterated function system, we have that $K = \Lambda_A$. Let $\alpha > 1$ and consider the sets

$$W(\alpha) = \Big\{ x \in [0, 1] : \Big| x - \frac{p}{q} \Big| < q^{-\alpha} \text{ for infinitely many } p \in \mathbb{Z}, q = 3^k \Big\}.$$

These sets were studied in the paper [5] by Levesley, Salp and Velani. They proved that the Hausdorff dimension of $W(\alpha) \cap K$ is $\frac{1}{\alpha} \frac{\log 2}{\log 3}$. We will strengthen this result and prove the following theorem.

Theorem 3. *$W(\alpha) \cap K$ contains sets from $\mathcal{G}^t(K)$ for all $t < \frac{1}{\alpha}\frac{\log 2}{\log 3}$.*

The proof of this theorem is in Section 8.

As a corollary of Theorem 2 and 3 we get that the set of points in $W(\alpha) \cap K$ such that the frequency of any finite word from $\{0,2\}^{\mathbb{N}}$ is undefined, has Hausdorff dimension $\frac{1}{\alpha}\frac{\log 2}{\log 3}$.

4.3 Julia sets

In this section we will apply Corollary 1 to non-typical points of the Julia set $J(f)$ of a polynomial $f(z)$.

Corollary 3. *Let $f(z)$ be a polynomial of degree larger than 1, such that $|f'| > 1$ on the Julia set $J(f)$. Let $(g_k)_{k=1}^{\infty}$ be a sequence of Hölder continuous functions $g_k \colon J(f) \to \mathbb{R}$, such that no g_k is cohomologous to a constant with respect to $(f, J(f))$. Then the set of points in $J(f)$ such that the ergodic averages of g_k do not converge for any g_k, has the same Hausdorff dimension as $J(f)$.*

Proof. Since f is of degree $d > 1$, the Julia set is compact and non-empty. There are only finitely many critical points since f is a polynomial, and by assumption none of them are in $J(f)$. Since $J(f)$ is compact and invariant, there exists an $R > 0$ such that the distance from $J(f)$ to any image of a critical point is a least R, and $|f'| > 1$ on any ball $B_R(x)$, $x \in J(f)$. Moreover, the compactness of $J(f)$ implies that there exists a constant c such that $|f'(x)| > c > 1$ for all $x \in J(f)$. We can even choose $c > 1$ and $0 < r < R$ such that if $x \in J(f)$, then the pre-image of the ball $B_r(x)$ of radius r around x satisfies

$$f^{-1}(B_r(x)) \subset \bigcup_{y \in f^{-1}(x)} B_{r/c}(y),$$

and $|f'| > c$ on $B_r(x)$.

We let U be an open neighborhood around $J(f)$ defined by

$$U = \bigcup_{x \in J(f)} B_r(x),$$

and put $X = \overline{U}$.

Now X contains no image of a critical point, so we can define inverse branches $(f_j)_{j=1}^{d}$ of f on X, and put $f_{i,j} = f_j$. It is clear that f_j are conformal contractions on X. Moreover $f_j \colon X \to X$. Indeed we have

$$f^{-1}(X) = \overline{f^{-1}(U)} \subset \overline{\bigcup_{x \in J(f)} B_{r/c}(x)} \subset \overline{U} = X,$$

so, $f_j(X) \subset X$.

Note that some f_j may be discontinuous at the curves $f(\partial f_j(X))$. However $f(\partial f_j(X))$ are piecewise smooth curves and if necessary, we can cut up X along these curves, so that f_j satisfies the assumptions (1), (2) and (3). Now, Corollary 1 finishes the proof. □

5 Proof of Theorem 1

Let us first comment on the relation between the classes $\mathcal{G}^t(\Lambda_A)$ and $\mathcal{G}^t(\Sigma_A)$. If $U \subset X$ is an open set, then $\pi^{-1}(U) \subset \Sigma_A$ is an open set. Hence, if $F \subset X$ is a G_δ-set, then so is $\pi^{-1}(F) \subset \Sigma_A$, since π is continuous. We are going to show that the class $\mathcal{G}^t(\Sigma_A)$ is closed under countable intersections. Since countable intersections of G_δ-sets are G_δ-sets, this implies that the class $\mathcal{G}^t(\Lambda_A)$ is also closed under countable intersections. Hence, to prove Theorem 1, we only need to prove the intersection statement for the class $\mathcal{G}^t(\Sigma_A)$. This is also true for the statement on the Hausdorff dimension in Theorem 1.

Define the potential $\phi\colon \Sigma_A \to \mathbb{R}$ by $\phi\colon \boldsymbol{i} = (i_k)_{k=1}^\infty \mapsto \log\|\mathrm{d}_{\pi(\boldsymbol{i})} f_{i_1,i_2}\|$. We also let

$$S_n\phi(\boldsymbol{i}) = \log\|\mathrm{d}_{\pi(\boldsymbol{i})} f_{i_1}\| + \sum_{k=1}^{n-1} \phi(\sigma^k(\boldsymbol{i})).$$

Then

$$e^{\inf_{\boldsymbol{i}\in C_{j_1\cdots j_n}} S_n\phi(\boldsymbol{i})} \leq d(C_{j_1\ldots j_n}) \leq e^{\sup_{\boldsymbol{i}\in C_{j_1\cdots j_n}} S_n\phi(\boldsymbol{i})}.$$

Since ϕ is a Hölder continuous potential, we can write the pressure of $s\phi$ as

$$\begin{aligned} P(s\phi) &= \lim_{n\to\infty} \frac{1}{n} \log\biggl(\sum_{C_{j_1\cdots j_n}} e^{\inf_{\boldsymbol{i}\in C_{j_1\cdots j_n}} sS_n\phi(\boldsymbol{i})} \biggr) \\ &= \lim_{n\to\infty} \frac{1}{n} \log\biggl(\sum_{C_{j_1\cdots j_n}} e^{\sup_{\boldsymbol{i}\in C_{j_1\cdots j_n}} sS_n\phi(\boldsymbol{i})} \biggr), \end{aligned}$$

where the sums run over all cylinders $C_{j_1\ldots j_n}$ of generation n, see [6] and [7]. It is known, see for instance [7], that $P(s\phi) = 0$ is equivalent to $\dim_{\mathrm{H}}(\Lambda_A) = s$.

We note that for each non-empty cylinder C we have

$$\begin{aligned} P(s\phi) &= \lim_{n\to\infty} \frac{1}{n} \log\biggl(\sum_{C_{j_1\cdots j_n}\subset C} e^{\inf_{\boldsymbol{i}\in C_{j_1\cdots j_n}} sS_n\phi(\boldsymbol{i})} \biggr) \\ &= \lim_{n\to\infty} \frac{1}{n} \log\biggl(\sum_{C_{j_1\cdots j_n}\subset C} e^{\sup_{\boldsymbol{i}\in C_{j_1\cdots j_n}} sS_n\phi(\boldsymbol{i})} \biggr), \end{aligned}$$

so given $s < \dim_{\mathrm{H}}(\Lambda_A)$, we have $P(s\phi) > 0$ and there are constants $m = m(s)$ and $c > 0$ such that

$$\left(\sum_{C_{j_1 \cdots j_n} \subset C_a} e^{\inf_{\boldsymbol{i} \in C_{j_1 \cdots j_n}} sS_n\phi(\boldsymbol{i})} \right) > e^{cn} > 1, \tag{5}$$

for all generation 1 cylinders C_a and all $n \geq m$.

We observe that if C is a cylinder, then it need not be true that $M_\infty^t(C) = d(C)^t$, since C might not be the optimal cover of C. However, because of (5), we only need to consider covers of C with cylinders that are at most of $m = m(t)$ generations higher than C. Since there are only finitely many such covers, we conclude that there is a largest constant $0 < c_t \leq 1$ such that

$$M_\infty^t(C) \geq c_t d(C)^t \tag{6}$$

holds for all cylinders C. A similar statement is of course true for the measure $\mathfrak{M}_\infty^t$. Note also that we always have $M_\infty^t(C) \leq d(C)^t$.

The proof of Theorem 1, will be divided into a sequence of lemmata. The proofs of these lemmata are in most cases quite similar to the corresponding ones in Falconer's paper [1]. The following lemma lets us extend the property that defines $\mathcal{G}^t(\Lambda_A)$ to any open set.

Lemma 1. *If $c > 0$ and F is a set such that*

$$M_\infty^t(F \cap C) \geq c M_\infty^t(C)$$

holds for all cylinders C, then

$$M_\infty^t(F \cap U) \geq c M_\infty^t(U)$$

holds for all open sets U.

Proof. Let U be open. Then we can write U as a disjoint union $U = \bigcup_i^\infty C_i$ where each C_i is a cylinder. Let (D_j) be cylinders covering $F \cap U$. We can assume that this cover is disjoint.

By the net property of cylinders, for any i, either there are no $D_j \subset C_i$ and instead $C_i \subset D_j$ for some j, or there are $D_j \subset C_i$ and

$$\bigcup_{D_j \cap C_i \neq \emptyset} D_j \subset C_i.$$

In case there are $D_j \subset C_i$ we have that

$$\sum_{D_j \subset C_i} d(D_j)^t \geq M_\infty^t(F \cap C_i) \geq c M_\infty^t(C_i). \tag{7}$$

If we take (D_j) and for each i such that there are $D_j \subset C_i$, we replace all these D_j by one copy of C_i, then we get a disjoint cover (E_r) of U, where all E_r are cylinders. Indeed, any C_i from $U = \bigcup_i^\infty C_i$ is either a subset of some element $E_r = D_j$, or it is itself an element E_r.

Using (7) and the fact that $c \leq 1$, we get

$$\sum d(D_j)^t \geq c \sum M_\infty^t(E_r) \geq c M_\infty^t(U).$$

Since (D_j) is arbitrary, this proves the lemma. $\square$

We will also need the following lemma.

Lemma 2. *Let F be a set such that there is a constant $c > 0$ such that*

$$M_\infty^{t_0}(F \cap C) \geq c M_\infty^{t_0}(C)$$

holds for all cylinders C. Then

$$M_\infty^t(F \cap C) \geq M_\infty^t(C)$$

holds for all cylinders C, and $0 < t \leq t_0$.

Proof. We start by proving that $M_\infty^t(F \cap C) \geq M_\infty^t(C)$ holds for $t = t_0$. We then use this result to obtain $M_\infty^t(F \cap C) \geq M_\infty^t(C)$ when $t < t_0$.

Let $(C_i)_i$ be a collection of cylinders covering $F \cap C$. We may assume that the sets C_i are pairwise disjoint. Since $M_\infty^{t_0}(F \cap C)$ is finite, we may assume that $\sum d(C_i)^{t_0}$ is finite. We let $I(m) = \{\, i : C_i \text{ is of generation } m \,\}$. Then, for any $\varepsilon > 0$, there is an m_0 such that

$$\sum_{m \geq m_0} \sum_{i \in I(m)} d(C_i)^{t_0} = \sum_i d(C_i)^{t_0} - \sum_{m < m_0} \sum_{i \in I(m)} d(C_i)^{t_0} < \varepsilon. \tag{8}$$

We let (D_j) denote a finite cover of the cylinder C by cylinders, such that for any j holds either

i) $D_j = C_i$ for some i and D_j is of generation less than m_0.

or

ii) D_j is of generation m_0, and those C_i that intersect $D_j \cap F$ are contained in D_j.

Let $Q(j) = \{\, i : C_i \subset D_j \,\}$. If j satisfies i) then

$$\sum_{i \in Q(j)} d(C_i)^{t_0} = d(D_j)^{t_0}. \tag{9}$$

For those j that satisfies ii), we have that

$$\sum_{i \in Q(j)} d(C_i)^{t_0} \geq M_\infty^{t_0}(F \cap D_j) \geq c M_\infty^{t_0}(D_j) \geq c c_{t_0} d(D_j)^{t_0}, \tag{10}$$

where c_{t_0} is defined by (6).

Using (8), (9) and (10), we conclude that

$$\begin{aligned}\sum_i d(C_i)^{t_0} &= \sum_{m<m_0}\sum_{i\in I(m)} d(C_i)^{t_0} + \sum_{m\geq m_0}\sum_{i\in I(m)} d(C_i)^{t_0} \\ &= \sum_{m<m_0}\sum_{i\in I(m)} d(C_i)^{t_0} + c^{-1}c_{t_0}^{-1}\sum_{m\geq m_0}\sum_{i\in I(m)} d(C_i)^{t_0} \\ &\qquad + (1-c^{-1}c_{t_0}^{-1})\sum_{m\geq m_0}\sum_{i\in I(m)} d(C_i)^{t_0} \\ &\geq \sum_j d(D_j)^{t_0} + (1-c^{-1}c_{t_0}^{-1})\varepsilon \geq M_\infty^{t_0}(C) + (1-c^{-1}c_{t_0}^{-1})\varepsilon.\end{aligned}$$

Hence, $M_\infty^{t_0}(C\cap F) \geq M_\infty^{t_0}(C) + (1-c^{-1}c_{t_0}^{-1})\varepsilon$. Since ε is arbitrary, we conclude that $M_\infty^{t_0}(C\cap F) \geq M_\infty^{t_0}(C)$.

We now turn to the case $0 < t < t_0$. Using what we proved above, we note that it is sufficient to prove that there is a constant c' such that $M_\infty^t(C\cap F) \geq c' M_\infty^t(C)$ holds for all cylinders C.

Let C be an n-cylinder. Take $n_0 \geq n$ such that $\lambda_2^{n_0+1} \leq \lambda_1^{n+1}$, or equivalently

$$\lambda_2^{(n_0+1)(t-t_0)} \geq \lambda_1^{(n+1)(t-t_0)}. \tag{11}$$

Let $(C_i)_i$ be a collection of cylinders covering $F\cap C$. We may assume that the sets C_i are pairwise disjoint. We let (D_j) denote a finite cover of the cylinder C by cylinders, such that for any j holds either

i) $D_j = C_i$ for some i and D_j is of generation less than n_0.

or

ii) D_j is of generation n_0, and those C_i that intersect $D_j\cap F$ are contained in D_j.

Let $Q(j) = \{\, i : C_i \subset D_j \,\}$. If j satisfies i) then

$$\sum_{i\in Q(j)} d(C_i)^t = d(D_j)^t = d(D_j)^{t-t_0} d(D_j)^{t_0} \geq d(C)^{t-t_0} d(D_j)^{t_0}.$$

If j satisfies ii) then, for $i\in Q(j)$, holds

$$\begin{aligned}d(C_i)^t &= d(C_i)^{t-t_0} d(C_i)^{t_0} \geq (\lambda_2^{n_0+1})^{t-t_0} d(C_i)^{t_0} \\ &\geq (\lambda_1^{n+1})^{t-t_0} d(C_i)^{t_0} \geq d(C)^{t-t_0} d(C_i)^{t_0},\end{aligned}$$

by (11). Hence, if j satisfies ii), then

$$\begin{aligned}\sum_{i\in Q(j)} d(C_i)^t &\geq d(C)^{t-t_0} \sum_{i\in Q(j)}^{t_0} d(C_i)^{t_0} \geq d(C)^{t-t_0} M_\infty^{t_0}(F\cap D_j) \\ &\geq d(C)^{t-t_0} M_\infty^{t_0}(D_j) \geq c_{t_0} d(C)^{t-t_0} d(D_j)^{t_0}.\end{aligned}$$

It follows that

$$\sum_i d(C_i)^t \geq c_{t_0} d(C)^{t-t_0} \sum_j d(D_j)^{t_0} \geq c_{t_0} d(C)^{t-t_0} M_\infty^{t_0}(C)$$
$$\geq c_{t_0}^2 d(C)^t \geq c_{t_0}^2 M_\infty^{t_0}(C)$$

This proves that

$$M_\infty^t(F \cap C) \geq c_{t_0}^2 M_\infty^t(C)$$

holds for all cylinders C, and $0 < t < t_0$. □

The proof of Theorem 1 is finished, if we prove the following two propositions.

Proposition 1. *If* $F \in \mathcal{G}^t(\Sigma_A)$*, then* $\dim_{\mathrm{H}}(F) \geq t$*, and so* $\dim_{\mathrm{H}}(\pi(F)) \geq t$ *with respect to Euclidean metric on* Λ_A*.*

Proposition 2. *If* $F_i \in \mathcal{G}^t(\Sigma_A)$ *for all* $i \in \mathbb{N}$*, then*

$$M_\infty^t(\bigcap_{i=1}^\infty F_i \cap U) = M_\infty^t(U),$$

for all open U*, and thereby* $\bigcap_{i=1}^\infty F_i \in \mathcal{G}^t(\Sigma_A)$*.*

Proof of Proposition 1. Clearly $M_\infty^t(F) > 0$. Hence $M^t(F) > 0$, and since M^t is equivalent to the t-dimensional Hausdorff-measure H^t, we conclude that $H^t(F) > 0$ and $\dim_{\mathrm{H}}(F) \geq t$. □

Proof of Proposition 2. The proof of this proposition is very similar to the corresponding proof in [1]. It is based on an increasing set lemma from Roger's book [8].

Let $E \subset \Sigma_A$ and $\delta > 0$. We will denote by $E_{(-\delta)}$ the set

$$E_{(-\delta)} = \{\, x \in E : \inf_{y \notin E} d(x,y) > \delta \,\},$$

and we note that $E_{(-\delta)}$ is an open set.

Take $\varepsilon > 0$. Let first $\{F_i\}_{i=1}^\infty$ be a decreasing sequence of open sets, with the property that

$$M_\infty^t(F_i \cap U) \geq M_\infty^t(U)$$

holds for any open set U. We will choose a sequence of numbers (δ_k) and open sets (U_k), such that

$$U_0 = U,$$
$$U_k = (F_k \cap U_{k-1})_{(-\delta_k)},$$

and $M_\infty^t(U_k) > M_\infty^t(U) - \varepsilon$.

We will choose δ_k and U_k inductively. Assume that they have been chosen for $k = 1, \ldots, n$. Since $F_n \in \mathcal{G}^t(\Lambda_A)$, Lemma 1 implies that

$$M^t_\infty(F_n \cap U_{n-1}) \geq M^t_\infty(U_{n-1}) > M^t_\infty(U) - \varepsilon.$$

The set $(F_n \cap U_{n-1})_{(-\delta_n)}$ is open and increases to $F_n \cap U_{n-1}$ as δ_n vanishes. Using Theorem 52 from [8], there is a δ_n such that

$$M^t_\infty(U_n) = M^t_\infty((F_n \cap U_{n-1})_{(-\delta_n)}) > M^t_\infty(U) - \varepsilon.$$

We have that $\overline{U}_k \subseteq F_k$. Let (C_l) be a cover by cylinders of $\cap \overline{U}_k$. Since $(\overline{U}_k)$ is a nested sequence of compact sets and C_l is open, there is an m such that $\overline{U}_m \subseteq \cup C_l$.

Hence

$$\sum_l d(C_l)^t \geq M^t_\infty(\overline{U}_m) \geq M^t_\infty(U) - \varepsilon.$$

As ε is arbitrary we conclude that

$$M^t_\infty\Big(\bigcap_i F_i \cap U\Big) \geq M^t_\infty(U).$$

We now note that any finite intersection of open sets in $\mathcal{G}^t(\Lambda_A)$ is in $\mathcal{G}^t(\Lambda_A)$ and that any countable intersection of G_δ sets can be expressed as the intersection of a countable decreasing sequence of open sets. Together with the result proved above, this finishes the proof of the lemma. □

6 Proof of Theorem 2 and Corollary 1

Let $G_g(p)$ be the set of points in Σ_A for which p is an accumulation point of the ergodic averages of g, see (4), and let $G_g(p, n, \varepsilon)$ be defined as

$$G_g(p, n, \varepsilon) = \Big\{ \boldsymbol{i} \in \Sigma_A : \; p - \varepsilon < \frac{1}{n} \sum_{k=0}^{n-1} g(\sigma^k \boldsymbol{i}) < p + \varepsilon \Big\}.$$

We also introduce the notation $\hat{G}_g(p)$ for the set of points for which the ergodic averages of g converges to p. Hence $\hat{G}_g(p)$ is a subset of $G_g(p)$ and so $\dim_{\mathrm{H}} G_g(p) \geq \dim_{\mathrm{H}} \hat{G}_p(p)$.

The key step in proving Theorem 2 and Corollary 1 is the following proposition.

Proposition 3. *For any real valued continuous function g on Σ_A, it holds that $G_g(p) \in \mathcal{G}^t(\Sigma_A)$ for all $t < \dim_{\mathrm{H}} (G_g(p))$.*

Proof of Theorem 2. According to Proposition 3, the set $G_{g_i}(x_i) = \{\, y \in \Lambda_A : x_i \in A_{g_i}(y) \,\}$ is in $\mathcal{G}^t(\Sigma_A)$ for $t < \dim_{\mathrm{H}} G_{g_i}(x_i)$. Now Theorem 2 follows from Theorem 1. □

Proof of Corollary 1. Let $\varepsilon > 0$. We will use Theorem 9 of [9]. This theorem implies that there is a non-empty open interval I_{g_i} such that $p \mapsto \dim_{\mathrm{H}} \hat{G}_{g_i}(p)$ is a real analytic function.

Since there are only finitely many maps f_i and $f_{i,j}$ the conformal graph directed markov system is regular and there exists an ergodic measure of full dimension, see [7]. By the Birkhoff ergodic theorem, it is therefore clear that $p \mapsto \dim_{\mathrm{H}} \hat{G}_{g_i}(p)$ attains the value $\dim_{\mathrm{H}} \Lambda_A$, somewhere in the interval I_{g_i}. Hence, for each g_i there are $x_{1,i} \neq x_{2,i}$ such that $\dim_{\mathrm{H}} \hat{G}_{g_i}(x_{1,i}) > \dim_{\mathrm{H}} \Lambda_A - \varepsilon$ and $\dim_{\mathrm{H}} \hat{G}_{g_i}(x_{2,i}) > \dim_{\mathrm{H}} \Lambda_A - \varepsilon$. We then have $\dim_{\mathrm{H}} G_{g_i}(x_{1,i}) > \dim_{\mathrm{H}} \Lambda_A - \varepsilon$ and $\dim_{\mathrm{H}} G_{g_i}(x_{2,i}) > \dim_{\mathrm{H}} \Lambda_A - \varepsilon$.

Proposition 3 implies that $G_{g_i}(x_{1,i})$ and $G_{g_i}(x_{2,i})$ are in $\mathcal{G}^{\dim_{\mathrm{H}} \Lambda_A - \varepsilon}(\Sigma_A)$. Thereby $G_{g_i}(x_{1,i}) \cap G_{g_i}(x_{2,i})$ is in $\mathcal{G}^{\dim_{\mathrm{H}} \Lambda_A - \varepsilon}(\Sigma_A)$. This set consists of points for which both $x_{1,i}$ and $x_{2,i}$ are accumulation points of the ergodic averages of g_i. Hence the set of points where the ergodic averages do not converge for any g_i is in $\mathcal{G}^{\dim_{\mathrm{H}} \Lambda_A - \varepsilon}(\Sigma_A)$. Let $\varepsilon \to 0$. □

The proof of Proposition 3 is broken down to a series of lemmata. The following lemma will be used to show that $G_g(p)$ is in the class $\mathcal{G}^t(\Sigma_A)$.

Lemma 3. *Let $(F_k)_{k=1}^{\infty}$ be a sequence of open sets in Σ_A. If there is a $c > 0$ such that*

$$\limsup_{k \to \infty} M_\infty^t(F_k \cap C) \geq c M_\infty^t(C)$$

for all cylinders C, then

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_k \in \mathcal{G}^{t'}(\Sigma_A), \ \ \forall t' \leq t.$$

Proof. It follows from Lemma 2 that $\bigcup_{k=n}^{\infty} F_k \in \mathcal{G}^{t'}(\Sigma_A)$ for all n. The Lemma then follows from Theorem 1. □

We need to prove that the condition of Lemma 3 is satisfied for $G_g(p)$. Of course, the sets F_k in Lemma 3 will be the sets $G_g(p, k, \varepsilon)$. Hence, we will need that the sets $G_g(p, k, \varepsilon)$ are open. This is guarantied if g is continuous. From now on we assume without mentioning that g is a continuous function on Σ_A.

Before we continue we fix an $s < \dim_{\mathrm{H}} \Lambda_A$, and an $m = m(s)$ according to (5). We define new outer measures $N_\infty^{m,t}$, $0 < t < s$, defined as M_∞^t, but instead of considering all covers by cylinders, we only consider covers by cylinders of generation km, where $k \in \mathbb{N}$. We note that (5) ensures that $N_\infty^{m,t}(C) = d(C)^t$ for all cylinders C of generation km, where $k \in \mathbb{N}$. This makes it easier to work with $N_\infty^{m,t}$, than with M_∞^t.

One observes that the outer measures M_∞^t and $N_\infty^{m,t}$ are equivalent. Hence there is some constant c_1, depending on m and t, such that $M_\infty^t(U) \leq N_\infty^{m,t}(U) \leq c_1 M_\infty^t(U)$ holds for any open set U. Therefore, by Lemma 2, to prove that $G_g(p)$ is in the class $\mathcal{G}^t(\Sigma_A)$, it is sufficient to consider the measures $N_\infty^{m,t}$ instead of M_∞^t.

Lemma 4. *Assume that there is a $c < 1$ and an integer M such that*

$$N_\infty^{m,t}(C_{i_1\dots i_m} \cap G_g(p, M, \varepsilon)) < cN_\infty^{m,t}(C_{i_1\dots i_m}),$$

holds for the cylinder $C_{i_1\dots i_m}$. Then for each cylinder $C = C_{c_1\dots c_{jm}}$ such that $j \in \mathbb{N}$ and $c_{(j-1)m+1}\dots c_{jm} = i_1 \dots i_m$, there is a constant K_C, depending only on C and g, such that

$$N_\infty^{m,t}(C \cap G_g(p, M, \varepsilon/2)) < \kappa c N_\infty^{m,t}(C)$$

holds if $M \geq K_C$.

Proof. Let C and $C_{i_1\dots i_m}$ be as in the statement of the lemma. Take A such that $A > |g|$.

By assumption, there is a cover of $C_{i_1\dots i_m} \cap G_g(p, M, \varepsilon)$ by cylinders (U_i) of generation km, with value less than $cN_\infty^{m,t}(C_{i_1\dots i_m})$. For each U_i there is a corresponding set $\tilde{U}_i$, created by appending $c_1 \dots c_{(j-1)m}$ to the beginning of the coding of each element in U_i.

We note that $d(\tilde{U}_i)^t \leq \kappa \frac{d(C)^t d(U_i)^t}{d(C_{i_1\dots i_m})^t} = \kappa \frac{d(C)^t d(U_i)^t}{N_\infty^{m,t}(C_{i_1\dots i_m})}$, by (3). Moreover, $(\tilde{U}_i)_{i=1}^\infty$ is a cover of

$$C \cap G_g(p, M, \varepsilon - 2A(j-1)m/M).$$

It now follows that

$$\begin{aligned} N_\infty^{m,t}(C \cap G_g(p, M, \varepsilon - 2A(j-1)m/M) &\leq \sum_i d(\tilde{U}_i)^t \\ &\leq \frac{\kappa d(C)^t}{N_\infty^{m,t}(C_{i_1\dots i_m})} \sum_i d(U_i)^t < \kappa c d(C)^t = \kappa c N_\infty^{m,t}(C). \end{aligned}$$

The lemma now follows if we choose K_C so large that $2A(j-1)m/K_C < \varepsilon/2$. □

Lemma 5. *Assume that there is a $c > 0$ and an integer M such that*

$$N_\infty^{m,t}(C_{i_1\dots i_m} \cap G_g(p, M, \varepsilon)) > cN_\infty^{m,t}(C_{i_1\dots i_m}),$$

holds for the cylinder $C_{i_1\dots i_m}$. Then for each cylinder $C = C_{c_1\dots c_{jm}}$ such that $j \in \mathbb{N}$ and $c_{(j-1)m+1}\dots c_{jm} = i_1 \dots i_m$, there is a constant K_C, depending only on C and g, such that

$$N_\infty^{m,t}(C \cap G_g(p, M, 2\varepsilon)) > \kappa^{-1} c N_\infty^{m,t}(C),$$

holds if $M \geq K_C$.

Proof. Let C and $C_{i_1 \dots i_m}$ be as in the statement of the theorem.

Let $A > |g|$ and consider the set $C_{i_1 \dots i_m} \cap G_g(p, M, \varepsilon)$. We can write this set as a union of cylinders. If we append $c_1 \dots c_{(j-1)m}$ to the beginning of the coding of each of these cylinders, we get a collection of cylinders in

$$C \cap G_g(p, M, \varepsilon + 2A(j-1)m/M).$$

Let $(\tilde{U}_i)$ be a cover of $C \cap G_g(p, M, \varepsilon + 2A(j-1)m/M)$. For each $\tilde{U}_i$ there is a corresponding set U_i, created by removing the first $(j-1)m$ symbols in the coding.

We note that $d(\tilde{U}_i)^t \geq \kappa^{-1} \frac{d(C)^t d(U_i)^t}{d(C_{i_1 \dots i_m})^t} = \kappa^{-1} \frac{d(C)^t d(U_i)^t}{N_\infty^{m,t}(C_{i_1 \dots i_m})}$, by (3), and that $(U_i)_{i=1}^\infty$ is a cover of $C_{i_1 \dots i_m} \cap G_g(p, M, \varepsilon)$.

It now follows that

$$\sum_i d(\tilde{U}_i)^t \geq \frac{\kappa^{-1} d(C)^t}{N_\infty^{m,t}(C_{i_1 \dots i_m})} \sum_i d(U_i)^t > \kappa^{-1} c d(C)^t = \kappa^{-1} c N_\infty^{m,t}(C).$$

Since the cover $(\tilde{U}_i)$ was arbitrary, we get

$$N_\infty^{m,t}(C \cap G_g(p, M, \varepsilon + 2A(j-1)m/M)) > \kappa^{-1} c N_\infty^{m,t}(C).$$

The lemma now follows if we choose K_C so large that $2A(j-1)m/K_C < \varepsilon$. □

Lemma 6. *There is a constant $L > 0$ such that if there is a $c > 0$ and an M such that*

$$N_\infty^{m,t}(C_{i_1 \dots i_m} \cap G_g(p, M, \varepsilon)) > c N_\infty^{m,t}(C_{i_1 \dots i_m}),$$

for one cylinder $C_{i_1 \dots i_m}$ of generation m, then for each cylinder C, of generation m, there is a number K_C, depending only on C and g, such that

$$N_\infty^{m,t}(C \cap G_g(p, M, 2\varepsilon)) > L c N_\infty^{m,t}(C),$$

if $M \geq K_C$.

Proof. Since Σ_A is transitive, there is a number $N \in \mathbb{N}$ such that any cylinder $C = C_{c_1 \dots c_m}$ contains a cylinder $C_{c_1 \dots c_{Nm}}$ of generation Nm for which $c_{(N-1)m+1} \dots c_{Nm} = i_1 \dots i_m$. Since Σ_A is of finite type and the maps $f = f_i$ or $f = f_{i,j}$ all satisfy $\lambda_1 \leq |f'| \leq \lambda_2$, it is clear that there is a uniform constant $L' > 0$ such that

$$\frac{d(C_{c_1 \dots c_{Nm}})^t}{d(C_{c_1 \dots c_m})^t} > L',$$

regardless of which cylinders $C_{i_1 \dots i_m}$ and $C_{c_1 \dots c_{Nm}}$ we started with. (For instance, $L' = \lambda_1^{t(N-1)m}$ will do.) Using Lemma 5 we can find K_C such that

$$\begin{aligned} N_\infty^{m,t}(C_{c_1 \dots c_m} \cap G_g(p, M, 2\varepsilon)) &\geq N_\infty^{m,t}(C_{c_1 \dots c_{Nm}} \cap G_g(p, M, 2\varepsilon)) \\ &\geq \kappa^{-1} c N_\infty^{m,t}(C_{c_1 \dots c_{Nm}}) > L' \kappa^{-1} c N_\infty^{m,t}(C_{c_1 \dots c_m}) \end{aligned}$$

if $M \geq K_C$. Let $L = L' \kappa^{-1}$. □

Lemma 7. *For each $t < \dim_{\mathrm{H}}(G_g(p))$ there is a constant $c > 0$ such that for any cylinder $C_{i_1\ldots i_m}$ of generation m and any $\varepsilon > 0$, it holds that*

$$\liminf_{M\to\infty} N_\infty^{m,t}(C_{i_1\ldots i_m} \cap G_g(p, M, \varepsilon)) \geq cN_\infty^{m,t}(C_{i_1\ldots i_m}),$$

for any $\varepsilon > 0$.

Proof. Let L be given by Lemma 6. Assume on the contrary that there exist a cylinder $C_{i_1\ldots i_m}$ and a strictly increasing sequence $(M_k)_{k=1}^\infty$ such that

$$N_\infty^{m,t}(C_{i_1\ldots i_m} \cap G_g(p, M_k, 2\varepsilon)) < \kappa^{-1}\frac{L}{2}N_\infty^{m,t}(C_{i_1\ldots i_m}), \quad \forall k. \tag{12}$$

By Lemma 6 we get

$$N_\infty^{m,t}(C_{c_1\ldots c_m} \cap G_g(p, M_k, \varepsilon)) \leq \kappa^{-1}\frac{1}{2}N_\infty^{m,t}(C_{c_1\ldots c_m}), \quad \forall M_k \geq K_{C_{c_1\ldots c_m}}$$

for each $C_{c_1\ldots c_m}$. Indeed, if would have

$$N_\infty^{m,t}(C_{c_1\ldots c_m} \cap G_g(p, M_k, \varepsilon)) > \kappa^{-1}\frac{1}{2}N_\infty^{m,t}(C_{c_1\ldots c_m}),$$

for some $C_{c_1\ldots c_m}$ and $M_k \geq K_{C_{c_1\ldots c_m}}$, then Lemma 6 gives us a contradiction to (12).

By Lemma 4 we get that

$$N_\infty^{m,t}(C \cap G_p(M_k, \varepsilon/2)) \leq \frac{1}{2}N_\infty^{m,t}(C), \ \forall k \geq \max\{K_C, K_{C_{c_1\ldots c_m}}\}$$

for each cylinder C of any generation jm, $j \in \mathbb{N}$. Thus, there is a number k_1 and finite cover $(C_i)_i$ of $C_{i_1\ldots i_m} \cap G_g(p, M_{k_1}, \varepsilon/2)$ such that

$$\sum_i d(C_i)^t \leq \frac{2}{3}N_\infty^{m,t}(C_{i_1\ldots i_m}).$$

Consider each C_i and its intersection with $G_g(p, M_k, \varepsilon)$ for $k > k_1$. There is now a $k_2 > k_1$ and a finite cover $(C_{i,j})$ such that $(C_{i,j})_{j=1}^\infty$ covers $C_i \cap G_g(p, M_{k_2}, \varepsilon)$ and

$$\sum_j d(C_{i,j})^t \leq \frac{2}{3}N_\infty^{m,t}(C_i) = \frac{2}{3}d(C_i)^t.$$

We get $\sum_{i,j} d(C_{i,j})^t \leq \left(\frac{2}{3}\right)^2 N_\infty^{m,t}(C_{i_1\ldots i_m})$.

Continuing like this we get $\dim_{\mathrm{H}}(G_g(p)) < t$ which contradicts the assumptions. □

Lemma 8. *For each $t < \dim_{\mathrm{H}}(G_g(p))$ there is a constant $c > 0$ such that for any cylinder C and any $\varepsilon > 0$, it holds that*

$$\liminf_{M\to\infty} N_\infty^{m,t}(C \cap G_g(p, M, \varepsilon)) \geq cN_\infty^{m,t}(C).$$

Proof. For cylinders of generation jm, where $j \in \mathbb{N}$, this follows from Lemma 7 and Lemma 5. Since any other cylinder contains at least one cylinder of some generation jm, and since there is a uniform bound on the relative size of these two sets, the lemma follows. □

We are now ready to prove Proposition 3.

Proof of Proposition 3. We first note that

$$G_g(p) = \bigcap_{\varepsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{M=n}^{\infty} G_g(p, M, \varepsilon)$$

and we recall that the outer measures M^t_∞ and $N^{m,t}_\infty$ are equivalent. By Theorem 1, Lemma 8 and Lemma 3 it follows that $G_g(p)$ is in $\mathcal{G}^t(\Sigma_A)$ for any $t < \dim_{\mathrm{H}}(G_g(p))$. □

7 Proof of Corollary 2

Given $\varepsilon > 0$, by Proposition 1 in [4], we can approximate Σ_f from the inside by a subshift of finite type Σ_A, such that the set $E \subset [0, 1]$, corresponding to Σ_A, has Hausdorff dimension at least $1 - \varepsilon$. The subshift Σ_A is created from Σ_f by forbidding some words of length $n(\varepsilon)$, corresponding to cylinders close to the endpoints of the intervals (a_k, a_{k+1}).

By assumption, the functions $g_i \circ \pi$ are all in $C^{K_0}_\alpha(\Sigma_f)$, for some constants K_0 and $\alpha > 0$. We need to show that Σ_A can be chosen such that no g_i restricted to Σ_A is cohomologous to a constant.

Lemma 9. *For each function $g_i \circ \pi$ there is a subshift Σ_A such that $g_i \circ \pi$ is not cohomologous to a constant on Σ_A.*

Proof. To get a contradiction, assume g_i is cohomologous to a constant on each Σ_A. It means that there is a constant C_A and a continuous function $\chi_A \colon \Sigma_A \to \mathbb{R}$ such that

$$g \circ \pi - C_A = \chi_A \circ \sigma - \chi_A \tag{13}$$

on Σ_A. We claim that the function $\chi_A \colon \Sigma_A \to \mathbb{R}$ is in $C^K_\alpha(\Sigma_A)$, where K only depends on f and K_0.

The assumption that $(0, 1) \setminus \cup_{k=0}^{q-1} f((a_k, a_{k+1}))$ does not contain isolated points implies that any small enough interval in $\cup_{k=0}^{q-1} f((a_k, a_{k+1}))$ is contained in some $f((a_k, a_{k+1}))$. Since f is uniformly expanding and there are only q inverse branches of f, it is clear that for some $n_0 \in \mathbb{N}$, each cylinder $C_{x_1 \dots x_n}$ in Σ_f, where $n \geq n_0$, satisfies $\pi(C_{x_1 \dots x_n}) = f(\pi(C_{a x_1 \dots x_n}))$ for some $a \in \{0, 1, \dots q - 1\}$. Hence, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are such that $x_1 \dots x_n = y_1 \dots y_n$, then if $n \geq n_0$, there is always a digit a such that $a\boldsymbol{x} \in \Sigma_f$ and $a\boldsymbol{y} \in \Sigma_f$.

Since Σ_A is created from Σ_f by removing words corresponding to points close to the endpoints of the intervals (a_k, a_{k+1}), we can make sure that we always have $a\boldsymbol{x} \in \Sigma_A$ and $a\boldsymbol{y} \in \Sigma_A$ by choosing Σ_A and n_0 large enough.

Define

$$A_n = \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Sigma_A \\ x_1\ldots x_n = y_1 \ldots y_n}} |\chi_A(\boldsymbol{x}) - \chi_A(\boldsymbol{y})|.$$

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be elements of Σ_A, with $x_1 \ldots x_n = y_1 \ldots y_n$, such that there is a digit a with $a\boldsymbol{x}, a\boldsymbol{y} \in \Sigma_A$. Then by (13), we have

$$\begin{aligned} |\chi_A(\boldsymbol{x}) - \chi_A(\boldsymbol{y})| &\leq |\chi_A(a\boldsymbol{x}) - \chi_A(a\boldsymbol{y})| + |g(a\boldsymbol{x}) - g(a\boldsymbol{y})| \\ &\leq |\chi_A(a\boldsymbol{x}) - \chi_A(a\boldsymbol{y})| + K_0 2^{-\alpha(n+1)}. \end{aligned}$$

Taking supremum, we get that

$$|\chi_A(\boldsymbol{x}) - \chi_A(\boldsymbol{y})| \leq A_{n+1} + K_0 2^{-\alpha(n+1)}$$

holds for all $\boldsymbol{x}$ and $\boldsymbol{y}$ with $x_1 \ldots x_n = y_1 \ldots y_n$, such that there is a digit a with $a\boldsymbol{x}, a\boldsymbol{y} \in \Sigma_A$. If $n \geq n_0$ there is always such a digit a if $x_1 \ldots x_n = y_1 \ldots y_n$. Hence we get

$$A_n \leq A_{n+1} + K_0 2^{-\alpha(n+1)},$$

for large enough n. Since χ_A is continuous on the compact set Σ_A, we know that $A_n \to 0$ as $n \to 0$. This implies that

$$A_n \leq K 2^{-\alpha n}, \quad \forall n,$$

for some constant K that does not depend on Σ_A. This shows the claim that χ_A is in $C_\alpha^K(\Sigma_A)$.

It is clear that we can use the same constant $C = C_A$ on each subshift $\Sigma_A \subset \Sigma_f$. Since the functions χ_A are uniformly Hölder continuous, they can be extended to a function χ in $C_\alpha^K(\Sigma_f)$. We get that

$$g \circ \pi - C = \chi \circ \sigma - \chi$$

on Σ_f, which is a contradiction. □

By Lemma 9 we can choose Σ_A such that no $g_i \circ \pi$, restricted to Σ_A, is cohomologous to a constant. Now Corollary 2 follows from Corollary 1 by letting $\varepsilon \to 0$.

8 Proof of Theorem 3

In this section we will prove Theorem 3, that $W(\alpha) \cap K$ is in $\mathcal{G}^t(K)$ for all $t < \frac{1}{\alpha} \frac{\log 2}{\log 3}$. We emphasise that $W(\alpha) \cap K$ is not in $\mathcal{G}^t(K)$ for any $t > \frac{1}{\alpha} \frac{\log 2}{\log 3}$, since $\dim_{\mathrm{H}} (W(\alpha) \cap K) \leq \frac{1}{\alpha} \frac{\log 2}{\log 3}$, as is easily shown by standard arguments.

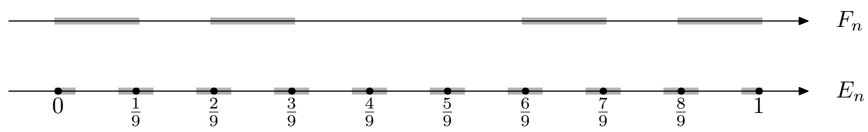

Figure 1: Picture of F_n and E_n for $n = 2$.

Let

$$E_n = \{ x \in [0,1) : |x - p/3^n| \leq 3^{-\alpha n}, \text{ for some } p \in \mathbb{Z} \},$$

and let F_n be the n-th step in the construction of the middle third Cantor set. There is a picture showing F_n and E_n for $n = 2$ in Figure 1.

Let n be so large that $2/3^{\alpha n} < 1/3^n$. Then E_n consists of 3^n+1 intervals of length $2/3^{\alpha n}$ (apart from two, which are of length $1/3^{\alpha n}$) and F_n consists of 2^n intervals of length $1/3^n$. For any interval from E_n that intersects K, the midpoint of the interval is an endpoint of an interval in F_n. Moreover, any endpoint of an interval in F_n is also a midpoint of an interval in E_n. It follows that exactly 2^{n+1} intervals from E_n intersect K and that all these intersections are congruent, that is, they are translations and reflections of each other.

Let C be a cylinder in K of generation m. Then C is an interval of length $1/3^m$. (Rather, the convex hull of C is an interval of length $1/3^m$.)

We need to estimate $\mathfrak{M}^t_\infty(C \cap E_n \cap K)$ from below, for large n. For this purpose we consider a cover of $C \cap E_n \cap K$ by cylinders $\{U_i\}$ in K. (This means that each U_i is a projection by π from a cylinder in the symbolic space Σ_A.) We may assume that this cover consists of sets with pairwise disjoint interior, all contained in C. Obviously $\mathfrak{M}^t_\infty(C \cap E_n \cap K) \leq |C|^t$, so to get a lower bound we can assume that the generations of the cylinders in the cover are strictly larger than m.

Consider one of the cylinders U_i and let n_i be its generation. We have two cases. Either $n_i \leq n$ or $n_i > n$. In both cases we have m smaller than n and n_i.

If $n_i \leq n$ then U_i intersects 2^{n-n_i+1} intervals from E_n.

If $n_i > n$, then U_i intersects at most one interval from E_n. We may then assume that U_i intersects exactly one interval and that $|U_i| \leq 3/3^{\alpha n}$.

Let μ be a uniform mass distribution of mass 1 on $C \cap K \cap E_n$. This means that μ is the normalised restriction of the Hausdorff measure $\mathfrak{H}^s$, for $s = \frac{\log 2}{\log 3}$, on $C \cap K \cap E_n$. (The measure $\mathfrak{H}^s$ is finite for this s.)

In the first case when $n_i \leq n$ we have

$$\mu(U_i) = \frac{2^{n-n_i+1}}{2^{n-m+1}} = \frac{2^{-n_i}}{2^{-m}} = \left(\frac{|U_i|}{|C|} \right)^{\frac{\log 2}{\log 3}} \leq \frac{|U_i|^t}{|C|^t},$$

if $t \leq \frac{\log 2}{\log 3}$.

In case $n_i > n$ we first observe that if I is the interval from E_n that intersects U_i, then $\mu(U_i)/\mu(I) \leq 6(|U_i|/|I|)^{\frac{\log 2}{\log 3}}$. (The constant 6 is not optimal.) We have that $|I| = 1/3^{\alpha n}$ since the intervals of E_n are of length $2/3^{\alpha n}$, and we loose half of each interval when intersecting with K. Moreover, $\mu(I) = 1/2^{n-m+1}$, since there are 2^{n-m+1} such intervals of equal measure. Hence

$$\begin{aligned}\mu(U_i) &\leq \frac{6}{2^{n-m+1}}\Big(\frac{|U_i|}{3^{-\alpha n}}\Big)^{\frac{\log 2}{\log 3}} = \frac{3}{2^{n-m}}\Big(\frac{|U_i|}{3^{-\alpha n}}\Big)^{\frac{\log 2}{\log 3}} \\ &= \frac{3}{2^n}\Big(\frac{|U_i|}{3^{-\alpha n}|C|}\Big)^{\frac{\log 2}{\log 3}} = 3\frac{3^{\alpha n t}}{2^n}\frac{|U_i|^t}{|C|^t}\Big(\frac{|U_i|}{3^{-\alpha n}|C|}\Big)^{\frac{\log 2}{\log 3}-t}.\end{aligned}$$

If $t < \frac{1}{\alpha}\frac{\log 2}{\log 3}$ and n is sufficiently large, we have

$$3\frac{3^{\alpha n t}}{2^n}\Big(\frac{|U_i|}{3^{-\alpha n}|C|}\Big)^{\frac{\log 2}{\log 3}-t} \leq 3\frac{3^{\alpha n t}}{2^n}\Big(\frac{3}{|C|}\Big)^{\frac{\log 2}{\log 3}-t} \leq 2,$$

so

$$\mu(U_i) \leq 2\frac{|U_i|^t}{|C|^t}.$$

Summing over i we get

$$1 = \sum_i \mu(U_i) \leq \sum_i 2\frac{|U_i|^t}{|C|^t},$$

and $\sum_i |U_i|^t \geq \frac{1}{2}|C|^t$. Since U_i is an arbitrary cover we get that $\mathfrak{M}^t_\infty(C\cap E_n\cap K) \geq \frac{1}{2}|C|^t$, provided that $t < \frac{1}{\alpha}\frac{\log 2}{\log 3}$ and n is large. Now Lemma 3 implies that $\limsup(E_n \cap K)$ is in $\mathcal{G}^t(K)$ for $t < \frac{1}{\alpha}\frac{\log 2}{\log 3}$. This proves Theorem 3.

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