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A quartic system with 26 limit cycles

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A QUARTIC SYSTEM WITH TWENTY-SIX LIMIT CYCLES

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ABSTRACT. We construct a planar quartic system and demonstrate that it has at least 26 limit cycles. The vector field is symmetric and integrable, but non-Hamiltonian. The proof is based on a verified computation of zeros of pseudo-Abelian integrals, together with the symmetry properties.

1. INTRODUCTION

Bifurcations of limit cycles from planar polynomial vector fields is connected to the second part of Hilbert's sixteenth problem, which asks for an upper bound, $H(n)$, on the number of limit cycles that a nth degree planar polynomial vector field can have. The determination of $H(n)$ has turned out to be very difficult, in fact it is not even known whether $H(n)$ exists. A thorough review of this problem and the progress that has been made towards its solution can be found in [13, 23].

There are various restricted versions of the problem, the most well-studied is probably the weak Hilbert's sixteenth problem, formulated by Arnold [1, 5]. The weak problem asks for an upper bound, $Z(n)$, on the number of limit cycles that can bifurcate from a planar Hamiltonian vector field under first order perturbation. Obviously, $Z(n) \leq H(n)$. The weak problem has been solved for $n = 2$, $Z(2) = 2$ [3]. For $n > 2$ it is known that $Z(n)$ exists, see e.g. [26], but there are no realistic upper bounds on its growth.

Based on the difficulties to obtain upper bounds on the number of limit cycles of a planar polynomial vector field, there has been a large interest in the computation of lower bounds of $H(n)$ and $Z(n)$. Some such bounds are $Z(3) \geq 13$ [19], $Z(4) \geq 16$ [27], $Z(5) \geq 27$ [17], $Z(7) \geq 53$ [18], and in addition $H(4) \geq 22$ [5]. The results for the weak problem for odd degrees are proved by calculating the number of zeros of Abelian integrals, whereas the results for quartic vector fields are proved by computing focus values at a linear centre. In this paper we improve the result for quartic vector fields and prove that $H(4) \geq 26$.

The construction in this paper is based on first order perturbation of an integrable, non-Hamiltonian planar vector field with maximum number of centres and $\mathbb{Z}_2$ symmetry. The method, however, is general, and could be applied to the study of perturbations around any centre where one is able to parametrise the level curves. A detailed analysis of the impact of symmetry on cubic and quartic vector fields is done in [24]. Most studies on the bifurcation of limit cycles under first order perturbations are done for perturbations of Hamiltonian vector fields; some other papers considering integrable, non-Hamiltonian, systems are [12, 25]. Our proof is done using verified numerical computations based on interval analysis [20, 21]. The details of the numerical computations, which are very delicate, are described

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in Section 3. Other computer-aided approaches to the determination of phase portraits of planar vector fields include [9, 10, 14, 15, 16]. A computational algebra approach to the centre problem can be found in [22].

The outline of this paper is as follows: in Section 2 we describe our construction, and the theoretical backround to our results. Some information about computer-aided proofs and the numerical issues involved is given in Section 3. Finally, in Section 4, we describe the results of the computations.

We end this introduction by stating our main theorem.

Theorem 1.1. *There exists a quartic integrable planar vector field, and a quartic perturbation, such that the perturbed system has at least* 26 *limit cycles when the perturbation is sufficiently small.*

2. The construction

As in [17, 18], we study vector fields with maximal number of centres and $\mathbb{Z}_2$ symmetric first integrals. We study bifurcations from a quartic system, whose first integral is equal to the Hamiltonian of a cubic Hamiltonian system with maximal number of centres.

$$\left\{ \begin{array}{lcl} \dot{x} & = & -y^2(y^2-1.1) \\ \dot{y} & = & xy(x^2-0.9) \end{array} \right. \tag{1}$$

Clearly, an integrating factor of (1) is $\mu = 1/y$, and the corresponding first integral is

$$F(x,y) = \frac{1}{4}\left(x^4+y^4\right) - \frac{1}{20}\left(9x^2+11y^2\right), \tag{2}$$

which is $\mathbb{Z}_2$ symmetric. The system (1) has 6 annuli of periodic orbits, appearing in two classes, Γ_1 (multiplicity 4) and Γ_2 (multiplicity 2), see Figure 1.

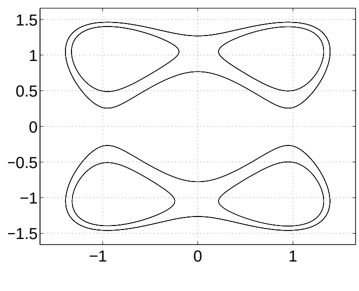

Figure 1. The phase portrait of system (1).

To use the symmetry of the system, we consider perturbations with the same kind of symmetry. By using perturbations that respect the symmetry, any limit cycle that we detect will imply the existence of 3 additional ones, if it belongs to the first class of periodic annuli, and 1 additional limit cycle if it belongs to the second class. The perturbed system that we consider is the following:

$$\left\{ \begin{array}{lcl} \dot{x} & = & -y^2(y^2-1.1) \\ \dot{y} & = & xy(x^2-0.9) + \epsilon\left(x^2y^2 + \frac{\alpha_1}{3}y^4 - \alpha_2 - \alpha_3x^2 - \alpha_4x^4\right) \end{array} \right., \tag{3}$$

where $\epsilon > 0$ is a small parameter. We denote the perturbation by $\epsilon g(x,y)$.

Each annulus of periodic orbits correspond to a continuous family of level curves of the first integral $\gamma_h \subset F^{-1}(h)$. The classical method to study perturbations from such an annulus in the Hamiltonian case is to study Abelian integrals, see e.g. [5, 8]. In the present non-Hamiltonian setting, the corresponding objects are pseudo-Abelian integrals, which have similar properties. Given a perturbation as in Equation (3) we define the pseudo-Abelian integral (in general multivalued) as:

$$I(h) = \int_{\Gamma_h} -\mu(x,y)g(x,y)\,dx. \tag{4}$$

Which in our case reads:

$$I(h) = \int_{\Gamma_h} \left(-x^2y - \frac{\alpha_1}{3}y^3 + \alpha_2\frac{1}{y} + \alpha_3\frac{x^2}{y} + \alpha_4\frac{x^4}{y}\right) dx. \tag{5}$$

The most important property of (pseudo-)Abelian integrals is described by the Poincaré-Pontryagin theorem.

Theorem 2.1 (Poincaré-Pontryagin). *Let P be the return map defined on some section transversal to the level curves of F, parametrised by the values h of F, where h is taken from some bounded interval (a,b). Let $d(h) = P(h) - h$ be the displacement function. Then, $d(h) = \epsilon(I(h) + \epsilon\phi(h,\epsilon))$,* as $\epsilon \to 0$, *where $\phi(h,\epsilon)$ is analytic and uniformly bounded on a compact neighbourhood of $\epsilon = 0,\ h \in (a,b)$.*

Proof. see e.g. [5]. □

As a consequence of the above theorem, one can prove that a simple zero of $I(h)$ corresponds to a unique limit cycle bifurcating from the integrable system as $\epsilon \to 0$. In fact, to prove the existence of a limit cycle, it suffices to have a zero of odd order.

Clearly, if (5) is zero on one level curve in one of the annuli of periodic orbits in a class, it is zero on the other ones, since the different annuli are reflections of each other in the $x-$ or $y-$ axes. This yields the claimed multiplicity of limit cycles.

3. Numerical issues

The proof of Theorem 1.1 is computer-aided. In order to use a computer to prove mathematical statements, the results of a computation must be guaranteed to be correct. We need to prove that a given mathematical statement can be reduced to a finite number of computable conditions, and construct an algorithm which checks these conditions.

A numerical algorithm is said to be *auto-validating* if it produces a mathematically correct result. The basic object in any such algorithm is an interval, whose endpoints are computer-representable floating points. Replacing numbers with intervals, yields an arithmetic for sets. Computing with sets, rather than points, we can quantify all discretisation errors of a numerical algorithm. Since any bounded subset of the plane can be covered by a finite number of axis-parallel boxes, we can, e.g., do computations that are valid for the entire domain of a function, or all functions in a finitely parametrised family. In addition to the discretisation errors of the numerical method, an auto-validated numerical algorithm also incorporates the computer's internal representation of the floating point numbers and its rounding procedures. All mathematical operations are performed in interval arithmetic with directed rounding to ensure the correctness of the result, see e.g. [20, 21] for details. Our algorithms use the interval arithmetic package `C-XSC` [6, 11].

In the case at hand, we need to rigorously enclose the value of $I(h)$ in Equation 5. We do this using the interval extension of Simpson's method, i.e.

$$\int_a^b f(x)\,dx \in \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)) - \frac{1}{2880}(b-a)^5 f^{(4)}([a,b]).$$

Note that $f^{(4)}([a,b])$ in the above formula denotes an interval enclosure of the range of $f^{(4)}$ on the interval $[a,b]$. In order to enclose the error term in Simpson's method we need differentiable information about the function. To get the enclosures of the required derivatives we compute with *Taylor arithmetic* (the necessary software is provided by [2]) which uses *automatic differentiation* (see e.g. [7]) to compute the derivatives of a function.

The simplicity of the first integral (2) implies that we can determine, by hand, the parts of a given level curve γ_h where $y = y(x)$ and $x = x(y)$, respectively. This reduces the computation of $I(h)$ to one-dimensional integration. With this parametrisation, however, the function we are integrating contains a factor of the form $\sqrt{\cdots + \sqrt{\cdots}}$, which is numerically unstable. When we compute higher order derivatives, even more accuracy is lost. The choice of Simpson's method rather than a higher order Taylor integration scheme, is a compromise between order and accuracy of enclosures. The size of the enclosures of higher order derivatives computed with automatic differentiation will grow very fast with the order.

In addition, the values of a pseudo-Abelian integral with many zeros is very small; in our case the sum in (5) is up to 14 orders of magnitude smaller than each of its terms. This huge demand for accuracy together with the inaccuracy of implicitly defined functions, and the fact that the square root is non-Lipschitz at zero, which implies that the relative size of the interval enclosures grow without bound close to zero, make the computations extremely delicate. In fact, it forces us to compute with 256-bit arithmetic. With lower precision computations we only get enclosures of the values of (5) of the form $(-\eta, \eta)$, for some small number η. This illustrates the strength of validated numerical methods; we are able to tell when we have enough accuracy. Using a standard numerical experiment for a problem this sensitive, any result is possible.

4. Numerical results

The idea is to generate a perturbation with as many zeros as possible in Γ_1, and afterwards check whether zeros in Γ_2 are also implied. In contrast with an algebraic approach, such as e.g. [22], we do not compute focus quantities. Instead, we compute the zeros of the pseudo-Abelian integral, for some fixed perturbation. This means that the limit cycles will have some distance from the centre.

In trying to locate a perturbation with as many zeros as possible, we start by – non-rigorously – sampling the value of each of the five terms in (5) at 100 uniformly distributed level curves in $-0.505 \le h \le -0.5$, where -0.505 corresponds to the centre. Since we have four coefficients in our perturbation, we can solve the linear system constructed by requiring that $I(h) = 0$ at four different h-values. We do this with the four values as close as possible to the centre, and evenly spaced. After some small corrections, we get the choice of parameters given in Table 1, yielding six limit cycles in Γ_1 and one limit cycle in Γ_2. These are our candidate coefficients, and the final step is to run the validated integration scheme to prove that $I(h)$ has validated sign changes.

α_1	-44.1527847886279
α_2	-64.7283580682510
α_3	155.6409265346564
α_4	-28.2884027418255

Table 1. The generated coefficients of the perturbation (3).

The level curves from which the limit cycles bifurcate are shown in Figure 2, the graphs of the pseudo-Abelian integrals on Γ_1 and Γ_2 are presented in Figure 3, and the validated enclosures of the values are given in Table 2.

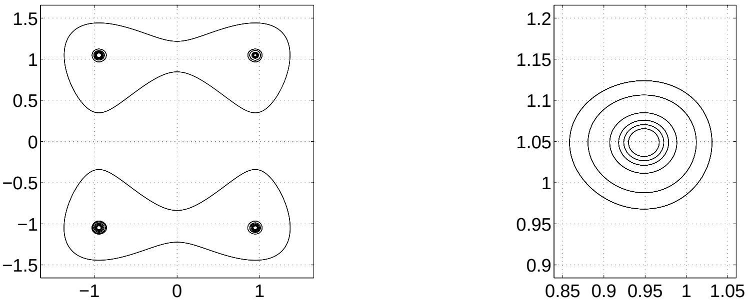

FIGURE 2. The level curves, from which the limit cycles bifurcate. On the right is a close up of Γ_1.

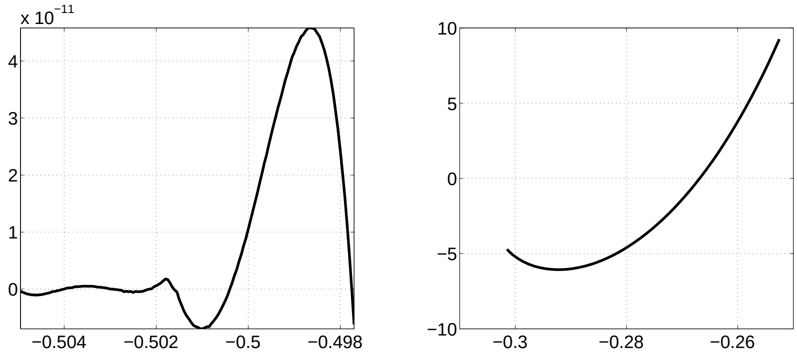

FIGURE 3. The graphs of $I(h)$ on Γ_1 and Γ_2.

Periodic annulus	h	$I(h)$
1	-0.5045	[-9.616E-0013,-9.542E-0013]
1	-0.5035	[4.982E-0013, 5.027E-0013]
1	-0.5025	[-4.766E-0013,-4.690E-0013]
1	-0.5018	[1.730E-0012, 1.742E-0012]
1	-0.5010	[-6.946E-0012,-6.931E-0012]
1	-0.5000	[1.052E-0011, 1.055E-0011]
1	-0.4500	[-9.858E-0005,-9.857E-0005]
2	-0.2900	[-6.018E+0000,-6.017E+0000]
2	-0.2500	[1.145E+0001, 1.146E+0001]

TABLE 2. The computed enclosures of the pseudo-Abelian integrals.

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