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# Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps

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# LINEAR RESPONSE FOR SMOOTH DEFORMATIONS OF GENERIC NONUNIFORMLY HYPERBOLIC UNIMODAL MAPS

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Abstract. We consider $C^2$ families $t \mapsto f_t$ of $C^3$ unimodal maps $f_t$ whose critical point is slowly recurrent, and we show that the unique absolutely continuous invariant measure $\mu_t$ of $f_t$ depends differentiably on $t$, as a distribution of order 1. The proof uses transfer operators on towers whose level boundaries are mollified via smooth cutoff functions, in order to avoid artificial discontinuities. This gives a new representation of $\mu_t$ for a Benedicks-Carleson map $f_t$, in terms of a single smooth function and the inverse branches of $f_t$ along the postcritical orbit. Along the way, we prove that the twisted cohomological equation $v = \alpha \circ f - f' \alpha$ has a continuous solution $\alpha$, if $f$ is Benedicks-Carleson and $v$ is horizontal for $f$.

## 1. Introduction

The linear response problem for discrete-time dynamical systems can be posed in the following way. Suppose that for each parameter $t$ (or many parameters $t$) in a smooth family of maps $t \mapsto f_t$ with $f_t \colon M \to M$, ($M$ a compact Riemann manifold, say) there exists a unique physical (or SRB) measure $\mu_t$. (See [61] for a discussion of SRB measures.) One can ask for conditions which ensure the differentiability, possibly in the sense of Whitney, of the function $\mu_t$ in a weak sense (in the weak $*$-topology, i.e., as a distribution of order 0, or possibly as a distribution of higher order). Ruelle has discussed this problem in several survey papers [45], [47], [49], to which we refer for motivation.

The case of smooth hyperbolic dynamics has been settled over a decade ago ([26], [44]), although recent technical progress in the functional analytic tools (namely, the introduction of anisotropic Sobolev spaces on which the transfer operator has a spectral gap) has allowed for a great simplification of the proofs (see e.g. [18]): For smooth Anosov diffeomorphisms $f_s$ and a $C^1$ observable $A$, letting

$$X_s = \partial_t f_t|_{t=s} \circ f_s^{-1} \,,$$

Ruelle [44][46] obtained the following explicit *linear response formula* (the derivative here is in the usual sense)

$$\partial_t \int A \, d\mu_t|_{t=0} = \Psi_A(1)$$

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where $\Psi_A(z)$ is the *susceptibility function*

$$\Psi_A(z) = \sum_{k=0}^{\infty} \int z^k \langle X_0, \mathrm{grad}\,(A \circ f_0^k) \rangle \, d\mu_0 \, ,$$

and the series $\Psi_A(z)$ at $z = 1$ converges exponentially. In fact, in the Anosov case, the susceptibility function is holomorphic in a disc of radius larger than 1. This is related to the fact (see [8] for a survey and references) that the transfer operator of each $f_s$ has a spectral gap on a space which contains not only the product of the distribution $\mu_s$ and the smooth vector field $X_s$, but also the derivative of that product, that is, $\langle X_s, \mathrm{grad}\, \mu_s \rangle + \mathrm{div}\, X_s \mu_s$.

One feature of smooth hyperbolic dynamics is structural stability: Each $f_t$ for small $t$ is topologically conjugated to $f_0$ via a homeomorphism $h_t$, which turns out to depend smoothly on the parameter $t$. With the exception of a deep result of Dolgopyat [21] on rapidly mixing partially hyperbolic systems (where structural stability may be violated, but where there are no critical points and shadowing holds for a set of points of large measure, so that the bifurcation structure is relatively mild), the study of linear response in the absence of structural stability, or in the presence of critical points, has begun only recently.

However, the easier property of *continuity* of $\mu_t$ with respect to $t$ (in other words, *statistical stability*) has been established also in the presence of critical points: For piecewise expanding unimodal interval maps, Keller [27] proved in 1982 that the density $\phi_t$ of $\mu_t$, viewed as an element of $L^1$, has a modulus of continuity at least $t \log t$, so that $t \mapsto \phi_t$ is $r$-Hölder, for any exponent $r \in (0, 1)$. For nonuniformly smooth unimodal maps, in general not all nearby maps $f_t$ admit an SRB measure even if $f_0$ does. Therefore, continuity of $t \mapsto \mu_t$ can only be proved in the sense of Whitney. This was done by Tsujii [55] and Rychlik–Sorets [51] in the 90's. More recently, Alves et al. [1], [2] proved that for Hénon maps, $t \mapsto \mu_t$ is continuous in the sense of Whitney in the weak $*$-topology. (We refer e.g. to [9] for more references.)

Differentiability of $\mu_t$, even in the sense of Whitney, is a more delicate issue, even in dimension one. For nonuniformly hyperbolic smooth unimodal maps $f_t$ with a quadratic critical point ($f_t''(c) < 0$), it is known [59], [31] that the density $\phi_t$ of the absolutely continuous invariant measure $\mu_t$ of $f_t$ has singularities of the form $\sqrt{x - c_{k,t}}^{-1}$, where the $c_{k,t} = f_t^k(c)$ are the points along the forward orbit of the critical point $c$. (Following Ruelle[50], we call these singularities *spikes*.) Thus, the derivative $\phi_t'$ of the invariant density has nonintegrable singularities, and the transfer operator cannot have a spectral gap in general on a space containing $(X_t \phi_t)'$. In fact, the spectral radius itself is strictly larger than 1 on any such space, and the radius of convergence of the susceptibility function $\Psi_A(z)$ is very likely strictly smaller than 1 in general. Ruelle [48] observed however that, in the case of a subhyperbolic (preperiodic) critical point for a real analytic unimodal map, $\Psi_A(z)$ is meromorphic in a disc of radius larger than 1, and that 1 is not a pole of $\Psi_A(z)$. He expressed the hope that the value $\Psi_A(1)$ obtained by analytic continuation could correspond to the actual derivative of the SRB measure, at least in the sense of Whitney.

This analytic continuation phenomenon in the subhyperbolic smooth unimodal case (where a finite Markov partition exists) could well be a red herring, in view of the linear response theory for the "toy model" of piecewise expanding interval maps that we recently established in a series of papers [8], [11], [13], [14]: Unimodal piecewise expanding interval maps $f_t$ have a unique SBR measure, whose density $\phi_t$ is a function of bounded variation (since $\phi_t'$ is a measure, the situation is much easier than for smooth unimodal maps). In [8], [11], we showed that Keller's [27] $t \log t$ modulus of continuity was optimal (see also [36]): In fact, there exist smooth families $f_t$ so that $t \mapsto \mu_t$ is not Lipschitz (all sequences $t_n \to 0$ so that the critical point is not periodc under $f_{t_n}$ are allowed), even when viewed as a distribution of arbitrarily high order, and even in the sense of Whitney. Such counter-examples $f_t$ are transversal to the topological class of $f_0$. If, on the contrary, the family $f_t$ is tangent at $t = 0$ to the topological class of $f_0$ (we say that $f_t$ is *horizontal*) then [11], [13] we proved that the map $t \mapsto \mu_t$ is differentiable for the weak $*$-topology. The series for $\Psi_A(1)$ is divergent, but can be resummed under the horizontality condition [8], [11]. This gives an explicit linear response formula. In fact, the susceptibility function $\Psi_A(z)$ is holomorphic in the open unit disc, and, under a condition slightly stronger than horizontality, $\partial_t \int A \, d\mu_t|_{t=0}$ is the Abel limit of $\Psi_A(z)$ as $z \to 1$.

Worrying about lack of differentiability of the SRB measure is not just a mathematician's pedantry: Indeed, this phenomenon can be observed numerically, for example in the guise of fractal transport coefficients. We refer, e.g., to the work of Keller et al. [29] (see also references therein), who obtained a $t \ln(t)$ modulus of continuity compatible with the results of [27], for drift and diffusion coefficients of models related to those analysed in [11].

Let us move on now to the topic of the present work, linear response for smooth unimodal interval maps: Ruelle recently obtained a linear response formula for real analytic families of analytic unimodal maps of Misiurewicz type [50], that is, assuming $\inf_k |f^k(c) - c| > 0$, a nongeneric condition which implies the existence of a hyperbolic Cantor set. (Again, this linear response formula can be viewed as a resummation of the generally divergent series $\Psi_A(1)$.) In [12], we showed that $t \mapsto \mu_t$ is real analytic in the weak sense for complex analytic families of Collet-Eckmann quadratic-like maps (the – very rigid – holomorphicity assumption allowed us to use tools from complex analysis). Both these recent results are for families $f_t$ in the conjugacy class of a single (analytic) unimodal map, and the assumptions were somewhat ungeneric.

The main result of the present work, Theorem 2.13, is a linear response formula for $C^2$ families $t \mapsto f_t$ of $C^3$ unimodal maps with quadratic critical points satisfying the so-called *topological slow recurrence* (TSR) condition ([52],[54],[33]). (We assume that the maps have negative Schwarzian and are symmetric, to limit technicalities.) The topological slow recurrence condition is much weaker than Misiurewicz, so that we give a new proof of Ruelle's result [50] (this may shed light on the informal study in §17 there). Topological slow recurrence implies the well-known Benedicks-Carleson and Collet-Eckmann conditions. Furthermore, the work of Tsujii [54] and Avila-Moreira [6] gives that unimodal maps with a quadratic critical point satisfying this condition are *generic.* (See Remark (2.3).) If all maps in a family of unimodal maps $f_t$ satisfy the topological slow recurrence condition then [56] this family is a *deformation,* that is, the family $\{f_t\}$ lies entirely in the topological class of $f_0$ (there exist homeomorphisms $h_t$ such that $h_t(c) = c$ and $h_t \circ f_0 = f_t \circ h_t$). In particular, horizontality holds.

We next briefly discuss a few new ingredients of our arguments, as well as a couple of additional results we obtained along the way. A first remark is that we need uniformity of the hyperbolicity constants of $f_t$ for all small $t$. We deduce this uniformity from previous work of Nowicki, making use of the TSR assumption (Section 5).

When one moves the parameter $t$, the orbit of the critical point also moves, and so do the spikes. Therefore, in order to understand $\partial_t \mu_t$, we need upper bounds on

$$\partial_t c_{k,t}|_{t=0} = \partial_t f_t^k(c)|_{t=0} = \partial_t h_t(f_0^k(c))|_{t=0} = \partial_t h_t(c_{k,0})|_{t=0}\,,$$

uniformly in $k$. It is not very difficult to show (Lemma 2.10, see also Proposition 2.15) that $\partial_t c_{k,t}|_{t=0} = \alpha(c_{k,0})$, if $\alpha$ solves the twisted cohomological equation (TCE) for $v = \partial_t f_t|_{t=0}$, given by,

$$v = \alpha \circ f_0 + f_0' \cdot \alpha\,, \quad \alpha(c) = 0\,.$$

(Such a function $\alpha$ is called an *infinitesimal conjugacy.)* In fact, we prove in Theorem 2.4 that if $f_0$ is Benedicks-Carleson and $v$ satisfies a horizontality condition for $f_0$, then the TCE above has a unique solution $\alpha$. In addition, $\alpha$ is continuous.

In the case of piecewise expanding maps on the interval, the invariant density $\phi_t$ is a fixed point of a Perron-Frobenius type transfer operator $\mathcal{L}_t$ in an appropriate space, where 1 is a simple isolated eigenvalue. So if we are able to verify some (weak) smoothness in the family $t \to \mathcal{L}_t$, then we can show (weak) differentiability of $\mu_t$ by using perturbation theory. (We may use different norms in the range and the domain, in the spirit of Lasota-Yorke or Doeblin-Fortet inequalities.) This is, roughly speaking, what was done in [11] and [14] (as already mentioned, a serious additional difficulty in the presence of critical points, which had to be overcome even in the toy model, is the absence of a spectral gap on a space containing the derivative of the invariant density). For Collet-Eckmann unimodal maps $f_t$, however, an inducing procedure or a tower construction [31], [59], [60] is needed to obtain good spectral properties for the transfer operator and to properly analyse the density $\phi_t$, even for a single map.

We use the tower construction from [15], under a Benedicks-Carleson assumption. However, when we consider a one-parameter family of maps $f_t$, the phase space of the tower moves with $t$. To compare the operators for $f_t$ and $f_0$, it is convenient to work with a finite part of the tower, the height of which goes exponentially to infinity as $t \to 0$. (We use results of Keller and Liverani [30] to control the spectrum of the truncated operator.) The uniform boundedness of $\alpha(c_k)$ is instrumental in working with such truncated towers and operators. In fact, the tower construction in [15] also has a key role in the proof of boundedness for $\alpha$: the natural candidate for the solution is a divergent series, but, under the horizontality condition, we devise a dynamical resummation (the mantra being: "don't perform a partial sum for the series while you are climbing the tower, unless you are ready to fall").

The tower from [15] has a drawback: The orbits of the edges of the tower levels apparently create "artificial discontinuities" in the functions. To eliminate these potential discontinuities, we modify the construction of the Banach spaces and transfer operators on the towers by introducing smooth cutoff functions (called $\xi_k$ below, see Section 4). As a consequence, we obtain a new expression for the invariant density of a Benedicks-Carleson unimodal map (Proposition 2.7), in terms of a smooth function and of the dynamics.

We would like to list now a few directions for further work. Several of them can be explored by exploiting the techniques developed in the present article (see [9] for other open problems):

• In the setting of the present paper, e.g., can one show that $\partial_t \mu_t(A)|_{t=0}$ is a resummation of the divergent series $\Psi_A(z)$ at $z = 1$? (Presumably, a dynamical resummation is possible, maybe using the operator $\mathcal{P}(\psi) = \psi \circ f$ dual to $\mathcal{L}$ acting on dual Banach spaces, and using e.g. the proof of the main result in [24].) Can one get an Abelian limit along the real axis? The radius of convergence of $\Psi_A(z)$ is strictly smaller than 1 in general. There appears to be an essential boundary, except in the subhyperbolic cases when the critical point is preperiodic. Analytic continuation in the usual sense is thus probably not available, some kind of Borel or Abelian continuation seems necessary. (In subhyperbolic cases $\Psi_A(z)$ is meromorphic, and horizontality very likely implies vanishing of the residue of the pole in $[0, 1]$.)

• Can one replace the topological slow recurrence condition on $f_0$ by Benedicks-Carleson, Collet-Eckmann, or possibly just a summability condition on the inverse of the postcritical derivative (see [42] and [17]), and still get differentiability [1] of $t \mapsto \mu_t$, as a distribution of order 1, at $t = 0$?

• If $f_t$ is a smooth family of quadratic unimodal maps, with $f_0$ a good map (summable, or Collet-Eckmann, or Benedicks-Carleson, or TSR), and if $v = \partial f_t|_{t=0}$ is horizontal for $f_0$, that is, (10) holds [2], is $t \mapsto \mu_t$ differentiable, as a distribution of order 1, in the sense of Whitney, at $t = 0$?

• If $f_t$ is a smooth (possibly transversal, that is, not horizontal) family of quadratic unimodal maps, with $f_0$ a good map, is $t \mapsto \mu_t$ always $r$-Hölder in the sense of Whitney for $r \in (0, 1/2)$ at $t = 0$? Which is the strongest topology one can use in the image? (Possibly, one could show Hölder continuity in the sense of Whitney of the Lyapunov exponent.)

• Can one construct a (non-horizontal) smooth family $f_t$ of quadratic unimodal maps, with $f_0$ a good map, so that $t \mapsto \mu_t$, as a distribution of any order, is not differentiable (even in the sense of Whitney, at least for large subsets) at $t = 0$? So that it is not Hölder for any exponent $> 1/2$?

• What about Hénon-like maps? Note that even the formula defining horizontality is not available in this case, see [9] (Numerical results of Cessac [19] indicate that $\Psi_A(z)$ has a singularity in the interior of the unit disc. In view of the above discussion, we expect that this singularity is not an isolated pole in general.)

• The dynamical zeta function associated to a Collet-Eckmann map $f$ and describing part of the spectrum of $\mathcal{L}$ was studied by Keller and Nowicki [31]. Can one study the analytic properties of a *dynamical determinant* for $\mathcal{L}$ in the spirit of what was done for subhyperbolic analytic maps [10]? (Analyticity would hold only in a disc of finite radius, and the correcting rational factor from [10, Theorem B] would be replaced by an infinite product, corresponding to the essential boundary of convergence within this disc.) Can one find and describe a dynamical determinant playing for $\Psi_A(z)$ the part that $\mathcal{L}$ plays for the Fourier transform of the correlation function of the SRB measure of $f$? (See [8] for piecewise expanding interval maps.)

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[1] Perturbation theory of isolated eigenvalues cannot be used if there is no spectral gap, but the analysis in Hairer-Majda [24] e.g. indicates that existence of the resolvent $(\mathrm{id} - \mathcal{L})^{-1}$ (up to replacing $\mathcal{L}$ by $\mathcal{P}$ if necessary) should be enough.

[2] By [5], we can heuristically view $f_t$ as tangent to the topological class of $f_0$.

The structure of paper is as follows. In Section 2, we give precise definitions and state our main results formally. Section 3 is devoted to the proof (by dynamical resummation) that horizontality implies that the TCE has a continuous solution $\alpha$ (Theorem 2.4). In particular, we recall in Subsection 3.1 the construction of the tower map $\hat{f}: \hat{I} \to \hat{I}$ from [15] which will be used in later sections. We also show (Subsection 3.5) that the formal candidate for $\alpha$ diverges at countably many points (Proposition 2.5). In Section 4, we revisit the tower construction, introducing Banach spaces and a transfer operator $\widehat{\mathcal{L}}$ involving the smooth cut-off functions discussed above. In particular, Proposition 4.10, which immediately implies our new expression for the invariant density (Proposition 2.7), is proved in Subsection 4.1. Also, we study truncations $\widehat{\mathcal{L}}_M$ on finite parts of the tower in Subsection 4.2. Uniformity in $t$ of the hyperbolicity constants of $f_t$ involved in the construction of Sections 3 and 4, is the topic of Section 5, the main result of which is Lemma 5.8 (proved by exploiting previous work of Nowicki). Finally, our linear response result, Theorem 2.13, is proved in Section 6. The argument borrows some ideas from [11], but their implementation required several nontrivial innovations, as explained above. The three appendices contain proofs of a more technical nature.

## 2. Formal statement of our results

### 2.1. **Collet-Eckmann, Benedicks-Carleson, and topologically slowly recurrent (TSR) unimodal maps.**

We start by formally defining the classes of maps that we shall consider. Note for further use that we shall sometimes write $[a, b]$ with $b < a$ to represent $[b, a]$. Another frequent abuse of notation is that we sometimes use $C > 0$ to denote different (uniform) constants in the same formula.

Let $I = [-1, 1]$. We say that $f$ is *S-unimodal* if $f : I \to I$ is a $C^3$ map with negative Schwarzian derivative such that $f(-1) = f(1) = -1$, $f'|_{[-1,0)} > 0$, $f'|_{(0,1]} < 0$, and $f''(0) < 0$ (i.e., we only consider the quadratic, case). The following notation will be convenient throughout: For $k \geq 1$, we let $J_+$ be the monotonicity interval of $f^k$ containing $c$ and to the right of $c$, $J_-$ be the monotonicity interval of $f^k$ containing $c$ and to the left of $c$, and we put

$$f_+^{-k} := (f^k|_{J_+})^{-1}\,,\ f_-^{-k} := (f^k|_{J_-})^{-1}\,. \tag{1}$$

*Remark* 2.1. It is likely that the negative Schwarzian derivative assumption is not needed for our results, see [32]. Note however that we cannot apply trivially the work of Graczyk-Sands-Świątek [23] to study linear response: If $f_t$ is a one-parameter family of $C^3$ unimodal maps, the smooth changes of coordinates which make their Schwarzian derivative negative will depend on $t$, and this dependency will require a precise study. In view of keeping the length of this paper within reasonable bounds, we refrained from considering the more general case.

Let $c = 0$ be the critical point of $f$, and put $c_k = f^k(c)$ for all $k \geq 0$. We say that an $S$-unimodal map $f$ is $(\lambda_c, H_0)$-*Collet-Eckmann (CE)* if $\lambda_c > 1$, $H_0 \geq 1$, and

$$|(f^k)'(f(c))| \geq \lambda_c^k\,, \quad \forall k \geq H_0\,. \tag{2}$$

All periodic orbits of Collet-Eckmann maps are repelling, and [41, Theorem B] gives that for *any* $C^2$ unimodal (or multimodal) map without periodic attractors there exists $\gamma > 0$ so that $|f^n(c) - c| \geq e^{-\gamma n}$ for all large enough $n$. Benedicks and Carleson [16] showed that $S$-unimodal Collet-Eckmann maps which satisfy the following *Benedicks-Carleson* assumption

$$\exists 0 < \gamma < \frac{\log(\lambda_c)}{4} \text{ so that } |f^k(c) - c| \geq e^{-\gamma k}\,, \quad \forall k \geq H_0 \tag{3}$$

form a positive measure set of parameters of non degenerate families. The Benedicks-Carleson assumption will suffice for some of our results, sometimes up to replacing 4 in the denominator by a larger constant.

A stronger condition, topologically slow recurrence (TSR), will allow us to obtain linear response. To define TSR, we shall use the following auxiliary sequence: Let $f$ be an $S$-unimodal map whose critical point is not preperiodic. The itinerary of a point $x \in I$ is the sequence $\operatorname{sgn}(f^i(x)) \in \{-1, 0, 1\}$. We put

$$R_f(x) := \min\{j \mid k_j(c) \neq \operatorname{sgn}(f^j(x)),\ j \geq 1\}\,. \tag{4}$$

We say that an $S$-unimodal map $f$ with non preperiodic critical point satisfies the *topological slow recurrence (TSR)* condition if

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{1\leq j\leq n \\ R_f(f^j(c))\geq m}} R_f(f^j(x)) = 0\,. \tag{5}$$

It follows from the definition that any $S$-unimodal map topologically conjugated with a map $f$ satisfying TSR also satisfies TSR. (Indeed, $R_f(c_n) = j$ if and only if $f^j$ is a diffeomorphism on $(c, c_n)$ and $c \in f^j(c, c_n)$.) We also have the much less trivial result below:

**Proposition 2.2** ([52], [58]. See also [33]). *An $S$-unimodal map $f$ with non preperiodic critical point satisfies the TSR condition if and only if $f$ is a Collet-Eckmann map and*

$$\lim_{\eta\to 0^+} \liminf_{n\to\infty} \frac{1}{n} \sum_{\substack{1\leq j\leq n \\ |f^j(c)-c|<\eta}} \log|f'(f^j(c))| = 0\,. \tag{6}$$

In Section 5, we shall prove that TSR implies Collet-Eckmann and Benedicks-Carleson-type conditions, uniformly in a subset of small enough $C^3$ diameter of a topological class.

*Remark* 2.3 (TSR is generic). Avila and Moreira [4] proved that for almost every parameter $s$ in a non-degenerate analytic family of quadratic unimodal maps $f_s$, the map $f_s$ is either regular or Collet-Eckmann with subexponential recurrence of its critical orbit (i.e., for every $\gamma > 0$, there is $H_0$ so that $|c_k - c| > \exp(-\gamma k)$ for all $k \geq H_0$). (Non-degenerate, or transversal, means that the family is not contained in a topological class.) Tsujii [54] had previously proved that the set of Collet-Eckmann and subexponentially recurrent parameters $s$ in a transversal family $f_s$ of $S$-unimodal maps has positive Lebesgue measure. By combining the results of Avila and Moreira [4] and Tsujii [54], we can see that TSR is a generic condition: In a nondegenerate analytic family $f_s$ of $S$-unimodal maps, almost every parameter is either regular or TSR.

2.2. **Boundedness and continuity of the infinitesimal conjugacy $\alpha$.** Let $f$ be an $S$-unimodal Collet-Eckmann map, and let $v : I \to \mathbb{C}$ be bounded. We want to find a bounded solution $\alpha : I \to \mathbb{C}$ of the *twisted cohomological equation (TCE):*

$$v(x) = \alpha(f(x)) - f'(x)\alpha(x)\,, \forall x \in I\,. \tag{7}$$

By analogy with the piecewise expanding unimodal case (that we studied in previous works [11], [13]), a candidate $\alpha_{cand}$ for the solution $\alpha$ of (7) is defined, for those $x \in I$ so that $f^j(x) \neq c$ for all $j \geq 0$, by the *formal series*

$$\alpha_{cand}(x) = -\sum_{j=0}^{\infty} \frac{v(f^j(x))}{(f^{j+1})'(x)}\,, \tag{8}$$

and, for those $x \in I$ so that there exists $j \geq 0$ with $f^j(x) = c$, but $f^\ell(x) \neq c$ for $0 \leq \ell \leq j-1$, by the sum

$$\alpha_{cand}(x) = -\sum_{\ell=0}^{j-1} \frac{v(f^\ell(x))}{(f^{\ell+1})'(x)}\,. \tag{9}$$

(In particular, $\alpha_{cand}(c) = 0$.) Clearly, the series (8) converges absolutely at every point $x$ for which the Lyapunov exponent

$$\Lambda(x) = \lim_{j\to\infty} \log |(f^j)'(x)|^{1/j}$$

is well-defined and strictly positive. In particular, (8) converges absolutely for $x$ in the forward orbit $\{c_k, k \geq 1\}$ of the critical point of the Collet-Eckmann $S$-unimodal map $f$, and also on the set of its preperiodic points.

We say that $v$ satisfies the *horizontality condition* if

$$v(c) = -\sum_{j=0}^{\infty} \frac{v(f^j(c_1))}{(f^{j+1})'(c_1)}\,, \tag{10}$$

(note that the right-hand-side of the above identity is just $\alpha_{cand}(c_1)$). If $v$ satisfies the horizontality, then it is easy to see that whenever the formal series (8) for $\alpha_{cand}(x)$ converges absolutely, then the corresponding series $\alpha_{cand}(f(x))$ also converges absolutely, and $\alpha_{cand}$ satisfies the twisted cohomological equation (7) at $x$. Violation of horizontality (that is, $v(c) \neq \alpha_{cand}(c_1)$) is a *transversality* condition which has been used for a long time in one-parameter families $f_t$ with $v(x) = \partial_t f_t|_{t=0}$ of smooth unimodal maps (see e.g. [53] for the transversality condition, see [54] for the transversality condition expressed as a postscritical sum, see e.g. [13, 5] for the link between the two expressions, see [5] for a recent occurrence, and see [50] for its use in linear response).

Nowicki and van Strien [42] showed that the absolutely continuous invariant probability measure $\mu$ of a quadratic Collet-Eckmann map satisfies

$$\mu(A) \leq Cm(A)^{1/2}\,,$$

where $m$ is the Lebesgue measure. In particular $\log|f'|$ is $\mu$-integrable, and for Lebesgue almost every point $x$ the Lyapunov exponent $\Lambda(x)$ is well-defined and positive and coincides with $\int \log|f'|\, d\mu$ (see Keller [28]). So the series $\alpha_{cand}(x)$ converges absolutely at Lebesgue almost every point $x$, and if $v$ is horizontal then $\alpha_{cand}$ satisfies the TCE (7) along the forward orbit of each such good $x$. However it is not clear a priori that there exists an upper bound for $|\alpha_{cand}(x)|$ on the set where $\alpha_{cand}(x)$ converges absolutely (for example, $c_k$ may be very close to $c$).

One can ask whether the formal series $\alpha_{cand}(x)$ converges everywhere. We shall show in Proposition 2.5 that for fairly general $v$ (see Remark 2.6), the series (8) for $\alpha_{cand}(x)$ diverges on a uncountable and dense subset (this set has Lebesgue measure zero, however, by the observations in the previous paragraph). This lack of convergence is a new phenomenon with respect to [11], [13]. In order to prove that the TCE nevertheless has a bounded solution in the horizontal case, we shall make a Benedicks-Carleson assumption (3) on $f$, and we shall group the terms of the formal series $\alpha_{cand}(x)$ to obtain an absolutely convergent series. The resummation procedure depends on $x$ through its dynamics with respect to an induced map on the tower introduced in [15] using strong expansion properties available in the Benedicks-Carleson case. This dynamical resummation will allow us to prove our first main result:

**Theorem 2.4** (Boundedness and continuity of $\alpha$). *Assume that $f$ is a $(\lambda_c, H_0)$-Collet-Eckmann $S$-unimodal map satisfying the Benedicks-Carleson condition* (3)*.*

*For any bounded function $v : I \to \mathbb{C}$, if the TCE* (7) *admits a bounded solution $\alpha : I \to \mathbb{C}$ with $\alpha(c) = 0$, then this solution is unique and $v$ satisfies the horizontality condition* (10)*.*

*Let $X : I \to \mathbb{C}$ be Lipschitz, and let $v = X \circ f$. If $v$ satisfies the horizontality condition* (10)*, then there exists a continuous function $\alpha : I \to \mathbb{C}$ with $\alpha(c) = 0$ solving the TCE* (7)*. In addition, $\alpha(x) = \alpha_{cand}(x)$ for all $x$ so that $f^j(x) = c$ for some $j \geq 0$, or such that the infinite series $\alpha_{cand}(x)$ in* (8) *converges absolutely*

The condition $v = X \circ f$ can be weakened but the term $k = 0$ of $(I)$ in (46) in the proof of Proposition 3.9 shows that we need something like $v'(c) = 0$. (If we allowed $f''(c) = 0$, then we would need $v''(c) = 0$, etc.)

We do not know whether Theorem 2.4 holds for all $S$-unimodal Collet-Eckmann maps, i.e., whether the Benedicks-Carleson assumption is needed. In any case, we shall use the stronger, but still generic (recall Remark 2.3), TSR assumption in Section 5 to show uniformity of the various hyperbolicity constants. This uniformity is required to prove linear response, the other main result of this paper.

The proof of Theorem 2.4 is given in Section 3 and organised as follows: In Section 3.1, we recall the tower construction from Baladi and Viana [15]. We study its properties in Section 3.2, which also contains two new (and key) estimates, Proposition 3.7 and its Corollary 3.8. In Section 3.3, we define a function $\alpha(x)$ by grouping the terms of the formal series (8) to obtain an absolutely convergent series (Definition 3.10 and Proposition 3.9). The resummation procedure for $\alpha(x)$ depends on the dynamics of $x$ on the tower. Finally, in Section 3.4 we complete the proof of Theorem 2.4: we show that $\alpha(x)$ is a continuous function, that it satisfies the TCE, and that if the TCE admits a bounded solution then it is unique.

We end this section with a result on the lack of convergence of the formal power series for $\alpha_{cand}(x)$ (recall that it converges at Lebesgue almost every $x$):

**Proposition 2.5.** *Let $f$ be an $S$-unimodal map, with all its periodic points repelling and an infinite postcritical orbit. Let $v$ be a $C^1$ function on $I$, with $v'(c) = 0$, such that $v(f^{i_0}(c)) \neq 0$ for some $i_0$. Let $\Sigma$ be the set of points $x$ such that $f^n(x) \neq c$ for every $n \geq 0$ and so that the series $\alpha_{cand}(x) = -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{(f^{i+1})'(x)}$ diverges. Then for every non empty open set $A \subset I$, the intersection $A \cap \Sigma$ contains a Cantor set.*

*Remark* 2.6. If $f$ is a Collet-Eckmann map whose critical orbit is not preperiodic, an open and dense set of horizontal vectors $v$ satisfies the conditions of Proposition 2.5. Indeed, the set $\{v \mid v(f^i(c)) = 0, \forall\, i\}$ is a subspace of infinite codimension, and the subspace of horizontal directions $v$ has codimension one.

The proof of Proposition 2.5 is to be found in Section 3.5.

2.3. **A new expression for the a.c.i.m. of a Benedicks-Carleson unimodal map.** It is well known that an $S$-unimodal map which is Collet-Eckmann admits an absolutely continuous invariant measure. The following more precise expression for the invariant density of a Benedicks-Carleson unimodal map appears to be new. It is a byproduct of our proof, and follows immediately from Proposition 4.10 and the definitions in Section 4 (the case when the critical point is preperiodic can be obtained by a much more elementary proof). The remarkable feature of (11) is that the data function, i.e. $\psi$ is *smooth,* and that the square-root singularities are injected dynamically, through the inverse interates of $f$ and their jacobians.

**Proposition 2.7.** *Let $f$ be a $(\lambda_c, H_0)$-Collet-Eckmann $S$-unimodal map satisfying the Benedicks-Carleson condition* (69)*, with $c$ not preperiodic. Then there exist*

- *a $C^1$ function $\psi_0 : I \to \mathbb{R}_+$,*
- *neighbourhoods $V_k \subset W_k$ of $c = 0$ so that $f^k|_{W_k \cap [0,1]}$ $f^k|_{[-1,0]\cap W_k}$ are injective, for each $k \geq 1$,*
- *$C^\infty$ functions $\xi_k : I \to [0,1]$, supported in $W_k$ and $\equiv 1$ on $V_k$,*

*so that the density $\phi$ of the unique absolutely continuous invariant probability measure of $f$ satisfies*

$$\phi(x) = \psi_0(x) + \sum_{k=1}^{\infty} \sum_{\varsigma \in \{+,-\}} \frac{\prod_{j=0}^{k-1} \xi_j(f_\varsigma^{-k}(x))}{|(f^k)' f_\varsigma^{-k}(x)|} \chi_k(x) \psi_0(f_\varsigma^{-k}(x)) \, , \tag{11}$$

*where $\chi_k = [-1, c_k]$ if $f^k$ has a local maximum at $c$, while $\chi_k = [c_k, 1]$ if $f^k$ has a local minimum at $c$. If the stronger condition* (86) *holds, then $\psi_0$ is in fact $C^2$.*

The length of $W_k$ must decay exponentially, but there is some flexibility in choosing the $V_k$, $W_k$ and $\xi_k$, see Definition 4.7 for details, noting also the parameter $\delta$ used in the construction of the tower. The function $\psi_0$ depends on these choices.

By Lemma 4.1, which describes the nature of the singularities of $|(f^k)' f_\pm^{-k}(x)|$ on the support of $\xi_k(f_\pm^{-k}(x))$, the expression for $\phi$ belongs to $L^p(I)$ for all $p < 2$. In fact, Lemma 4.1 implies that the invariant density of $f$ can be written as

$$\psi_0 + \sum_{k \geq 1} \phi_k \frac{\chi_k}{\sqrt{|x - c_k|}} \, ,$$

where the derivatives of the $\phi_k$ decay exponentially with $k$. (A slightly weaker version of this result, replacing differentiable by bounded variation, was first proved by L.S.Young [59]. Ruelle obtained a formula involving differentiable objects in the analytic Misiurewicz case [50], but his expression is somewhat less dynamical.)

2.4. **Uniformity of hyperbolicity constants in deformations of slowly recurrent maps.** We shall study one-parameter families $t \mapsto f_t$ of $S$-unimodal maps which stay in a topological class, i.e., *deformations:*

**Definition 2.8.** ($C^r$ deformations $f_t$. Notations $v_t$, $X_t$, $h_t$.) Let $f : I \to I$ be an $S$-unimodal Collet-Eckmann map. For $r \geq 1$, a $C^r$ *one-parameter family* through $f$ is a $C^r$ map

$$t \mapsto f_t \, , \; t \in [-\epsilon, \epsilon] \, ,$$

(taking the topology of $C^3$ endomorphisms of $I$ in the image) , with $f_0 = f$, and so that each $f_t$ is $S$-unimodal. We use the notations:

$$c_{k,t} = f_t^k(c) \, , k \geq 1 \, , \; v_s := \partial_t f_t|_{t=s} \, , v = v_0 \, , \; c_k = c_{k,0} \, .$$

A $C^r$ *deformation of* $f$ is a $C^r$ one-parameter family through $f$ so that, in addition, for each $|t| \le \epsilon$, there exists a homeomorphism $h_t : I \to I$ with

$$h_0(x) \equiv x\,,\ \text{and}\ f_t \circ h_t = h_t \circ f_0\,, \forall t \in [-\epsilon, \epsilon]\,,$$

and [3] $v_s = X_s \circ f_s$ for each $|s| \le \epsilon$, with $X_s : I \to \mathbb{R}$ a $C^2$ function. (We write $X = X_0$.)

*Remark* 2.9. If $f_t$ is a deformation then each $v_s$ is horizontal. (This was proved by Tsujii [54].)

Given Theorem 2.4, the next lemma is easy to prove. It is essential in our argument:

**Lemma 2.10.** *Let $f_t$ be a $C^1$ one-parameter family of Collet-Eckmann $S$-unimodal maps through $f = f_0$. Assume that $v$ is horizontal, that is, $\alpha_{cand}(c_1) = v(c)$.*

*Then, we have for all $k \ge 1$*

$$\lim_{t \to 0} \frac{c_{k,t} - c_k}{t} = \alpha_{cand}(c_k)\,.$$

*If, in addition, $f_t$ is a $C^1$ deformation of $f_0$ then*

$$\partial_t h_t(c_k)|_{t=0} = \alpha_{cand}(c_k) = \alpha(c_k)\,, \quad \forall k \ge 1\,.$$

*Proof.* Our assumptions ensure that for each $k \ge 1$ the limit

$$a(c_k) = \lim_{t \to 0} \frac{c_{k,t} - c_k}{t}$$

exists. Clearly, $a(c_1) = v(c)$. More generally, it is easy to check that we have

$$a(c_k) = \sum_{j=0}^{k-1} (f^j)'(c_{k-j}) v(c_{k-j-1})\,,$$

so that that $a$ satisfies the TCE (7). By the horizontality condition, this implies that $a = \alpha_{cand}$ on $\{c_k\}$.

The additional assumption that $f_t$ and $f_0$ are conjugated via $h_t$ implies that $h_t(c_k) = c_{k,t}$, for all $t$ and all $k \ge 0$. The last statement of Theorem 2.4 implies that $\alpha_{cand}(c_k) = \alpha(c_k)$. $\square$

The following fact is an immediate consequence of van Strien's remark on "robust chaos" [56, Theorem 1.1], using the well-known fact that Collet-Eckmann maps do not have any attracting periodic orbit:

**Lemma 2.11.** *Let $f : I \to I$ be a $S$-unimodal $(\lambda_c, H_0)$ Collet-Eckmann map and let $f_t$ be a $C^1$ one-parameter family through $f$. If each $f_t$ is Collet-Eckmann for some parameters $\lambda_c(t)$ and $H_0(t)$, then $f_t$ is a $C^1$ deformation of $f$.*

In the other direction, although topological invariance of the Collet-Eckmann condition is known, we do not know how to prove topological invariance of Benedicks-Carleson conditions (3) (or variants of the type (69)–(86)). Since we also need uniformity of the various constants in the definitions, we shall work with the stronger, but still generic (see Remark 2.3), assumption of topological slow recurrrence TSR (recall (5)). In Section 5, assuming for simplicity that the maps are symmetric, we shall prove that if $f_s$ is a $C^0$ deformation of a TSR map $f_0$

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[3] This is mostly a technical assumption also the remark after Theorem 2.4.

then the various hyperbolicity constants of $f_s$ (that is $\lambda_c(f_s)$ and $H_0(f_s)$, but also $\sigma(f_s)$, $c_{f_s}(\delta)^{-1}$, $\rho(f_s)$ from Subsection 3.2, and, especially, $\gamma(f_s)$ ) are uniform in small $s$. We refer to Lemma 5.8 for a precise statement. Also, it will follow from Propositions 5.1 and 5.2 that (TSR) implies (CE) and (BeC).

Uniformity of constants implies the following result, essential in many places in the proof of Theorem 2.13:

**Lemma 2.12.** *Let $f_t$ be a $C^1$ deformation of symmetric $S$-unimodal maps so that $f_0$ enjoys topological slow recurrence TSR. Then there exists $\epsilon > 0$ and $L < \infty$ so that*

$$\sup_x \sup_{|s|<\epsilon} |\alpha_s(x)| \le L\,, \tag{12}$$

*and*

$$|c_k - c_{k,t}| \le L|t| \quad \forall k \ge 1\,, \quad \forall |t| < \epsilon\,. \tag{13}$$

*Proof.* We may assume that the critical point is not preperiodic. (If it is, the proof is much easier.)

Denote by $\alpha_s$ the continuous solution to the TCE given by Theorem 2.4 applied to each $f_s$ (the assumptions of the theorem are satisfied because of Lemma 5.8). The proof of Proposition 3.9 shows that for each fixed $s$ the supremum $\sup_x |\alpha_s(x)|$ may be estimated in terms of $c_{f_s}(\delta)^{-1}$, $\sup|v_s|$, $\mathrm{Lip} X_s$, $(1-\sigma(f_s)^{-1})^{-1}$, $(1-(\rho(f_s))^{-1})^{-1}$, in the notation of Lemma 3.1. By Lemma 5.8, this implies (12).

Next, applying Lemma 2.10 to each $f_s$, we get

$$\lim_{t\to 0} \frac{c_{k,s+t} - c_{k,s}}{t} = \alpha_s(h_s(c_k)) = \alpha_s(c_{k,s})\,.$$

In other words, $t \mapsto h_t(c_k)$ is differentiable on $[-\epsilon, \epsilon]$ (with $\epsilon$ independent of $k$), with derivative $\alpha_s(c_{k,s})$.

Then, for each $k \ge 1$ and each $|t| < \epsilon$ the mean value theorem gives $s$ with $|s| \le |t|$ so that

$$\frac{c_{k,t} - c_k}{t} = \frac{h_t(c_k) - h_0(c_k)}{t} = \alpha_s(c_{k,s})\,.$$

□

2.5. **Linear response.** Our main result will be proved in Section 6:

**Theorem 2.13** (Linear response and linear response formula)**.** *Let $f_t$ be a $C^2$ deformation of an $S$-unimodal map $f_0$ which satisfies TSR. Assume that all maps $f_t$ are symmetric. Write $\mu_t = \phi_t\, dx$ for the unique absolutely continuous invariant probability of $f_t$. Then, letting $(C^1(I))^*$ be the dual of $C^1(I)$, the map*

$$t \mapsto \mu_t \in (C^1(I))^*\,,\ t \in [-\epsilon, \epsilon] \tag{14}$$

*is differentiable. In addition, for any $A \in C^1(I)$,*

$$\int_I (A - A\circ f_s)\partial_t \mu_t|_{t=s} = \int_I A' X_s \phi_s\, dx\,, \tag{15}$$

The formula (15) is an easy consequence of the differentiability of (14), as was pointed out to us by Ruelle [43].

We next give an explicit formula for $\partial_t \mu_t|_{t=0}$ (we choose $t = 0$ for definiteness). For this, we need further notation. Introduce $Y_{k,s} = \lim_{t\to s} \frac{f_t^k - f_s^k}{t-s}$ for $k \geq 1$ (we write $Y_k$ instead of $Y_{k,0}$). Then $Y_{1,s} = X_s \circ f_s$, $Y_{2,s} = X_s \circ f_s^2 + (f_s' \circ f_s) X_s \circ f_s$, and

$$Y_{k,s} = \sum_{j=1}^{k} ((f_s^{k-j})' \circ f_s^j) \cdot X_s \circ f_s^j \,,\ k \geq 1 \,. \tag{16}$$

Put (note the shift in indices!)

$$\hat{Y}_s = (\hat{Y}_s(x,k) = Y_{k+1,s}(x), k \geq 0)\,, \quad \hat{Y} = \hat{Y}_0 \,. \tag{17}$$

Referring to Section 4 for the definitions of $\lambda$, $\widehat{\mathcal{L}}$, $\hat{\phi}$, $\Pi$, and $\mathcal{T}_0$, and summing (130) and (151) from the proof of Theorem 2.13, we get

$$\int A \partial_t \mu_t|_{t=0} = -\int A \cdot \Pi((\mathrm{id} - \widehat{\mathcal{L}})^{-1} \mathcal{T}_0(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi}))' \, dx - \lambda \int A' \cdot \Pi((\mathrm{id} - \mathcal{T}_0)(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi}))) \, dx \,.$$

Using the definitions of $\widehat{\mathcal{L}}$, $\hat{\phi}$, $\Pi$, and $\mathcal{T}_0$, the linear response formula above can be rewritten in terms of $f$ and the functions $\psi$, $\xi_k$ and $\chi_k$ from Proposition 2.7, and can then be compared to the expression in [50, §17 and §18], obtained under the additional assumptions that $f_0$ is Misiurewicz and all the $f_t$ are real analytic. See also Remark 2.14 below.

Using (132), we find

$$\int (A - A\circ f) \partial_t \mu_t|_{t=0} = \int A' \mathcal{T}_0(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi})) \, dx - \lambda \int (A' - (A \circ f)') \cdot \Pi((\mathrm{id} - \mathcal{T}_0) \widehat{\mathcal{L}}(\hat{Y}\hat{\phi})) \, dy \,.$$

By Theorem 2.13, the right-hand-side above coincides with the expression (15). We sketch here a direct proof of this fact: The left-hand-side above being independent of the parameter $\delta$ used in the construction of the tower (note however that $\Pi$, $\widehat{\mathcal{L}}$, and $\hat{\phi}$ depend on $\delta$), we can let $\delta \to 0$.

We expect that, when $\delta \to 0$, the function $\phi_0$ converges to $\phi$ in the $L^1$ topology, and that for any continuous function $B$

$$\lim_{\delta\to 0} \int_I B \cdot \Pi((\mathrm{id} - \mathcal{T}_0)(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi}))) \, dy = 0 \,, \tag{18}$$

while, using in particular (133) and the fact that [4] $Y_1 = X_0 \circ f_0$ and $\widehat{\mathcal{L}}(\hat{\phi}) = \hat{\phi}$,

$$\lim_{\delta\to 0} \int B \mathcal{T}_0(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi})) \, dx = \int B X \phi \, dx \,. \tag{19}$$

*Remark* 2.14. If $\mathcal{L}$ denotes the transfer operator defined on distributions $\nu$ of order one by $\int A \mathcal{L}\nu = \int (A \circ f)\nu$ for all $C^1$ functions $A$, expression (15) can be written as a left inverse expression (both sides should be viewed as distributions of order one)

$$\partial_t \mu_t|_{t=0} = -(\mathrm{id} - \mathcal{L})_L^{-1} (X\phi)' \, dx \,. \tag{20}$$

[4] Note also that, as $\delta \to 0$, the smallest level from which points may fall from the tower tends to infinity.

Indeed

$$\int (A - A \circ f) \partial_t \mu_t|_{t=0} = \int A(\mathrm{id} - \mathcal{L}) \partial_t \mu_t|_{t=0} ,$$

and, in the sense of distributions (writing $\mu = \mu_0$ as usual),

$$\int A' X \phi \, dx = - \int A (X\phi)' \, dx = - \int A \mathrm{div}_\mu (X) \, d\mu .$$

In situations where more information is available (such as smooth expanding circle maps), the following formal manipulations become licit (they are not licit in the present case of smooth unimodal maps, in particular the sum below diverges in general):

$$\begin{aligned}\int A(\mathrm{id} - \mathcal{L})_L^{-1} (X\phi)' \, dx &= \int A \sum_{j=0}^{\infty} \mathcal{L}^j (X\phi)' \, dx \\ &= \int \sum_{j=0}^{\infty} \mathcal{L}^j ((A \circ f^j)(X\phi)') \, dx = - \sum_{j=0}^{\infty} \int (A \circ f^j)' X \phi \, dx ,\end{aligned}$$

so that $\int A \partial_t d\mu_t|_{t=0} = \sum_{j=0}^{\infty} \int (A \circ f^j)' X d\mu$.

We end this section with a result that we shall not need, but which is of independent interest (the proof is given in Appendix A):

**Proposition 2.15** (The solution of the TCE is an infinitesimal conjugacy)**.** *Let $t \mapsto f_t$ be a $C^1$ deformation of the $S$-unimodal $(\lambda_c, H_0)$ Collet-Eckmann map $f_0$. Assume furthermore that for each $|t| \le \epsilon$ there exists a unique continuous function $\alpha_t$ on $I$ which solves the TCE (7) for $v = v_t := \partial_s f_s|_{s=t}$ and $f = f_t$, and in addition that the family $\{\alpha_t\}_{|t|\le\epsilon}$ of [5] continuous maps is equicontinuous. Then for each $x \in I$ the function $t \mapsto h_t(x)$ is $C^1$, and*

$$\partial_s h_s(x)|_{s=t} = \alpha_t(h_t(x)) , \qquad \forall\, t \in (-\epsilon, \epsilon) .$$

Note that if each $f_t$ satisfies the Benedicks-Carleson assumption (3) then Theorem 2.4 ensures that the TCE associated to $f_t$ and $v_t$ has a unique solution $\alpha_t$, which is continuous (recall Remark 2.9). We expect that equicontinuity of the family $\alpha_t$ can be obtained, possibly under the topological slow recurrence condition TSR.

## 3. Proof of Theorem 2.4: Boundedness and continuity of the solution $\alpha$ of the TCE

**3.1. The tower map $\hat{f}$, the times $S_i(x)$ and $T_i(x)$, and the intervals $I_j$.** Before recalling the tower construction from [15], we mention crucial expansion properties of Collet-Eckmann maps which improve [6] over [15, Lemma 1]:

[5] By compactness of $I$ and $[-\epsilon, \epsilon]$, continuity is equivalent to uniform continuity here.

[6] The improvements are: the Benedicks-Carleson condition is not needed, the expansion factor $\rho$ in (22) can be taken arbitrarily close to $\sqrt{\lambda_c}$, and $c(\delta) > C|\delta|$. The flexibility on $\rho$ means we can take any $\rho \in (e^\gamma, e^{-\gamma}\sqrt{\lambda_c})$ in Lemma 3.5. This makes (3) sufficient for Proposition 3.7, with no condition relating $\sigma$ and $\gamma$.

**Lemma 3.1** (Collet-Eckmann maps expansion)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map.*

*There exist $\sigma > 1$ and $C > 0$ and for every small $\delta > 0$ there exists $c(\delta) > C\delta$ such that*

$$|(f^i)'(x)| \geq c(\delta)\sigma^i \, , \, \forall 0 \leq i \leq j \, , \, \forall x \text{ so that } |f^k(x)| > \delta \, , \, \forall 0 \leq k < j \, . \tag{21}$$

*For every $1 < \rho < \sqrt{\lambda_c}$ there exists $C_1 = C_1(\rho) \in (0, 1]$ and for each $\delta_0 > 0$ there exists $\delta \in (0, \delta_0)$ such that*

$$|(f^j)'(x)| \geq C_1\rho^j \, , \, \forall x \text{ so that } |f^i(x)| > \delta \, , \, \forall 0 \leq i < j \, , \, |f^j(x)| \leq \delta \, . \tag{22}$$

*In addition, we can assume that either $\pm\delta$ are preperiodic points, or that they have infinite orbits and that their Lyapunov exponents exist and are strictly positive.*

*Remark* 3.2. Except in remarks (18) and (19), we do not use that $c(\delta) \geq C\delta$, only that $c(\delta) > 0$ if $\delta > 0$.

*Proof.* By Theorem 7.7 in [40], there exist $\sigma > 1$ and $K > 0$ such that

$$|(f^i)'(x)| \geq K\sigma^i \min_{0 \leq k < i} |f'(f^k(x))| \, .$$

Since $|f'(y)| \geq \widetilde{K}|y|$ and $|f^k(x)| \geq \delta$ for $k < j$ and $i \leq j$, we have (21).

To prove (22), we use Proposition 3.2(6) in [39] which says that for every $1 < \rho < \lambda_c^{1/2}$ there exists $\widetilde{C}$ such that if $f^j(y) = c$ then

$$|(f^j)'(y)| \geq \widetilde{C}\rho^j \, . \tag{23}$$

By Theorem 3.2 in [35] and the Koebe lemma, there exist $K > 0$ and arbitrarily small $\delta > 0$ such that the following holds: if $|f^i(x)| > \delta$ for $0 \leq i < j$ and $|f^j(x)| \leq \delta$ then there exists an interval $J$, with $x \in J$, such that $f^j(J) = [-\delta, \delta]$, $f^j$ is a diffeomorphism on $J$, and

$$\frac{1}{K} \leq \frac{|(f^j)'(y)|}{|(f^j)'(x)|} \leq K \, , \, \forall y \in J \, . \tag{24}$$

Let $y \in J$ be such that $f^j(y) = c$. By (23) and (24) it follows that

$$|(f^j)'(x)| \geq \frac{\widetilde{C}}{K}\rho^j \, .$$

By the principal nest construction in [34], we can choose $\delta$ so that $\pm\delta$ are preperiodic points, or $\pm\delta$ non preperiodic with $\Lambda(\pm\delta)$ well-defined and positive. □

We now recall the tower $\hat{f} : \hat{I} \to \hat{I}$ associated in [15] to a $(\lambda_c, H_0)$-Collet Eckmann $S$-unimodal map $f$ satisfying the Benedicks-Carleson assumption (3). *As the (so-called subhyperbolic) case of a finite postcritical orbit is much simpler, we shall assume in this construction that this orbit is infinite.* Choose $\rho$ so that

$$e^\gamma < \rho < e^{-\gamma}\sqrt{\lambda_c} \, , \tag{25}$$

and fix [7] two constants

$$\frac{3}{2}\gamma < \beta_1 < \beta_2 < 2\gamma \, . \tag{26}$$

[7]Our lower bound on $\beta_1$ is stronger than the one in [15] because we use some estimates in [57].

The *tower* $\hat{I}$ is the union $\hat{I} = \cup_{k\geq 0} E_k$ of levels $E_k = B_k \times \{k\}$ satisfying the following properties: The ground floor interval $B_0 = [a_0, b_0]$ is just the interval $I$. For $k \geq 1$, the interval $B_k = [a_k, b_k]$ is such that

$$[c_k - e^{-\beta_2 k}, c_k + e^{-\beta_2 k}] \subset B_k \subset [c_k - e^{-\beta_1 k}, c_k + e^{-\beta_1 k}] . \tag{27}$$

(Observe that $0 = c \notin B_k$ for all $k \geq H_0$.) Fix $\delta > 0$ such that the Lyapunov exponents $\Lambda(\pm\delta)$ are well defined and strictly positive, so that both claims of Lemma 3.1 hold (for our present choice of $\rho$), and small enough so that

$$|f^j(x) - c_j| < \min\{|c_j| e^{-\gamma j}, \, e^{-\beta_2 j}\} \quad \text{ for all } 1 \leq j \leq H_0 \text{ and } |x| \leq \delta . \tag{28}$$

(Just after (30), and later on, in Section 3.2, we may need to take a smaller choice of $\delta$ still assuming that $\Lambda(\pm\delta) > 0$ and that both claim of Lemma 3.1 hold.)

We may assume that the Lyapunov exponents $\Lambda(a_k) > 0$ and $\Lambda(b_k) > 0$ for all $k$, recalling that the set of points with a positive Lyapunov exponent has full Lebesgue measure. Let us write

$$\{0, \pm\delta\} \cup \{a_j \mid j \geq 0\} \cup \{b_j \mid j \geq 0\} = \{e_0 = c, e_1 = \delta, e_2 = -\delta, e_3, \dots\} .$$

We may and do require additionally that

$$f^j(e_k) \neq e_k \text{ and } f^j(e_k) \neq f^i(e_\ell) \quad \forall i, j \geq 1 \, , k \neq \ell \geq 0 \, . \tag{29}$$

(Indeed, (29) is a co-countable set of conditions, while the set of points $x$ with Lyapunov exponent $\Lambda(x)$ well defined and strictly positive has full Lebesgue measure, as recalled in Section 2.2.) The positivity condition on the Lyapunov exponents $\Lambda(e_k)$ ($k \neq 0$) ensures that $\alpha_{cand}(e_k)$ converges absolutely for each $k \geq 1$, and this will be used in the proof of Theorem 2.4.

For $(x, k) \in E_k$ we set [8]

$$\hat{f}(x, k) = \begin{cases} (f(x), k+1) & \text{if } k \geq 1 \text{ and } f(x) \in B_{k+1} \, , \\ (f(x), k+1) & \text{if } k = 0 \text{ and } x \in [-\delta, \delta] \, , \\ (f(x), 0) & \text{otherwise.} \end{cases} \tag{30}$$

Denoting $\pi : \hat{I} \to I$ the projection to the first factor, we have $f \circ \pi = \pi \circ \hat{f}$ on $\hat{I}$.

Define $H(\delta)$ to be the minimal $k \geq 1$ such that there exists some $x \in (-\delta, \delta)$ such that $\hat{f}^{k+1}(x, 0) \in E_0$. By continuity, $H(\delta)$ can be made arbitrarily large by choosing small enough $\delta$, and we assume that $H(\delta) \geq \max(2, H_0)$.

Having defined the tower $\hat{f}$, we next introduce notations $I_j$, $T_i(x)$ and $S_i(x)$ which will play a key part in the proof. We decompose $(-\delta, \delta) \setminus \{0\}$ as a disjoint union of intervals

$$\begin{aligned} &(-\delta, \delta) \setminus \{0\} = \cup_{j \geq H(\delta)} I_j \, , \quad I_j := I_j^+ \cup I_j^- \, , \\ &I_j^\pm := \{|x| < \delta, \pm x > 0, \hat{f}^\ell(x, 0) \in E_\ell, 0 \leq \ell < j, \hat{f}^j(x, 0) \in E_0\} \, . \end{aligned} \tag{31}$$

(Note that $I_j^\pm$ can be empty for some $j$.) For any $k \geq H(\delta)$ both sets $J_k^+ := \cup_{H(\delta) \leq j \leq k} I_j^+$ and $J_k^- := \cup_{H(\delta) \leq j \leq k} I_j^-$ are intervals.

For each $x \in I$ we next define inductively an infinite non decreasing sequence

$$0 = S_0(x) \leq T_1(x) < S_1(x) \leq \cdots < S_i(x) \leq T_{i+1}(x) < S_{i+1}(x) \leq \dots ,$$

---

[8] With respect to the definition in [15], note that we replaced $(-\delta, \delta)$ by $[-\delta, \delta]$, this is not essential but convenient e.g. in (76).

with $S_i(x), T_i(x) \in \mathbb{N} \cup \{\infty\}$ as follows: Put $T_0(x) = S_0(x) = 0$ for every $x \in I$. Let $i \geq 1$ and assume recursively that $S_j(x)$ and $T_j(x)$ have been defined for $j \leq i-1$. Then, we set (as usual, we put $\inf \emptyset = \infty$)

$$T_i(x) = \inf\{j \geq S_{i-1}(x) \mid \ |f^j(x)| \leq \delta\}\,.$$

If $T_i(x) = \infty$ for some $i \geq 1$ then we set $S_i(x) = \infty$. Otherwise, either $f^{T_i(x)}(x) = c$, and then we put $S_i(x) = \infty$, or $f^{T_i(x)}(x) \in I_j$, for some $j \geq H(\delta)$, and we put $S_i(x) = T_i(x) + j$.

Note that if $T_i(x) < \infty$ for some $i \geq 1$ then

$$\begin{aligned} &\hat{f}^j(x,0) \notin E_0\,,\ T_i(x)+1 \leq j \leq S_i(x) - 1\,,\\ &\hat{f}^\ell(x,0) \in E_0\,,\ S_{i-1}(x) \leq \ell \leq T_i(x)\,. \end{aligned}$$

If $T_{i_0}(x) = \infty$ for $i_0 \geq 1$, minimal with this property, we have $\hat{f}^\ell(x,0) \in E_0$ for all $\ell \geq S_{i_0-1}$ (that is, $|f^\ell(x)| > \delta$ for all $\ell \geq S_{i_0-1}$).

In other words, $T_i$ is the beginning of the $i$-th *bound period* and $S_i - 1$ is the end of the $i$-th bound period, [9] and if $S_i < T_{i+1}$ then $S_i$ is the beginning of the $i+1$-th *free period* (which ends when the $i+1$-th bound period starts).

In order to give a meaning to some expressions below, e.g. when $S_i = \infty$ or $T_i = \infty$, we set

$$S_i - T_i = 0 \text{ if } S_i = T_i = \infty\,, \quad T_i - S_{i-1} = 0 \text{ if } S_{i-1} = T_i = \infty\,,$$

and, for all $x \in I$, we set $(f^\infty)'(x) := \infty$ and $f^\infty(x) := c_1$.

### 3.2. **Properties of the tower map.**

After recalling in Proposition 3.3 and Lemma 3.4 some results of [15], we shall state in Lemma 3.5 expansion and distorsion control properties of the tower map $\hat{f}$ (invoking Lemma 3.1 instead of [15, Lemma 1]). Then we shall prove two new estimates (Proposition 3.7 and its Corollary 3.8) which will play a key part in the resummation argument of Proposition 3.9.

For the sake of completeness (we shall use an estimate from the proof later on), we first recall how to obtain distorsion bounds (see [15] or [57, Lemma 5.3(1)]):

**Lemma 3.3** (Bounded distortion in the bound period)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, with non preperiodic critical point. Then, if $\delta$ is small enough, there exists $C > 0$ such that for every $j \geq 1$, and every $k \leq j-1$, recalling* (31)

$$C^{-1} \leq \frac{|(f^k)'(x)|}{|(f^k)'(y)|} \leq C\,,\ \forall x, y \in U_j := f\big(\{c\} \cup \bigcup_{m \geq j} I_m\big)\,. \tag{32}$$

Note that $U_j$ is the set of points in $(-\delta, \delta) \times \{0\} \subset E_0$ which climb at least up to level $j-1$ before their first return to $E_0$.

---

[9] Bound period refers to the fact that the orbit is bound, i.e., sufficiently exponentially close, to the postcritical orbit.

*Proof.* For $1 \le \ell \le k \le j-1$, pick $x_\ell$ and $y_\ell$ in $\cup_{m\ge j}(f^\ell(I_m)) \subset B_\ell$. We have

$$\prod_{\ell=1}^{k} \frac{|f'(x_\ell)|}{|f'(y_\ell)|} \le \prod_{\ell=1}^{k}\Big(1+\frac{\sup|f''|}{|f'(y_\ell)|}|x_\ell-y_\ell|\Big) \le \prod_{\ell=1}^{k}\Big(1+C\sup|f''|\frac{|x_\ell-y_\ell|}{|y_\ell|}\Big)$$
$$\le \prod_{\ell=1}^{\infty}(1+\tilde{C}C\sup|f''|e^{-\beta_1\ell}) < \infty\,, \tag{33}$$

uniformly in $m \ge j$. We used that $|x_\ell - y_\ell| \le e^{-\beta_1\ell}$ and $|y_\ell| \ge e^{-\gamma\ell} - e^{-\beta_1\ell}$ with $\beta_1 > 3\gamma/2$, but any summable condition would be enough here. If we choose $y_\ell = f^{\ell-1}(y)$ and $x_\ell = f^{\ell-1}(x)$ we get the upper bound in (33). If we pick $y_\ell = f^{\ell-1}(x)$ and $x_\ell = f^{\ell-1}(y)$ then we obtain the lower bound in (33). □

The following upper and lower bounds from [15], about points which climb for exactly $j-1$ steps, will be used several times:

**Lemma 3.4** (The $j$-bound intervals $I_j^\pm$)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3) *and with non preperiodic critical point. Then there exist $C$ and $C_2$ so that for any $j \ge H(\delta)$, recalling* (31)*, we have*

$$|x-c| \le Ce^{\frac{-3\gamma(j-1)}{4}}|(f^{j-2})'(c_1)|^{-1/2}\,,\ \forall x \in I_j\,, \tag{34}$$

*and*

$$|(f^j)'(x)| \ge C_2 e^{-\frac{\beta_2 j}{2}}|(f^{j-1})'(c_1)|^{1/2}\,, \forall x \in I_j\,, \tag{35}$$

*and, finally,*

$$|f'(x)| \ge C^{-1}e^{-\gamma j}|(f^{j-1})'(c_1)|^{-1/2}\,, \forall x \in I_j\,. \tag{36}$$

*Proof.* If $I_j$ is empty, there is nothing to prove. Otherwise, our definitions and the mean value theorem imply that there exists $y$ with $f(y) \in [f(x), c_1]$ so that

$$|(f^{j-2})'(f(y))||f(x)-c_1| \le Ce^{-\beta_1(j-1)}\,.$$

Therefore, since $\beta_1 > 3\gamma/2$ (recall (26)) the lower bound in (32) and the fact that $|f(x)-c_1| \ge C^{-1}|x-c|^2$ yield (34).

The bound (35) follows from [15, (3.10), Proof of Lemma 2] (see top of p. 495 there, or see [57, Lemma 5.3, eq. after (5.15)]).

For (36), use (35) and that $\beta_2 < 2\gamma$ from (26). □

Recall the times $S_i$, $T_i$ from Subsection 3.1, for suitably small $\delta$. The following lemma gives expansion at the end of the free period $T_i - 1$ (just before climbing the tower), at the end $S_i - 1$ of the bound period (after falling from the tower), and during the free period (when staying at level zero):

**Lemma 3.5** (Tower expansion for Benedicks-Carleson maps)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, with non preperiodic critical point, and let $\rho$ satisfy* (25)*. For every small enough $\delta_0 > 0$, if $\delta < \delta_0$, $\sigma > 1$, $C_1 = C_1(\rho) \in (0,1]$, and $c(\delta) > 0$ are as in Lemma 3.1, letting $S_i(x)$ and $T_i(x)$ be the times associated to the tower for $\delta$, then*

$$|(f^{S_i(x)})'(x)| \ge \rho^{S_i(x)}\,, \qquad |(f^{T_i(x)})'(x)| \ge C_1\rho^{T_i(x)}\,, \quad \forall x \in I\,, \quad \forall i \ge 0\,, \tag{37}$$

*and*

$$|(f^{S_i(x)+j})'(x)| \ge c(\delta)\rho^{S_i(x)}\sigma^j\,, \qquad \forall x \in I\,, \qquad \forall i \ge 0\,, \forall 0 \le j < T_{i+1}(x) - S_i(x)\,.$$

*Remark* 3.6. An immediate consequence of Lemma 3.5 is that, for every $x$ such that $f^n(x) \neq c$, for every $n$, we have $\limsup_n |(f^n)'(x)|^{1/n} \geq \xi > 1$, where $\xi = \min(\rho, \sigma)$.

*Proof of Lemma 3.5.* The lemma will easily follow from Lemma 3.1 and (35) from Lemma 3.4.

Choose $\delta < \delta_0$ as in the second claim of Lemma 3.1, small enough so that

$$C_1 C_2 \cdot e^{-\frac{\beta_2 j}{2}} \lambda_c^{\frac{j-1}{2}} \geq \rho^j \,, \quad \forall j \geq H(\delta) \,.$$

Let now $x \in I$. Recall that for any $\ell \geq 1$, the definitions imply $f^{S_{\ell-1}(x)+k}(x) \in I \setminus [-\delta, \delta]$ for all $0 \leq k < T_\ell(x) - S_{\ell-1}(x)$ and $f^{T_\ell(x)}(x) \in I_j$ with $j = S_\ell(x) - T_\ell(x) \geq H(\delta)$. Therefore, the second claim of Lemma 3.1 and (35) give for all $i \geq 0$

$$|(f^{S_i})'(x)| = \prod_{\ell=1}^{i} |(f^{S_\ell(x) - T_\ell(x)})'(f^{T_\ell(x)} x)||(f^{T_\ell(x) - S_{\ell-1}(x)})'(f^{S_{\ell-1}(x)} x)| \geq \rho^{S_i(x)} \,,$$

and

$$|(f^{T_i})'(x)| = |(f^{T_i(x) - S_{i-1}(x)})'(f^{S_{i-1}(x)} x)||(f^{S_i})'(x)| \geq C_1 \rho^{T_i(x) - S_i(x)} \rho^{S_i(x)} \,.$$

Using in addition the first claim of Lemma 3.1, we get, for $0 \leq j \leq T_{i+1}(x) - S_i(x)$,

$$|(f^{S_i(x)+j})'(x)| = |(f^j)'(f^{S_i(x)}(x))||(f^{S_i})'(x)| \geq c(\delta) \sigma^j \rho^{S_i(x)} \,.$$

□

The information on the tower will allow us to prove the next proposition, which is a crucial ingredient [10] to show that $\alpha_{cand}$ can be resummed to a bounded function (Proposition 3.9 and Definition 3.10):

**Proposition 3.7** (Key estimate for Benedicks Carleson maps)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, with non preperiodic critical point. There exists $C > 0$ such that for every $j \geq 0$ we have*

$$\sum_{k=j+1}^{\infty} \frac{1}{|(f^{k-j})'(f^j(c_1))|} \leq C e^{\gamma j} \,. \tag{38}$$

The proof shows that $C = O((c(\delta))^{-1})$, where $\delta$ is the parameter used in the construction of the tower and $c(\delta)$ is given by Lemma 3.1. More importantly, the proposition implies that $|\alpha_{cand}(c_j)| \leq C \sup |v| e^{\gamma j}$. This bound is of course not uniform in $j$, but it will act as a bootstrap for the proof of the Proposition 3.9 which performs the resummation.

*Proof.* Fix $j \geq 1$. Since the coefficients of the series are all positive, we may (and shall) group them in a convenient way, using the times $T_i := T_i(c_{j+1})$ and $S_i := S_i(c_{j+1})$ defined in the tower construction for a small enough $\delta$. We have

$$\sum_{k=j+1}^{\infty} \frac{1}{|(f^{k-j})'(f^j(c_1))|}$$
$$= \sum_{i=0}^{\infty} \frac{1}{|(f^{S_i})'(c_{j+1})|} u_{T_{i+1} - S_i}(f^{S_i}(c_{j+1})) + \sum_{i=1}^{\infty} \frac{1}{|(f^{T_i})'(c_{j+1})|} u_{S_i - T_i}(f^{T_i}(c_{j+1})) \,,$$

[10] Proposition 3.7 will also be used in an essential way in the proof of Theorem 2.13, in particular in the proof of Lemma 4.1, and also in Lemma 4.11.

where we use the notation

$$u_n(y) = \sum_{\ell=1}^{n} \frac{1}{|(f^\ell)'(y)|} \, .$$

(In particular $u_0 \equiv 0$.) Since $T_{i+1} - S_i = T_1(f^{S_i}(c_{j+1}))$, Lemma 3.5 implies

$$u_{T_{i+1}-S_i}(f^{S_i}(c_{j+1})) \le \frac{C}{c(\delta)(1-\sigma^{-1})} \, ,$$

(in particular the series converges if $n = T_{i+1} - S_i = \infty$). Since $f''(c) \neq 0$, the Benedicks-Carleson assumption (3) implies for all $i$

$$|f'(f^{T_i}(c_{j+1}))| = |f'(f^{T_i+j}(c_1))| \ge C^{-1} e^{-\gamma(T_i+j)} \, .$$

Therefore, the bounded distortion estimate (33) in the proof of Lemma 3.3 gives, together with the Collet-Eckmann assumption, [11]

$$u_{S_i-T_i}(f^{T_i}(c_{j+1})) \le \frac{1}{|f'(f^{T_i}(c_{j+1}))|} \sum_{\ell=0}^{\infty} \frac{C}{|(f^\ell)'(c_1)|} \le \frac{C^3}{(1-\lambda_c^{-1})} e^{\gamma(T_i+j)} \, .$$

By Lemma 3.5 we have $|(f^{S_i})'(c_{j+1})| \ge C\rho^{S_i}$ and $|(f^{T_i})'(c_{j+1})| \ge C_1 \rho^{T_i}$. Therefore, there exists constants $K_1(\delta)$, $K_2(\delta)$ so that

$$\sum_{k=j+1}^{\infty} \frac{1}{|(f^{k-j})'(f^j(c_1))|} \le K_1(\delta) e^{\gamma j} \left[ \sum_{i=0}^{\infty} \rho^{-S_i} + \sum_{i=1}^{\infty} \rho^{-T_i} e^{\gamma T_i} \right] \le K_2(\delta) e^{\gamma j} \, .$$

(We used that $\rho > e^\gamma$.) □

We end this section of preparations by a consequence of Proposition 3.7 which will also be needed in our resummation Proposition 3.9:

**Corollary 3.8.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, with non preperiodic critical point. Then there exists $C$ so that for any $j \ge 1$*

$$\frac{1}{|f'(x)|} \sum_{k=1}^{j-1} \left| \frac{1}{|(f^k)'(c_1)|} - \frac{1}{|(f^k)'(f(x))|} \right| \le C \frac{e^{5\gamma j/4}}{|f^{j-2}(c_1)|^{1/2}} \, , \forall x \in I_j \, . \tag{39}$$

[11] The constant $C$ above depends on $[\sup_{1\le j<H_0} \lambda_c / |(f^j)'(c_1)|^{1/j}]^{H_0}$. By Lemma 5.8, this expression is uniform for suitable families $f_t$.

*Proof.* For any $j \geq 1$, $x \in I_j$, and $1 \leq k < j$ we get by using (33) from the proof of Lemma 3.3 that

$$\begin{aligned}
&\left|\frac{1}{(f^k)'(f(x))} - \frac{1}{(f^k)'(c_1)}\right| \\
&\leq \sum_{n=0}^{k}\left|\frac{1}{f'(f^n(f(x)))} - \frac{1}{f'(f^n(c_1))}\right| \prod_{\ell=0}^{n-1} \frac{1}{|f'(f^\ell(c_1))|} \prod_{i=n+1}^{k} \frac{1}{|f'(f^i(f(x)))|} \\
&\leq \sum_{n=0}^{k} \sup_{y\in f^n[f(x),c_1]} \frac{|f''(y)|}{|f'(y)|^2} \sup_{z\in[f(x),c_1]} |(f^n)'(z)||f(x)-c_1| \\
&\qquad\qquad \cdot \prod_{\ell=0}^{n-1} \frac{1}{|f'(f^\ell(c_1)|} \prod_{i=n+1}^{k} \frac{1}{|f'(f^i(f(x)))|} \\
&\overset{(33)}{\leq} C|f(x)-c_1| \sum_{n=0}^{k} \sup_{y\in f^n[f(x),c_1]} \frac{|f''(y)|}{|f'(y)|} \frac{|(f^n)'(c_1)|}{|(f^k)'(c_1)|} \\
&\leq \frac{C}{1-e^{-\gamma/2}} |f(x)-c_1| \sum_{n=0}^{k} e^{\gamma n} \frac{1}{|(f^{k-n})'(f^n(c_1))|} .
\end{aligned}$$

(In the last inequality, we used that $f^n[f(x), c_1] \subset B_{n+1}$, and that the Benedicks-Carleson assumption (3) implies $|y - c| > e^{-\gamma(n+1)} - e^{-\beta_1(n+1)}$ for $y \in B_{n+1}$, together with (26).)

Therefore, for any $j \geq 1$ and $x \in I_j$, since $|f(x) - c_1| \leq C|x - c|^2$, Proposition 3.7 implies

$$\begin{aligned}
&\frac{1}{|f'(x)|} \sum_{k=0}^{j-1}\left|\frac{1}{(f^k)'(f(x))} - \frac{1}{(f^k)'(c_1)}\right| \qquad (40)\\
&\leq C|x-c| \sum_{k=0}^{j-1}\sum_{n=0}^{k} e^{\gamma n} \frac{1}{|(f^{k-n})'(f^n(c_1))|} \\
&\leq C|x-c| \sum_{n=0}^{j-1} e^{\gamma n} \sum_{i=0}^{j-n-1} \frac{1}{|(f^i)'(f^n(c_1))|} \\
&\leq C|x-c| \sum_{n=0}^{j-1} e^{\gamma n} \sum_{i=0}^{\infty} \frac{1}{|(f^i)'(f^n(c_1))|} \\
&\overset{(38)}{\leq} C|x-c| \sum_{n=0}^{j-1} e^{2\gamma n} \leq C|x-c|e^{2\gamma j} \leq C\frac{e^{-3\gamma(j-1)/4}}{|(f^{j-2})'(c_1)|^{1/2}} e^{2\gamma j} ,
\end{aligned}$$

where we used (34) from Lemma 3.4 in the last inequality. □

### 3.3. Resummation: Definition and boundedness of $\alpha$ for horizontal $v$.

Proposition 3.9 is the heart of the proof of Theorem 2.4. This is where we define the dynamical resummation for the series $\alpha_{cand}$, under a horizontality condition.

**Proposition 3.9** (Resummation). *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, with non preperiodic critical point. Lett $v = X \circ f$, for $X$ a Lipschitz function, and assume that $v$ satisfies the horizontality condition* (10) *for $f$.*

*If $\delta$ is small enough, then for every $x \in I$, letting $T_i = T_i(x)$ and $S_i = S_i(x)$ be the times associated to the tower for $\delta$, the following series converges:*

$$\sum_{i=1}^{\infty}\Big(\frac{1}{|(f^{T_i})'(x)|}|w_{S_i-T_i}(f^{T_i}(x))| + \frac{1}{|(f^{S_{i-1}})'(x)|}|w_{T_i-S_{i-1}}(f^{S_{i-1}}(x))|\Big)\,, \tag{41}$$

*where $w_\infty(c) = 0$ and* [12]

$$w_n(y) := \sum_{\ell=0}^{n-1} \frac{v(f^\ell(y))}{(f^{\ell+1})'(y)}\,, \quad y \neq c\,, n \in \mathbb{Z}_+ \cup \{\infty\}\,.$$

*Moreover the sum of the series (41) is bounded uniformly in $x \in I$.*

The proposition allows us to give the following definition:

**Definition 3.10.** We define $\alpha(x)$ for any $x \in I$ by

$$\alpha(x) = -\sum_{i=1}^{\infty}\Big(\frac{1}{(f^{T_i})'(x)}w_{S_i-T_i}(f^{T_i}(x)) + \frac{1}{(f^{S_{i-1}})'(x)}w_{T_i-S_{i-1}}(f^{S_{i-1}}(x))\Big)\,. \tag{42}$$

If the formal series (8) is absolutely convergent at $x$, then (42) is just the sum of this series. If $f^j(x) = c$ for some $j \geq 0$, minimal with this property, then our notation ensures that (42) is just the finite sum (9). In both these cases, $\alpha(x) = \alpha_{cand}(x)$.

*Proof of Proposition 3.9.* Choose $\delta$ small enough, as in Lemma 3.5, and $C_1(\rho)$, $\sigma > 1$, and $c(\delta)$ from Lemma 3.1.

For any $i \geq 1$ so that $S_{i-1} < \infty$, since $T_i - S_{i-1} = T_1(f^{S_{i-1}}(x))$, Lemma 3.5 implies

$$\begin{aligned} |w_{T_i-S_{i-1}}(f^{S_{i-1}}(x))| &\leq \sum_{\ell=0}^{T_i-S_{i-1}-1}\left|\frac{v(f^\ell(f^{S_{i-1}}(x)))}{(f^{\ell+1})'(f^{S_{i-1}}(x))}\right| \\ &\leq \frac{C}{c(\delta)(1-\sigma^{-1})}\sup|v|\,,\ \forall i \geq 0\,. \end{aligned} \tag{43}$$

(Note that $T_i = \infty$ is allowed in the previous estimate.)

If $S_i = \infty$ then either $f^{T_i}(x) = c$ or $T_i = \infty$, in both cases $w_{S_i-T_i}(f^{T_i}(x)) = 0$. Next, we claim that whenever $S_i \neq \infty$, we have,

$$|w_{S_i-T_i}(f^{T_i}(x))| \leq C\max\{\sup|v|, \mathrm{Lip}X\}\frac{e^{2\gamma(S_i-T_i)}}{|(f^{S_i-T_i-1})'(c_1)|^{1/2}}\,, \forall i \geq 0\,. \tag{44}$$

We shall prove (44), which requires horizontality as well as the key Proposition 3.7 and its Corollary 3.8, at the end of this proof.

Putting together (43) and (44), and recalling Lemma 3.5, we find the following upper bound for (41) :

$$\sum_{i=1}^{\infty}\frac{C\rho^{-S_{i-1}}}{c(\delta)(1-\sigma^{-1})}\sup|v| + C\frac{\rho^{-T_i}}{C_1(\rho)}\max(\sup|v|, \mathrm{Lip}X)\frac{e^{2\gamma(S_i-T_i)}}{|(f^{S_i-T_i-1})'(c_1)|^{1/2}}\,,$$

Using again (3), we are done.

[12] In particular, we claim that $w_n(y)$ converges if $n = S_i - T_i = \infty$ and $y = f^{T_i}(x) \neq c$, or if $n = T_i - S_{i-1} = \infty$ and $y = f^{S_{i-1}}(x)$.

It remains to prove (44). The definitions imply $f^{T_i}(x) \in I_j$ for $j = S_i - T_i$, for all $i \geq 1$. So it suffices to show that

$$|w_j(y)| \leq C \max(\sup |v|, \mathrm{Lip} X) \frac{e^{2\gamma j}}{|(f^{j-1})'(c_1)|^{1/2}}\,, \quad \forall y \in I_j\,. \tag{45}$$

We shall use the decomposition

$$\begin{aligned} |w_j(y)| \leq & \underbrace{\frac{1}{|f'(y)|}\left|\sum_{k=0}^{j-1} \frac{v(c_k)}{(f^k)'(c_1)}\right|}_{I} + \underbrace{\frac{1}{|f'(y)|}\sum_{k=0}^{j-1} |v(c_k)| \left|\frac{1}{(f^k)'(f(y))} - \frac{1}{(f^k)'(c_1)}\right|}_{II} \\ & + \underbrace{\frac{1}{|f'(y)|}\sum_{k=0}^{j-1}\left|\frac{v(f^k(y)) - v(c_k)}{(f^k)'(c_1)}\right|}_{III}\,. \end{aligned} \tag{46}$$

We first consider $I$. By the horizontality condition (10) for $v$, we have

$$\sum_{k=0}^{j-1} \frac{v(c_k)}{(f^k)'(c_1)} + \frac{1}{(f^j)'(c_1)} \sum_{k=j}^{\infty} \frac{v(c_k)}{(f^{k-j})'(c_{j+1})} = 0\,.$$

Therefore, (36) in Lemma 3.4 and Proposition 3.7 imply

$$\begin{aligned} I &= \frac{1}{|f'(y)|} \frac{1}{|(f^j)'(c_1)|} \left| \sum_{k=j}^{\infty} \frac{v(c_k)}{(f^{k-j})'(c_{j+1})} \right| \\ &\leq C e^{\gamma j} |(f^{j-1})'(c_1)|^{1/2} \frac{1}{|(f^j)'(c_1)|} \sup |v| e^{\gamma j} \\ &\leq C \sup |v| \frac{e^{2\gamma j}}{|(f^{j-1})'(c_1)|^{1/2}}\,. \end{aligned} \tag{47}$$

Next, by Corollary 3.8 we find

$$\begin{aligned} II &\leq \sup |v| \frac{1}{|f'(y)|} \sum_{k=0}^{j-1} \left| \frac{1}{(f^k)'(f(y))} - \frac{1}{(f^k)'(c_1)} \right| \\ &\leq C \sup |v| \frac{e^{5\gamma j/4}}{|(f^{j-2})'(c_1)|^{1/2}}\,. \end{aligned} \tag{48}$$

Recalling our assumption $v = X \circ f$, we consider now

$$III = \frac{1}{|f'(y)|} \sum_{k=0}^{j-1} \left| \frac{X(f^{k+1}(y)) - X(c_{k+1})}{(f^k)'(c_1)} \right|\,. \tag{49}$$

Since $X$ is Lipschitz and $f$ is $C^1$, for any $0 \leq k \leq j-1$ there exists $z$ between $c_1$ and $f(y)$ so that

$$\begin{aligned} |X(f^{k+1}(y)) - X(f^{k+1}(c))| &\leq \mathrm{Lip}(X) |f^k(f(y)) - f^k(c_1)| \\ &\leq \mathrm{Lip}(X) |f(y) - c_1| |(f^k)'(z)|\,. \end{aligned} \tag{50}$$

Then (50), together with (32) from the proof of Lemma 3.3 (recall $k \le j-1$ and $y \in I_j$), imply

$$\frac{|X(f^{k+1}(y)) - X(f^{k+1}(c))|}{|(f^k)'(c_1)|} \le \mathrm{Lip}X|f(y) - c_1| \sup_{z\in[f(y),c_1]} \frac{|(f^k)'(z)|}{(f^k)'(c_1)}$$
$$\le C\mathrm{Lip}X|f(y) - c_1|$$
$$\le C\mathrm{Lip}X|y - c| \sup_{z\in[c,y]} |f'(z)| \, .$$

Since $|f'(y)| \ge C|y-c|$ and $|f'(z)| \le C|z-c|$, the bound (34) in Lemma 3.4 gives

$$III \le C\frac{j}{|f'(y)|}\mathrm{Lip}X|y-c| \sup_{z\in[0,y]} |f'(z)| \le Cj\mathrm{Lip}X \sup_{z\in[0,y]} |z-c|$$

$$(51) \qquad \le Cj\mathrm{Lip}X e^{-\frac{3\gamma(j-1)}{4}} |(f^{j-2})'(c_1)|^{-1/2} \le C\mathrm{Lip}X \frac{1}{|(f^{j-2})'(c_1)|^{1/2}} \, .$$

Putting (47), (48), and (51) together, we get (45). □

3.4. **Proof of Theorem 2.4: Continuity of $\alpha$ and checking the TCE.** We prove that Proposition 3.9 implies Theorem 2.4:

*Proof of Theorem 2.4.* The so-called subhyperbolic case when the critical point is preperiodic is easier and left to the reader. For small enough $\delta$ (recall Sections 3.1 and 3.2), we construct a tower map and associated times $T_i(x)$ and $S_i(x)$.

To show the uniqueness statement for bounded $v$, suppose that $\beta : I \to \mathbb{C}$ is a bounded function such that $v = \beta \circ f - f' \cdot \beta$ on $I$. It is easy to see that for every $x$ and $n \ge 1$ such that $(f^n)'(x) \ne 0$ we have

$$(52) \qquad \beta(x) = -\sum_{i=0}^{n-1} \frac{v(f^i(x))}{(f^{i+1})'(x)} + \frac{\beta(f^n(x))}{(f^n)'(x)} \, .$$

If $f^n(x) \ne c$ for every $n$, Remark 3.6 implies that $\limsup_n |(f^n)'(x)| = \infty$, so that there exists[13] $n_i(x) \to_i \infty$ with $\lim_k |(f^{n_i(x)})'(x)| = \infty$. Since $\beta$ is bounded, it follows from (52) that

$$(53) \qquad \beta(x) = -\lim_{i\to\infty} \sum_{j=0}^{n_i} \frac{v(f^j(x))}{(f^{j+1})'(x)} \, .$$

This proves that $\beta$ is uniquely defined on $\{x \mid f^n(x) \ne c \, , \forall n\}$. In particular, $\beta(c_1) = \alpha_{cand}(c_1)$, so that $v$ is horizontal (using the TCE, that $f'(c) = 0$, and that $\beta$ is bounded).

Now, if $f^i(x) \ne c$ for $0 \le i < n$ and $f^n(x) = c$, then (52) and $\beta(c) = 0$ give

$$(54) \qquad \beta(x) = -\sum_{i=0}^{n-1} \frac{v(f^i(x))}{(f^{i+1})'(x)} \, .$$

Therefore, $\beta(x) = \alpha_{cand}(x)$. This ends the proof of uniqueness.

From now on, we assume that $v = X \circ f$ with $X$ Lipschitz. If $v$ is horizontal, Proposition 3.9 implies that the function $\alpha(x)$ defined by the series (42) is bounded uniformly in $x \in I$. It remains to show that $\alpha$ is continuous and satisfies the TCE.

[13]Note that if $T_i(x) < \infty$ for all $i$, we can take $n_i(x) = T_i(x)$.

The definitions easily imply that for every $x \in I$ and all $i \geq 1$ the following limits exist:

$$S_i^+(x) = \lim_{y \to x^+} S_i(y)\,, \quad S_i^-(x) = \lim_{y \to x^-} S_i(y)\,,$$
$$T_i^+(x) = \lim_{y \to x^+} T_i(y)\,, \quad T_i^-(x) = \lim_{y \to x^-} T_i(y)\,.$$

Writing $T_i^\pm$ and $S_i^\pm$ for $T_i^\pm(x)$ and $S_i^\pm(x)$, define

$$\alpha^\pm(x) := -\sum_{i=1}^\infty \left( \frac{1}{(f^{T_i^\pm})'(x)} w_{S_i^\pm - T_i^\pm}(f^{T_i^\pm}(x)) + \frac{1}{(f^{S_{i-1}^\pm})'(x)} w_{T_i^\pm - S_{i-1}^\pm}(f^{S_{i-1}^\pm}(x)) \right).$$

We claim that for every $x \in I$ we have

$$\lim_{y \to x^+} \alpha(y) = \alpha^+(x)\,, \text{ and } \lim_{y \to x^-} \alpha(y) = \alpha^-(x)\,. \tag{55}$$

We shall show (55) at the end of the proof of this theorem.

Let now $\mathcal{S}$ be the set of $x \in I$ so that there exists $\ell \geq 0$ with $\hat{f}^\ell(x, 0) \in \partial E_k$ for some $k \geq 1$, or $\hat{f}^\ell(x, 0) = (\pm\delta, 0)$, or $\hat{f}^\ell(x, 0) = (c, 0)$. Clearly, if $x \notin \mathcal{S}$, then $S_i(x) = S_i^+(x) = S_i^-(x)$ and $T_i(x) = T_i^+(x) = T_i^-(x)$, for every $i \geq 1$. Consequently $\alpha$ is continuous at $x \notin \mathcal{S}$. If $x \in \mathcal{S}$ but $\hat{f}^\ell(x, 0) \neq (c, 0)$ for all $\ell \geq 0$, then the conditions on $\{e_k\}$ in Section 3.1 imply that the series (8) converges absolutely at $x$. If $\hat{f}^\ell(x, 0) = (c, 0)$ for some $\ell \geq 0$, then $\alpha(x)$ is the finite sum (9). Let now $x \in \mathcal{S}$. The three series, or finite sums, which define $\alpha(x)$, $\alpha^+(x)$, and $\alpha^-(x)$ are obtained by grouping together in different ways the terms of the absolutely convergent series (8), or of the sum (9). Therefore, $\alpha(x) = \alpha^+(x) = \alpha^-(x)$, and (55) implies that $\alpha$ is continuous at $x$.

To show that $\alpha$ satisfies the twisted cohomological equation, note that if $x$ is a repelling periodic point then $f^\ell(x) \neq c$ for all $\ell$, and the series (8) is absolutely convergent at $x$. Therefore, this series coincides with $\alpha(x)$. In particular one can easily check that $v(x) = \alpha(f(x)) - f'(x)\alpha(x)$ for repelling periodic points $x$. Since all periodic points of a Collet-Eckmann map are repelling, since the set of periodic points is dense, and since $\alpha$ is continuous, it follows that $\alpha$ satisfies the twisted cohomological equation everywhere.

The fact that $\alpha(x) = \alpha_{cand}(x)$ for all $x$ so that $f^j(x) = c$ or such that the series $\alpha_{cand}(x)$ converges absolutely follows from the remark after Definition 3.10.

It remains to prove (55). We shall consider the limit as $y$ approaches $x$ from above (the proof of the other one-sided limit is identical).

Before we start, note that for any $\epsilon > 0$, the uniform constants and exponential rates in the proof of Proposition 3.9 (see (43), where $T_i = \infty$ is allowed, and (44), (37)) imply that there is $n_0 = n_0(\epsilon) \geq 1$ such that for all $x \in I$

$$\sum_{i:\, T_i \geq n_0} \frac{1}{|(f^{T_i})'(x)|} |w_{S_i - T_i}(f^{T_i}(x))| + \sum_{i:\, S_i \geq n_0} \frac{1}{|(f^{S_i})'(x)|} |w_{T_{i+1} - S_i}(f^{S_i}(x))| < \frac{\epsilon}{4}\,. \tag{56}$$

Fix $x \in I$, and let $T_i = T_i(x)$, $S_i = S_i(x)$, $T_i^+ = T_i^+(x)$, $S_i^+ = S_i^+(x)$. There are three cases to consider to prove (55) when $y \downarrow x$. The first case occurs when $T_i < \infty$ and $S_i < \infty$ for every $i$. This means that the forward $\hat{f}$-orbit of $x$ never hits the critical point $(c, 0)$ and never gets trapped inside the base $E_0$. Then observe that there exists $\epsilon_1 > 0$ such that if $x < y < x + \epsilon_1$ then, for every $i$ so that $S_i^+ < n_0$, we have $S_i(y) = S_i^+$, and for every $i$ so that $T_i^+(x) < n_0$, we have $T_i(y) = T_i^+$. Clearly, for any $n_0 \ge 1$, the function

$$\tilde\alpha(y) = \tilde\alpha_{n_0}(y) = -\sum_{i:\, T_i^+ < n_0} \frac{1}{(f^{T_i^+})'(y)} w_{S_i^+ - T_i^+}(f^{T_i^+}(y)) \\ - \sum_{i:\, S_i^+ < n_0} \frac{1}{(f^{S_i^+})'(y)} w_{T_{i+1}^+(x) - S_i^+}(f^{S_i^+}(y))$$

is continuous on $[x, x+\epsilon_1)$. So for any $\epsilon > 0$ there exists $0 < \epsilon_2 < \epsilon_1$ (depending also on $n_0$) such that if $x \le y < x + \epsilon_2$ then $|\tilde\alpha(y) - \tilde\alpha(x)| < \epsilon/2$. Clearly, $\tilde\alpha(x)$ is just the $n_0$-truncation of $\alpha^+(x)$ while the observation above implies that $\tilde\alpha(y)$ is the $n_0$-truncation of $\alpha(y)$. Thus, taking $n_0(\epsilon)$ as in (56), we get that

$$|\alpha(y) - \alpha^+(x)| < |\tilde\alpha(y) - \tilde\alpha(x)| + \frac{2\epsilon}{4} < \epsilon\,, \quad \forall y \in [x, x+\epsilon_2)\,.$$

The second case occurs when the forward $\hat f$-orbit of $x$ gets trapped in the first level $E_0$. That is, there exists $i_0 \ge 0$ such that $S_{i_0} < \infty$ but $T_{i_0+1}(x) = \infty$. Then, for any $n_0 \ge 1$, there exists $\epsilon_1 > 0$ such that if $x < y < x + \epsilon_1$ then $S_i(y) = S_i^+$ and $T_i(y) = T_i^+$ for every $i \le i_0$, and $T_{i_0+1}(y) \ge n_0$. Clearly, the function

$$\tilde\alpha(y) = -\sum_{i<i_0} \frac{1}{(f^{S_i^+(x)})'(y)} w_{T_{i+1}^+ - S_i^+}(f^{S_i^+}(y)) - \frac{1}{(f^{S_{i_0}})'(y)} w_{n_0 - S_{i_0}^+}(f^{S_{i_0}^+}(y)) \\ - \sum_{i \le i_0} \frac{1}{(f^{T_i^+})'(y)} w_{S_i^+ - T_i^+}(f^{T_i}(y))$$

is continuous on $[x, x+\epsilon_1)$. Choose $\epsilon_2 < \epsilon_1$ such that if $x \le y < x+\epsilon_2$ then $|\tilde\alpha(y) - \tilde\alpha(x)| < \epsilon/4$. Using again the uniformity of the estimates in the proof of Proposition 3.9 (in particular of the exponentially decaying term of the series (43) for $i = i_0$), it is easy to see that if $n_0$ is large enough then $|\alpha(y) - \alpha^+(x)| < \epsilon$ for $y \in [x, x+\epsilon_2)$.

The third case occurs when $\hat f^{i_0}(x, 0) = (c, 0)$ for some $i_0 \ge 0$. That is, there exists $i_0 \ge 1$ such that $T_{i_0}(x) < \infty$ but $S_{i_0}(x) = \infty$, and $\alpha(x)$ is just a finite sum. (If $x = c$ then $i_0 = 1$.) Then there exists $\epsilon_2 > 0$ such that if $x < y < x + \epsilon_2$ then $T_i(y) = T_i^+$ and $S_{i-1}(y) = S_{i-1}^+$ for every $i \le i_0$, while $S_{i_0}(y) \ge n_0$. To finish, define

$$\tilde\alpha(y) = -\sum_{i<i_0} \frac{1}{(f^{S_i^+})'(y)} w_{T_{i+1}^+ - S_i^+}(f^{S_i^+}(y)) - \sum_{i<i_0} \frac{1}{(f^{T_i^+})'(y)} w_{S_i^+ - T_i^+}(f^{T_i^+}(y))\,.$$

and adapt the arguments from the first two cases. This ends the proof of (55) and of Theorem 2.4. $\square$

3.5. **Divergence of the formal power series (Proposition 2.5).** In this final subsection, we show that the formal power series $\alpha_{cand}(x)$ diverges for many $x$.

*Proof of Proposition 2.5.* It is enough to show that the set of points such that $\liminf_i \left|\frac{v(f^i(x))}{(f^{i+1})'(x)}\right| > 0$ has the desired property. We shall build a decreasing sequence of closed sets

$$A \supset \mathcal{K}_n \supset \mathcal{K}_{n+1}\,,$$

where each connected component of $\mathcal{K}_n$ is a closed interval with positive length, and a sequence of functions

$$\Gamma_{n+1} : \{A \mid \text{ connected comp. of } \mathcal{F}_n\} \to \mathcal{P}(\{1, \dots, n+i_0\}),$$

where $\mathcal{P}(\{1, \dots, n\})$ stands for the family of all subsets of $\{1, \dots, n+i_0\}$, with the following properties:

i. If $C$ is a connected component of $\mathcal{K}_n$ and $x \in C$, then $|\frac{v(f^j(x))}{(f^{j+1})'(x)}| \geq 2$ for every $j \in \Gamma_n(C)$.
ii. If $C$ is a connected component of $\mathcal{K}_n$, then $f^n$ is a diffeomorphism on $C$.
iii. If $C_{n+1} \subset C_n$ are connected components of $KK_n$ and $\mathcal{K}_{n+1}$ respectively, then $\Gamma_n(C_n) \subset \Gamma_{n+1}(C_{n+1})$.
iv. If $C_n$ is a connected component of $\mathcal{K}_n$, then there exists $m > n$ such that $\mathcal{K}_m$ has at least two connected components contained in $C_n$.
v. If $x \in \cap_n C_n$, where $C_n$ is a connected component of $\mathcal{K}_n$, then $\{x\} = \cap_n C_n$ and $\lim_n \#\Gamma_n(C_n) = \infty$.

Note that (iv) and (v) imply that $\cap_n \mathcal{K}_n$ is a Cantor set. If $x \in \cap_n C_n$ then

$$|\frac{v(f^j(x))}{(f^{j+1})'(x)}| \geq 2$$

holds for every $j \in \cup_n \Gamma_n(C_n)$. Due to (v), there are infinitely many $j$'s. Denote

$$\mathcal{O}(c) = \{x \in I \mid \ f^i(x) = c, \text{ for some } i \geq 0\}.$$

Let $\mathcal{K}_0 \subset A$ be a closed interval $[a, b]$, $a \neq b$, with $a, b \notin \mathcal{O}(c)$, and $\Gamma_0(\mathcal{K}_0) = \emptyset$. Suppose that we have defined $\mathcal{K}_n$. Let $C$ be a connected component of $\mathcal{K}_n$. If $c \notin C$ then $C$ is the unique connected component of $\mathcal{K}_{n+1}$ which intersects $C$ and $\Gamma_{n+1}(C) = \Gamma_n(C)$. Otherwise, let $x \in C$ be such that $f^n(x) = c$. Since $c$ is the critical point we have $(f^{n+i_0+1})'(x) = 0$. Moreover $v(f^{n+i_0}(x)) = v(f^{i_0}(x)) \neq 0$, so

$$\lim_{y \to x} |\frac{v(f^{n+i_0}(y))}{(f^{n+i_0+1})'(y)}| = \infty.$$

Let $\epsilon > 0$ be such that if $0 < |y - x| \leq \epsilon$ then $y \in C$ and

$$|\frac{v(f^{n+i_0}(y))}{(f^{n+i_0+1})'(y)}| \geq 2.$$

Choose two closed disjoint intervals $J_1$ and $J_2$ with positive lengths, such that $J_1 \cup J_2 \subset (x - \epsilon, x + \epsilon) \setminus \{x\}$ and $\mathcal{O}(c) \cap \partial(J_1 \cup J_2) = \emptyset$. Then $J_1$ and $J_2$ will be the unique connected components of $\mathcal{K}_n$ that intercept $C$ and $\Gamma_{n+1}(J_1) = \Gamma_{n+1}(J_2) = \Gamma_n(C) \cup \{n + i_0\}$. Note that in this case $\Gamma_{n+1}(J_1) = \Gamma_{n+1}(J_2) > \Gamma_n(C)$.

Properties (i)–(iii) follow from the definition of $\mathcal{K}_n$. To show (iv) and (v), consider $C_\infty = \cap_n C_n$, where the $C_n$ are the connected components of $\mathcal{K}_n$. The set $C_\infty$ is either a closed interval of positive length or $\{x\}$. In the first case, in particular we have that $f^n$ is a diffeomorphism on $C_\infty$, for every $n$. This is not possible, since $f$ has neither wandering intervals nor periodic attractors. Furthermore, note that if $\lim_n \#\Gamma_n(C_n) < \infty$, then there exists $n_0$ such that $\Gamma_n(C_n) = \Gamma_{n_0}(C_{n_0})$ for every $n \geq n_0$, so by the construction of $\mathcal{K}_i$ this occurs only if $f^i$ is a diffeomorphism on $C_{n_0}$ for every $i$, which is not possible, as we saw above. The proof of (iv) is similar. If (iv) does not hold for certain $C_{n_0}$, then by the construction of $\mathcal{K}_i$ we have that $f^i$ is a diffeomorphism on $C_{n_0}$ for every $i$, which contradicts the non-existence of wandering intervals and periodic attractors. □

# 4. Transfer operators $\widehat{\mathcal{L}}$ and their spectra

In this section, we study a transfer operator associated to a a Collet-Eckmann $S$-unimodal map $f$ satisfying the Benedicks-Carleson condition. More precisely, in Subsection 4.1, we introduce a Banach space $\mathcal{B} = \mathcal{B}^1$ of smooth ($C^1$) functions (see Definition 4.3) on the tower $\hat{I}$ defined in the previous section, maps $\Pi : \mathcal{B} \to L^1(I))$ (see (72)), as well as a transfer operator $\widehat{\mathcal{L}}$ acting on $\mathcal{B}$ (Definition 4.8). We shall prove that $\widehat{\mathcal{L}}$ has essential spectral radius strictly smaller than 1 (Proposition 4.9), and then (Proposition 4.10) that 1 is a simple eigenvalue, and that the fixed point $\hat{\phi}$ of $\widehat{\mathcal{L}}$ is such that $\Pi(\hat{\phi})$ is the invariant density of $f$. In Subsection 4.2, we present results of truncated versions of $\widehat{\mathcal{L}}$, acting on finite parts of the tower.

Although the methods in this section are inspired from [15], we would like to point out here nontrivial modifications were needed, in view of proving Theorem 2.13. (See Remark 4.4 below for the comparison with [50].) First, our Banach spaces $\mathcal{B}$ are not exactly the same as the space $\widehat{\mathcal{B}}$ used in [15]: the functions in $\mathcal{B}$ are smooth and locally supported at each level. Also, the transfer operator we use here is slightly different from the one in [15]: First, and this is the most original ingredient, we introduce a smooth cutoff function $\xi_k$ at each level on which there exist points which "fall" to the ground level; second, we do not compose with the dynamics until we fall (this strategy was used by L.S. Young in [59], [60]). These are the main new ideas in the present section, and they allow to limit the effect of the discontinuities and square root singularities (called "spikes" in [50]) to the maps $\Pi$ and $\Pi_t$ (see Step 3 in the proof of Theorem 2.13). See also the comments after Definition 4.8.

## 4.1. Spectral gap for a transfer operator $\widehat{\mathcal{L}}$ associated to the tower map.

Let $f$ be a $(\lambda_c, H_0)$-Collet-Eckmann $S$-unimodal map, with a non preperiodic critical point (the preperiodic case is easier and left to the reader) satisfying the Benedicks-Carleson condition (3). (We will need to strenghen the condition slightly later on, see (69) and (86), and also (112).) We consider the tower map $\hat{f} : \hat{I} \to \hat{I}$ from Section 3.1, for some small enough fixed $\delta$. We shall not require the fact that the Lyapunov exponents of $\pm\delta$, or of the endpoints $a_k$ and $b_k$ of the tower levels $B_k$, are positive, and we *remove* [14] this assumption. (The positivity of the exponents was useful only when proving that $\alpha_t$ is continuous, and one can use a different tower for $f_t$ when studying $\alpha$ and when considering the transfer operator.) In particular, we may take for all $k \geq H_0$

$$B_k = [c_k - e^{-\beta_1 k}, c_k + e^{-\beta_1 k}] \, .$$

The following refinement of the estimates in Subsection 3.2 will play an important part in our argument (see Proposition 4.9, and – in view of (72) – Proposition 4.10, see also Step 2 in the proof of Theorem 2.13):

**Lemma 4.1.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the Benedicks-Carleson condition* (3)*, and with a non-preperiodic critical point. Then there exists $C$ so that for any $k \geq 1$, we have*

$$\frac{1}{|(f^k)'(f_\pm^{-k}(x))|} \leq C \frac{1}{|(f^{k-1})'(c_1)|^{1/2}\sqrt{|x-c_k|}} \, , \quad \forall x \in \pi(E_k \cap \hat{f}^k(E_0)) \, . \tag{57}$$

[14] This additional freedom will also be used in Subsection 5.2.

*In addition, recalling the intervals $I_k$ defined in* (31)*, we have for any $k \ge H(\delta)$*

$$\sup_{x\in f^k(I_k)}\left|\partial_x \frac{1}{|(f^k)'(f_\pm^{-k}(x))|}\right| \le C\frac{e^{3\gamma k}}{|(f^{k-1})'(c_1)|^{1/2}}\,, \tag{58}$$

*and*

$$\sup_{x\in f^k(I_k)}\left|\partial_x^2 \frac{1}{|(f^k)'(f_\pm^{-k}(x))|}\right| \le C\frac{e^{5\gamma k}}{|(f^{k-1})'(c_1)|^{1/2}}\,. \tag{59}$$

*Proof.* We consider the case $\varsigma = +$, the other case is identical. Let us first show (57). Putting $z = f_+^{-(k-1)}(x)$, we decompose

$$|(f^k)'(f_+^{-k}(x))| = |(f^{k-1})'(z)||f'(f_+^{-k}(x))|\,.$$

By Lemma 3.3, the first factor can be estimated by [15]

$$|(f^{k-1})'(z)| \ge C^{-1}|(f^{k-1})'(c_1)|\,. \tag{60}$$

For the second factor, we have

$$|f'(f_+^{-k}(x))| \ge C^{-1}|f_+^{-k}(x)|\,.$$

Put $w = f(f_+^{-k}(x))$. Then, Lemma 3.3 and the mean value theorem imply

$$|w-c_1| \ge \frac{|x-c_k|}{C|(f^{k-1})'(c_1)|}\,.$$

Next, noting that $w \in \pi(E_1\cap \hat f(E_0))$, we have

$$|f_+^{-k}(x)| = |f_+^{-1}(w)| \ge C^{-1}\sqrt{|w-c_1|}$$

(Just use that $f(y) = c_1 + f''(c)y^2 + f'''(\tilde y)y^3$ for some $|\tilde y|\le \delta$, if $|y|\le\delta$.) The three previous inequalities give

$$|f'(f_+^{-k}(x))| \ge C^{-1}\frac{\sqrt{|x-c_k|}}{|(f^{k-1})'(c_1)|^{1/2}}\,. \tag{61}$$

Putting together (60) and (61), we get (57).

To prove (58), we first note that there is $C\ge 1$ so that

$$|f'(f^j(y))| \ge C^{-1}e^{-\gamma j}\,,\ \forall y\in I_k\,,\ \forall 1\le j\le k\,. \tag{62}$$

Indeed, $|f^j(y)-c_j| \le e^{-\beta_1 j}$ and $|c_j - c| \ge e^{-\gamma k}$ for $j\ge H_0$, with $\beta_1 \ge 3\gamma/2$, and we assumed that $f$ is $C^2$ with $f'(c)=0$ and $f''(c)\neq 0$. Next, Lemma 3.3 and Lemma 3.7 give $C>0$ so that

$$\sup_{y\in I_k}\frac{1}{|(f^{k-j})'(f^j(y))|} \le C\sum_{\ell=1}^{\infty}\frac{1}{|(f^\ell)'(f^{j-1}(c_1))|} \le Ce^{\gamma j}\,,\ \forall 1\le j\le k-1\,. \tag{63}$$

Then, (36) from Lemma 3.4 and Lemma 3.3 give

$$\sup_{y\in I_k}\frac{1}{|(f^m)'(y)|} \le C_2^{-1}e^{\gamma k}\frac{|(f^{k-1})'(c_1)|^{1/2}}{|(f^{m-1})'(c_1)|}\,,\quad \forall 1\le m\le k\,. \tag{64}$$

[15]The constant $C$ depends on $H_0$, by Lemma 5.8 it will be uniform within our families $f_t$.

Applying (62), (63), and (64) for $m = k$ and $m = 1$, we get $C > 0$ so that

$$\sup_{y \in I_k} \partial_y \frac{1}{|(f^k)'(y)|} \le \sup_{y \in I_k} \sum_{j=0}^{k-1} \frac{|f''(f^j(y))|}{|(f^k)'(y)|} \frac{|(f^j)'(y)|}{|f'(f^j(y))|}$$

$$= \sup_{y \in I_k} \sum_{j=0}^{k-1} \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))|} \frac{1}{|f'(f^j(y))|} \le C e^{2\gamma k} \,. \tag{65}$$

Finally,

$$\partial_x \frac{1}{|(f^k)'(f_+^{-k}(x))|} = \frac{1}{|(f^k)'(f_+^{-k}(x))|} \cdot \partial_y \frac{1}{|(f^k)'(y)|} \,. \tag{66}$$

The first factor in (66) is bounded by (64) for $m = k$, the second by (65), so that we have proved (58).

To prove (59), we start from the decomposition (66), and we deduce from (65) that for any $x \in f^k(I_k)$, setting $y = f_+^{-k}(x)$,

$$\partial_x^2 \frac{1}{|(f^k)'(f_+^{-k}(x))|} \le C e^{2\gamma k} \partial_x \frac{1}{|(f^k)'(f_+^{-k}(x))|}$$

$$+ \frac{1}{|(f^k)'(f_+^{-k}(x))|} \cdot \sum_{j=0}^{k-1} \partial_x \left( \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \right) . \tag{67}$$

By (58), the first term in the right-hand-side is bounded by $C e^{5\gamma k} |(f^{k-1})'(c_1)|^{-1/2}$. For the second term, we have, for $0 \le j \le k-1$,

$$\partial_x \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \le \frac{1}{|(f^k)'(f_+^{-k}(x))|} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \,.$$

Since $f$ is $C^3$, the Leibniz formula gives for $0 \le j \le k-1$,

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|}$$

$$\le \frac{1}{|(f^k)'(f_+^{-k}(x))|} \frac{1}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|} \cdot \big[ |f'''(f^j(y))||(f^j)'(y)|$$

$$+ |f''(f^j(y))| \Big[ \frac{|f''(f^j(y))||(f^j)'(y)|}{|f'(f^j(y))|} + \sum_{\ell=j}^{k-1} \frac{|f''(f^\ell(y))||(f^\ell)'(y)|}{|f'(f^\ell(y))|} \Big] \big] \tag{68}$$

$$\le \frac{C}{|(f^{k-j})'(f^j(y))|} \cdot \Big[ \frac{1}{|(f^{k-j})'(f^j(y))|} \Big( \frac{1}{|f'(f^j(y))|} + \frac{1}{|f'(f^j(y))|^2} \Big)$$

$$+ \frac{1}{|(f^{k-j})'(f^j(y))|} \sum_{\ell=j}^{k-1} \frac{|(f^\ell)'(y)|}{|f'(f^\ell(y))|} \Big] \,.$$

If $j \ge 1$, we may apply (62). Then, (68), together with (64) for $m = k - \ell$ and (63) imply that

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|} \sum_{j=1}^{k-1} \partial_y \frac{|f''(f^j(y))|}{|(f^{k-j})'(f^j(y))||f'(f^j(y))|}$$

$$\le C e^{\gamma k} \cdot \big[ e^{2\gamma k} + e^{3\gamma k} + e^{2\gamma k} |(f^{k-1})'(c_1)|^{1/2} \big] \,.$$

If $j = 0$, then (68) together with (64) for $m = k$ imply (distinguish between $\ell = 0$ and $\ell \geq 1$)

$$\frac{1}{|(f^k)'(f_+^{-k}(x))|}\partial_y \frac{|f''(y)|}{|(f^k)'(y)||f'(y)|} \leq Ce^{3\gamma k}(e^{\gamma k} + \frac{1}{|(f^k)'(c_1)|^{1/2}} + |(f^{k-1})'(c_1)|^{1/2})\,.$$

Putting the two above inequalities together with (67) and (64) for $m = k$, we get (59). $\square$

In view of the definition of our Banach space $\mathcal{B}$, we need further preparations. First, we assume from now on that $f$ satisfies the following strengthened Benedicks-Carleson condition:

$$\exists 0 < \gamma < \frac{\log(\lambda_c)}{8} \text{ so that } |f^k(c) - c| \geq e^{-\gamma k}\,, \quad \forall k \geq H_0\,. \tag{69}$$

*Remark* 4.2. In [15] we needed to assume that $f$ was $C^4$ or symmetric (see the comments before [15, Lemma 5]) because of the more complicated form of the cocycle used in the transfer operator there.

In view of (69) we may choose $\lambda$ so that

$$1 < \lambda < e^{\gamma}\,, \text{ and } e^{4\gamma}\lambda < \sqrt{\lambda_c}\,. \tag{70}$$

**Definition 4.3** (The main Banach space $\mathcal{B} = \mathcal{B}^1$)**.** Let $\mathcal{B} = \mathcal{B}^1$ be the space of sequences $\hat\psi = (\psi_k : I \to \mathbb{C},\, k \in \mathbb{Z}_+)$, so that each $\psi_k$ is $C^1$ and, in addition,

$$\mathrm{supp}(\psi_0) \subset (-1, 1)\,, \qquad \mathrm{supp}(\psi_k) \subset [-\delta, \delta]\,, \forall 1 \leq k \leq \max(2, H_0)\,,$$

$$\mathrm{supp}(\psi_k) \subset \cap_{H_0 \leq j \leq k}(f_+^{-j}(B_j) \cup f_-^{-j}(B_j))\,, \forall k > \max(2, H_0)\,, \tag{71}$$

endowed with the norm

$$\|\hat\psi\|_{\mathcal{B}} = \sup_{k\geq 0}\|\psi_k\|_{C^0} + \sup_{k\geq 0} e^{-2\gamma k}\|\psi_k'\|_{C^0}\,.$$

We sometimes write $\hat\psi(x, k)$ instead of $\psi_k(x)$.

*Remark* 4.4. In contradistinction to the piecewise expanding case treated in [11], or to the Misiurewicz *and analytic* case studied in [50], the postcritical data is not given here by a finite set of complex numbers for each $c_k$ with $k \geq 1$: We need a full "germ" $\psi_k$, which is supported in a neighbourhood of $c$. Since we will later consider $(\psi_k \chi_k \circ f_\pm^{-k})$ (see (72)), we can view $\psi_k$ as the contribution in a one-sided neighbourhood of $c_k$. In fact, since the space of sequences in $\mathcal{B}$ so that $\psi_k = \lambda^{-1}\xi_{k-1}\psi_{k-1}$ for all $k \geq 1$ (see below for the definition of $\xi_k$ and the transfer operator) is a $\widehat{\mathcal{L}}$-invariant Banach subspace, what counts is the germ at the critical point $c$. (We could define the transfer operator as acting on the space of $C^1$ functions $\psi_0$ on $I$, and we would get the same results.) For the sake of comparison with previous papers, we have kept the current tower description.

**Definition 4.5** (The projection $\Pi$)**.** Define $\Pi(\hat\psi)$ for $\hat\psi \in \mathcal{B}$ by

$$\Pi(\hat\psi)(x) = \sum_{k\geq 0, \varsigma\in\{+,-\}} \frac{\lambda^k}{|(f^k)'(f_\varsigma^{-k}(x))|}\psi_k(f_\varsigma^{-k}(x))\chi_k(x)\,. \tag{72}$$

(We set $\chi_0 \equiv 1$. When the meaning is clear, we sometimes omit the factor $\chi_k$ in the formula.)

By (57) in Lemma 4.1 and our construction [16], setting $[c_k, d_k] = \pi(E_k \cap \hat{f}^k(E_0))$,

$$\int_{[c_k,d_k]} \frac{\lambda^k}{|(f^k)' f_+^{-k}(x)|} |\psi_k(f_+^{-k}(x))|\, dx \le C\lambda^k \lambda_c^{-k/2} \sqrt{|d_k - c_k|} \sup |\psi_k|$$
$$\le C(\lambda \lambda_c^{-1/2} e^{-3\gamma/4})^k \sup |\psi_k| \, .$$

Since $\lambda < \sqrt{\lambda_c}$, the above bound, and its analogue for the branch $f_-^{-k}$, imply that $\Pi(\hat{\psi}) \in L^1(I)$.

As usual, we will need weak norms in order to exploit Lasota-Yorke inequalities (there will be two of them, of $C^0$ and $L^1$ type, as is often the case in such settings, and we shall pair a $BV$ norm to the $L^1$ norm, see e.g. Lemma 4.11). We will also need a stronger norm, of $C^2$ type, in order to exploit the higher smoothness properties of the fixed point of the transfer operator. For the sake of easy reference, we list here the other norms (an additional Sobolev norm [17] will be defined in Step 1 of the proof of Theorem 2.13 in Section 6):

**Definition 4.6** (Spaces $\mathcal{B}^0$, $\mathcal{B}^2$, $\mathcal{B}^{BV}$, $\mathcal{B}^{L^1}$)**.** Let $\mathcal{B}^0 = \mathcal{B}^0$ be the space of sequences $\hat{\psi} = (\psi_k : I \to \mathbb{C},\, k \in \mathbb{Z}_+)$ of continuous functions $\psi_k$ satisfying (71), endowed with the norm

$$\|\hat{\psi}\|_{\mathcal{B}^0} = |\hat{\psi}|_{C^0} = \sup_{k \ge 0} \|\psi_k\|_{C^0} \, . \tag{73}$$

Let $\mathcal{B}^2$ be the space of sequences $\hat{\psi}$ of $C^2$ functions $\psi_k$ satisfying (71), with norm

$$\|\hat{\psi}\|_{\mathcal{B}^2} = \sum_{j=0,1,2} \sup_{k \ge 0} e^{-2j\gamma k} \|\psi_k^{(j)}\|_{C^0} \, .$$

Let $\mathcal{B}^{BV}$ be the space of sequences $\hat{\psi}$ of regular functions of bounded variation $\psi_k$ satisfying (71), with the norm

$$\|\hat{\psi}\|_{\mathcal{B}^{BV}} = \sup_{k \ge 0} \operatorname{var} \psi_k \, .$$

Let $\mathcal{B}^{L^1}$ be the space of sequences $\hat{\psi}$ of functions $\psi_k \in L^1(I)$ satisfying (71), endowed with the norm

$$|\hat{\psi}|_{L^1} = \sup_{k \ge 0} \|\psi_k\|_{L^1(I)} \, . \tag{74}$$

Clearly

$$|\cdot|_{L^1} \le |\cdot|_{C^0} \le \|\cdot\|_{BV} \le \|\cdot\|_{\mathcal{B}} \le \|\cdot\|_{\mathcal{B}^2} \, .$$

In order to define the transfer operator $\widehat{\mathcal{L}}$, we next introduce smooth cutoff functions $\xi_k$. Recall the constants $3\gamma/2 < \beta_1 < \beta_2 < 2\gamma$ from (26) and (27).

**Definition 4.7** (The cutoff functions $\xi_k$)**.** For each $k \ge 0$, let $\xi_k : I \to [0,1]$ be a $C^\infty$ function with

$$\operatorname{supp}(\xi_0) = [-\delta, \delta] \, , \qquad \xi_0|_{[-\frac{\delta}{2}, \frac{\delta}{2}]} \equiv 1 \, ,$$
$$k \ge 1 \, : \begin{cases} \operatorname{supp}(\xi_k) = f_+^{-(k+1)}(B_{k+1}) \cup f_-^{-(k+1)}(B_{k+1}) \, , \\ \xi_k|_{f_+^{-(k+1)}[c_{k+1} - e^{-\beta_2(k+1)}, c_{k+1} + e^{-\beta_2(k+1)}]} \equiv 1 \, . \end{cases}$$

---

[16]Uniformity of $C$ within our families is important here, it will follow from Lemma 5.8.

[17]It is conceivable that one can work exclusively with Banach spaces based on Sobolev spaces $H_p^r$ for $r = 0, 1, 2$, avoiding $BV$ and $C^k$ spaces.

We require in addition that for some $C > 0$,

$$\sup |\xi_k'| \le C e^{2\gamma k}\,,\ \sup |\xi_k''| \le C e^{4\gamma k}\,,\ \forall k\,. \tag{75}$$

Note that $\xi_k(y) > 0$ if and only if $\hat f(f^k(y), k) \in B_{k+1} \times (k+1)$, and $\xi_k(y) = 1$ implies that $\pi \hat f(f^k(y), k) \in [c_{k+1} - e^{-\beta_2(k+1)}, c_{k+1} + e^{-\beta_2(k+1)}]$. The low levels ($k \le H_0$) will be taken care of by the condition $\mathrm{supp}(\psi_k) \subset [-\delta, \delta]$.

**Definition 4.8** (Transfer operator)**.** The transfer operator $\widehat{\mathcal{L}}$ is defined on on $\mathcal{B}$ by

$$(\widehat{\mathcal{L}}\widehat{\psi})(x,k) = \begin{cases} \frac{\xi_{k-1}(x)}{\lambda} \cdot \hat\psi(x, k-1) & k \ge 1\,,\\ \sum_{j\ge 0, \varsigma \in \{+,-\}} \frac{\lambda^j (1 - \xi_j(f_\varsigma^{-(j+1)}(x)))}{|(f^{j+1})'(f_\varsigma^{-(j+1)}(x))|} \cdot \hat\psi(f_\varsigma^{-(j+1)}(x), j) & k = 0\,. \end{cases} \tag{76}$$

Note that some $j$-terms in the sum for $(\widehat{\mathcal{L}}\widehat{\psi})(x, 0)$ vanish, in particular, for all $1 \le j < H_0$ because of our choice of small $\delta$.

As already mentioned in the beginning of this section, there are two differences between the present definition and the one used in [15]. First, $\widehat{\mathcal{L}}$ does not act via the dynamics when climbing the tower, only when falling. (This strategy was used also, e.g., in [59] and [60].) Secondly, if $0 < \xi_j(y) < 1$, then $y$ will contribute to both $(\widehat{\mathcal{L}}\widehat{\psi})(y, j+1)$ and $(\widehat{\mathcal{L}}\widehat{\psi})(f^{j+1}(y), 0)$. In other words, the transfer operator just defined is associated to a multivalued (probabilistic-type) tower dynamics. For this multivalued dynamics, some points may fall from the tower a little earlier than they would for $\hat f$. However, the conditions on the functions $\xi_k$s guarantee that they do not fall *too* early. More precisely, if we define "fuzzy" analogues of the intervals $I_k$ from (31) as follows

$$\widetilde{I}_k := \{x \in I \mid \xi_k(x) < 1\,,\ \xi_j(x) > 0\,, \forall 0 \le j < k\}\,, \tag{77}$$

then we can replace $I_k$ by $\widetilde{I}_k$ in the previous estimates, in particular in Lemma 4.1. (Note however that the intervals $\widetilde{I}_k$ are not pairwise disjoint.) Indeed, just observe that if a point "falls" according to our fuzzy dynamics, it would have fallen for some choice of intervals $\widetilde{B}_k$ so that

$$[c_k - e^{-\beta_2 k}, c_k + e^{-\beta_2 k}] \subset \widetilde{B}_k \subset B_k\,.$$

These two modifications allow us to work with the Banach space $\mathcal{B}$ of $C^1$ functions (as opposed to the BV functions in [15], where the jump singularities corresponding to the edges of the levels are an artefact of the construction), and will simplify our spectral perturbation argument in Section 6. (The argument in [11] could be modified similarly.)

Before we continue, let us note that if we introduce the ordinary (Perron-Frobenius) transfer operator

$$\mathcal{L} : L^1(I) \to L^1(I)\,, \qquad \mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|f'(y)|}\,,$$

then one easily shows that

$$\mathcal{L}(\Pi(\hat\psi)) = \Pi(\widehat{\mathcal{L}}(\hat\psi))\,,\ \forall \hat\psi \in \mathcal{B}^{L^1}\,. \tag{78}$$

(just decompose $1 = (\xi_k + 1) - \xi_k$, see (87) below for a similar computation). In particular, if $\widehat{\mathcal{L}}(\hat\phi) = \hat\phi$ then $\mathcal{L}(\Pi(\hat\phi)) = \Pi(\hat\phi)$.

**Proposition 4.9** (Essential spectral radius of $\widehat{\mathcal{L}}$)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the strengthened Benedicks-Carleson condition* (69)*, with a non-preperiodic critical point. Let $\lambda$ satisfy* (70)*. Then the operator $\widehat{\mathcal{L}}$ is bounded on $\mathcal{B} = \mathcal{B}$. Let $\rho$ satisfy* (25) *and let $\sigma > 1$ be the constant from Lemma 3.1, then for any*

$$1 < \Theta < \min\left(\frac{\lambda_c^{1/2}}{e^{4\gamma}\lambda}, \lambda, \sigma, \rho\right) \tag{79}$$

*the essential spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}$ is bounded by $\Theta^{-1}$.*

In fact, we could replace $\sqrt{\lambda_c}$ by $\lambda_c$ in the right-hand-side of (79). This is not very useful in view of $\rho < \sqrt{\lambda_c}$. In addition, the present upper bound for $\Theta$ will also work for Lemma 4.11, see (154).

*Proof.* Let $c(\delta)$ be the constant from Lemma 3.1.

Setting $\widehat{\mathcal{L}}(\hat{\psi}) = (\tilde{\psi}_k)$, our assumptions on $\psi_j$ and $\xi_j$ ensure that each $\tilde{\psi}_k$ for $k \geq 1$ is $C^1$, and supported in the desired interval, and that $\tilde{\psi}_0$ is supported in the desired interval.

Next, it easily follows from the choice of $\xi_k$ (in particular (75)) that for all $k \geq 1$

$$\begin{aligned} &\|\tilde{\psi}_k\|_{C^0} \leq \frac{1}{\lambda}\|\psi_{k-1}\|_{C^0}\,, \\ &\|\tilde{\psi}_k'\|_{C^0} \leq \frac{1}{\lambda}\|\psi_{k-1}'\|_{C^0} + \frac{C}{\lambda}\max(e^{2\gamma k}, \frac{1}{c(\delta)})\|\psi_{k-1}\|_{C^0}\,. \end{aligned} \tag{80}$$

If $\psi_k(y) > 0$ then $|f^j(y) - c_j| \leq e^{-\beta_1 j}$ for all $j \leq k$. If $\xi_k(f_\pm^{-(k+1)}(x)) < 1$ then $|x - c_{k+1}| \geq e^{-\beta_2(k+1)}$. Thus, recalling (35) from Lemma 3.4 and (58) from Lemma 4.1, and using again (75), we have that $\tilde{\psi}_0$ is $C^1$ and

$$\begin{aligned} \|\tilde{\psi}_0\|_{C^0} &\leq 2c(\delta)^{-1}\|\psi_0\|_{C^0} + C\sum_{k\geq H_0}\frac{\lambda^k e^{\gamma k}}{|(f^{k+1})'(c_1)|^{1/2}}\|\psi_k\|_{C^0} \\ \|\tilde{\psi}_0'\|_{C^0} &\leq C\Big(c(\delta)^{-1}\|\psi_0'\|_{C^0} + c(\delta)^{-2}\|\psi_0\|_{C^0} \\ &\qquad + \sum_{k\geq H_0}\Big[\frac{\lambda^k e^{2\gamma k}}{|(f^{k+1})'(c_1)|}\|\psi_k'\|_{C^0} + \frac{\lambda^k e^{4\gamma k}}{|(f^{k+1})'(c_1)|^{1/2}}\|\psi_k\|_{C^0}\Big]\Big)\,. \end{aligned} \tag{81}$$

In view of (70), we have proved that $\widehat{\mathcal{L}}$ is bounded on $\mathcal{B}$, with $\|\widehat{\mathcal{L}}\|_{\mathcal{B}} \leq C(\delta)$.

To estimate the essential spectral radius of $\widehat{\mathcal{L}}$ acting on $\mathcal{B}$, we shall use that if $\widehat{\mathcal{L}}$ is a bounded operator on a Banach space $\mathcal{B}$, and $\mathcal{K}$ is a compact operator on $\mathcal{B}$, then the essential spectral radii of $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}} - \mathcal{K}$ coincide (see e.g. [22] or [25, Theorem IV.5.35]), and in particular the essential spectral radius of $\widehat{\mathcal{L}}$ is not larger than the spectral radius of $\widehat{\mathcal{L}} - \mathcal{K}$.

We introduce for each $M \geq 0$ the *truncation operator* $\mathcal{T}_M$ defined by

$$\mathcal{T}_M(\hat{\psi})_k = \begin{cases} \psi_k & k \leq M \\ 0 & k > M\,. \end{cases} \tag{82}$$

By definition $\mathcal{T}_M$ is a bounded operator on $\mathcal{B}$, with $\|\mathcal{T}_M\|_{\mathcal{B}} \leq 1$ for any $M$. Also, it is easy to see that for any $\tilde{\theta} > 1$ there exists $C$ so that for all $N$

$$\|(\mathrm{id} - \mathcal{T}_M)\widehat{\mathcal{L}}^N\| \leq C(\tilde{\theta}/\lambda)^N\,, \quad \forall M > N\,.$$

Therefore, we only need to study $\mathcal{T}_M \widehat{\mathcal{L}}^N$ for some $M > N$.

The Leibniz formula gives

$$((\mathcal{T}_M \widehat{\mathcal{L}}^N)(\hat\psi))' = \mathcal{T}_M \mathcal{Q}_N(\hat\psi) + \mathcal{T}_M \mathcal{M}_N(\hat\psi)\,, \tag{83}$$

where $\mathcal{T}_M \mathcal{Q}_N$ does not involve any derivatives of the $\psi_k$, and can be easily shown to be bounded on $\mathcal{B}$ by using (75), (58), and Lemma 3.5. Since $\mathcal{T}_M \mathcal{Q}_N$ lives on a bounded part of the tower, Arzelà-Ascoli implies that it is compact from $\mathcal{B}$ to $\mathcal{B}^0$.

Finally, $\mathcal{M}_N$ is the $N$th iterate of the operator $\mathcal{M}$ defined by

$$(\mathcal{M}\hat\psi)_k = \frac{\xi_{k-1}}{\lambda}\psi'_{k-1}\,,\ k \ge 1\,,$$

$$(\mathcal{M}\hat\psi)_0(x) = \sum_{k\ge H_0}\sum_{\varsigma\in\{+,-\}} \frac{\lambda^k(1-\xi_k(f_\varsigma^{-(k+1)}(x)))}{|(f^{k+1})'(f_\varsigma^{-(k+1)}(x))|}\frac{\psi'_k(f_\varsigma^{-(k+1)}(x))}{(f^{k+1})'(f_\varsigma^{-(k+1)}(x))}\,.$$

Keeping in mind (80) and (81), as well as the properties of the supports of the $\xi_j$ and $\psi_j$, the conditions (70) on $\lambda$, and exploiting Lemma 3.5 to see that we get at most one factor $c(\delta)^{-2}$, it is not difficult [18] to show that

$$\|\mathcal{M}_N\hat\psi\|_{\mathcal{B}^0}^{1/N} \le \frac{C}{c(\delta)^{2/N}}\max(\frac{1}{\lambda}, \frac{e^{4\gamma}\lambda}{\lambda_c}, \frac{1}{\sigma}, \frac{1}{\rho})\|\hat\psi\|_{\mathcal{B}}^{1/N}\,, \tag{84}$$

finishing the proof of Proposition 4.9. □

To state further spectral properties of $\widehat{\mathcal{L}}$, set

$$w(x,k) = \begin{cases}\lambda^k & (x,k)\in \hat f^k(E_0)\\ 0 & \text{otherwise}\end{cases} \qquad k\ge 0\,, \tag{85}$$

and define $\nu$ to be the nonnegative measure on $\widehat{I}$ whose density with respect to Lebesgue is $w(x,k)$.

**Proposition 4.10** (Maximal eigenvalue of $\widehat{\mathcal{L}}$ and its dual)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the strengthened Benedicks-Carleson condition* (69)*, with a non-preperiodic critical point.*

*The spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}$ is equal to one, and $1$ is a simple eigenvalue of $\widehat{\mathcal{L}}$, for a nonnegative eigenvector $\hat\phi$. The fixed point of the dual of $\widehat{\mathcal{L}}$ is $\nu$. If $\nu(\hat\phi) = 1$, then $\phi := \Pi(\hat\phi)$ is the density of the unique absolutely continuous $f$-invariant probability measure. Finally, if the Benedicks-Carleson condition* (69) *is strengthened to*

$$0 < \gamma < \frac{\log\lambda_c}{14}\,, \tag{86}$$

*then one can choose the parameter $\lambda$ so that $\hat\phi \in \mathcal{B}^2$.*

*Proof.* Observe that $\sum_k \int_{B_k} \hat\psi(x,k)w(x,k)\,dx$ is finite if $\hat\psi \in \mathcal{B} \subset \mathcal{B}^0$ (just recall that $|B_k| \le 2e^{-\beta_1 k} \le 2e^{-3\gamma k/2}$ and use the bound $\lambda < e^{3\gamma/2}$ from (70)). So $\nu$ is an element of the dual of $\mathcal{B}$ (and also of $\mathcal{B}^0$ and $\mathcal{B}^{BV}$).

[18] This is much simpler than the proof of [15, Variation lemma], since we need only consider the supremum norm. A further simplification may perhaps be obtained by considering the operator dual to $\widehat{\mathcal{L}}$ acting on the dual space of $\mathcal{B}$, in the spirit of [24].

The fact that $\widehat{\mathcal{L}}^*(\nu) = \nu$ can easily be proved using the change of variables formula. Indeed,

(87)
$$\begin{aligned}\int \widehat{\mathcal{L}}(\hat\psi)\, d\nu &= \int_{B_0} \widehat{\mathcal{L}}(\hat\psi)(y,0)\, dy + \sum_{k\ge 0}\int_{B_{k+1}} \widehat{\mathcal{L}}(\hat\psi)(y,k+1)\, w(y,k+1)\, dy\\
&= \sum_{j\ge 0, \varsigma\in\{+,-\}} \int \frac{\lambda^j}{|(f^{j+1})' f_\varsigma^{-(j+1)}(y)|} \psi_j(f_\varsigma^{-(j+1)}(y))(1-\xi_j)(f_\varsigma^{-(j+1)}(y))\, dy\\
&\qquad\qquad + \sum_{k\ge 0}\int \frac{1}{\lambda}\psi_k(x)\xi_k(x)\, w(x,k+1)\, dx\\
&= \sum_{j\ge 0}\int \psi_j(x)\,(1-\xi_j)(x)\, w(x,j)\, dx + \sum_{k\ge 0}\int \psi_k(x)\xi_k(x)\, w(x,k)\, dx\,.\end{aligned}$$

By Proposition 4.9, the essential spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}$ is strictly smaller than one, so the fact that the spectral radius is equal to 1 and that 1 is a simple eigenvalue for a nonnegative eigenvector follow standard arguments (see e.g. [15, Corollaries 1, 2] or [57, Propositions 5.13, 5.14]). The normalisation $\nu\hat\phi = 1$ implies that $\int_I \phi\, dx = \int_I \Pi(\hat\phi)\, dx = 1$. Recalling (78), we get that $\mathcal{L}(\Pi(\hat\phi)) = \hat\phi$ so that $\Pi(\hat\phi) \in L^1$ is indeed the invariant density of $f$ (which is known to be unique and ergodic).

It only remains to show that, under a stronger Benedicks-Carleson condition, the fixed point $\hat\phi$ is in $\mathcal{B}^2$, i.e., that each $\phi_k \in C^2$, with appropriate bounds of the $C^2$ norms. For this, one can let $\widehat{\mathcal{L}}$ act on the Banach space $\mathcal{B}^2$, up to a suitable modification of the conditions $\lambda$ in (70). Indeed, in view of (86), one can exploit (in addition to the properties already used) the bound (59) and the condition on $\xi_k''$ in (75). Then, differentiating one more time in the proof of Proposition 4.9, one easily shows that the $\widehat{\mathcal{L}}$ is bounded on on $\mathcal{B}^2$ with essential spectral radius is strictly smaller than 1, and that 1 is still a simple eigenvalue on this space. Details are straightforward, although tedious, and left to the reader. (We do not claim that the factor 14 in (86) is optimal. In any case, we shall need to work with the TSR condition soon.) □

4.2. **Truncated transfer operators $\widehat{\mathcal{L}}_M$.** Let $\mathcal{B}$ be as defined in the previous subsection. Recalling the truncation operator $\mathcal{T}_M$ from (82), we consider the bounded operator defined on $\mathcal{B}$ by

$$\widehat{\mathcal{L}}_M = \mathcal{T}_M \widehat{\mathcal{L}} \mathcal{T}_M\,.$$

By using the results of Keller and Liverani [30], and by exploiting the weak norm $|\cdot|_{L^1}$, we shall prove the following result:

**Lemma 4.11** (Spectrum of the truncated operators)**.** *Let $f$ be an $S$-unimodal $(\lambda_c, H_0)$-Collet-Eckmann map satisfying the strengthened Benedicks-Carleson condition* (69)*, with a non-preperiodic critical point. Recall the condition* (79) *on* $\Theta$*.*

*The essential spectral radius of $\widehat{\mathcal{L}}_M$ acting on $\mathcal{B}$ is not larger than $\Theta^{-1} < 1$.*

*In addition, there exists $M_0 \ge 1$ so that for all $M \ge M_0$ the operator $\widehat{\mathcal{L}}_M$ has a real nonnegative maximal eigenfunction $\hat\phi_M$, for an eigenvalue $\kappa_M > \Theta$, the dual* *operator* $\widehat{\mathcal{L}}_M$ *has a nonnegative maximal eigenfunction* $\nu_M$*, and, setting*

$$\tau_M = e^{3\gamma M}|(f^M)'(c_1)|^{-1/2} < 1\,, \tag{88}$$

*there exist* $C > 0$ *and* $\eta > 0$ *so that, normalising by* $\nu(1) = \nu_M(1)$ *and* $\int \hat{\phi}_M \, d\nu_M = 1$*, we have*

$$|\hat{\phi} - \hat{\phi}_M|_{L^1} \le C\tau_M^\eta\,,\ \|\nu - \nu_M\|_{\mathcal{B}^*} \le C\tau_M^\eta\,,\ |\kappa_M - 1| \le C\tau_M^\eta\,. \tag{89}$$

*Finally,* $\sup_M \|\hat{\phi}_M\|_{\mathcal{B}} < \infty$*, and if* (86) *holds, then we may choose* $\lambda$ *(uniformly in* $M$*) so that* $\sup_M \|\hat{\phi}_M\|_{\mathcal{B}^2} < \infty$.

*Proof.* Recall the Banach norms $\|\hat{\psi}\|_{\mathcal{B}^{BV}}$ and $|\hat{\psi}|_{L^1}$ from Definition 4.6. Clearly, $|\hat{\psi}|_{L^1} \le C\|\hat{\psi}\|_{\mathcal{B}^{BV}} \le C^2\|\hat{\psi}\|_{\mathcal{B}}$ for all $\hat{\psi}$. It is easy to see that each $\mathcal{T}_M$ is also bounded for the norms $|\cdot|_{L^1}$ and $\|\cdot\|_{\mathcal{B}^{BV}}$.

The claim about the essential spectral radius can be obtained by going over the proof of Proposition 4.9 and checking that it applies to $\widehat{\mathcal{L}}_M$. The reader is invited to do this, and to check that we have the following uniform Lasota-Yorke estimates for $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_M$: there exist $C \ge 1$, $C_0 \ge 1$ so that for all $N$ and all $M$

$$\max(\|\widehat{\mathcal{L}}^N(\hat{\psi})\|_{\mathcal{B}^{BV}}, \|\widehat{\mathcal{L}}_M^N(\hat{\psi})\|_{\mathcal{B}^{BV}}) \le C\Theta^{-N}\|\hat{\psi}\|_{\mathcal{B}^{BV}} + CC_0^N|\hat{\psi}|_{L^1}\,. \tag{90}$$

To prove the above Lasota-Yorke estimates (which imply that the essential spectral radii of $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}_M$ on $\mathcal{B}^{BV}$ are $\le \Theta^{-1}$), it suffices to see that the analogue in the $BV$ setting of the compact term $\mathcal{Q}_N$ from the proof of Proposition 4.9 can be bounded by $CC_0^N|\hat{\psi}|_{L^1}$. This can be done by using Lemma 4.1 together with the fact that for any bounded $\varphi$ we have

$$|\int_{-1}^y \varphi\psi_k(x)\, dx| \le \sup|\varphi| \int |\psi_k|\, dx\,. \tag{91}$$

(See Appendix B for more details on the proof of (90).)

The bounds (89) for some $\eta > 0$ then follow from [30, Theorem 1, Corollary 1], combined with the fact that there exists $C_0 \ge 1$ and $C \ge 1$ so that

$$|\widehat{\mathcal{L}}^N|_{L^1} \le CC_0^N\,, \quad |(\widehat{\mathcal{L}}_M)^N|_{L^1} \le CC_0^N\,, \forall M\,, \forall N\,,$$

and that there exists $C$ so that for all large enough $M$

$$|\widehat{\mathcal{L}} - \widehat{\mathcal{L}}_M|_{L^1} \le C\tau_M\|\hat{\psi}\|_{\mathcal{B}^{BV}}\,.$$

The last inequality is an easy consequence of

$$|\ (\mathrm{id} - \mathcal{T}_M)\hat{\psi}|_{L^1} \le C\tau_M\|\hat{\psi}\|_{\mathcal{B}^0} \le C\tau_M\|\hat{\psi}\|_{\mathcal{B}^{BV}}\,,$$

which follows from Lemma 4.1 and Lemma 3.7 .

Note for use in Step 1 of the proof of Theorem 2.13 in Section 6 that the first claim of [30, Theorem 1] gives a small disc $B$ around 1 so that

$$\sup_{M \ge M_0} \sup_{z \notin B} \|z - \widehat{\mathcal{L}}_M\|^{-1}_{\mathcal{B}^{BV}} < \infty\,, \tag{92}$$

while, letting $\mathbb{P}_M(\hat{\psi}) = \hat{\phi}_M\nu_M(\hat{\psi})$ be the spectral projector corresponding to the maximal eigenvalue of $\widehat{\mathcal{L}}_M$, and setting

$$\mathcal{N}_M := (\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M) - (\mathrm{id} - \widehat{\mathcal{L}}_0)^{-1}(\mathrm{id} - \hat{\phi}\nu(\cdot))\,. \tag{93}$$

the second claim of [30, Theorem 1] with the first lines of [11, Appendix B] give

$$|\mathcal{N}_M(\hat{\psi})|_{L^1} \le \widehat{C}\tau_M^\eta\|\hat{\psi}\|_{\mathcal{B}^{BV}}\,, \tag{94}$$

and

$$\Delta := \|(\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)\|_{\mathcal{B}^{BV}} < \infty\,. \tag{95}$$

The last claim of Lemma 4.11 can be proved just like the analogous statement in Proposition 4.10. □

## 5. Topological invariance and uniformity of constants for various recurrence conditions

It is well-known that the Collet-Eckmann property is an invariant of topological conjugacy, and the fact that $\lambda_c(f_t)$ can be estimated uniformly in $t$ for a smooth deformation $f_t$ of $f_0$ is explained e.g. in [12, Appendix]. Our argument requires more: we need a Benedicks-Carleson-type condition of the form (69) or (86), and uniform estimates on the constants

$$\lambda_c(f_t)\,,\ H_0(f_t)\,,\ \gamma(f_t)\,,\ \text{and also } \sigma(f_t)\,,\ C_1(f_t)\,,\ c(\delta, f_t)\,,\ \rho(f_t) \tag{96}$$

(recall Lemma 3.1), as $t$ varies. The constant $\sigma(f_t)$ is bounded away from 1 uniformly in small $t$, by the proof of [20, Theorem III.3.3], in particular the choice of $m$ and $\lambda$ there, and noting that all $f_t$ have only repelling periodic orbits and are $S$-unimodal. However, if $f_t$ is a smooth deformation of a Benedicks-Carleson $S$-unimodal map, we do not know how to estimate $\gamma(f_t)$ in general.

Lemma 5.8, the main result of this section, is proved in Subsection 5.1: It says that all constants in (96) are uniform, for deformations $f_t$ which satisfy the TSR condition (5). In Subsection 5.2, we exploit a consequence of this uniformity which will play an important part in the proof of Theorem 2.13: If $f_t$ is a deformation, one can use the same lower part of the tower for all operators $\widehat{\mathcal{L}}_t$ with $|t| \le t_0$, up to some level depending on $t_0$.

In order to apply directly the results of Nowicki, we shall assume that $f$ is symmetric, i.e.,

$$f(x) = f(-x)\,. \tag{97}$$

### 5.1. Uniformity of constants.

Recall that our definition of $S$-unimodal includes the condition $f''(c) \neq 0$, that is, all our $S$-unimodal maps are quadratic. Let $R_f(x)$ be the function from (4) in the definition of the TSR condition.

**Proposition 5.1** (Uniform Collet-Eckmann condition [33])**.** *Let $f_0$ be a $S$-unimodal map satisfying the topological slow recurrence condition* (5)*. Then there exist $\lambda_c > 1$, $\underline{\kappa} > 0$, $K > 0$, and $\epsilon > 0$ such that for every $S$-unimodal map $f$ in the topological class of $f_0$ such that $|f - f_0|_{C^3} < \epsilon$, we have*

$$R_f(f^j(c)) \ge -\underline{\kappa} \log |f^j(c) - c|\,, \quad \forall j \ge 0\,, \tag{98}$$

*and*

$$|(f^j)'(f(c))| \ge K\lambda_c^j\,, \quad \forall j \ge 0\,. \tag{99}$$

*Proof.* Except for the explicit statement on the dependence of $\underline{\kappa}$, (98) is Lemma 2 in [33]. We say that $f \in V(D, L, \theta)$ if

$$D_f = \max_{x\in I} |f'(x)| < D\,,\ L_f = \sup_{x\in I} \frac{|f(x) - f(c)|}{|x-c|^2} < L\,,\ \theta_f = \sup_{f(x)=f(y)} \frac{|x-c|}{|y-c|} < \theta\,.$$

For $\epsilon$ small enough, we have $f \in V(D, L, \theta)$, with $D = 2D_{g_0}$, $L = 2L_{f_0}$ and $\theta = 2\theta_{f_0}$. The proof of Lemma 2 relies on Sublemmas 2.1 and 2.2 in [33]. The constants $C$ and $\underline{\kappa}$ in [33, Sublemma 2.1] depend only on $D$, $L$ and $\theta$. The constant $N_\varepsilon$ in [33, Sublemma 2.2] depends only on the topological class of $f$. In the proof of Lemma 2 in [33], since $f$ has a unique critical point we can take $\delta_0 = |I|$ and $N_0 = 1$ in (5) and (6) of [33]. Moreover, we can find $\epsilon > 0$ such that

$$\inf_{|f-f_0|_{C^3}<\epsilon} \min\{|f^i(c) - c| \ s.t.\ 0 < i \le \max\{N_0, N_\varepsilon\}\} > 0\,.$$

This shows (98).

Except for the explicit statement on the dependence of $K$ and $\lambda_c$, (99) is Corollary 5.1 in [33]. The proof of this result relies on (98) above, and on Lemmas 3, 4, 5 and Sublemma 5.1 in [33]. The estimates obtained in Lemma 3 depend only on the topological class of $f$. Given $T > 0$, we can find $\epsilon > 0$ such that

$$\inf_{|f-f_0|_{C^3}<\epsilon} \min\{|x - y| \ s.t.\ x \neq y,\ x, y \in \{f^i(c)\}_{i\le T} \cup \{z\colon\ f^i(z) = c\}_{i\le T}\} > 0\,,$$

so we can see from the proof of [33, Lemma 4] that there exists $\gamma(T)$ that satisfies the estimates obtained in Lemma 4 for every $S$-unimodal map $f$ in the topological class of $f_0$ satisfying $|f - f_0|_{C^3} < \epsilon$. Sublemma 5.1 in [33] follows directly from Lemmas 3 and 4 for every $f$ satisfying the same conditions, with the same constants $\eta$ and $\gamma$. Finally the proof of the estimates in Lemma 5 in [33] depends only on the topological class of $f$, estimates in Sublemma 2.2, (98) above and $D$. □

**Proposition 5.2** (Uniform Benedicks-Carleson type conditions)**.** *Let $f_0$ be a $S$-unimodal map satisfying the topological slow recurrence condition* (5)*. Then for every $\gamma > 0$ there exist $H_0 > 0$ and $\epsilon > 0$ such that for every $S$-unimodal map $f$ in the topological class of $f_0$ such that $|f - f_0|_{C^3} < \epsilon$, we have*

$$|f^k(c) - c| \ge e^{-\gamma k}\,, \quad \forall k \ge H_0\,.$$

*Proof.* Let $\underline{\kappa}$ and $\epsilon$ be as in Proposition 5.1. Choose $m_0$, $n_0$ large enough so that

$$\frac{1}{n} \sum_{\substack{1\le j\le n \\ R_{f_0}(f_0^j(c))\ge m_0}} R_{f_0}(f_0^j(c)) < \underline{\kappa}\gamma\,, \forall n \ge n_0\,.$$

Consequently, we have the same estimate for every map $f$ topologically conjugate to $f_0$, that is

$$\frac{1}{n} \sum_{\substack{1\le j\le n \\ R_f(f^j(c))\ge m_0}} R_f(f^j(c)) < \underline{\kappa}\gamma\,, \forall n \ge n_0\,.$$

In particular, if $R_f(f^k(c)) \ge m_0$ and $k \ge n_0$, we have

$$\frac{R_f(f^k(c))}{k} < \underline{\kappa}\gamma\,,$$

so by (98) we obtain

$$-\frac{\log|f^k(c) - c|}{k} < \gamma\,,$$

so $|f^k(c) - c| \ge e^{-\gamma k}$. Since $c$ is not periodic for $f_0$, we can find $\eta, \epsilon > 0$ such that for each $S$-unimodal map $f$ such that $|f - f_0|_{C^3} < \epsilon$ and for every $x \in (c - \eta, c + \eta)$ we have $|f^i(x) - c| > 0$ for $1 \le i \le 2m_0$. In particular, $\mathrm{dist}\,(\Omega_f, c) > \eta$, where

$$\Omega_f = \{x \in I\colon\ R_f(x) < m_0\}\,.$$

Let $H_0 > n_0$ be large enough such that $\eta > e^{-\gamma H_0}$. Then $|f^k(c) - c| \ge e^{-\gamma k}$ for every $k \ge H_0$. $\square$

We are going to use some results by Nowicki [38]. An interval $[a_1, a_2]$ is a nice interval if $c \in (a_1, a_2)$ and $f^j(a_i) \notin (a_1, a_2)$, for every $j \ge 1$ and $i = 1, 2$. We say that an interval $(c, b)$ is a $*(n)$ interval if $f^n$ is a diffeomorphism on $(c, b)$ and $f^n(b) = c$.

**Proposition 5.3** (Lemma 9 and Proposition 11 in [38])**.** *Let $(c, b)$ be an $*(n)$ interval of a symmetric $S$-unimodal map $f$ satisfying (99). Then $|f^n(c) - f^n(b)| > |c - b|$. Furthermore*

$$|f^n(c) - f^n(b)| \ge K_1 \lambda_c^{n/4} |c - b| ,$$

*where $K_1 = (Km/4M)^{1/2}$, where $K$ is as in (99), and $m$ and $M$ satisfy $m|x - c| \le |f'(x)| \le M|x - c|$.*

**Proposition 5.4** (Proposition 13 in [38])**.** *Let $f$ be a symmetric $S$-unimodal map $f$ satisfying (99). Let $b \in [-1, 1]$ be such that $f^n(b) = c$. Then*

$$|(f^n)'(b)| \ge \rho^n ,$$

*for every* [19]

$$\rho \le \min(\inf_n \inf_{(c,b)\ \text{an} *(\text{n})\ \text{interval}} \left| \frac{f^n(c) - f^n(b)}{c - b} \right|^{1/n}, |f'(-1)|^{1/2}) . \tag{100}$$

*(The right-hand-side above is $> 1$ by Proposition 5.3.)*

**Proposition 5.5.** *Let $f_0$ be a $S$-unimodal map satisfying the topological slow recurrence condition* (5)*. Then for every $\beta \in (0, 1)$ there exist $\epsilon, \delta > 0$ and $\tilde{K} > 0$ with the following property: Let $f$ be a symmetric $S$-unimodal map in the topological class of $f_0$ such that $|f - f_0|_{C^3} < \epsilon$, let $[-q, q] \subset [-\delta, \delta]$ be a nice interval for $f$, and let $x \in [-1, 1] \setminus [-q, q]$ be such that $f^n(x) \in [-q, q]$ for some $n \ge 1$. Define*

$$n_0(x) = \min\{n \ge 1 \text{ s.t. } f^n(x) \in [-q, q]\} .$$

*Then there exist intervals $I_{n_0(x)} \subset J_{n_0(x)}$ such that*

1. *For every $\in I_{n_0(x)}$ we have $n_0(y) = n_0(x)$ and $f^{n_0(x)} I_{n_0(x)} = [-q, q]$.*
2. *The map $f^{n_0(x)} \colon J_{n_0(x)} \to f^{n_0(x)} J_{n_0(x)}$ is a diffeomorphism, and each connected component of $f^{n_0(x)} J_{n_0(x)} \setminus \{c\}$ is larger than $\tilde{K} q^\beta$.*

*Proof.* The existence of $I_{n_0(x)}$ satisfying Claim 1 follows from the fact that $[-q, q]$ is a nice interval. Let $[a, b] = J_{n_0(x)} \supset I_{n_0(x)}$ be the largest interval such that $f^{n_0(x)}$ is a diffeomorphism on $(a, b)$. In particular there are $n_a, n_b < n_0(x)$ such that $f^{n_a}(a) \in \{1, -1, c\}$ and $f^{n_b}(b) \in \{1, -1, c\}$. Suppose $f^{n_b}(b) = c$. We are going to show that $|f^{n_0(x)} b - c| \ge \tilde{K} q^\beta$. The proof of the analogous statement for $a$ is similar. By Claim 1 there is $d \in I_{n_0(x)}$ such that $f^{n_0(x)}(d) = c$ and, moreover, $f^{n_b}(d) \notin [-q, q]$, so either $[-q, c] \subset [f^{n_b}(d), c] = f^{n_b}[d, b]$ or $[c, q] \subset [c, f^{n_b}(d)] = f^{n_b}[b, d]$. Since $(f^{n_b}(d), c)$ is a $*(n_0(x) - n_b)$ interval [38], by Proposition 5.3 and Proposition 5.1, we have

$$|f^{n_0(x)}(b) - c| = |f^{n_0(x) - n_b}(c) - c| \ge K_1 \lambda_c^{(n_0(x) - n_b)/4} |c - f^{n_b}(d)| \ge K_1 \lambda_c^{(n_0(x) - n_b)/4} q , \tag{101}$$

[19] Note that $\rho$ is called $\lambda_T$ in [38].

where $K_1$ is uniform on a $C^3$ neighbourhood of $f_0$. Choose

$$0 < \gamma < \frac{\beta \log \lambda_c}{4(1-\beta)} .$$

Reducing this neighbourhood, if necessary, we have by Proposition 5.2 that

$$|f^{n_0(x)}(b) - c| = |f^{n_0(x)-n_b}(c) - c| \geq K e^{-\gamma(n_0(x)-n_b)} . \tag{102}$$

We have two cases. If $n_0(x) - n_b > -4(1-\beta)\log q/\log \lambda_c$ then, by (101), we easily obtain $|f^{n_0(x)}(b) - c| \geq K_1 q^\beta$. Otherwise $n_0(x) - n_b \leq -4(1-\beta)\log q/\log \lambda_c$, so by (102), we get

$$|f^{n_0(x)}(b)-c| = |f^{n_0(x)-n_b}(c)-c| \geq K e^{-\gamma(n_0(x)-n_b)} \geq K e^{-\frac{\beta \log \lambda_c}{4(1-\beta)}(n_0(x)-n_b)} \geq K q^\beta .$$

Choose $\tilde{K} = \min(K, K_1)$. If $|f^{n_b}(b)| = 1$, then $f^{n_0(x)}(b) = -1$. Choose $\delta$ such that $\delta^{-\beta} \geq \tilde{K}$. Then

$$|f^{n_0(x)}(b) - c| = 1 \geq \tilde{K}\delta^\beta \geq \tilde{K} q^\beta .$$

□

**Corollary 5.6** (Uniformity of $C_1$ and $\rho$)**.** *Let $f_0$ be a symmetric $S$-unimodal map satisfying the topological slow recurrence condition* (5)*. There exist $\rho > 1$ and $\epsilon > 0$ with the following property: For every $C_1 > 0$ there exists $\delta > 0$ so that, for every symmetric $S$-unimodal map $f$ in the topological class of $f_0$ such that $|f - f_0|_{C^3} < \epsilon$, and for every nice interval $[-q, q]$ of $f$ such that $q < \delta$, if $x \notin [-q, q]$ and $n \geq 1$ is the first entrance time of $x$ in $[-q, q]$, then*

$$|(f^n)'(x)| \geq C_1 \rho^n .$$

*Proof.* By Proposition 5.1, we can find $\epsilon_0 > 0$ and $\rho > 1$ such that Proposition 5.4 holds for every $f$ such that $|f - f_0|_{C^3} < \epsilon$, with $f$ in the topological class of $f_0$. Take $\beta = 1/2$, and let $\epsilon, \delta$ be as in Proposition 5.5. Reducing $\delta$ if necessary, we have that if $q < \delta$ then each connected component of $f^n(J_n(x)) \setminus [-q, q]$ is far larger than $q$. In particular by the Koebe lemma we have

$$\frac{(f^n)'(z)}{(f^n)'(w)} < C_1 , \ \forall z, w \in I_n(x) .$$

But there exists $b \in I_n(x)$ such that $f^n(b) = c$, so $(f^n)'(b) \geq \rho^n$. We conclude that $|(f^n)'(x)| \geq C_1 \rho^n$. □

Finally, we will need the following result:

**Corollary 5.7** (Uniformity of $c(\delta)$ and $\sigma$)**.** *Let $f_0$ be a symmetric $S$-unimodal map satisfying the topological slow recurrence condition* (5)*. There exists*[20] *$\sigma > 1$ such that for every $\delta > 0$ there exist $c(\delta) > 0$ and $\epsilon > 0$ with the following property: for every symmetric $S$-unimodal map $f$ in the topological class of $f_0$ such that $|f - f_0|_{C^3} < \epsilon$, if $|f^i(x)| > \delta$ for $0 \leq i < n$ then*

$$|(f^n)'(x)| \geq c(\delta)\sigma^n .$$

*Proof.* By Proposition 5.1, we can find $\epsilon_0 > 0$ and $\rho > 1$ such that Proposition 5.4 holds for every $f$ such that $|f - f_0|_{C^3} < \epsilon_0$, with $f$ in the topological class of $f_0$. Using the same argument as in Proposition 3.9 in [37], we can show that for every periodic point $q$ such that $f^n(q) = q$ we have $|(f^n)'(q)| \geq \rho^n$. Note that since $c$ is recurrent by $f_0$, there exists a sequence of periodic points for $f_0$ converging to

[20]The analogue of $\sigma$ is called $\lambda_M$ in [37].

$c$. So given $\delta > 0$ there exists a periodic point $p$ for $f_0$ such that $|p| < \delta$. Let $n_0$ be the prime period of $p$. There exists $\epsilon_1 < \epsilon_0$ such that every map $f$ such that $|f - f_0|_{C^3} < \epsilon_1$ has an analytic continuation $p_f$ for $p$ such that $|p_f| < \delta$ and

$$\eta_{per} = \inf_{|f-f_0|_{C^3} < \epsilon_1} |p_f| > 0 \,.$$

Without loss of generality, we can assume that $|f^i(p_f)| \geq |p_f|$ for every $i$. So $[-p_f, p_f]$ is a nice interval. Let $x \notin [-\delta, \delta]$ be such that $|f^i(x)| > \delta$ for $0 \leq i < n$. If $f^n(x) \in [-p_f, p_f]$ we can use Corollary 5.6 to conclude that $|(f^n)'(x)| \geq C_1 \rho^n$. So assume that $f^n(x) \notin [-p_f, p_f]$. Let $(a, b)$ be the largest interval such that $x \in (a, b)$ and $f^i(y) \notin [-p_f, p_f]$ for every $0 \leq i \leq n$ and $y \in (a, b)$. In particular, $f^n$ is a diffeomorphism on $(a, b)$, and there exist $n_a, n_b \leq n$ such that $|f^{n_a}(a)|, |f^{n_b}(b)| \in \{|p_f|, 1\}$. Without loss of generality, we can assume that $|f^i(a)|, |f^j(b)| \notin \{|p_f|, 1\}$ for every $i < n_a$, $j < n_b$. If $f^{n_a}(a) \in \{-1, 1\}$, then indeed $a \in \{-1, 1\}$, so $|(f^n)'(a)| = |f'(-1)|^n$. We have a similar statement for $b$. Otherwise either $f^{n_a}(a)$ (respectively $f^{n_b}(b)$) or $-f^{n_a}(a)$ (respectively $-f^{n_b}(b)$) is a periodic point with period $n_0$. Then $n_a$ and $n_b$ are the first entry times of $a$ and $b$ in $[-p_f, p_f]$. By Corollary 5.6, we have

$$|(f^{n_a})'(a)| \geq C_1 \rho^{n_a} \text{ and } |(f^{n_b})'(b)| \geq C_1 \rho^{n_b} \,.$$

Since $p_f$ is a periodic point of period $n_0$, $|(f^{n_0})'(p_f)| \geq \rho^{n_0}$ and $f$ is symmetric and quadratic, we have

$$\begin{aligned}|(f^{n-n_a})'(f^{n_a}(a))| &\geq \rho^{n-n_a-n_0} \min\{|f'(f^i(p_f))|,\ 0 \leq i < n_0\}^{n_0} \\ &\geq C^{n_0} |p_f|^{n_0} \rho^{n-n_a-n_0} \,,\end{aligned}$$

so

$$|(f^n)'(a)| \geq C^{n_0} |p_f|^{n_0} \rho^{n-n_0} \geq C^{n_0} \eta_{per}^{n_0} \rho^{-n_0} \rho^n = c(\delta) \rho^n \,.$$

We can obtain similarly $|(f^n)'(b)| \geq c(\delta)\rho^n$. In any case

$$\min(|(f^n)'(a)|, |(f^n)'(b)|) \geq \min(1, c(\delta)) \cdot \min(\rho, |f'(-1)|)^n \,.$$

By the minimum principle

$$|(f^n)'(x)| \geq \min(1, c(\delta)) \cdot \min(\rho, |f'(-1)|)^n \,.$$

So choose $\sigma = \min(\rho, |f'(-1)|) > 1$. □

Summarising the results of this section, we have proved:

**Lemma 5.8** (Uniformity of constants in topological classes of TSR maps)**.** *If $f_0$ is a symmetric $S$-unimodal $(\lambda_c(f_0), H_0(f_0))$-Collet-Eckmann map satisfying topological slow recurrence* (5)*, then for every $C_1 > 0$ there exists $\lambda_c \in (1, \lambda_c(f_0))$ so that for any $\gamma > 0$ there exists $H_0 > H_0(f_0)$ so that for each $\rho \in (1, \lambda_c^{1/2})$, there exists $\sigma > 1$ and $\delta_0 > 0$ so that for every $\delta \in (0, \delta_0)$ there exist $c(\delta) > 0$ and $\epsilon > 0$ so that the following holds for each symmetric $S$-unimodal map $f$ topologically conjugated to $f_0$ and so that $|f - f_0|_{C^3} < \epsilon$:*

*The map $f$ is $(\lambda_c, H_0)$-Collet-Eckmann and satisfies* (86) *for $\gamma$, $\lambda_c$, and $H_0$. For any $y \in I$, if $j \geq 0$ is minimal satisfying $|f^j(y)| \leq \delta$, then*

$$|(f^j)'(y)| \geq C_1 \rho^j \,, \tag{103}$$

*for any $x \in I$, if $j \geq 1$ is such that $|f^k(x)| > \delta$ for all $0 \leq k < j$, then*

$$|(f^i)'(x)| \geq c(\delta) \sigma^i \,,\ \forall 0 \leq i \leq j \,. \tag{104}$$

Comparing the above result to Lemma 3.1 we emphasize that *we do not claim* that $\lambda_c$ can be taken arbitrarily close to $\lambda_c(f_0)$ or $H_0$ close to $H_0(f_0)$, where $\lambda_c(f_0)$, $H_0(f_0)$ are the best possible constants for $f_0$. So (86) cannot be viewed strictly as a Benedicks-Carleson assumption. (This is mostly because of the infimum in the right-hand-side of (100) from Proposition 5.4.) However, this does not matter since we are assuming the much stronger TSR assumption in any case (see also (112) below), which implies that we can take $\gamma$ arbitrarily close to 0 after $\lambda_c$ has been fixed. The advantage of the notation introduced in Lemma 5.8 is that we can use the estimates from Sections 3 and 4 directly, with the same notation for the constants, for deformations of maps $f_0$ satisfying the assumptions of Lemma 5.8.

5.2. **Transfer operators $\widehat{\mathcal{L}}_t$, $\widehat{\mathcal{L}}_{t,M}$ for a (TSR) smooth deformation $f_t$.** If $f_t$ is a $C^1$ one-parameter family of $S$-unimodal symmetric Collet-Eckmann maps $f_t$, with a non preperiodic critical point, Lemma 5.8 implies that all $f_t$ satisfy estimates for uniform parameters $\lambda_c$ and $H_0$, and satisfy the strengthened Benedicks-Carleson condition (86) for some $\gamma$. We can associate a tower $\hat f_t : \hat I_t \to \hat I_t$ to each $f_t$, choosing small $\delta_t$ and intervals $B_{k,t}$ by using the parameters $\lambda_c$, $H_0$, $\gamma$ as in Subsection 3.1, replacing $c_k$ by $c_{k,t}$. Then, we can define spaces $\mathcal{B}_t$, $\mathcal{B}^2_t$, and an operator $\widehat{\mathcal{L}}_t$ in Subsection 4.1, replacing $f^k$ by $f_t^k$ and $c_k$ by $c_{k,t}$ in (71) and the definition of $\xi_{k,t}$. We summarize first the results which follow from applying Proposition 4.9 and Proposition 4.10 to each $f_t$, in order to fix notation (note however that we shall modify slightly the lower parts of the tower maps $\hat f_t$ in Proposition 5.9): There is $\Theta < 1$ so that each operator $\widehat{\mathcal{L}}_t$ has essential spectral radius bounded by $\Theta$ on $\mathcal{B}_t$. Outside of a disc of radius $\theta_t < 1$ the spectrum of $\widehat{\mathcal{L}}_t$ on $\mathcal{B}_t$ consists in a simple eigenvalue at 1, with a nonnegative eigenfunction $\hat\phi_t$, which belongs to $\mathcal{B}^2_t$. Define $\Pi_t : \mathcal{B}_t \to L^1(I)$ by

$$\Pi_t(\hat\psi)(x) = \sum_{k\geq 0, \varsigma\in\{+,-\}} \frac{\lambda^k}{|(f_t^k)'(f_{t,\varsigma}^{-k}(x))|} \psi_k(f_{t,\varsigma}^{-k}(x))\chi_{k,t}(x)\,, \tag{105}$$

where $\chi_{k,t}$ is defined like $\chi_k$ (see Proposition 2.7), replacing $f^k$ by $f_t^k$ and $c_k$ by $c_{k,t}$. The fixed point of the dual of $\widehat{\mathcal{L}}_t$ is the nonnegative measure $\nu$ on $\widehat{I}_t$, absolutely continuous with respect to Lebesgue on $\widehat{I}_t$ whose density is $w(x,k)$. If we normalise by requiring $\nu(\hat\phi_t) = 1$, the invariant density of $f_t$ is just $\phi_t = \Pi_t(\hat\phi_t)$. Lemma 4.11 also holds for $\widehat{\mathcal{L}}_{t,M}$, using the weak norm $|\cdot|_{L^1}$. This gives [21] $\hat\phi_{t,M}$, $\nu_{t,M}$, and $\kappa_{t,M}$. If $\sigma(f_t)$, $C_1(f_t)$ and $c(\delta, f_t)$ from Lemma 3.1 applied to $f_t$ are uniform in $t$, then all objects constructed are uniform in $t$ (including the $\mathcal{B}^2_t$ norm of $\hat\phi_t$ and $\hat\phi_{t,M}$).

There is of course some flexibility in choosing the intervals $B_{k,t}$ and the functions $\xi_{k,t}$. It is tempting, in order to get conjugated tower dynamics $\hat f_t : \hat I_t \to \hat I_t$, to choose $B_{k,t} = h_t(B_{k,0})$ and $\xi_{k,t} = \xi_{k,0} \circ h_t^{-1}$, where the homeomorphisms $h_t$ are given by Lemma 2.11. Then, in order to prove Theorem 2.13 on linear response, one would need additional information on the $h_t$ (for example, but not only, the fact that $\partial_t h_t(x) = \alpha(x)$ at all points $x$). We shall work instead with truncated operators $\mathcal{L}_{t,M}$, disregarding the top part of the tower via Lemma 4.11, and *artificially forcing the lower parts of the towers* associated to the various $f_t$ to *coincide.* This is going to be possible in view of the following consequence of Lemmas 2.10 and 2.12:

[21] In general, $\nu_{t,M}$ depends on $t$.

**Proposition 5.9** (Controlling the truncated tower)**.** *Let $f_t$ be a $C^1$ deformation of $S$-unimodal maps $f_t$ satisfying the Benedicks-Carleson condition* (86) *for $\gamma_0 = \gamma$. Let $\hat f_0 : \hat I \to \hat I$ be a tower associated to $f_0$ as in Section 3.1, for some $\delta > 0$ and $3\gamma_0/2 < \beta_1 < \beta_2 < 2\gamma_0$. Let $\alpha_s$ be the solution of the TCE* (7) *for $f_s$ and $v_s = \partial_t f_t|_{t=s}$, with $\alpha_s(c) = 0$, given by*[22] *Theorem 2.4. Fix*

$$\frac{3\gamma_0}{2} < \tilde\beta_1 < \beta_1 < \beta_2 < \tilde\beta_2 < 2\gamma_0 \, .$$

*Then, for any $M \ge 1$, and for any $t$ so that*

$$\sup_{|s|\le t} |\alpha_s(c_k)||t| < \min((e^{-\tilde\beta_1 k} - e^{-\beta_1 k}), (e^{-\beta_2 k} - e^{-\tilde\beta_2 k}))\, , 1 \le k \le M \, , \tag{106}$$

*one can construct the towers $\hat f_t : \hat I_t \to \hat I_t$, the Banach spaces $\mathcal{B}_t$, $\mathcal{B}_t^2$, $\mathcal{B}_t^{BV}$, etc, and the transfer operators $\widehat{\mathcal{L}}_t$, using parameters $\delta_t > 0$, intervals $B_{k,t}$ admissible for $\tilde\beta_1$, and $\tilde\beta_2$, and smooth cutoff functions $\xi_{k,t}$ in such a way as to ensure*

$$\delta_t = \delta \, , \;\; B_{k,t} = B_k \, , \; \xi_{k,t} = \xi_k \, , \; \forall k \le M \, ,$$

*and, in addition, so that all results of Section 4 hold for $\widehat{\mathcal{L}}_t$.*

If $f_0$ enjoys TSR then, up to taking smaller $\epsilon$, Lemma 2.12 implies that

$$\sup_{|x|\le\delta, |s|\le\epsilon} |\alpha_s(x)||\epsilon| < \infty \, ,$$

so that we can exploit the above proposition.

*Proof of Proposition 5.9.* Recall $h_t$ as given by (2.11) and recall Lemma 2.10. By the proof of Lemma 2.12, we have

$$\begin{aligned} |a_k - h_t(c_k)| &\le |a_k - c_k| + |h_t(c_k) - h_0(c_k)| \\ &\le e^{-\beta_1 k} + \sup_{|s|\le|t|} |\alpha_s(c_k)||t| \, , \end{aligned}$$

and

$$\begin{aligned} |a_k - h_t(c_k)| &\ge |a_k - c_k| - |h_t(c_k) - h_0(c_k)| \\ &\ge e^{-\beta_2 k} - \sup_{|s|\le|t|} |\alpha_s(c_k)||t| \, . \end{aligned}$$

The claim of Proposition 5.9 follows. □

## 6. Proof of linear response

In this section, we prove Theorem 2.13. Let $f_t$ satisfy the assumptions of the theorem. We suppose in addition that the critical point is not preperiodic (the proof is much easier if it is). Applying Lemma 5.8, we fix $\epsilon > 0$ and constants $\gamma$, $\lambda_c$, $\sigma$, $C_1$, $\rho$, $\delta$, and $c(\delta)$ and we choose $3\gamma/2 < \beta_1 < \beta_2 < 2\gamma$. By Lemma 5.8, we may assume that the strong Benedicks-Carleson condition (86) holds, and in several places in the proof below (see in particular above (122)) we shall require a stronger upper bound on $\gamma$.

Constructing a tower, Banach space, and transfer operator for each $f_t$ as in Section 4, the invariant density of $f_t$ can be written as $\phi_t = \Pi_t(\hat\phi_t)$, where $\Pi_t$ was defined in (105), and where $\hat\phi_t$ is the nonnegative and normalised fixed point

[22]Recall Remark 2.9.

of $\widehat{\mathcal{L}}_t$ on $\mathcal{B}_t$ given by Proposition 4.10 applied to $f_t$. It will be convenient to work with truncated transfer operators $\widehat{\mathcal{L}}_{t,M}$, recalling Subsection 4.2, in particular Lemma 4.11, which gives $\hat\phi_{t,M}$. We shall in fact require the lower part of the towers of $f_t$ for small enough $t$, up to $M = M(t)$ as given by Proposition 5.9 to coincide with that of $f_0$.

When the meaning is clear, we shall remove 0 from the notation, writing, e.g., $\Pi$, $\hat\phi$, and $\hat\phi_M$, instead of $\Pi_0$, $\hat\phi_0$, and $\hat\phi_{0,M}$.

We start with the decomposition

$$\phi_t - \phi = \Pi_t(\hat\phi_t - \hat\phi_{t,M}) + \Pi(\hat\phi_M - \hat\phi) + \Pi_t(\hat\phi_{t,M}) - \Pi(\hat\phi_M)\,. \tag{107}$$

Lemma 4.11 implies that

$$|\Pi_t(\hat\phi_t - \hat\phi_{t,M})|_{L^1} \le M\lambda^M \|\hat\phi_t - \hat\phi_{t,M}\|_{L^1} \le CM\lambda^M \tau_M^\eta\,, \forall |t| < \epsilon\,. \tag{108}$$

Note that if $f^k|_{[c,y]}$ is injective then

$$\int_{c_k}^{f^k(y)} \frac{|\psi_k(f_+^{-k}(y))|}{|(f^k)'(f_+^{-k}(y))|}\,dy = \int_c^y |\psi_k(x)|\,dx\,. \tag{109}$$

Fix $\zeta > 0$, then (108) and (109) imply

$$\begin{aligned}&\max(\|\Pi_t(\hat\phi_t - \hat\phi_{t,M})\|_{L^1(I))}, \|\Pi(\hat\phi - \hat\phi_M)\|_{L^1(I))}) \le C|t|^{1+\zeta}\,,\\ &\qquad\qquad\qquad\qquad \forall t \text{ so that } (M\lambda^M\tau_M^\eta)^{\frac{1}{1+\zeta}} < |t| < \epsilon\,.\end{aligned} \tag{110}$$

It is now sufficient to estimate the third term in the right-hand-side of (107) for $t$ and $M = M(t)$ satisfying (110).

For this, in order to apply Proposition 5.9, and noting that the right-hand-side of (106) is $\ge Ce^{-2\gamma M}$, we want $t$ and $M$ to satisfy

$$\sup_{k,|s|<\epsilon} |\alpha_s(c_k)||t| < Ce^{-2\gamma M}\,. \tag{111}$$

(Recall that $\sup_{k,|s|<\epsilon} |\alpha_s(c_k)| \le L$ by Lemma 2.12.) In several places below (see for example (122)) we shall require a stronger version of (111), of the form

$$|t| < e^{\Gamma\gamma M}\,, \tag{112}$$

where $\Gamma > 2$ is large (but uniformly bounded over the argument). Since $\tau_M < \lambda_c^{-M/2} e^{3M\gamma}$ (recalling Lemma 4.11) and $\lambda < e^\gamma$ (by (70)), we see that (110) and (112) are compatible if $\epsilon$ is small enough and

$$\frac{1}{2}\log\lambda_c > \gamma\big(\frac{1+\Gamma(1+\zeta)}{\eta} + 3\big)\,. \tag{113}$$

By our TSR assumption and Lemma 5.8, we may indeed require that $\gamma$ is small enough for the above Benedicks-Carleson condition to hold, even if $\eta > 0$ is small and $\Gamma > 2$ is large.

We shall call pairs $(M, t)$ so that $|t| < \epsilon$ and (110) and (112) hold *admissible pairs.* In the remainder of this section, $(M, t)$ will always be an admissible pair, and we will work with the towers and operators given by Proposition 5.9 for a given such pair.

The key decomposition for an admissible pair is then

$$\Pi_t(\hat\phi_{t,M}) - \Pi(\hat\phi_M) = \Pi_t(\hat\phi_{t,M} - \hat\phi_M) + \Pi_t(\hat\phi_M) - \Pi(\hat\phi_M)\,. \tag{114}$$

Before we start with the proof, let us briefly sketch it: The term $\hat\phi_{t,M}-\hat\phi_M$, will be handled using spectral perturbation-type methods. This is the content of Steps 1 and 2 below, the outcome of which are claims (126), (130), and (132). (Horizontality is used here to get uniform estimates, in view of Proposition 5.9 but also, e.g., in Lemma 6.2.)

The other term requires the analysis of $\Pi_t-\Pi_0$. This will produce derivatives of the 'spikes," i.e., of functions of the type $(x-c_k)^{-1/2}$ (recall the definition of $\Pi_t$ and see Lemma 4.1). Since $\partial_t(x-h_t(c_k))^{-1/2}$ is not integrable, this will require working with $\int A(\Pi_t\hat\phi_t-\Pi_0\hat\phi_t)\,dx$, with $A$ a $C^1$ function and integrating by parts, as well as using again horizontality. We perform this analysis in Step 3 of the proof, which yields (151).

**Step 1: The first term of (114): perturbation theory via resolvents**

(Recall that $(M,t)$ is an admissible pair.) In order to get a formula for the limit as $t\to 0$ (in a suitable norm) of the first term of (114) divided by t, we shall first analyse $(\hat\phi_{t,M}-\hat\phi_M)/t$, and then see how $\Pi_t$ enters in the picture.

Since $\nu_M(\hat\phi_M)=1$, we have

$$\hat\phi_{t,M}\nu_{t,M}(\hat\phi_M)-\hat\phi_M=\hat\phi_{t,M}-\hat\phi_M+\hat\phi_{t,M}(\nu_{t,M}(\hat\phi_M)-\nu_M(\hat\phi_M))\,.$$

Now, Lemma 4.11 applied to $f$ and $f_t$ implies that

$$\begin{aligned}|\nu_{t,M}(\hat\phi_M)-\nu_M(\hat\phi_M)|&\le \max(\|\nu_{t,M}-\nu\|_{(\mathcal{B}^{BV})^*},\|\nu_M-\nu\|_{(\mathcal{B}^{BV})*})\|\hat\phi_M\|_{\mathcal{B}^{BV}}\\&\le C\tau_M^\eta\|\hat\phi_M\|_{\mathcal{B}^{BV}}\end{aligned}$$

(we used that $\nu=\nu_t$ for all $t$). Our choices imply that $C\tau_M^\eta=O(|t|^{1+\zeta})$ while $\|\hat\phi_M\|_{\mathcal{B}^{BV}}$ is uniformly bounded, e.g. by the proof of Lemma 4.11. Therefore, to study $\hat\phi_{t,M}-\hat\phi_M$, it suffices to estimate $\hat\phi_{t,M}\nu_{t,M}(\hat\phi_M)-\hat\phi_M$, which we will express as a difference of spectral projectors.

Next, set

$$\widehat{\mathcal{Q}}_{t,M}=\widehat{\mathcal{Q}}_{t,M}(z)=z-\widehat{\mathcal{L}}_{t,M}\,,\quad \widehat{\mathcal{Q}}_M=\widehat{\mathcal{Q}}_M(z)=z-\widehat{\mathcal{L}}_M\,,$$

recall $\mathbb{P}_M$ from the proof of Lemma 4.11, and denote by

$$\mathbb{P}_{t,M}(\hat\psi)=\hat\phi_{t,M}\nu_{t,M}(\hat\psi)\,,$$

the spectral projector corresponding to the maximal eigenvalue of $\widehat{\mathcal{L}}_{t,M}$. Using

$$\widehat{\mathcal{Q}}_{t,M}^{-1}-\mathcal{Q}_M^{-1}=\widehat{\mathcal{Q}}_{t,M}^{-1}(\widehat{\mathcal{L}}_{t,M}-\widehat{\mathcal{L}}_M)\widehat{\mathcal{Q}}_M^{-1}\,,\text{ and }\mathcal{Q}_M^{-1}(\hat\phi_M)=\frac{\hat\phi_M}{z-\kappa_M}\,,$$

we rewrite $\hat\phi_{t,M}\nu_{t,M}(\hat\phi_M)-\hat\phi_M=(\mathbb{P}_{t,M}-\mathbb{P}_M)(\hat\phi_M)$ as follows:

$$\begin{aligned}\hat\phi_{t,M}\nu_{t,M}(\hat\phi_M)-\hat\phi_M&=-\frac{1}{2i\pi}\oint\frac{\widehat{\mathcal{Q}}_{t,M}^{-1}(z)}{z-\kappa_M}(\widehat{\mathcal{L}}_{t,M}-\widehat{\mathcal{L}}_M)(\hat\phi_M)\,dz\\&=(\kappa_M-\widehat{\mathcal{L}}_{t,M})^{-1}(\mathrm{id}-\mathbb{P}_{t,M})(\widehat{\mathcal{L}}_{t,M}-\widehat{\mathcal{L}}_M)(\hat\phi_M)\,,\end{aligned}\tag{115}$$

where the contour is a circle centered at 1, outside of the disc of radius $\max(\theta_0,\theta_t)$ (using the notation from Subsection 5.2).

We are going to use the weak norm (73) and adapt the arguments in [30] [23].

[23]In view of our use of (142), working with the weak norm (74) would not be enough for the estimate (129) below, since we would not get the sup norm in (122).

By uniformity of the constants in Lemma 5.8, the estimates [24] in the proofs of Proposition 4.9 and Lemma 4.11 give $\epsilon > 0$ and $C \ge 1$ so that, for all $|t| \le \epsilon$, all $M$, all $j$, and all $\hat\psi \in \mathcal{B}$,

$$|\widehat{\mathcal{L}}^j_{t,M}(\hat\psi)|_{C^0} \le C^j |\hat\psi|_{C^0}\,, \quad \|\widehat{\mathcal{L}}^j_{t,M}(\hat\psi)\|_{\mathcal{B}} \le C\Theta^{-j}\|\hat\psi\|_{\mathcal{B}} + C^j|\psi|_{C^0}\,, \tag{116}$$

$$|\widehat{\mathcal{L}}^j_{M}(\hat\psi)|_{C^0} \le C^j |\hat\psi|_{C^0}\,, \quad \|\widehat{\mathcal{L}}^j_{M}(\hat\psi)\|_{\mathcal{B}} \le C\Theta^{-j}\|\hat\psi\|_{\mathcal{B}} + C^j|\psi|_{C^0}\,. \tag{117}$$

In Step 2 we shall find $\widetilde C \ge 1$ and $\tilde\eta > 0$ so that for each admissible pair $(M, t)$

$$|\widehat{\mathcal{L}}_{t,M}(\hat\psi) - \widehat{\mathcal{L}}_M(\hat\psi)|_{C^0} \le \widetilde C |t|^{\tilde\eta}\|\hat\psi\|_{\mathcal{B}}\,, \forall \hat\psi \in \mathcal{B}\,. \tag{118}$$

We are not exactly in the setting of [30], since we have a "moving target" $\widehat{\mathcal{L}}_{M(t)}$ as $t \to 0$. However, since the right-hand-side of (118) does not depend on $M$, setting

$$\mathcal{N}_{t,M} := (\kappa_{t,M} - \widehat{\mathcal{L}}_{t,M})^{-1}(\mathrm{id} - \mathbb{P}_{t,M}) - (\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)\,,$$

then (116)–(117) and (118) imply by a small modification of the proofs of [30, Theorem 1, Corollary 1] that there exist $\widehat C \ge 1$ and $\hat\eta > 0$ so that for all admissible pairs $(M, t)$

$$|\mathcal{N}_{t,M}(\hat\psi)|_{C^0} \le \widehat C |t|^{\tilde\eta\hat\eta}\|\hat\psi\|_{\mathcal{B}}\,, \ \forall \hat\psi \in \mathcal{B}\,. \tag{119}$$

In Step 2, we shall show that there exist $C > 0$ and $\mathcal{D}_M \in \mathcal{B}$ with

$$\int_I \mathcal{D}_0\, dx = 0\,,\ \mathcal{D}_{M,k} = 0\,, \forall k \ge 1\,,\ \text{and } \|\mathcal{D}_M\|_{\mathcal{B}} \le Ce^{12\gamma M}\,, \quad |\mathcal{D}_M|_{L^1} \le C\,,$$

and $\tilde\zeta > 0$, so that for all admissible pairs $(M, t)$

$$\|\widehat{\mathcal{L}}_{t,M}(\hat\phi_M) - \widehat{\mathcal{L}}_M(\hat\phi_M) - t\mathcal{D}_M\|_{\mathcal{B}} \le C|t|^{1+\tilde\zeta}\,. \tag{120}$$

Writing

$$(\kappa_{t,M} - \widehat{\mathcal{L}}_{t,M})^{-1}(\mathrm{id} - \mathbb{P}_{t,M}) = \mathcal{N}_{t,M} + (\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)\,,$$

we see that (120) together with (95) from the proof of Lemma 4.11 and (115) imply

$$\begin{aligned}\hat\phi_{t,M} - \hat\phi_M &= [\mathcal{N}_{t,M} + (\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)](t\mathcal{D}_M + O_{\mathcal{B}}(|t|^{1+\tilde\zeta}))\\ &= t\mathcal{N}_{t,M}(\mathcal{D}_M) + t(\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)(\mathcal{D}_M) + \Delta O_{\mathcal{B}^{BV}}(|t|^{1+\tilde\zeta})\,,\end{aligned} \tag{121}$$

where we used that $\|\cdot\|_{\mathcal{B}^{BV}} \le \|\cdot\|_{\mathcal{B}}$.

Note for further use (in (129) below) that, since $\|\cdot\|_{\mathcal{B}^0} \le \|\cdot\|_{\mathcal{B}^{BV}}$, the bounds (121) with (119) imply that, up to choosing a large enough $\Gamma$ in (112) in order to ensure that $e^{12\gamma M}|t|^{\tilde\eta\hat\eta} = O(1)$ for admissible pairs $(M, t)$, we have

$$|\hat\phi_{t,M} - \hat\phi_M|_{C^0} = O(t)\,,\ \text{as } t \to 0, \text{ uniformly in admissible pairs } (M, t). \tag{122}$$

We need more Banach spaces. Let $\mathcal{B}^{H^r_p} \subset \mathcal{B}^{L^1}$ for $r \ge 0$ and $p \ge 1$, be the space of sequences $\psi_k$ satisfying (71) and endowed with the norm

$$\|\hat\psi\|_{\mathcal{B}^{H^r_p}} = \sup_k \|\psi_k\|_{H^r_p}\,,$$

where, denoting by $\mathcal{F}$ the continuous Fourier transform on $\mathbb{R}$, we recall that the generalised Sobolev norm of $\psi : I \to \mathbb{C}$ is just

$$\|\psi\|_{H^r_p} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{r/2}\mathcal{F}(\psi)]\|_{L^p(\mathbb{R})}\,.$$

[24]We use that the constant $c(\delta, f_t)$ associated to $f_t$ by Lemma 5.8 does not depend on $t$ and that $\inf_{\delta\ge\delta_0} c(\delta) > 0$ for any $\delta_0 > 0$.

Note that for any $r \in [0,1]$ and $p \geq 1$,

$$\|\hat\psi\|_{\mathcal{B}^{L^1}} \leq \|\hat\psi\|_{\mathcal{B}^{H^0_p}} \leq \|\hat\psi\|_{H^r_p} \leq \|\hat\psi\|_{\mathcal{B}}\,.$$

It is not difficult to adapt the proofs of Proposition 4.9 and Proposition 4.10 to show that for any $r \in (0,1]$ and $p \geq 1$ the essential spectral radius of $\widehat{\mathcal{L}}$ on $\mathcal{B}^{H^r_p}$ is strictly smaller than 1, and that 1 is a simple eigenvalue. (The eigenvector is, of course, $\hat\phi$, while $\nu$ is the fixed point of $\widehat{\mathcal{L}}^*$.) We shall in fact only need the easiest cases $r=1$ and $p=1$, the cases $r \in (0,1)$ can be obtained by interpolation.

Recalling from (17) the definition $\hat Y$, we shall see in Step 2 that the following expression defines an element of $\mathcal{B}^{H^1_1}$

$$\mathcal{D} = -(\mathcal{T}_0(\widehat{\mathcal{L}}(\hat Y \hat\phi))'\,, \tag{123}$$

and that, in addition $\lim_{M\to\infty} \|\mathcal{D} - \mathcal{D}_M\|_{H^1_1} = 0$. More precisely, there is $\zeta' > 0$ so that for admissible pairs $(M,t) \to (\infty, 0)$

$$\|\mathcal{D} - \mathcal{D}_M\|_{H^1_1} = \|\mathcal{D}_0 - \mathcal{D}_{M,0}\|_{H^1_1(I)} = 0(|t|^{\zeta'})\,. \tag{124}$$

Note that $\mathcal{D}_k = 0$ for $k \geq 1$, and that $\int \mathcal{D}_0\, dx = 0$, so that $\nu(\mathcal{D}) = 0$. Therefore, by the spectral properties of $\widehat{\mathcal{L}}$ on $\mathcal{B}^{H^1_1}$ , we have that

$$(\mathrm{id} - \widehat{\mathcal{L}})^{-1}(\mathcal{D}) \in \mathcal{B}^{H^1_1}\,.$$

*Remark* 6.1. We do not claim that $\mathcal{D} \in \mathcal{B}$ or $\mathcal{D} \in \mathcal{B}^{BV}$. We therefore use generalised Sobolev spaces, in order to show that $(\mathrm{id} - \widehat{\mathcal{L}})^{-1}(\mathcal{D})$ is well-defined in $\mathcal{B}^{H^1_1} \subset \mathcal{B}^{L^1}$. (Since we are in dimension 1, the Sobolev embedding theorem also gives $\mathcal{B}^{H^1_1} \subset \mathcal{B}^0$.)

Recalling $\mathcal{N}_M$ from (93) in the proof of Lemma 4.11, the estimate (94) and our condition (110) on $M$ give

$$|\mathcal{N}_M(\hat\psi)|_{L^1} \leq \widehat{C}|t|^\eta \|\hat\psi\|_{\mathcal{B}^{BV}} \leq \widehat{C}|t|^\eta \|\hat\psi\|_{\mathcal{B}}\,. \tag{125}$$

In particular, up to choosing $\Gamma$ in (112) large enough so that $\lim_{t\to0} e^{12\gamma M}|t|^\eta = 0$, we get $\lim_{t\to0} |\mathcal{N}_M(\mathcal{D}_M)|_{L^1} = 0$, exponentially in $M$. Therefore, recalling that $\nu(\mathcal{D}) = 0$, using (124), there exists $\zeta'' > 0$ so that for admissible $(M,t) \to (\infty, 0)$,

$$\big|(\kappa_M - \widehat{\mathcal{L}}_M)^{-1}(\mathrm{id} - \mathbb{P}_M)(\mathcal{D}_M) - (\mathrm{id} - \widehat{\mathcal{L}})^{-1}(\mathcal{D})\big|_{\mathcal{B}^{L^1}} = |\mathcal{N}_M(\mathcal{D}_M)|_{\mathcal{B}^{L^1}} = 0(|t|^{\zeta''})\,.$$

We may now conclude the first part of Step 1: Dividing (121) by $t$, letting $t \to 0$, and applying again (119) gives $\hat\zeta > 0$ so that (recall that $\nu(\mathcal{D}) = 0$)

$$\Big|\frac{1}{t}(\hat\phi_{t,M(t)} - \hat\phi_{M(t)}) - (\mathrm{id} - \widehat{\mathcal{L}})^{-1}(\mathcal{D})\Big|_{L^1} = 0(|t|^{\hat\zeta}) \text{ as } t \to 0. \tag{126}$$

It remains to assess the effect of composition by $\Pi_t$ in the first term of (114) divided by $t$. We claim that (126) implies

$$\lim_{t\to0} \|\frac{1}{t}\Pi_t[\hat\phi_{t,M} - \hat\phi_M] - \Pi_0((\mathrm{id} - \widehat{\mathcal{L}})^{-1}(\mathcal{D}))\|_{L^1(I)} = 0\,. \tag{127}$$

To prove (127), we start from the decomposition

$$\frac{1}{t}\Pi_t(\hat\phi_{t,M} - \hat\phi_M) = \frac{1}{t}\Pi_0(\hat\phi_{t,M} - \hat\phi_M) + \frac{1}{t}(\Pi_t - \Pi_0)(\hat\phi_{t,M} - \hat\phi_M)\,. \tag{128}$$

Note that (109) implies $|\Pi_0 \mathcal{T}_M(\hat\psi)|_{L^1} \leq C\lambda^M \|\hat\psi\|_{\mathcal{B}^{L^1}}$. Therefore, since $\nu(\mathcal{D}) = 0$, (126) takes care of the first term in (128), and it suffices to show that the second term in (128) tends to zero in $L^1(I)$ as $t \to 0$.

Recall once more that Lemma 5.8 allows us to take a larger value of $\Gamma$ in (112) if necessary.) Estimate (142) in Step 3 implies

$$\|(\Pi_t - \Pi_0)(\hat\psi)\|_{L^1(I)} \le C\sqrt{t}|\hat\psi|_{C^0}\,.$$

Therefore, by (122), we have

$$\|(\Pi_t - \Pi_0)(\hat\phi_{t,M} - \hat\phi_M)\|_{L^1} \le C\sqrt{t}|\hat\phi_{t,M} - \hat\phi_M|_{C^0} = o(t)\,, \tag{129}$$

proving (127).

Finally, (127) immediately implies that

$$\lim_{t\to 0}\frac{1}{t}\int A\Pi_t[\hat\phi_{t,M} - \hat\phi_M]\,dx = -\int A\Pi_0((\mathrm{id} - \widehat{\mathcal{L}})^{-1}[\mathcal{T}_0(\widehat{\mathcal{L}}(\hat Y\hat\phi))'])\,dx\,. \tag{130}$$

In other words,

$$\lim_{t\to 0}\frac{1}{t}\int (A - A\circ f)\Pi_t[\hat\phi_{t,M} - \hat\phi_M]\,dx = -\int A(\mathcal{T}_0(\widehat{\mathcal{L}}(\hat Y\hat\phi)))'\,dx\,, \tag{131}$$

where we used (78). If $A$ is $C^1$, we can integrate by parts, and we find

$$\lim_{t\to 0}\frac{1}{t}\int (A - A\circ f)\Pi_t[\hat\phi_{t,M} - \hat\phi_M]\,dx = \int A'\mathcal{T}_0(\widehat{\mathcal{L}}(\hat Y\hat\phi))\,dx\,. \tag{132}$$

**Step 2: The first term of** (114)**: Computing** $\lim\frac{1}{t}(\widehat{\mathcal{L}}_{t,M} - \widehat{\mathcal{L}}_M)(\hat\phi_M)$**.**

In this step, we prove (118), (120), and (123), (124), for admissible pairs $(M,t)$. The following estimates will play a crucial part in the argument (their proof is given in Appendix C, it uses the fact that $t\mapsto f_t\in C^3$ is a $C^2$ map):

**Lemma 6.2** (Taylor series for $f_\pm^{-k}(x) - f_{t,\pm}^{-k}(x)$)**.** *Let $f_t$ satisfy the assumptions of Theorem 2.13. Recall the functions $Y_{k,t}$ from* (16)*, the maps $f_{t,\pm}^{-k}$ from* (1)*, and the smooth cutoff functions $\xi_k$ from Definition 4.7. Then there is $C>0$ so that for any $k\ge H_0$ and any $|s|\le\epsilon$, if* (111) *holds, then*

$$\sup_{y\in\widetilde I_k}\frac{|Y_{k,s}(y)|}{|(f_s^k)'(y)|} < C\frac{e^{2\gamma k}}{|(f_s^{k-1})'(c_{1,s})|^{1/2}}\,. \tag{133}$$

*In addition, for all $k\ge 1$, we have*

$$\sup_{x\in f_s^k(\widetilde I_k)}\left|\partial_x\Big(\frac{Y_{k,s}(f_{s,\pm}^{-k}(x))}{(f_s^k)'(f_{s,\pm}^{-k}(x))}\Big)\right| \le Ce^{3\gamma k}\,, \tag{134}$$

$$\int_{f_s^k(\widetilde I_k)}\left|\partial_x\Big(\frac{Y_{k,s}(f_{s,\pm}^{-k}(x))}{(f_s^k)'(f_{s,\pm}^{-k}(x))}\Big)\right|dx \le C\frac{e^{3\gamma k}\log(|(f_s^{k-1})'(c_{1,s})|^{1/2})}{|(f_s^{k-1})'(c_{1,s})|^{1/2}}\,, \tag{135}$$

$$\sup_{x\in f_s^k(\widetilde I_k)}\left|\partial_x^2\Big(\frac{Y_{k,s}(f_{s,\pm}^{-k}(x))}{(f_s^k)'(f_{s,\pm}^{-k}(x))}\Big)\right| \le Ce^{5\gamma k}\,, \tag{136}$$

$$\int_{f_s^k(\widetilde I_k)}\left|\partial_x^2\Big(\frac{Y_{k,s}(f_{s,\pm}^{-k}(x))}{(f_s^k)'(f_{s,\pm}^{-k}(x))}\Big)\right|dx \le C\frac{e^{5\gamma k}}{|(f_s^{k-1})'(c_{1,s})|^{1/2}}\,. \tag{137}$$

*Finally, for all $k\le M$ and all $x\in f_s^k(\widetilde I_k)$*

$$\left|f_\pm^{-k}(x) - f_{t,\pm}^{-k}(x) - t\frac{Y_k(f_\pm^{-k}(x))}{(f^k)'(f_\pm^{-k}(x))}\right| \le C|t|^2e^{4\gamma k}\,, \tag{138}$$

*and, for the same $k$ and $x$*

$$\Big|\frac{1}{(f_t^k)'(f_\pm^{-k}(x))}-\frac{1}{(f^k)'(f_{t,\pm}^{-k}(x))}-t\Big(\frac{Y_k(f_\pm^{-k}(x))}{(f^k)'(f_\pm^{-k}(x))}\Big)'\Big|\le C|t|^2e^{7\gamma k}\,. \tag{139}$$

We first prove (118). If $j>M$ then $\widehat{\mathcal{L}}_{t,M}(\hat\psi)(x,j)=\widehat{\mathcal{L}}_M(\hat\psi)(x,j)=0$. If $1\le j\le M$, since $\xi_j=\xi_{j,t}$ (recall the construction in Proposition 5.9),

$$\widehat{\mathcal{L}}_{t,M}(\hat\psi)(x,j)-\widehat{\mathcal{L}}_M(\hat\psi)(x,j)=0\,.$$

Therefore, we need only worry about $j=0$.

Recall the definition (76) of $\widehat{\mathcal{L}}_{t,M}(\hat\psi)(x,0)$. The definition (17) of $\hat Y_s$ (the shift in indices there mirrors that in (76)) together with (164) and (167) from the proof of Lemma 6.2 imply the following: Assume that $\varphi$ is $C^1$ and supported in $\widetilde I_k$. Then there exists $s(t)\in[0,t]$ so that

$$\begin{aligned}&\Big|\frac{\varphi(f_+^{-k}(x))}{|(f^k)'(f_+^{-k}(x))|}-\frac{\varphi(f_{t,+}^{-k}(x))}{|(f_t^k)'(f_{t,+}^{-k}(x))|}\Big|\\&\le|t||\varphi(f_+^{-k}(x))|\Big|\Big(\frac{Y_{k,s}(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}\Big)'\Big|+|t|\frac{|\varphi'(f_{s,+}^{-k}(x))|}{|(f_t^k)'(f_{t,+}^{-k}(x))|}\Big|\frac{Y_{k,s}(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}\Big|\,.\end{aligned}$$

Of course, the branch $f_-^{-k}$ is handled similarly. Therefore, summing over the inverse branches, and taking into account the contribution of $(1-\xi_k)(f_{t,+}^{-k}(x))-(1-\xi_k)(f_+^{-k}(x))$ (our assumptions imply that each $\psi_k'$ and $\xi_k'$ vanishes at the boundary of its support), we get $C>0$ so that for any $\hat\psi\in\mathcal{B}$ and any admissible pair $(M,t)$, using (133) and (134) from Lemma 6.2 and the upper bound (70) on $\lambda$

$$|(\widehat{\mathcal{L}}_{t,M}(\hat\psi)-\widehat{\mathcal{L}}_M(\hat\psi))|_{C_0}\le C|t|e^{5\gamma M}\lambda^M\|\psi\|_{\mathcal{B}}\le C|t|e^{6\gamma M}\|\psi\|_{\mathcal{B}}\,. \tag{140}$$

In view of (112), this proves (118).

Next, we show (120). Note that

$$\begin{aligned}\varphi(f_+^{-k}(x))\Big(\frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}\Big)'+\frac{\varphi'(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}\frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}&\\=\Big(\frac{\varphi(f_+^{-k}(x))Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}\Big)'&,\end{aligned}$$

and set

$$\mathcal{D}_M:=-(\mathcal{T}_0(\widehat{\mathcal{L}}_M(\hat Y\hat\phi_M)))'\in\mathcal{B}\,.$$

Clearly $\nu(\mathcal{D}_M)=0$, integrating by parts, and Lemma 6.2 implies that $\|\mathcal{D}_M\|_{\mathcal{B}}\le Ce^{12\gamma M}$ for all $M$.

Using that $\hat\phi_M'\in\mathcal{B}$, we may write the $t$-Taylor series of order two of

$$\frac{\hat\phi_{M,k-1}(f_+^{-k}(x))}{|(f^k)'(f_+^{-k}(x))|}-\frac{\hat\phi_{M,k-1}(f_{t,+}^{-k}(x))}{|(f_t^k)'(f_{t,+}^{-k}(x))|}\,,$$

and of its $x$-derivative. By (136), (138), and (139), this gives

$$\|\widehat{\mathcal{L}}_{t,M}(\hat\phi_M)-\widehat{\mathcal{L}}_M(\hat\phi_M)-t\mathcal{D}_M\|_{\mathcal{B}}\le Ce^{11\gamma M}|t|^2\,. \tag{141}$$

Since we can take $\Gamma$ in (112) as large as necessary, this establishes (120).

Since Lemma 4.11 implies that $\hat\phi_M$ is an eigenvector of $\widehat{\mathcal{L}}_M$ for an eigenvalue $\kappa_M$ close to 1 (so that the $\lambda^k$ factor can be replaced by $\kappa_M^k$, which is strictly smaller than

$Ce^{2\gamma k}$ by our choices), we get, using (133) and (135) (for $s=0$) from Lemma 6.2, as well as Lemma 4.1,

$$\sup_M \|\mathcal{D}_M\|_{\mathcal{B}^{L^1}} < \infty \,.$$

We do not claim that $\sup_M \|\mathcal{D}_M\|_{\mathcal{B}} < \infty$ or even that $\sup_M \|\mathcal{D}_M\|_{\mathcal{B}^0} < \infty$. However (up to increasing $\Gamma$ in (112)), (135) and (137) (for $s=0$) imply that

$$\sup_M \|\mathcal{D}_M\|_{\mathcal{B}^{H^1_1}} < \infty \,.$$

Set $\mathcal{D} = -\mathcal{T}_0(\widehat{\mathcal{L}}(\hat{Y}\hat{\phi})))'$. Clearly, $\nu(\mathcal{D}) = 0$, integrating by parts. The estimates we proved imply that $|\mathcal{D}_0 - \mathcal{D}_{M,0}|_{H^1_1} \to 0$, exponentially fast as $M \to \infty$, and that $\mathcal{D} \in \mathcal{B}^{H^1_1}$. This shows (123) and (124).

**Step 3: The second term of** (114)**: Estimating** $\frac{1}{t}(\Pi_t - \Pi_0)(\hat{\phi}_M) \in (C^1(I))^*$
In this step, the points are not necessarily falling from the tower, so that the analogues of the derivatives in Lemma 6.2 have nonintegrable spikes. Therefore, as already mentioned, we shall not only require horizontality, but we shall also need to perform integration by parts, using that the observable $A$ is $C^1$.

As before, the index $k$ ranges between 1 and $M$, where $(M,t)$ is an admissible pair. We focus on the branch $f_+^{-k}$, the other one is handled in a similar way.

Note for further use that in view of Proposition 5.9, Lemma 4.1 implies that there exists $C$ so that for any admissible pair $(M,t)$ and any $1 \le k \le M$

$$\Psi_{k,t} := \sup_{z\in[c_k,c_{k,t}]} \int_{c_k}^{z} \frac{\lambda^k}{|(f^k)'(f_+^{-k}(z))|}\,dx \le \int_{c_k}^{c_{k,t}} \frac{C}{\sqrt{x-c_k}}\,dx \le CL|t|^{1/2} \,. \tag{142}$$

(We used (13) to get $|c_{k,t} - c_k| = |h_t(c_k) - h_0(c_k)| \le L|t|$.) In particular,

$$|f_+^{-k}(c_{k,t}) - f_+^{-k}(c_k)| \le \Psi_{k,t} \le C|t|^{1/2} \,. \tag{143}$$

Assume to fix ideas that $c_k > c_{k,t}$, with $c_k$ and $c_{k,t}$ local maxima for $f^k$ and $f_t^k$, respectively (the other possibilities are treated similarly and left to the reader).

We first study the points in $f^k(\mathrm{supp}(\phi_{M,k}))$ for which $f_+^{-k}(x)$ exists but not $f_{t,+}^{-k}(x)$, i.e., the interval $[c_{k,t}, c_k]$. This gives the following contribution:

$$\begin{aligned}
&\int_{c_{k,t}}^{c_k} A(x)\frac{\lambda^k}{|(f^k)'(f_+^{-k}(x))|}\phi_{M,k}(f_+^{-k}(x))\,dx && (144)\\
&\qquad = \int_{f_+^{-k}(c_{k,t})}^{c_0} [(A(f^k(y)) - A(c_k)) + A(c_k)]\lambda^k \phi_{M,k}(y)\,dy \\
&\qquad = \int_{f_+^{-k}(c_{k,t})}^{c_0} A'(z_{k,t}(y))(c_k - c_{k,t})\lambda^k \phi_{M,k}(y)\,dy \\
&\qquad\qquad + A(c_k)\int_{f_+^{-k}(c_{k,t})}^{c_0} \lambda^k \phi_{M,k}(y)\,dy \,, && (145)
\end{aligned}$$

where $z_{k,t}(y) \in [c_{k,t}, c_k]$. Now

$$
\begin{aligned}
&|\int_{f_+^{-k}(c_{k,t})}^{c_0} A'(z_{k,t}(y))(c_k - c_{k,t})\lambda^k \phi_{M,k}(y)\, dy| \\
&\qquad \le \sup |A'||c_k - c_{k,t}| \int_{c_{k,t}}^{c_k} \frac{\lambda^k}{|(f^k)'(f_+^{-k}(x))|} \phi_{M,k}(f_+^{-k}(x))\, dx \\
&\qquad \le C \sup |A'||c_k - c_{k,t}| \sqrt{c_k - c_{k,t}} \\
&\qquad \le C \sup |A'| \sup_s |\alpha_s(c_k)||t|^{3/2} ,
\end{aligned}
$$

where we used Lemma 2.12 together with (142). Since there are $M$ terms and since $\lim_{t\to 0} M\sqrt{|t|} = 0$ for admissible pairs $(M,t)$, the relevant contribution of (144) is fully contained in the last line (145) of (144). (We shall see in a moment that (145) cancels out exactly with another term.)

Second, we need to consider

$$
\pm \int_{-1}^{c_{k,t}} \lambda^k A(x) \Big( \frac{\phi_{M,k}(f_+^{-k}(x))}{|(f^k)'(f_+^{-k}(x))|} - \frac{\phi_{M,k}(f_{t,+}^{-k}(x))}{|(f_t^k)'(f_{t,+}^{-k}(x))|} \Big)\, dx \tag{146}
$$

$$
= - \int_{-1}^{c_{k_t}} \lambda^k A'(x) [\tilde\phi_{M,k}(f_+^{-k}(x)) - \tilde\phi_{M,k}(f_{t,+}^{-k}(x))]\, dx \tag{147}
$$

$$
+ \lambda^k [A(x)(\tilde\phi_{M,k}(f_+^{-k}(x)) - \tilde\phi_{M,k}(f_{t,+}^{-k}(x)))]|_{-1}^{c_{k,t}} . \tag{148}
$$

where $\tilde\phi'_{M,k} = \phi_{M,k}$, $\tilde\phi_{M,k}(-1) = 0$ and the sign $\pm$ in line (146) depends on the sign of $(f^k)'(f_+^{-k}(x))$. One term in (148) vanishes because of the support of $\tilde\phi_{M,k}$. The other term is

$$
\lambda^k A(c_{k,t})(\tilde\phi_{M,k}(f_+^{-k}(c_{k,t})) - \tilde\phi_{M,k}(f_{t,+}^{-k}(c_{k,t}))) . \tag{149}
$$

Since $A(c_{k,t}) = A(c_k) + A'(z_{k,t})(c_{k,t} - c_k)$ for $z_{k,t} \in [c_{k,t}, c_k]$, and since $f_{t,+}^{-k}(c_{k,t}) = c_0$, (149) can be written as

$$
\lambda^k A(c_k)(\tilde\phi_{M,k}(f_+^{-k}(c_{k,t})) - \tilde\phi_{M,k}(c_0)) + H_{k,t} ,
$$

with $|H_{k,t}| \le C|t|^{3/2}$, uniformly in $k \le M$ for admissible pairs $(M,t)$ (recall (13) and (143)). Summing over $1 \le k \le M$ and dividing by $t$, we have proved that, as $t \to 0$, the contributions from (149) cancel out exactly with the singular terms from line (145). (Recall that $(M,t)$ are admissible, in particular (112) holds.)

The other term, (147), is

$$
\begin{aligned}
&- \int_{-1}^{c_{k,t}} \lambda^k A'(x) [\tilde\phi_{M,k}(f_+^{-k}(x)) - \tilde\phi_{M,k}(f_{t,+}^{-k}(x))]\, dx \\
&= - \int_{-1}^{c_{k,t}} \lambda^k A'(x) \phi_{M,k}(f_{u(t,x),+}^{-k}(x))[f_+^{-k}(x) - f_{t,+}^{-k}(x)]\, dx , \qquad (150)
\end{aligned}
$$

for $u = u(t,x) \in [0,t]$. To finish, we shall next prove that the sum over $1 \le k \le M$ of (150) divided by $t$ converges as $t \to 0$ and $(M,t)$ is an admissible pair.

Recalling the definition (16) of $Y_{k,s}$, the proof of Lemma 6.2 (in particular (164)) implies that there is $s = s(t,x) \in [0,t]$ so that

$$
f_{t,+}^{-k}(x) - f_+^{-k}(x) = t \frac{Y_{k,s}(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))} = t \sum_{j=1}^{k} \frac{X_s(f_s^j(f_{s,+}^{-k}(x)))}{(f_s^j)'(f_{s,+}^{-k}(x))} .
$$

Since $X$ is $C_1$, and since bounded distorsion holds for points who climb (Lemma 3.3), we find (recall (45) in the proof of Proposition 3.9)

$$\Big|\sum_{j=1}^{k}\frac{X_s(f_{s,+}^{j-k}(x))}{(f_s^j)'(f_{s,+}^{-k}(x))}-\frac{1}{f_s'(f_{s,+}^{-k}(x))}\sum_{j=1}^{k}\frac{X_s(f_{s,+}^{j-1-k}(c_{1,s}))}{(f_s^{j-1})'(c_{1,s})}\Big| \le Ce^{2\gamma k}|(f_s^{k-1})'(c_{1,s})|^{-1/2}\,.$$

Then, using horizontality, we find

$$\sum_{j=1}^{k}\frac{X_s(f_{s,+}^{j-1-k}(c_{1,s}))}{(f_s^{j-1})'(c_{1,s})}=-\frac{1}{(f_s^k)'(c_{1,s})}\sum_{\ell=k}^{\infty}\frac{X_s(f_s^{\ell}(c_{1,s}))}{(f_s^{\ell-k})'(c_{k+1,s})}\,.$$

The proof of Proposition 3.9 implies that the above expression is bounded, uniformly in $k\le M$ and admissible pairs $(M,t)$. (Here we use the uniform bounds from Lemma 5.8.) Finally, recalling the properties of the support of $\phi_{M,k}$

$$\begin{aligned}|\int_{c_k-e^{-\beta_1 k}}^{c_{k,t}}\frac{1}{|(f_s')(f_{s,+}^{-k}(x))|}\,dx| &\le C\int_{c_k-Ce^{-\beta_1 k}}^{c_{k,t}}\frac{1}{\sqrt{|x-c_{k,s}|}}\,dx\\ &= C\sqrt{|x-c_{k,s}|}\Big|_{c_k-Ce^{-\beta_1 k}}^{c_{k,t}}\,.\end{aligned}$$

Summarizing, we have proved that (150) divided by $t$ satisfies

$$\begin{aligned}&|\frac{1}{t}\int_{-1}^{c_{k,t}}\lambda^k A'(x)\phi_{M,k}(f_{u(t,x),+}^{-k}(x))[f_+^{-k}(x)-f_{t,+}^{-k}(x)]\,dx|\\ &\qquad=|\int_{-1}^{c_{k,t}}\lambda^k A'(x)\phi_{M,k}(f_{u(t,x),+}^{-k}(x))\frac{Y_{k,s}(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}\,dx|\\ &\qquad\le C\kappa_M^{-k}\sup|A'|\sup|\phi_{M,0}|e^{-\beta_1 k/2}\,.\end{aligned}$$

The bound in the third line above is summable over $k\ge 1$, uniformly in $M$.

Since it is easy to check for each fixed $k$ that

$$\begin{aligned}\lim_{t\to 0}&\int_{-1}^{c_{k,t}}\lambda^k A'(x)\phi_{M,k}(f_{u(t,x),+}^{-k}(x))\frac{Y_{k,s}(f_{s,+}^{-k}(x))}{(f_s^k)'(f_{s,+}^{-k}(x))}\,dx\\ &=\int_{-1}^{c_k}\lambda^k A'(x)\phi_k(f_+^{-k}(x))\frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x)}\,dx\,,\end{aligned}$$

we have proved that

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\int_I A(x)\big(\Pi_t(\phi_M)-\Pi(\phi_M)\big)(x)\,dx\\ &=-\sum_{k=1}^{\infty}\sum_{\varsigma\in\{+,-\}}\pm\int_{-1}^{c_k}\lambda^k A'(x)\phi_k(f_+^{-k}(x))\frac{Y_k(f_\varsigma^{-k}(x))}{(f^k)'(f_\varsigma^{-k}(x)}\,dx\\ &=-\sum_{k=1}^{\infty}\sum_{\varsigma\in\{+,-\}}\int_{-1}^{c_{k-1}}A'(f(y))\frac{\lambda^{k-1}}{|(f^{k-1})'(f_\varsigma^{-(k-1)}(y)|}\lambda(\phi_k\cdot Y_k)(f_\varsigma^{-(k-1)}(y))\,dy\,.\end{aligned}$$

(The sign in the second line above comes from in (146), that is, it is the sign of $(f^{k-1})'(f_\varsigma^{-(k-1)}(x)$.) The fixed point property of $\hat\phi$ implies $\lambda\phi_{k+1}=\phi_k\xi_k$. Therefore, setting $\hat\xi=(\xi_k)$, and recalling the shift in indices in the definition (17) of $\hat Y$, we have

$$\lim_{t\to 0}\frac{1}{t}\int_I A(x)\big(\Pi_t(\hat\phi_M)-\Pi(\hat\phi_M)\big)(x)\,dx = -\int_I (A'\circ f)\cdot\Pi(\hat\xi\hat Y\hat\phi)\,dy$$

$$= -\lambda\int_I A'\cdot(\Pi\circ(\mathrm{id}-\mathcal{T}_0)\circ\widehat{\mathcal{L}})(\hat Y\hat\phi)\,dy\,, \tag{151}$$

where we used (78). This ends Step 3 and the proof of Theorem 2.13.

## Appendix A. Relating the conjugacies $h_t$ with the infinitesimal conjugacy $\alpha$

We show here that $\alpha$ deserves to be called an infinitesimal conjugacy.

*Proof.* (Proof of Proposition 2.15.) Let $\alpha_t\colon[-1,1]\to\mathbb{R}$ be the unique continuous solution for the TCE

$$v_t=\alpha_t\circ f_t - f_t'\cdot\alpha_t.$$

Since the family $\{\alpha_t\}_{|t|<\epsilon}$ is equicontinuous and the solutions are unique, it is easy to see that $(t,x)\to\alpha_t(x)$ is a continuous and bounded function in $(-\epsilon,\epsilon)\times[-1,1]$. Note that $\alpha_t(-1)=\alpha_t(1)=0$ for every $t$. For each $x_0\in[-1,1]$ $t_0\in(-\epsilon,\epsilon)$, the Peano theorem ensures that the ODE

$$\partial u_s(t_0,x_0)|_{t=s}=\alpha_t(u_t(t_0,x_0))\,,\quad u_{t_0}(t_0,x_0)=x_0 \tag{152}$$

admits a $C^1$ solution $u_t(t_0,x_0)$. It is not difficult to see that this solution is defined for every $t\in(-\epsilon,\epsilon)$. Since $f_t$ is a deformation, there exists an unique conjugacy $h_t$ such that

$$f_t\circ h_t=h_t\circ f_0\,.$$

If $x_{t_0}$ is an eventually periodic point for $f_{t_0}$, since all periodic points are hyperbolic there exists an analytic continuation $x_t$ for $x_0$. Then $x_t=h_t(x_0)$. An easy calculation shows that $u_t(t_0,x_0)=h_t(x_0)$ is a solution of the above ODE.

*Claim:* If $w_t$ is a solution of the ODE $\partial_t w_t=\alpha_t(w_t)$ and $w_{t_0}=h_{t_0}(x_0)$ for some $t_0$ and eventually periodic point $x_0$ then $w_t=h_t(x_0)$ for every $t$. Indeed, denote $w_t^n=f_t^n(w_t)$. Note that the TCE implies that $w_t^n$ and $f_t^n(h_t(x_0))$ are also solutions of the ODE above. Since $\alpha_t(x)$ is a bounded function

$$|w_t^n-f_t^n(h_t(x_0))|\le|w_t^n-w_{t_0}^n|+|f_{t_0}^n(h_{t_0}(x_0))-f_t^n(h_t(x_0))|\le 2\sup_{(t,x)}|\alpha_t(x)||t-t_0|\,.$$

So if $t$ is sufficiently close to $t_0$ then

$$|f_t^n(w_t)-f_t^n(h_t(x_0))|<\delta$$

for every $n$. If $\delta$ is small enough, since the orbit of $h_t(x_0)$ by $f_t$ eventually lands on a repelling periodic point, it follows that $w_t=h_t(x_0)$ for $t$ close enough to $t_0$. This argument implies that

$$\{t:\ w_t=h_t(x_0)\}$$

is an open set in $(-\epsilon,\epsilon)$. Since it is obviously a closed set, it follows that $w_t=h_t(x_0)$ for every $t$. This finishes the proof of the claim.

In particular this claim implies the uniqueness of the solution of the ODE when $x_0$ is an eventually periodic point.

Now let $x$ be a point that is not eventually periodic for $f_0$. We can find sequences $p_n$, $q_n$ of eventually periodic points for $f_0$ such that

$$p_n < x < q_n$$

and $\lim_n p_n = \lim_n q_n = x$. Let $w_t$ be a solution for $\partial_t w_t = \alpha_t(w_t)$ such that $w_0 = x$. The claim above implies that

$$h_t(p_n) < w_t < h_t(q_n)$$

for every $n$. Since $h_t$ is continuous we get $\lim_n h_t(p_n) = \lim_n h_t(q_n) = h_t(x)$, so $h_t(x) = w_t$, for every $t$. □

## Appendix B. The proof of Lemma 4.11 on truncated operators $\widehat{\mathcal{L}}_M$

In this appendix, we give more details regarding the proof of the Lasota-Yorke inequality (90) in the proof of Lemma 4.11.

Let $D$ be the derivative map (see e.g. [7, §3.2]) from $BV$ to Radon measures, so that $D(\mathcal{B}^{BV})$ is a subset of Radon measures on $\hat{I}$. (Radon measures are continuous linear functionals on the space of continuous functions with compact support. We are using $\mathcal{T}_M$, so we are only concerned with a finite, i.e. compact part of the tower.)

Consider $\mathbb{L}^N = D\widehat{\mathcal{L}}^N D^{-1}$ acting on $D(\mathcal{B}^{BV})$. Write $\mathcal{T}_M^D = D\mathcal{T}_M D^{-1}$. Then, we have

$$D\mathcal{T}_M \widehat{\mathcal{L}}^N D^{-1} = \mathcal{T}_M^D \mathbb{L}^N .$$

We shall decompose $\mathcal{T}_M^D \mathbb{L}^N$ on $D(\mathcal{B}^{BV})$ into a compact operator and a continuous operator with norm bounded by $C\Theta^{-N}$, with $C$ uniform in $M$: By the Leibniz formula (see e.g. [7, Lemma 3.2], noting that the $\psi_k$ are regular), we find, as operators acting on Radon measures (see [7, p. 155])

$$\mathbb{L}^N = \mathcal{Q}_N + \mathcal{M}_N , \tag{153}$$

where $\mathcal{Q}_N(m)$ only depends on $D^{-1}(m)$. It is not difficult to see that (58) and Lemma 3.1 together with the fact that $\sup_k \mathrm{var}\xi_k < \infty$ imply that $\mathcal{Q}_N$ is a bounded operator on $D(\mathcal{B}^{BV})$, whose norm is bounded by $C_0^N$ for some constant $C_0 \geq 1$ which is independent of $M$. Denoting by $\|m\|_{Radon}$ the norm Radon measure, note also that (91) implies that

$$\|D^{-1}(\hat{\psi})\|_{BV} = \|\hat{\psi}\|_{Radon} \leq C|\hat{\psi}|_{L^1} .$$

Next, using the fact that if $m_0 \in D(\mathcal{T}_M \mathcal{B}^{BV})$ then $m \mapsto (D^{-1}m)m_0$ is compact on $D(\mathcal{B}^{BV})$ (see e.g. [7, Proposition 3.3] for a proof of this Arzelà-Ascoli type result), it is easy to see that $\mathcal{T}_M^D \mathcal{Q}_N$ is compact on $D(\mathcal{B}^{BV})$ for any $M$ and $N$.

Finally, one shows that $\mathcal{M}_N$ is continuous on $D(\mathcal{B}^{BV})$, and by Lemma 3.5,

$$\|\mathcal{M}_N\|_{\mathcal{B}^{BV}}^{1/N} \leq c(\delta)^{-1/N} \max\left(\frac{1}{\lambda}, \frac{e^{2\gamma}\lambda}{\lambda_c^{1/2}}, \frac{1}{\sigma}, \frac{1}{\rho}\right) . \tag{154}$$

This ends the proof of (90).

## Appendix C. Proof of Lemma 6.2 on Taylor expansions

As usual, we consider $f_+^{-k}$, the other branch is similar. The assumptions imply that $\partial_t f_t|_{t=s} = X_s \circ f_s$, where $X_s \circ f_s$ is $C^2$ and horizontal for $f_s$.

We prove (133) and (134)–(136) for $s = 0$, the general case then follows from Lemma 5.8, using that for all $H_0 \le k \le M$ so that (111) holds, we may take $\xi_{k,t} = \xi_k$ by Proposition 5.9.

By horizontality, the estimate (45) in the proof of Proposition 3.9 (using the notation $w_k(y)$ introduced there) gives $C > 0$ so that for any $k \ge H_0$

$$\sup_{y \in I_k} \frac{|Y_k(y)|}{|(f^k)'(y)|} = \left| \sup_{y \in I_k} \sum_{j=1}^{k} \frac{X(f^j(y))}{(f^j)'(y)} \right| = \sup_{y \in I_k} |w_k(y)| \le C \frac{e^{2\gamma k}}{|(f^{k-1})'(c_1)|^{1/2}} , \tag{155}$$

proving (133).

For the claim (134) on the derivative, note that

$$\partial_x \frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))} = \frac{1}{(f^k)'(f_+^{-k}(x))} \partial_y \frac{Y_k(y)}{(f^k)'(y)} , \tag{156}$$

with

$$\begin{aligned} \partial_y \frac{Y_k(y)}{(f^k)'(y)} &= \sum_{j=1}^{k} \partial_y \frac{X(f^j(y))}{(f^j)'(y)} \\ &= \sum_{j=1}^{k} X'(f^j(y)) - \sum_{j=1}^{k} X(f^j(y)) \sum_{\ell=0}^{j-1} \frac{f''(f^\ell(y))}{(f^{j-\ell})'(y) f'(f^\ell(y))} . \end{aligned} \tag{157}$$

We shall use the estimates in the proof of Lemma 4.1: For $y \in I_k$, the bound (64) says that $|(f^m)'(y)| \ge C_2 e^{-\gamma k} |(f^{m-1})'(c_1)| |(f^{k-1})'(c_1)|^{-1/2}$ for $1 \le m \le k$, while (62) gives $|f'(f^\ell(y))| \ge C e^{-\gamma \ell}$ for $1 \le \ell \le k$. (These bounds do not use horizontality.)

The second term in the right-hand-side of (157) can be decomposed as

$$\begin{aligned} &- \sum_{j=1}^{k} X(f^j(y)) \sum_{\ell=0}^{j-1} \frac{f''(f^\ell(y))}{(f^{j-\ell})'(y) f'(f^\ell(y))} \\ &\qquad = - \frac{f''(f(y))}{f'(y)} \sum_{j=1}^{k} \frac{X(f^j(y))}{(f^j)'(y)} - \sum_{\ell=1}^{k-1} \frac{f''(f^\ell(y))}{f'(f^\ell(y))} \sum_{j=\ell+1}^{k} \frac{X(f^j(y))}{(f^{j-\ell})'(y)} . \end{aligned} \tag{158}$$

By (64) for $m = 1$, combined with (133) (which holds by horizontality), we find

$$\left| \frac{f''(f(y))}{f'(y)} \sum_{j=1}^{k} \frac{X(f^j(y))}{(f^j)'(y)} \right| \le C e^{3\gamma k} .$$

The second term in the right-hand-side of (158) does not require horizontality, only (64) and (62), which give

$$\sum_{\ell=1}^{k-1} \frac{|f''(f^\ell(y))|}{|f'(f^\ell(y))|} \sum_{j=\ell+1}^{k} \frac{|X(f^j(y))|}{|(f^{j-\ell})'(y)|} \le C e^{2\gamma k} |(f^k)'(c_1)|^{1/2} . \tag{159}$$

Remembering (156), and using again (64) (for $m = k$), we have proved (134).

Using the above bounds for (134), singling out the terms in (159) corresponding to $j = \ell + 1$, and noting that for $\epsilon_k > 0$

$$\int_{f^k[\epsilon_k,1]} \frac{1}{|f'(f_+^{-k}(x))||(f^k)'(f_+^{-k}(x))|}\, dx = \int_{\epsilon_k}^1 \frac{1}{|f'(y)|}\, dy \le C|\log(\epsilon_k)|\,,$$

we find (135).

The proof of (136) is similar. We start by noting that $\partial_x^2 \frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))} =$

$$\partial_x \frac{1}{(f^k)'(f_+^{-k}(x))} \partial_y \frac{Y_k(y)}{(f^k)'(y)} + \frac{1}{((f^k)'(f_+^{-k}(x))^2} \partial_y^2 \frac{Y_k(y)}{(f^k)'(y)}\,, \tag{160}$$

where $\partial_y^2 \frac{Y_k(y)}{(f^k)'(y)} =$

$$\sum_{j=1}^k X''(f^j(y))(f^j)'(y) - \sum_{j=1}^k X'(f^j(y)) \sum_{\ell=0}^{j-1} \frac{f''(f^\ell(y))(f^\ell)'(y)}{f'(f^\ell(y))} \tag{161}$$

$$- \sum_{j=1}^k X(f^j(y)) \Big[ \sum_{\ell=0}^{j-1} \frac{f'''(f^\ell(y))(f^\ell)'(y)}{(f^{j-\ell})'(y) f'(f^\ell(y))} - \frac{f''(f^\ell(y))^2 (f^\ell)'(y)}{(f^{j-\ell})'(y)(f'(f^\ell(y)))^2} \tag{162}$$

$$- f''(f^\ell(y)) \sum_{i=0}^{j-\ell-1} \frac{f''(f^i(y))}{(f^{j-\ell-i})'(y) f'(f^\ell(y)) f'(f^i(y))} \Big]\,. \tag{163}$$

The first term in (160) is bounded by $e^{5\gamma k}$ in view of (159) and (58) from Lemma 4.1. The first term in (161) is bounded by $C|(f^{k-1})'(c_1)|^{1/2}$, in view of Lemma 3.3 and Lemma 3.4. Hence, dividing by $((f^k)'(f_+^{-k}(x))^2$, and using Lemma 3.4, the contribution of this term is bounded by $Ce^{2\gamma k}|(f^{k-1})'(c_1)|^{-1/2}$.

Using again the same observations, we find that the second term in (161) is bounded by $Ce^{\gamma k}|(f^{k-1})'(c_1)|^{1/2}$. Dividing by $((f^k)'(f_+^{-k}(x))^2$, we get a contribution bounded by $Ce^{3\gamma k}|(f^{k-1})'(c_1)|^{-1/2}$.

For (162), one must distinguish between the terms for $\ell = 0$, for which horizontality gives a bound $Ce^{4\gamma k}|(f^{k-1})'(c_1)|$ (note the factor $(f'(y))^2$ in the denominator), and the terms where $\ell \ge 1$, for which a straightforward estimate, using the remarks above (and in particular Lemma 3.3), gives an upper bound of the form $Ce^{3\gamma k}|(f^{k-1})'(c_1)|$. Dividing by $((f^k)'(f_+^{-k}(x))^2$, we get a contribution bounded by $Ce^{5\gamma k}$.

For (163), if $\ell = 0$ and $i = 0$, horizontality gives a bound $Ce^{4\gamma k}|(f^{k-1})'(c_1)|^{1/2}$, while if $i + \ell \ge 1$, we get a bound $Ce^{3\gamma k}|(f^{k-1})'(c_1)|$. Dividing by $((f^k)'(f_+^{-k}(x))^2$, we get a contribution $\le Ce^{5\gamma k}$. This ends the proof of (136).

Using the bounds we just got for (136), the proof of (137) is easy: we need only be concerned with the contributions from (162) and (163). The fact that we are using an $L^1$ norm instead of a supremum produces an upper bound of $Ce^{3\gamma k}|(f^{k-1})'(c_1)|^{1/2}$ instead of $Ce^{3\gamma k}|(f^{k-1})'(c_1)|^1$ (use $\int_{\epsilon_k}^1 |f'(y)|^{-2}\, dy \le |\epsilon_k|^{-1}$). Dividing by $((f^k)'(f_+^{-k}(x))^2$ gives (137).

In view of the more complicated estimates to follow, we notice the following pattern: The dangerous factors in the above estimates are powers of $f'(y)$ in the denominator and factors $(f^\ell)'(y)$ for large $\ell$ in the numerator. The "white knight" available to fight them is a power of $(f^k)'(y)$ in the denominator. An additional such power appears each time we differentiate with respect to $x$. The terms for which the power of $f'(y)$ in the denominator exceeds that of $(f^k)'(y)$ in the numerator can be handled by horizontality. The price to be paid for the control is a power of $e^{\gamma k}$.

We turn to (138) and (139). If $(x,t)\mapsto \Phi_t(x)\in I$ is a $C^1$ map on $I\times[-\epsilon,\epsilon]$ so that $x\mapsto\Phi_t(x)$ is invertible, then we have

$$\partial_t \Phi_t^{-1}(x)|_{t=s} = \frac{(\partial_t\Phi_t|_{t=s})\circ\Phi_s^{-1}(x)}{(\partial_x\Phi_s)\circ\Phi_s^{-1}(x)}\,, \tag{164}$$

and

(165)

$$\begin{aligned}
\partial^2_{tt}\Phi_t^{-1}(x)|_{t=s} &= \partial_t \frac{(\partial_t\Phi_t)\circ\Phi_t^{-1}(x)}{(\partial_x\Phi_t)\circ\Phi_t^{-1}(x)}|_{t=s}\\
&= \frac{1}{(\partial_x\Phi_s)\circ\Phi_s^{-1}(x)}\cdot\left(\partial^2_{tt}\Phi_t|_{t=s}\circ\Phi_s^{-1}(x)+\partial^2_{xt}\Phi_t|_{t=s}\circ\Phi_s^{-1}(x)\cdot\partial_t\Phi_t^{-1}(x)|_{t=s}\right)\\
&- \frac{(\partial_t\Phi_t|_{t=s})\circ\Phi_s^{-1}(x)}{\partial_x\Phi_s)\circ\Phi_s^{-1}(x)}\,\frac{(\partial^2_{tx}\Phi_s|_{t=s})\circ\Phi_s^{-1}(x)+(\partial^2_{xx}\Phi_s)\circ\Phi_s^{-1}(x)\cdot\partial_t\Phi_t^{-1}(x)|_{t=s}}{\partial_x\Phi_s)\circ\Phi_s^{-1}(x)}\,.
\end{aligned}$$

Since $t\mapsto f_t\in C^2(I)$ is $C^2$, we may apply the above to $\Phi_t(x)=f_t^k(x)$ restricted to a suitable domain. The right-hand-side of (164) is just $Y_{k,s}(f_{s,+}^{-k}(x))/(f_s^k)'(f_{s,+}^{-k}(x))$. Then, a Taylor series of order 2 gives

$$f_+^{-k}(x)-f_{t,+}^{-k}(x) = t\frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}+t^2F_k(x,s)\,, \tag{166}$$

where $x$ is as in (138) and $s\in[0,t]$. In order to estimate $F_k(x,s)$, we look at the various terms in (165). The (identical) factors $\partial_t\Phi_t^{-1}$ and $(\partial_t\Phi_t/\partial_x\Phi_t)\circ\Phi_t^{-1}$ can be bounded by (133). Since $\partial^2_{xt}\Phi_t|_{t=0}=\partial_y Y_k$, the two terms containing this expression can be controlled, when divided by $(\partial_x\Phi_t)\circ\Phi_t^{-1}$, respectively by $e^{-2\gamma k}|(f^k)'(c_1)|^{1/2}$, by using the ideas to bound (157). Analysing the term containing $\partial^2_{xx}\Phi_t$ is of the same type as (but simpler than) what we did for $\partial^2_{yy}Y_k$, and the available factor $|(f^k)'(c_1)|^{-1}$ gives the right control. The only new expression is

$$\partial^2_{tt}\Phi_t|_{t=s}(x) = Z_{k,s}(x) := \partial_t Y_{k,t}(x)|_{t=s} = \lim_{t\to s}\frac{Y_{k,t}(x)-Y_{k,s}(x)}{t-s}\,.$$

This involves functions such as $f'$, $f''$, $f''$, $X_s$, and $X'_s$, but also $\partial_t X_t$. The dominant term contains a factor $|(f^k)'(y)$, which can be controlled by $(\partial_x\Phi_s)\circ\Phi_s^{-1}$ in the denominator.

Finally, using

$$\frac{1}{(f_t^k)'(f_+^{-k}(x))}-\frac{1}{(f^k)'(f_{t,+}^{-k}(x))} = (f_{t,+}^{-k}(x)-f_+^{-k}(x))'\,, \tag{167}$$

we find

$$\frac{1}{(f_t^k)'(f_+^{-k}(x))}-\frac{1}{(f^k)'(f_{t,+}^{-k}(x))} = t\left(\frac{Y_k(f_+^{-k}(x))}{(f^k)'(f_+^{-k}(x))}\right)'+t^2G_k(x,s)\,, \tag{168}$$

for $x$ as in (139) and $s\in[0,t]$. The new derivatives appearing in $G_k$ are $\partial^3_{xxx}\Phi$, $\partial^3_{txx}\Phi$ and $\partial^3_{ttx}\Phi$ (but not $\partial^3_{ttt}\Phi$, which is a priori undefined). The claimed estimates on $\sup|G_k|$ can be obtained by horizontality, similarly to those for $F_k$, using now the $x$-derivative of (165) and exploiting in addition to the previous remarks the bound

(136). The cancellation pattern described above emerges again. The computations are straightforward, although cumbersome to write, and left to the reader.

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