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Abstract

For a measure preserving transformation, entropy zero means that there is no increasing σ-algebra. In this note, we prove that a similar phenomenon occurs for C^2 diffeomorphisms when considering the increment between the partial entropies associated with different exponents.

1 Introduction

We consider in this note a $C^{1+\text{Hölder}}$ diffeomorphism preserving a Borel probability measure μ on a compact manifold M. Then smooth ergodic theory provides geometric objects and numbers that we quickly review (see [1], [10]). Firstly, Oseledets Theorem yields a μ-almost everywhere decomposition of the tangent space

$$T_xM \ = \ E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$$

and invariant functions $\lambda_1(x) > \cdots > \lambda_{r(x)}(x)$ such that

$$0 \neq v \in E_i(x) \iff \lim_{n \to \pm\infty} \frac{1}{n} \ln \|D_x f^n v\| = \lambda_i(x),$$

for some Riemannian metric on M. Then, Pesin theory says that for μ-almost every x, for all $\lambda > 0$, the set

$$W^\lambda(x) \ := \ \{y \in M; \limsup_{n\to\infty} \frac{1}{n} \ln d(f^{-n}x, f^{-n}y) \leq -\lambda\}$$

is an embedded d_λ-dimensional disk, where $d_\lambda(x) = \sum_{i;\lambda_i(x)\geq\lambda} \mathrm{Dim}\ E_i(x)$. For all $\lambda > 0$, the $W^\lambda(x)$ form a measurable lamination $\mathcal{W}^\lambda$. To each foliation $\mathcal{W}^\lambda$ is associated a family of conditional measures μ^λ, where for μ-a.e. x, μ^λ_x is a measure on $W^\lambda(x)$, up to a multiplicative constant (see e.g. [7]). Moreover, by Ledrappier-Young, conditional measures μ^λ_x associated to $\mathcal{W}^\lambda$ are exact-dimensional: there exists an invariant function $\delta_\lambda(x)$ such that for μ-almost every x,

$$\lim_{\varepsilon\to 0} \frac{\ln \mu^\lambda_x B^\lambda(x,\varepsilon)}{\ln \varepsilon} \ = \ \delta_\lambda(x),$$

where $B^\lambda(x,\varepsilon)$ is the ball of radius ε in $W^\lambda(x)$ for the induced metric. Finally, define $\gamma_i(x)$ by:

$$\delta_\lambda(x) = \sum_{i;\lambda_i(x)\geq\lambda} \gamma_i(x).$$

The number $\gamma_1(x)$ is the dimension of the conditional measures $\mu^{\lambda_1}_x$, i.e., $\gamma_1(x) = \delta_{\lambda_1}(x)$. In particular, if $\mu^{\lambda_1}_x$ is a Dirac measure, then $\gamma_1(x) = 0$. This property is not true any more for $\gamma_i(x)$ when $i \neq 1$. In Section 3 we describe a system in dimension 3 where the middle direction is integrable, the conditional measures $\mu^{\lambda_2}_x$ are Dirac measures, and $\gamma_2(x) = 1$. Our main result is that the vanishing of some $\gamma_i(x)$ corresponds to a stronger property: loosely speaking, the conditional measures associated to $\mathcal{W}^{\lambda_i}$ are carried by lower dimensional leaves.

Theorem 1 *Let f be a C^2 diffeomorphism preserving a Borel probability measure μ on a compact manifold M. With the above notations, for $\lambda < \lambda'$ and for an invariant set A of positive measure, the following properties are equivalent:*

1. *$\delta_\lambda(x) = \delta_{\lambda'}(x)$ for μ-almost every $x \in A$,*

2. $\sum_{i;\lambda\leq\lambda_i(x)<\lambda'} \gamma_i(x) = 0$ *for* μ*-almost every* $x \in A$

3. *for* μ*-almost every* $x \in A$, $\mu_x^\lambda = \mu_x^{\lambda'}$,

4. *for* μ*-almost every* $x \in A$, μ_x^λ *is carried by a single* $W^{\lambda'}$ *leaf.*

Theorem 1 answers a question by A. Katok and F. Rodriguez Hertz in [6]. Indeed, one expects that for the action of a higher rank abelian group which is truly of higher rank, either $\gamma_i(x) = 0$ or μ^{λ_i} is equivalent to the product of $\mu^{\lambda_{i+1}}$ and a transversal Lebesgue class (see [7], [4], [3], [5], [6] for surveys and results). Theorem 1 is relevant since it relates some global entropy property (the vanishing of some γ_i) to the geometry of the invariant measure. The proof of Theorem 1 is given in section 2.

2 Proof of Theorem 1

Step 1. Observe first that the equivalence between property 1) and property 2) is immediate from the definition of $\gamma_i(x)$. The equivalence between the geometric properties 3) and 4) is immediate as well. Moreover, it is also immediate that if the geometric property 3) is satisfied, then property 1) holds.

It remains to show that property 2) implies property 4). We may assume that the measure μ is ergodic. Indeed, invariant sets are $\mathcal{W}^\lambda$-saturated for any $\lambda > 0$ and the decomposition of μ into ergodic measures can be seen as a decomposition into conditional measures associated with the σ-algebra of invariant sets. So, for μ-almost every x, μ_x^λ is also the conditional measure associated to the ergodic component μ_x and 2) $\Rightarrow$ 4) is a property of the conditional measures μ_x^λ. Since μ is ergodic, the functions $r, \lambda_i, \text{Dim } E_i, \delta_\lambda$ and γ_i are almost everywhere constant and from here on, we will not write the dependence in x of these functions.

We assume that $0 \leq i < j, \lambda_i > \lambda_j > 0$ and we write $W^i, W^j, \mathcal{W}^i, \mathcal{W}^j$ for respectively

$W^{\lambda_i}, W^{\lambda_j}, \mathcal{W}^{\lambda_i}, \mathcal{W}^{\lambda_j}$. For $i = 0$, we mean $W^0_x = \{x\}$. By the above discussion, Theorem 1 follows from the following proposition:

Proposition 2 *With the above notations, assume that $\gamma_k = 0$ for $i < k \leq j$. Then, for μ-almost every $x \in A$, $\mu_x^{\lambda_j}$ is carried by a single W^i leaf.*

Step 2: Subordinate partitions and leafwise entropy. Recall that a measurable partition η is called *subordinate* to the lamination $\mathcal{W}$ if

1. η is increasing: $f^{-1}\eta$ refines η,
2. η generates: $f^{-n}\eta$ tends to the partition into points as $n \to \infty$,
3. for μ-almost every x, $\eta(x)$ is a neighborhood of x inside the leaf $W(x)$ and
4. for μ-almost every x, $\cup_{n\geq 0} f^n\big(\eta(f^{-n}x)\big)$ is the whole leaf $W(x)$.

Our proof relies on the following lemma:

Lemma 3 *With the above notations, there exist measurable partitions η_i subordinate to $\mathcal{W}^i$ and η_j subordinate to $\mathcal{W}^j$ such that*

$$\eta_i > \eta_j \quad \text{and} \quad f^{-1}\eta_i \;=\; \eta_i \vee f^{-1}\eta_j. \tag{2.1}$$

Subordinate partitions η_i, η_j are constructed following the line presented in [10] (where the corresponding constructions are for the random diffeomorphisms case) or the original construction of Ledrappier and Young [8, 9]. As observed in [12, Proposition IX.2.7] (where the corresponding constructions are for the endomorphisms case), a careful simultaneous construction of η_i and η_j ensures property (2.1). Take for η_0 the partition of M into points.

In the rest of the proof, we fix η_i, η_j given by Lemma 3 and two families μ^i_x, μ^j_x of conditional probability measures associated respectively with η_i and η_j. Following [9], we introduce the leafwise entropies h_i and h_j

$$h_i = H_\mu(f^{-1}\eta_i|\eta_i) = \int -\ln \mu_x^i(f^{-1}\eta_i(x))\mathrm{d}\mu,$$
$$h_j = H_\mu(f^{-1}\eta_j|\eta_j) = \int -\ln \mu_x^j(f^{-1}\eta_j(x))\mathrm{d}\mu$$

and recall the entropy formula: $h_\ell = \sum_{k\le\ell} \gamma_k\lambda_k$, for $\ell = i, j$, so that:

$$h_j - h_i = \sum_{k;i<k\le j} \gamma_k\lambda_k = \int \ln \frac{\mu_x^i(f^{-1}\eta_i(x))}{\mu_x^j(f^{-1}\eta_j(x))}\mathrm{d}\mu.$$

Using the property (2.1), we obtain

$$\int \ln \frac{\mu_x^i(f^{-1}\eta_j(x))}{\mu_x^j(f^{-1}\eta_j(x))}\mathrm{d}\mu = \sum_{k;i<k\le j} \gamma_k\lambda_k. \tag{2.2}$$

Remark 1 *In the limit case when $i = 0$, proposition 2 is well known: μ_x^0 is the Dirac measure at x and equation (2.2) reads: $-\int \ln \mu_x^j(f^{-1}\eta_j(x))\mathrm{d}\mu = 0$. This means that μ_x^j is carried by a single element of $f^{-1}\eta_j$. Since $f^{-n}\eta$ generates, the conditional measures μ_x^j are Dirac measures. Since $\bigcup_{n\ge0} f^n\big(\eta_j(f^{-n}x)\big)$ is the whole leaf $W^j(x)$, μ-a.e. conditional measure $\mu_x^{\lambda_j}$ is the Dirac measure μ_x^0. Our statement is that this property still hold for intermediate foliations ($i > 0$).*

Step 3: Proof of Proposition 2. For any element A of the partition η_j, define

$$I(A) := \int_A \ln \frac{\mu_y^i(f^{-1}\eta_j(y))}{\mu_y^j(f^{-1}\eta_j(y))}\mathrm{d}\mu_A^j(y).$$

Then $h_j - h_i = \int_{M/\eta_j} I(A)\mathrm{d}\mu_{\eta_j}(A)$, where μ_{η_j} is the quotient measure of μ on M/η_j.

Since $h_j = H_\mu(f^{-1}\eta_j|\eta_j) < +\infty$, there exists a countable partition with finite entropy $\{B_k, k\ge1\}$ such that $f^{-1}\eta_j = \eta_j \vee \{B_k, k\ge1\}$ (see [13, Lemma 10.2]). So for μ_{η_j}-almost every A, we partition A by $A_k := A\cap B_{k'}$ in such a way that $\mu_A(A_k) > 0$ and that the partition $\{A_k\}$ is the trace of $f^{-1}\eta_j$ on A (μ_A-mod 0).

For two probability measures ν_1, ν_2 with $\nu_2 \ll \nu_1$ and $\rho := \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1}$, we consider the relative entropy $H(\nu_2,\nu_1) = H(\rho\cdot\nu_1,\nu_1)$ given by

$$H(\rho\cdot\nu_1,\nu_1) = \int \rho\ln\rho\mathrm{d}\nu_1 \ge 0.$$

The relative entropy $H(\rho \cdot \nu_1, \nu_1)$ is zero iff $\rho \equiv 1$ holds ν_1-a.e. (or equivalently $\nu_1 = \nu_2$). Now noting the fact that

$$\mu^j_A(A_k) = \int_{A/\eta_i} \mu^i_B(A_k) \mathrm{d}(\mu^j_A)_{\eta_i}(B), \tag{2.3}$$

where $(\mu^j_A)_{\eta_i}$ is the measure on the quotient space A/η_i associated with the measure μ^j_A on A, we have the following (with the convention $0 \cdot \ln 0 = 0$):

$$\begin{aligned}
I(A) &= \sum_k \int_{A_k} \ln \frac{\mu^i_x(f^{-1}\eta_j(x))}{\mu^j_x(f^{-1}\eta_j(x))} \mathrm{d}\mu^j_A(x) \\
&= \sum_k \int_{A_k} \ln \frac{\mu^i_x(A_k)}{\mu^j_A(A_k)} \mathrm{d}\mu^j_A(x) \\
&= \sum_k \mu^j_A(A_k) \cdot \int_{A/\eta_i} \frac{\mu^i_B(A_k)}{\mu^j_A(A_k)} \cdot \ln \frac{\mu^i_B(A_k)}{\mu^j_A(A_k)} \mathrm{d}(\mu^j_A)_{\eta_i}(B) \\
&= \sum_k \mu^j_A(A_k) \cdot \int_{A/\eta_i} \rho_k(B) \ln \rho_k(B) \mathrm{d}(\mu^j_A)_{\eta_i}(B) \quad (\text{with } \rho_k(B) := \frac{\mu^i_B(A_k)}{\mu^j_A(A_k)}) \\
&= \sum_k \mu^j_A(A_k) \cdot H(\rho_k \cdot (\mu^j_A)_{\eta_i}, (\mu^j_A)_{\eta_i}).
\end{aligned}$$

Observe that $\nu_1 := (\mu^j_A)_{\eta_i}$ is a probability measure since it is a quotient of the probability measure μ_A and that $\nu_2 := \rho_k \cdot \nu_1$ is also a probability measure by equation (2.3). Therefore, $I(A) \geq 0$ for μ_{η_j}-a.e. A.

Under the hypothesis $\gamma_k = 0$ for $i < k \leq j$, then $h_i = h_j$ and $I(A) = 0$ for μ_{η_j}-a.e. A. This implies that for μ_{η_j}-a.e. A, $\rho_k(B) = 1$ for $(\mu^j_A)_{\eta_i}$-a.e. $B \in A/\eta_i$. Hence $\mu^j_A(A_k) = \mu^i_B(A_k)$ holds true for any $k \geq 1$ and μ_{η_j}-a.e. A, $(\mu^j_A)_{\eta_i}$-a.e. B. Equivalently,

$$\mu^j_A = \mu^i_B \text{ when restricted to the } \sigma - \text{algebra } \mathcal{F}_1 := \mathcal{B}(f^{-1}\eta_j)$$

for μ_{η_j}-a.e. A, $(\mu^j_A)_{\eta_i}$-a.e. B; here by $\mathcal{B}(\xi)$ we denote the σ-algebra generated by a partition ξ.

Noting the fact that $f^{-n}\eta_i = \eta_i \vee f^{-n}\eta_j$ (restricted on Λ) for any $n \geq 1$ (which can be deduced inductively from the property (2.1)) and also the fact that $nh_j - nh_i = H_\mu(f^{-n}\eta_j|\eta_j) - H_\mu(f^{-n}\eta_i|\eta_i)$, we can deduce

$$\mu^j_A = \mu^i_B \text{ when restricted to all the } \sigma - \text{algebras } \mathcal{F}_n := \mathcal{B}(f^{-n}\eta_j), n \geq 1.$$

Saying that η_j generates means that for any $X \in \mathcal{B}(M)$, one can find for all $n \geq 1$ a set $X_n \in \mathcal{F}_n$ such that $\mu(X \Delta X_n) \to 0$ as $n \to \infty$. For μ_{η_j} almost every A, we have $\mu_A(X \Delta X_n) \to 0$ as $n \to \infty$. Since $\mathcal{B}(M)$ is countably generated, for μ_{η_j} almost every A, the union $\bigcup_n \mathcal{F}_n$ generates $\mathcal{B}(M)$ (in sense of μ_A-mod 0). In the same way, for μ_{η_i} almost every B, the union $\bigcup_n \mathcal{F}_n$ generates also $\mathcal{B}(M)$ (in sense of μ_B-mod 0). It follows that for μ_{η_j}-a.e. A fixed,

$$\mu^j_{_A} = \mu^i_{_B} \tag{2.4}$$

holds true for $(\mu^j_{_A})_{\eta_i}$-a.e. B.

Equation (2.4) means that for μ-a.e. x, the measure μ^j_x is carried by a single element of η_i. By invariance, it follows that the conditional measure associated with $f^n \eta_j$ is carried by a single element of $f^n \eta_i$. Since $\bigcup_{n \geq 0} f^n\big(\eta_j(f^{-n}x)\big)$ is the whole leaf $W^j(x)$, μ-a.e. conditional measure $\mu_x^{\lambda_j}$ is carried by a single $\bigcup_{n \geq 0} f^n\big(\eta_i(f^{-n}x)\big)$, which is $W^i(x)$. □

3 An example

In this section, we give an example where the weak stable foliation E_j is uniquely integrable and defines a foliation $\mathcal{W}^j$, the associated conditional measures are Dirac, but the corresponding γ_j is Dim E_j.

Choose $0 > \lambda_2 > \ln 1/2 > \lambda_3$. Consider a smooth mapping f from $\mathbb{R}^3$ into itself which sends the cube $\Omega := [0,1]^3$ into a horseshoe $f(\Omega)$ with the direction e_1 expanding and the intersection $f(\Omega) \cap \Omega = [0,1] \times \Lambda$, where $\Lambda \subset [0,1]^2$ is made of two disjoint rectangles R_0, R_1, with sides parallel to the axes, with height e^{λ_3} and width e^{λ_2}. The intersections of R_0 with the plane $x_2 = 0$ and of R_1 with the plane $x_2 = 1$ are non empty.

There is a unique f-invariant measure μ on $\bigcap_n f^n \Omega$ such that

$$\mu\Big(\bigcap_{n=0}^{N-1} f^{-n}([0,1] \times R_{i_n})\Big) \;=\; 2^{-N}$$

for all sequences $\{i_n\} \in \{0,1\}^N$. Oseledets directions are $E_1(x)$ and the two other coordinate axes. Lyapunov exponents are some $\lambda_1 > 0$ and the negative numbers λ_2, λ_3 we chose. The direction E_2 is integrable: the local leaves of $\mathcal{W}^2$ are the lines with constant x_1, x_3. By construction, there is at most one point of $\bigcap_n f^n\Omega$ in each such local leaf.

On the other hand, the local stable manifolds are the planes with constant x_1. The conditional measures on these planes are all the same, the uniform measure m on Λ. The numbers γ_2, γ_3 can be read on the uniform measure m: γ_3 is the dimension of the conditional measures on the strong stable manifolds $x_2 =$ constant; γ_2 is the dimension of the projection of m along the strong stable foliation on the transversal $x_3 = 0$. Observe finally that the projection of m along the strong stable foliation is (up to affine reparametrization) the infinite Bernoulli convolution with parameter e^{λ_2} (see [11]). By a famous result of B. Solomyak, for almost every choice of $\lambda_2, 0 > \lambda_2 > \ln 1/2$, the infinite Bernoulli convolution of parameter e^{λ_2} is absolutely continuous, in particular $\gamma_2 = 1$.

Question. One may ask whether a companion of Theorem 1 holds for the case when the transverse dimensions are maximum. The analog of Remark 1 holds true: when $i = 0$ and $\gamma_\ell =$ Dim E_ℓ for $\ell = 1, \dots j$, then the measures μ^j are absolutely continuous (w.r.t. Lebesgue measures on $\mathcal{W}^j$); this can be proved similarly following [8] where Ledrappier and Young proved the equivalence between Pesin's entropy formula and SBR property. The generalized statement would be:

Statement 1 *Let f be a C^2 diffeomorphism preserving a Borel probability measure μ on a compact manifold M. With the notations of Theorem 1, for $\lambda < \lambda'$, for an invariant set A of positive measure, the following properties are equivalent:*

1. *$\gamma_i(x) =$ Dim $E_i(x)$ for i such that $\lambda \leq \lambda_i(x) < \lambda'$ and for μ-almost every $x \in A$,*
2. *for μ-almost every $x \in A$, the measure μ_x^λ is obtained by integrating the measures $\mu_y^{\lambda'}$ on $W^{\lambda'}$ leaves with respect to some transversal measure in the Lebesgue class.*

In the particular case of the example, which is not atypical, Statement 1 would imply that an infinite Bernoulli convolution of dimension 1 is absolutely continuous. This is compatible with known conjectures about infinite Bernoulli convolutions, but far from being understood at present.

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