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Typical points for one-parameter families of piecewise expanding maps of the interval

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TYPICAL POINTS FOR ONE-PARAMETER FAMILIES OF PIECEWISE EXPANDING MAPS OF THE INTERVAL

Abstract. Let $I \subset \mathbb{R}$ be an interval and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of piecewise expanding maps such that for each $a \in I$ the map T_a admits a unique absolutely continuous invariant probability measure μ_a. We establish sufficient conditions on such a one-parameter family such that a given point $x \in [0,1]$ is typical for μ_a for a full Lebesgue measure set of parameters a, i.e.

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T_a^i(x)} \overset{\text{weak-}*}{\longrightarrow} \mu_a, \qquad \text{as } n \to \infty,$$

for Lebesgue almost every $a \in I$. In particular, we consider $C^{1,1}(L)$-versions of β-transformations, skew tent maps, and Markov structure preserving one-parameter families. For the skew tent maps we show that the turning point is almost surely typical.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of maps of the unit interval such that, for every $a \in I$, T_a is piecewise C^2 and $\inf_{x\in[0,1]} |\partial_x T_a(x)| \geq \lambda > 1$, where λ is independent on a. Assume that, for all $a \in I$, T_a admits a unique (hence ergodic) absolutely continuous invariant probability measure (a.c.i.p.) μ_a. According to [8] and [9], for Lebesgue almost every $x \in [0,1]$, some iteration of x by T_a is contained in the support of μ_a. From Birkhoff's ergodic theorem we derive that Lebesgue almost every point $x \in [0,1]$ is *typical* for μ_a, i.e.

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{T_a^i(x)} \overset{\text{weak-}*}{\longrightarrow} \mu_a, \qquad \text{as } n \to \infty.$$

In this paper we are interested in the question if the same kind of statement holds in the parameter space, i.e. if a chosen point $x \in [0,1]$ is typical for μ_a for Lebesgue a.e. $a \in I$, or more general, if, for some given C^1 map $X : I \to [0,1]$, $X(a)$ is typical for μ_a for Lebesgue a.e. a in I. In Section 2 we try to establish some sufficient conditions on a one-parameter family such that the following statement is true.

For Lebesgue a.e. $a \in I$, $X(a)$ is typical for μ_a.

The method we use in this paper is a dynamical one. It is essentially inspired by the result of Benedicks and Carleson [1] where they prove that for the quadratic family $f_a(x) = 1 - ax^2$ on $(-1,1)$ there is a set $\Delta_\infty \subset (1,2)$ of a-values of positive Lebesgue measure for which f_a admits almost surely an a.c.i.p. and for which the critical point is typical with respect to this a.c.i.p. The main tool in their work is to switch from the parameter space to the dynamical interval by showing that the a-derivative $\partial_a f_a^j(1)$ is comparable to the x-derivative $\partial_x f_a^j(1)$. This will also be the essence of the basic condition on our one-parameter family T_a with an associated map X, i.e. we require that the a- and the x-derivatives of $T_a^j(X(a))$ are comparable (see condition (I) below).

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Some typicality results related to this paper can be found in [14], [4], and [6]. The one-parameter families T_a, $a \in I$, considered in these papers have in common that their slopes are constant for a fixed parameter value, i.e. for each $a \in I$ there is a constant $\lambda_a > 1$ such that $|T_a'| \equiv \lambda_a$ on $[0,1]$. The advantage of our method and the main novelty of this paper is that we can drop this restriction and, thus, we are able to consider much more general families. In the case when $T_a : S^1 \to S^1$ is a smooth expanding map of the circle, there are some recent results by Pollicott [12].

This paper consists mainly of two parts. In the first part, which corresponds to Sections 2-5, we establish a general criteria for typicality. In the second part, which corresponds to Sections 6-8, we apply this criteria to several well-studied one-parameter families and derive various typicality results for these families. Some of these results are presented in the following subsections of this introduction.

We will shortly give a motivation and an overview of our criteria for typicality. Let $B \subset [0,1]$ be a (small) interval and set $x_j(a) = T_a^j(X(a))$, $a \in I$, i.e. $x_j(a)$ is the forward iteration by T_a^j of the points we are interested in. For $h \geq 1$ fixed, the main estimate to be established in the method we apply is roughly of the form:

$$\frac{1}{|I|}|\{a \in I \ ; \ x_{j_1}(a) \in B, ..., x_{j_h}(a) \in B\}| \leq (C|B|)^h, \tag{1}$$

where $1 \leq j_1 < ... < j_h \leq n$ (n large) are h integers with large ($\geq \sqrt{n}$) gaps between each other and $C \geq 1$ is some constant. Such an estimate is easier to establish for a fixed map T_a in the family, i.e. it is easier to verify the estimate

$$|\{x \in [0,1] \ ; \ T_a^{j_1}(x) \in B, ..., T_a^{j_h}(x) \in B\}| \leq (C|B|)^h. \tag{2}$$

(See also inequality (12).) Hence, in order to prove (1), the main idea in the first part of this paper is to imitate the proof of (2). This leads to three rather natural conditions, conditions (I)-(III), on the sequence of maps $x_j : I \to [0,1]$. Condition (I) roughly says that T_a^j and x_j are comparable, i.e. there exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{|\partial_x T_a^j(X(a))|}{|D_a x_j(a)|} \leq C,$$

for all $j \geq 1$, and $a \in I$ for which the derivatives are defined. Condition (II) ensures that estimates of the following type apply: There exists a constant $C > 0$ such that for all intervals $B \subset [0,1]$ and $j \geq 1$ we have

$$|\{a \in I \ ; \ x_j(a) \in B\})| \leq C|B|.$$

Since maps in one-parameter families of piecewise expanding maps have in general no finite Markov partition, the image $x_j(\omega)$ of a (maximal) interval of smooth monotonicity ω for x_j might be arbitrarily small. Condition (III) provides us we a certain control of such 'too short' intervals of smooth monotonicity.

Apart from condition (I), in order to apply our result to the examples in Sections 6-8, we will not verify conditions (II) and (III) directly. Instead we will show that two other conditions, conditions (IIa) and (IIb), are satisfied. Conditions (IIa) and (IIb) are described in Section 4. Condition (IIa) should hold for general families. It only requires that the densities for the a.c.i.p. μ_a are uniformly (in a) bounded above and below away from zero. In contrast, condition (IIb) is more restrictive. It requires that there is a kind of order relation in the one-parameter family in the sense that for each two parameter values $a, a' \in I$ satisfying $a < a'$ the following holds. The symbolic dynamics of T_a is contained in the symbolic dynamics of $T_{a'}$ and, furthermore, if ω is a (maximal) interval of smooth monotonicity for T_a^j, then the image of the (maximal) interval of smooth monotonicity ω' for $T_{a'}^j$ with the same combinatorics as ω contains the image of

ω, i.e. $T_a^j(\omega) \subset T_{a'}^j(\omega')$. It is interesting to find other ways of verifying conditions (II) and (III). This might allow to treat more general examples of piecewise expanding one-parameter families than the ones studied in Sections 6-8. The main obstacle seems to be the verification of condition (II). As the labeling of conditions (IIa) and (IIb) suggest, they are aimed for verifying condition (II). More or less as a byproduct they also imply condition (III); see Section 5. For a simple example of a family not satisfying condition (IIb), see Remark 4.1.

1.1. **β-transformations.** The example in Section 6 is a $C^{1,1}(L)$-version of the β-transformation. By saying that a function is $C^{1,1}(L)$, we mean that it is C^1 and its derivative is in $\mathrm{Lip}(L)$, i.e. its derivative is Lipschitz continuous with Lipschitz constant L. For a sequence $0 = b_0 < b_1 < ...$ of real numbers such that $b_k \to \infty$ as $k \to \infty$ and a constant $L > 0$, let $T : [0, \infty) \to [0, 1]$ be a right continuous function which is $C^{1,1}(L)$ on each interval (b_k, b_{k+1}), $k \geq 0$. Furthermore, we assume that:

- $T(b_k) = 0$ for each $k \geq 0$.
- For each $a > 1$,

$$1 < \inf_{x\in[0,1]} \partial_x T(ax) \quad \text{and} \quad \sup_{x\in[0,1]} \partial_x T(ax) < \infty.$$

See Figure 1.

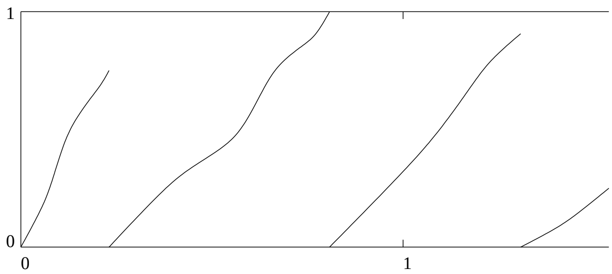

FIGURE 1. A possible beginning of a graph for $T : [0, \infty) \to [0, 1]$.

Given a map T as above, we obtain a $C^{1,1}(L)$-version of the β-transformation T_a : $[0, 1] \to [0, 1]$, $a > 1$, by defining $T_a(x) = T(ax)$, $x \in [0, 1]$. As we will see in Section 6, for each $a > 1$, T_a admits a unique a.c.i.p. μ_a. In Section 6 we will show the following.

Theorem 1.1. *If $X : (1, \infty) \to (0, 1]$ is C^1 and $X'(a) \geq 0$, then $X(a)$ is typical for μ_a for Lebesgue a.e. $a > 1$.*

If we choose $X(a) = b_1/a$ then $X'(a) < 0$ and $T_a^j(X(a)) = 0$ for all $j \geq 1$. Hence, if the condition $X'(a) \geq 0$ in Theorem 1.1 is not satisfied, we cannot any longer guarantee almost sure typicality for the a.c.i.p. For an illustration of the curves on which we have a.s. typicality see Figure 2 (when a is fixed, we can apply Birkhoff's ergodic theorem and get a.s. typicality on the associated vertical line). Observe that if we choose $T : [0, \infty) \to [0, 1]$ by $T(x) = x \bmod 1$, then $T_a(x) = ax \bmod 1$ is the usual β-transformation. Theorem 1.1 generalizes a result due to Schmeling [14] where it is shown that for the usual β-transformation the point 1 is typical for the associated a.c.i.p. for Lebesgue a.e. $a > 1$.

1.2. **Skew tent maps.** In Section 7 we investigate unimodal maps with slopes constant to the left and to the right of the turning point. Let these slopes be α and $-\beta$ where

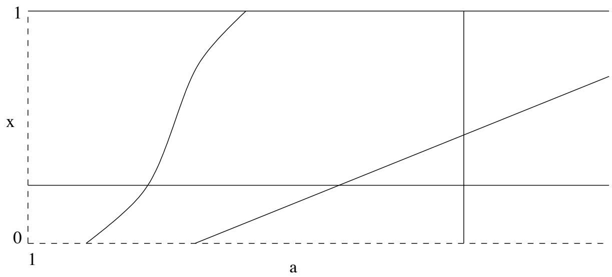

FIGURE 2. Lines and curves on which we have a.s. typicality for the $C^{1,1}(L)$-version of the β-transformation.

$\alpha, \beta > 1$. For instance, let $T_{\alpha,\beta} : [0,1] \to \mathbb{R}_0^+$ be defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \alpha x & \text{if } x \leq \frac{\beta}{\alpha+\beta}, \\ \beta(1-x) & \text{otherwise.} \end{cases}$$

In order that $T_{\alpha,\beta} : [0,1] \to [0,1]$, we have also to assume that $\alpha^{-1} + \beta^{-1} \geq 1$ (see, e.g., Lemma 3.1 in [11]). The map $T_{\alpha,\beta}$ is called a *skew tent map*. Fix two points (α_0, β_0) and (α_1, β_1) in the set $\{(\alpha, \beta) \ ; \ \alpha, \beta > 1 \text{ and } \alpha^{-1} + \beta^{-1} \geq 1\}$ such that $\alpha_1 \geq \alpha_0$, $\beta_1 \geq \beta_0$, and at least one of these two inequalities is sharp. Let

$$\alpha : [0,1] \to [\alpha_0, \alpha_1] \quad \text{and} \quad \beta : [0,1] \to [\beta_0, \beta_1]$$

be functions in $C^1([0,1])$ such that $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$, $(\alpha(1), \beta(1)) = (\alpha_1, \beta_1)$, and, for all $a \in [0,1]$, if $\alpha_0 \neq \alpha_1$, then $\alpha'(a) > 0$ and if $\beta_0 \neq \beta_1$, then $\beta'(a) > 0$. Consider the one-parameter family T_a, $a \in [0,1]$, where $T_a : [0,1] \to [0,1]$ is the skew tent map defined by $T_a = T_{\alpha(a),\beta(a)}$. By [9], since T_a has only two intervals of monotonicity, it follows that there exists a unique a.c.i.p. μ_a for T_a. In Section 7 we will show that the turning point is a.s. typical for the a.c.i.p.

Theorem 1.2. *The turning point for the skew tent map T_a is typical for μ_a for Lebesgue a.e. $a \in [0,1]$.*

Theorem 1.2 generalizes a result due to Bruin [4] where almost sure typicality is shown for the turning point of symmetric tent maps (i.e. when $\alpha(a) \equiv \beta(a)$). It is possible to extend the results in Section 7 to certain one-parameter families of $C^{1,1}(L)$ unimodal maps (see Remark 7.5). In Section 7 we will use a slightly different representation of skew tent maps.

1.3. **One-parameter families of Markov maps.** In Section 8 we consider one-parameter families, which preserve a certain Markov structure. A simple example for such a family are the maps $T_a : [0,1] \to [0,1]$ defined by

$$T_a(x) = \begin{cases} \frac{x}{a} & \text{if } x < a, \\ \frac{x-a}{1-a} & \text{otherwise,} \end{cases}$$

where the parameter $a \in (0,1)$. See Figure 3. By [9], since this map has only one point of discontinuity, it admits a unique a.c.i.p. μ_a (which coincides in this case with the Lebesgue measure on $[0,1]$). In Example 8.2 in Section 8 we will show the following.

Proposition 1.3. *If $X : (0,1) \to (0,1)$ is a C^1 map such that $X'(a) \leq 0$, then $X(a)$ is typical for μ_a for a.e. parameter $a \in (0,1)$.*

Observe that if $X(a) = p_a$ where p_a is the unique point of periodicity 2 in the interval $(0, a)$, then $X'(a) > 0$. Hence, if the condition $X'(a) \leq 0$ is violated in Proposition 1.3, we cannot any longer guarantee almost sure typicality for the a.c.i.p. The very simple structure of the example in this subsection makes it to a good candidate for serving the reader as a model along the paper. Example 8.2 in Section 8 is formulated slightly more general by composing T_a with a $C^{1,1}(L)$ homeomorphism $g : [0,1] \to [0,1]$.

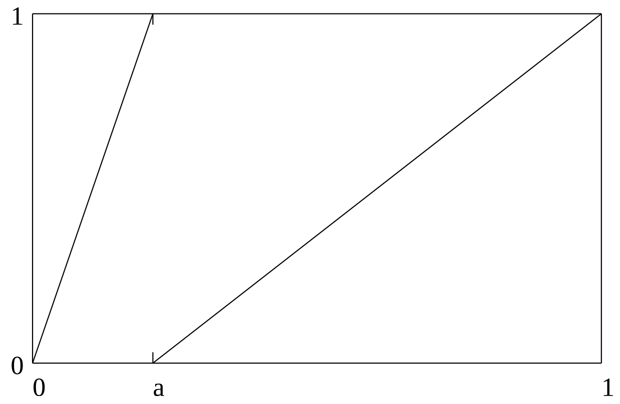

FIGURE 3. A Markov structure preserving one-parameter family T_a where $a \in (0,1)$.

2. Piecewise expanding one-parameter families

2.1. **Preliminaries.** In this subsection we introduce the basic notation and put up a general model for one-parameter families of piecewise expanding maps of the unit interval. A map $T : [0,1] \to \mathbb{R}$ will be called *piecewise* $C^{1,1}(L)$ if there exists a partition $0 = b_0 < b_1 < ... < b_p = 1$ of the unit interval such that for each $1 \leq k \leq p$ the restriction of T to the open interval (b_{k-1}, b_k) is a $C^{1,1}(L)$ function. Observe that, by the Lipschitz property, it follows that T restricted to (b_{k-1}, b_k) can be extended to the closed interval $[b_{k-1}, b_k]$ as a $C^{1,1}(L)$ function. Let $I \subset \mathbb{R}$ be an interval of finite length and $T_a : [0,1] \to [0,1]$, $a \in I$, a one-parameter family of piecewise $C^{1,1}(L)$ maps where the Lipschitz constant $0 < L < \infty$ is independent on the choice of the parameter a. We assume that there are real numbers $1 < \lambda \leq \Lambda < \infty$ such that for every $a \in I$,

$$\lambda \leq \inf_{x \in [0,1]} |\partial_x T_a(x)| \quad \text{and} \quad \sup_{x \in [0,1]} |\partial_x T_a(x)| \leq \Lambda. \tag{3}$$

The parameter dependence is assumed to be piecewise C^1, i.e. for all $x \in [0,1]$, there exists a partition $a_0 < a_1 < ... < a_{p(x)}$ (where a_0 is the left and $a_{p(x)}$ the right boundary point of I) of the parameter interval I such that for each $1 \leq k \leq p(x)$ the restriction of the map $a \mapsto T_a(x)$ to the open interval (a_{k-1}, a_k) is a C^1 function, which can be extended to the closed interval $[a_{k-1}, a_k]$ as a C^1 function. More precise requirements on the parameter dependence will be given shortly (see (i)-(iii) below).

In the sequel, instead of referring to [8] and [9], we will refer to a paper by S. Wong [17] who extended the results in [8] and [9] on piecewise C^2 maps to a broader class of maps containing also piecewise $C^{1,1}(L)$ maps. For a fixed $a \in I$, by [17], there is a finite collection of sets $K_1(a), ..., K_{p_0(a)}(a)$ such that each $K_k(a)$, $1 \leq k \leq p_0(a)$, is a union of finitely many disjoint closed intervals (each of positive length) and, for Lebesgue a.e. $x \in [0,1]$, the accumulation points of the forward orbit of x is identical with one of these $K_k(a)$'s, i.e. to every x in a full Lebesgue measure set of $[0,1]$, there is a $K_k(a)$ such that

$$K_k(a) = \bigcap_{N=1}^{\infty} \overline{\{T_a^n(x)\}_{n=N}^{\infty}}. \tag{4}$$

Furthermore, for each $K_k(a)$ there is a unique (hence ergodic) a.c.i.p. $\mu_{a,k}$ such that $\mathrm{supp}(\mu_{a,k}) = K_k(a)$. Since we are always interested in only one $K_k(a)$, we can without loss of generality assume that $p_0(a) \equiv 1$, $a \in I$. Henceforth, we write $K(a)$ and μ_a instead of $K_1(a)$ and $\mu_{a,1}$, respectively. So, we have that μ_a is the unique a.c.i.p. for T_a and

$$K(a) = \mathrm{supp}(\mu_a).$$

For $a \in I$, let $c_0(a) < c_1(a) < ... < c_{p_1(a)}(a)$ be the associated partition for the piecewise $C^{1,1}(L)$ map $T_a : K(a) \to K(a)$, i.e., if $D_1(a), ..., D_{p_2(a)}(a)$ ($p_2(a) \leq p_1(a)$) denote the (maximal) open intervals in $K(a)$ on which T_a is $C^{1,1}(L)$, then the $c_k(a)$'s are the boundary points of these $C^{1,1}(L)$ domains. For the sake of definition, assume that, for $1 \leq k < p_2$, the domain $D_k(a)$ is to the left of the domain $D_{k+1}(a)$.

We assume that the number of $c_k(a)$'s and $D_k(a)$'s are constant, i.e. $p_1(a) \equiv p_1$ and $p_2(a) \equiv p_2$. Furthermore, we make the following three natural assumptions on our one-parameter family.

(i) For all $0 \leq k \leq p_1$, the map $a \mapsto c_k(a)$ which maps I to $[0, 1]$ is $\mathrm{Lip}(L)$, and there is a constant $\delta_0 > 0$ such that
$$|D_k(a)| \geq \delta_0,$$
for all $1 \leq k \leq p_2$ and $a \in I$.

(ii) For each $x \in [0, 1]$ and $1 \leq k \leq p_2$, if J denotes the set of parameters $a \in I$ such that $x \in D_k(a)$, then if J is non-empty, it is an interval and the maps $a \mapsto T_a(x)$ and $a \mapsto \partial_x T_a(x)$ from J to $\mathbb{R}$ are $\mathrm{Lip}(L)$.

(iii) For each $x \in [0, 1]$ and $1 \leq k \leq p_2$, if J denotes the set of parameters $a \in I$ such that $T_a^{-1}\{x\}$ has a pre-image in $D_k(a)$, then if J is non-empty, it is an interval and the branch of $T_.^{-1}(x)$, which maps J to D_k is $\mathrm{Lip}(L)$.

2.2. **Partitions.** For a fixed parameter value $a \in I$, we denote by $\mathcal{P}_j(a)$, $j \geq 1$, the partition on the dynamical interval consisting of the maximal open intervals of smooth monotonicity for the map $T_a^j : K(a) \to K(a)$. In other words, $\mathcal{P}_j(a)$ denotes the set of open intervals ω in $K(a)$ such that $T_a^j : \omega \to K(a)$ is $C^{1,1}(L)$ and ω is maximal, i.e. for every other open interval $\tilde{\omega} \subset K(a)$ with $\omega \subsetneq \tilde{\omega}$, $T_a^j : \tilde{\omega} \to K(a)$ is no longer $C^{1,1}(L)$. Clearly, $\mathcal{P}_1(a) = \{D_1(a), ..., D_{p_2}(a)\}$. For an open set $H \subset K(a)$, we denote by $\mathcal{P}_j(a)|H$ the restriction of $\mathcal{P}_j(a)$ to the set H. For a set $J \subset K(a)$, which lies completely in one $D_k(a)$, $1 \leq k \leq p_2$, we denote by $\mathrm{symb}_a(J)$ the index (or symbol) k.

We will define similar partitions on the parameter interval I. Let $X : I \to [0, 1]$ be a C^1 map from I into the dynamical interval $[0, 1]$. The points $X(a)$, $a \in I$, will be our candidates for typical points. The forward orbit of a point $X(a)$ under the map T_a we denote as

$$x_j(a) := T_a^j(X(a)), \qquad j \geq 0.$$

Remark 2.1. Since a lot of informations for the dynamics of T_a is contained in the forward orbits of the partition points $c_k(a)$, $0 \leq k \leq p_1$, it is of interest to know how the forward orbits of these points are distributed. Hence, an evident choice of the map X would be

$$X(a) = \lim_{\substack{x \to c_k(a) \\ x \in \omega}} T_a(x),$$

where $\omega \in \mathcal{P}_1(a)$ is an interval adjacent to $c_k(a)$.

Let J be an open set in the parameter space I. By $\mathcal{P}_j|J$, $j \geq 1$, we denote the partition consisting of all open intervals ω in J such that for each $0 \leq i < j$, there exists $1 \leq k \leq p_2$, such that $x_i(a) \in D_k(a)$, for all $a \in \omega$, and such that ω is maximal, i.e. for every other open interval $\tilde{\omega} \subset J$ with $\omega \subsetneq \tilde{\omega}$, there exist $a \in \tilde{\omega}$, $0 \leq i < j$, and $1 \leq k \leq p_2$

such that $x_i(a) \in \partial D_k(a)$. Observe that this partition might be empty. This is, e.g., the case when $X(a) \notin K(a)$ for all $a \in I$ or when T_a is the usual β-transformation and the map X is chosen to be equivalently equal to 0. However, such trivial situations are excluded by condition (I) formulated in the next subsection. Knowing that condition (I) is satisfied, then the partition $\mathcal{P}_j|J$ can be thought of as the set of the (maximal) open intervals of smooth monotonicity for $x_j : J \to [0,1]$ (in order that this is really true one should also assume that $|x_j'(a)| > L$, for all $j \geq 0$ and parameter values $a \in I$ for which this derivative is defined). We set $\mathcal{P}_0|J = J$, and we will write $\mathcal{P}_j|I$ instead of $\mathcal{P}_j|\operatorname{int}(I)$. If for a set J' in the parameter space and for some integer $j \geq 0$ the symbol $\operatorname{symb}_a(x_j(a))$ exists for all $a \in J'$, then it is constant and we denote this symbol by $\operatorname{symb}(x_j(J'))$. Finally, in view of condition (I) below, observe that if a parameter $a \in I$ is contained in an element of $\mathcal{P}_j|I$, $j \geq 1$, then also the point $X(a)(= x_0(a))$ is contained in an element of $\mathcal{P}_j(a)$.

2.3. **Main statement.** In this subsection we will state our main result, Theorem 2.2. Let n be large. To ensure good distortion estimates we will, in the proof of Theorem 2.2, split up the interval I into smaller intervals $J \subset I$ of size $1/n$. The main idea in this paper is to switch from the map $x_j : J \to [0,1]$, $j \leq n$, to the map $T_a^j : [0,1] \to [0,1]$ where a is, say, the right boundary point of J. By this, since the dynamics of the map T_a is well-understood, we derive similar dynamical properties for the map x_j, which then can be used to prove Theorem 2.2. To be able to switch from x_j to T_a^j, we put further three rather natural conditions, conditions (I)-(III), on our one-parameter family.

In condition (I) we require that the derivatives of x_j and T_a^j are comparable. This is the very basic assumption in this paper. Of course, the choice of the map $X : I \to [0,1]$ plays here an important role. If, e.g., for every parameter $a \in I$, $X(a)$ is a periodic point for the map T_a, then x_j will have bounded derivatives and the dynamics of x_j is completely different from the dynamics of T_a. Henceforth, we will use the notations

$$T_a'(x) = \partial_x T_a(x) \qquad \text{and} \qquad x_j'(a) = D_a x_j(a), \quad j \geq 1.$$

(I) There is a constant $C_0 \geq 1$ such that for $\omega \in \mathcal{P}_j|I$, $j \geq 1$, we have

$$\frac{1}{C_0} \leq \left| \frac{x_j'(a)}{{T_a^j}'(X(a))} \right| \leq C_0,$$

for all $a \in \omega$. Furthermore, the number of $a \in I$, which are not contained in any element $\omega \in \mathcal{P}_j|I$ is finite.

We turn to condition (II). For $a \in I$, let φ_a denote the density for μ_a. Assume for the moment that φ_a is bounded from below and from above by a constant $C \geq 1$, i.e $C^{-1} \leq \varphi_a(x) \leq C$ for a.e. $x \in K(a)$. Note that, since the density φ_a is a fixed point of the Perron-Frobenius operator, we have, for $j \geq 1$,

$$\varphi_a(y) = \sum_{\substack{x \in K(a) \\ T_a^j(x) = y}} \frac{\varphi_a(x)}{|{T_a^j}'(x)|},$$

for a.e. $y \in K(a)$. This implies that

$$\sum_{\substack{x \in K(a) \\ T_a^j(x) = y}} \frac{1}{|{T_a^j}'(x)|} \leq C^2, \tag{5}$$

for a.e. $y \in K(a)$. We require a similar estimate for the map x_j.

(II) There exists a constant $C_1 \geq 1$ such that the following holds. Let $J \subset I$ be an open interval of length $1/n$. For $\tilde{\omega} \in \mathcal{P}_i|J$, $i \geq 1$, and $1 \leq j \leq n$, we have

$$\sum_{\substack{a \in \tilde{\omega} \\ x_{i+j}(a)=y}} \left| \frac{x_i'(a)}{x_{i+j}'(a)} \right| \leq C_1 \tag{6}$$

for a.e. $y \in [0,1]$.

Let $B \subset [0,1]$ be a (small) interval. Recall that in condition (I) it is required that, for $j \geq 1$, there are only finitely many parameter values $a \in I$ not contained in any element of $\mathcal{P}_j|I$. Thus, if in addition to condition (I), condition (II) is satisfied, then it follows that

$$|x_i(\{a \in \tilde{\omega} \ ; \ x_{i+j}(a) \in B\})| = \int_B \sum_{\substack{a \in \tilde{\omega} \\ x_{i+j}(a)=y}} \left| \frac{x_i'(a)}{x_{i+j}'(a)} \right| dy \leq C_1|B|, \tag{7}$$

which will be the main estimate in the proof of Theorem 2.2. Now, we want to pull back this estimate to the parameter interval I. Assuming that $x_i(\tilde{\omega})$ has a large size, for instance, assuming that $x_i(\tilde{\omega}) = (0,1)$, then, by a minor distortion estimate for x_i, it follows (see (15) below) that

$$|\{a \in \tilde{\omega} \ ; \ x_{i+j}(a) \in B\}| \leq C_1 C_3 |B||\tilde{\omega}|, \tag{8}$$

where C_3 is a bound for the distortion of x_i. If $\tilde{\omega}$ was a very small interval of smooth monotonicity for x_i then it might happen that $\tilde{\omega}$ is mapped by x_{i+j} entirely into the interval B and not just a $|B|$ fraction of it, as it is the case in (8). To avoid such cases we want that the total measure of partition elements with a too small image is negligible.

Condition (III) requires that we are able to exclude elements $\omega \in \mathcal{P}_j|I$, $j \geq 1$, whose length of $x_j(\omega)$ is below a fixed constant, say, $\delta_1 > 0$. However, if $\omega \in \mathcal{P}_j|I$ is an element such that $|x_j(\omega)| < \delta_1$, we will not exclude it immediately but, roughly speaking, we will give ω (or at least a part of it) a chance to grow during the following $\sqrt{n}$ iterations. The formulation of condition (III) is rather technical.

(III) There is a constant $\delta_1 > 0$ such that to every $\varepsilon > 0$ there is an integer n_ε growing at most polynomially in $1/\varepsilon$ such that for $n \geq n_\varepsilon$ the following holds. Let $J \subset I$ be an open interval of length $1/n$ and fix an integer $1 \leq j \leq 2n$. The exceptional set

$$E = \{\omega \in \mathcal{P}_{j+[\sqrt{n}]}|J \ ; \ \nexists\, \tilde{\omega} \in \mathcal{P}_{j+k}|J, \ \ 0 \leq k \leq [\sqrt{n}], \\ \text{such that } \tilde{\omega} \supset \omega \text{ and } |x_{j+k}(\tilde{\omega})| \geq \delta_1\},$$

satisfies

$$|E| \leq \frac{\varepsilon}{n}.$$

Finally, we state the main result of this paper.

Theorem 2.2. *Let $T_a : [0,1] \to [0,1]$, $a \in I$, be a piecewise expanding one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii), and let $X : I \to [0,1]$ be a C^1 map. If conditions (I)-(III) are fulfilled, then $X(a)$ is typical for μ_a for Lebesgue almost every $a \in I$.*

As already pointed out in the introduction, for all the examples considered in this paper we will not verify conditions (II) and (III) directly. Instead we will verify conditions (IIa) and (IIb) which are described in Section 4. Knowing that these two conditions are satisfied is then sufficient to deduce that also conditions (II) and (III) are satisfied (provided that the basic condition (I) holds); see Propositions 4.5 and 5.1. Furthermore, in the considered examples, we will usually not verify conditions (I), (IIa) and (IIb) for the whole interval I for which the corresponding family is defined. Instead we will cover I by a countable number of smaller intervals and verify these conditions on these smaller intervals.

3. Proof of Theorem 2.2

The idea of the proof of Theorem 2.2 is inspired by Chapter III in [1] where Benedicks and Carleson prove the existence of an a.c.i.p. for a.e. parameter in a certain parameter set (the set Δ_∞). Their argument implies that the critical point is in fact typical for this a.c.i.p.

Let

$$\mathcal{B} := \left\{ (q-r, q+r) \cap [0,1] \ ; \ q \in \mathbb{Q}, r \in \mathbb{Q}^+ \right\}.$$

We will show that there is a constant $C \geq 1$ such that for each $B \in \mathcal{B}$ the function

$$F_n(a) = \frac{1}{n} \sum_{j=1}^{n} \chi_B(x_j(a)), \quad n \geq 1,$$

fulfills

$$\overline{\lim_{n\to\infty}} F_n(a) \leq C|B|, \qquad \text{for a.e. } a \in I. \tag{9}$$

By standard measure theory (see, e.g., [10]), (9) implies that, for a.e. $a \in I$, every weak-$*$ limit point ν_a of

$$\frac{1}{n} \sum_{j=1}^{n} \delta_{x_j(a)}, \tag{10}$$

has a density which is bounded above by C. In particular, ν_a is absolutely continuous. Observe that, by the definition of $x_j(a)$, the measure ν_a is also invariant for T_a and, hence, ν_a is an a.c.i.p. for T_a. By the uniqueness of the a.c.i.p. for T_a, we finally derive that, for a.e. $a \in I$, the weak-$*$ limit of (10) exists and coincides with the a.c.i.p. μ_a. This concludes the proof of Theorem 2.2.

In order to prove (9), it is sufficient to show that for all (large) integers $h \geq 1$ there is an integer $n_{h,B}$, growing for fixed B at most exponentially in h, such that

$$\int_I F_n(a)^h da \leq \text{const}(C|B|)^h,$$

for all $n \geq n_{h,B}$ (see Lemma A.1 in [2]).

In the remaining part of this section, we assume that $B \in \mathcal{B}$ is fixed. For $h \geq 1$, we have

$$\int_I F_n(a)^h da = \sum_{1 \leq j_1, \ldots, j_h \leq n} \frac{1}{n^h} \int_I \chi_B(x_{j_1}(a)) \cdots \chi_B(x_{j_h}(a)) da. \tag{11}$$

For a fixed parameter a, there exists an integer k such that (T_a^k, μ_a) is exact and, hence, this system is mixing of all degrees (see [13] and [16]). It follows that for sequences of non-negative integers $j_1^r, \ldots, j_h^r$, $r \geq 1$, with

$$\lim_{r\to\infty} \inf_{i \neq l} |j_i^r - j_l^r| = \infty,$$

one has

$$\int_{[0,1]} \chi_B\Big(T_a^{kj_1^r}(x)\Big)\cdots\chi_B\left(T_a^{kj_h^r}(x)\right)d\mu_a(x) \tag{12}$$
$$= \mu_a\left(T_a^{-kj_1^r}(B)\cap...\cap T_a^{-kj_h^r}(B)\right) \stackrel{r\to\infty}{\longrightarrow} \mu_a(B)^h \le (\|\varphi_a\|_\infty |B|)^h\,.$$

Since the maps T_a^j and x_j are, by conditions (I)-(III), 'comparable', it is natural to expect similar mixing properties for the maps x_j. In fact, in the next subsection, we are going to prove the following statement.

Proposition 3.1. *Under the assumption that conditions (I)-(III) are satisfied, there is a constant $C \ge 1$ such that the following holds. For all $h \ge 1$, there is an integer $n_{h,B}$ growing at most exponentially in h such that, for all $n \ge n_{h,B}$ and for all integer h-tuples $(j_1,...,j_h)$ with $\sqrt{n} \le j_1 < j_2 < ... < j_h \le n$ and $j_l - j_{l-1} \ge \sqrt{n}$, $l = 2,...,h$,*

$$\int_I \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le 4|I|(C|B|)^h.$$

Seen from a more probabilistic point of view, Proposition 3.1 says that whenever the distances between consecutive j_i's are sufficiently large, the behavior of the $\chi_B(x_{j_i}(.))$'s is comparable to that of independent random variables. Now, the number of h-tuples $(j_1,...,j_h)$ in the sum in (11), for which $\min_i j_i < \sqrt{n}$ or $\min_{k\neq l}|j_k - j_l| < \sqrt{n}$, is bounded by $2h^2 n^{h-1/2}$. Hence, by Proposition 3.1,

$$\int_I F_n(a)^h da \le 4|I|(C|B|)^h + \frac{2h^2}{\sqrt{n}}|I| \le 5|I|(C|B|)^h,$$

whenever

$$n \ge \max\left\{ n_{h,B}, \left(\frac{2h^2}{(C|B|)^h}\right)^2\right\}.$$

Since both terms in this lower bound for n grow at most exponentially in h, this concludes the proof of Theorem 2.2.

3.1. **Proof of Proposition 3.1.** To be able to make use of conditions (II) and (III), we split up the integral in Proposition 3.1 and integrate over smaller intervals of length $1/n$. More precisely, under the assumptions of Proposition 3.1, we are going to show that there exists an integer $n_{h,B}$ growing at most exponentially in h such that, for $n \ge n_{h,B}$, we have

$$\int_J \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le \frac{1}{n}3(C|B|)^h, \tag{13}$$

where $J \subset I$ is an arbitrary interval of length $1/n$. This immediately implies that, for $n \ge n_{h,B}$,

$$\int_I \chi_B(x_{j_1}(a))\cdots\chi_B(x_{j_h}(a))da \le 4|I|(C|B|)^h,$$

(if $n_{h,B} \gg 1/|I|$), which concludes the proof of Proposition 3.1.

Note that by condition (I), for $j \ge 1$, there are only finitely many parameter values not contained in any element of $\mathcal{P}_j|I$. Hence, we can neglect such parameter values and focus on the partitions $\mathcal{P}_j|I$. Let $\tilde\omega \in \mathcal{P}_i|J$, $i \ge 1$, and $1 \le j \le n$. By condition (II) (see (7)), we have

$$|x_i(\{a \in \tilde\omega\ ;\ x_{i+j}(a) \in B\})| \le C_1|B|. \tag{14}$$

We will give a rough idea of how condition (III) can be used to conclude the proof of Proposition 3.1. If our one-parameter family satisfies condition (III), this means that we can neglect too small partition elements, and we can without loss of generality assume that the following preferable picture is true: If $\omega' \in \mathcal{P}_{j_i}|J$, $1 \le i \le h-1$, then we can write the subinterval of ω' which is mapped into B as a disjoint union of intervals $\tilde{\omega}$ such that each $\tilde{\omega}$ is an element of some partition $\mathcal{P}_{j_i+k}|J$, $1 \le k \le j_{i+1} - j_i$, having a large image, i.e. $|x_{j_i+k}(\tilde{\omega})| \ge \delta_1$. By Lemma 4.2 a), which is stated and proved in Section 4 and which follows essentially from condition (I), we have good distortion estimates on $\tilde{\omega}$, i.e.

$$\left|\frac{x'_{j_i+k}(a)}{x'_{j_i+k}(a')}\right| \le C_3,$$

for $a, a' \in \tilde{\omega}$. Hence, combined with (14), we get

$$\begin{aligned} |\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in B\}| &\le C_3 \frac{|x_{j_i+k}(\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in B\})|}{|x_{j_i+k}(\tilde{\omega})|} |\tilde{\omega}| \\ &\le \frac{C_1 C_3}{\delta_1} |B||\tilde{\omega}|. \end{aligned} \tag{15}$$

So, of each such 'large' $\tilde{\omega}$ only a fraction which is proportional to the length of B can be mapped by $x_{j_{i+1}}$ into B. Observe that the argument for deriving (15) also applies when $\tilde{\omega} \in \mathcal{P}_k|J$, $1 \le k \le j_1$, satisfying $|x_k(\tilde{\omega})| \ge \delta_1$, in which case we obtain

$$|\{a \in \tilde{\omega} \ ; \ x_{j_1}(a) \in B\}| \le \frac{C_1 C_3}{\delta_1} |B||\tilde{\omega}|. \tag{16}$$

From (16) combined with (15), applied $h-1$ times, we can derive Proposition 3.1. In the remaining part of this subsection, we will work this out in detail.

Fix an integer $\tau \ge 1$ such that $C_0^2 \lambda^{-\tau} \le |B|$, and assume that n is so large that $\sqrt{n} \ge \tau$, which ensures that there are at least τ iterations between two consecutive j_i's. Let $\Omega_0 = J$ and

$$\Omega_i = \{\omega \in \mathcal{P}_{j_i+\tau}|\Omega_{i-1} \ ; \ x_{j_i}(\omega) \cap B \neq \emptyset\},$$

for $1 \le i \le h$. Notice that, by (I) and the assumption on τ, we have $|x_{j_i}(\omega)| \le |B|$ for all $\omega \in \mathcal{P}_{j_i+\tau}|J$ and, thus,

$$\Omega_i \subset \{a \in \Omega_{i-1} \ ; \ x_{j_i}(a) \in 3B\},$$

where $3B$ denotes the interval being three times as long as B and sharing the same midpoint. In each step we will exclude partition intervals with too short images. To this end we define, for $0 \le i \le h-1$, the following exceptional sets (let $j_0 = 0$):

$$\begin{aligned} E_i = \{\omega \in \mathcal{P}_{j_{i+1}}|\Omega_i \ ; \ \nexists \, \tilde{\omega} \in \mathcal{P}_{j_i+k}|\Omega_i, \ \tau \le k \le j_{i+1} - j_i, \\ \text{s.t. } \tilde{\omega} \supset \omega \text{ and } |x_{j_i+k}(\tilde{\omega})| \ge \delta_1\}. \end{aligned}$$

As the $\varepsilon > 0$ in condition (III) we take

$$\varepsilon = \frac{(C|B|)^h}{h},$$

where C is the constant $3C_1C_3/\delta_1$. By (III) we derive that there is an integer $n_{\varepsilon,\tau}$ growing at most polynomially in $1/\varepsilon$ such that for each $0 \le i \le h-1$, $|E_i| \le \varepsilon/(\sqrt{n} - \tau)^2$, for $n \ge n_{\varepsilon,\tau}$. (The need to introduce the integer τ in this subsection is the reason why we require in the formulation of condition (III) that $1 \le j \le 2n$ instead of $1 \le j \le n$.) If $n_{\varepsilon,\tau} \ge (4\tau)^2$, we get that $|E_i| \le 2\varepsilon/n$. τ is only dependent on $|B|$. By the definition of ε, it follows that $n_{h,B} = \max\{n_{\varepsilon,\tau}, (4\tau)^2\}$ grows at most exponentially in h. Disregarding finitely many points, $\Omega_i \setminus E_i$ can be seen as a set of disjoint and open intervals $\tilde{\omega}$ such that each $\tilde{\omega}$ is an element of a partition $\mathcal{P}_{j_i+k}|\Omega_i$, $\tau \le k \le j_{i+1} - j_i$, and $|x_{j_i+k}(\tilde{\omega})| \ge \delta_1$. By (15) and (16), we obtain

$$|\{a \in \tilde{\omega} \ ; \ x_{j_{i+1}}(a) \in 3B\}| \le C|B||\tilde{\omega}|,$$

which in turn implies that, for $n \ge n_{h,B}$,

$$|\Omega_{i+1}| \le C|B||\Omega_i \setminus E_i| + |E_i| \le C|B||\Omega_i| + \frac{2\varepsilon}{n}.$$

Hence, we have

$$|\Omega_h| \le (C|B|)^h |\Omega_0| + h\frac{2\varepsilon}{n} \le \frac{1}{n} 3(C|B|)^h,$$

where in the last inequality we used the definitions of Ω_0 and ε. Since

$$\{a \in J \ ; \ x_{j_1}(a) \in B, ..., x_{j_h}(a) \in B\} \subset \Omega_h,$$

this implies (13).

4. Condition (II)

As already mentioned in Subsection 2.3, in the examples considered in this paper, we will not verify condition (II) directly. Instead we will verify two other conditions, conditions (IIa) and (IIb) described below. We will show in this section that conditions (IIa) and (IIb) imply condition (II). In fact, conditions (IIa) and (IIb) also imply condition (III), see next section.

4.1. **Conditions (IIa) and (IIb).** Recall that for inequality (5) we assumed that the density φ_a is bounded from below and from above. We require that this holds for each density φ_a, $a \in I$, and with a constant independent on a.

(IIa) There is a constant $C_2 \ge 1$ such that for each density φ_a, $a \in I$, we have

$$\frac{1}{C_2} \le \varphi_a(x) \le C_2,$$

for a.e. $x \in K(a)$.

Even if condition (IIa) appears to be a natural requirement on a one-parameter family of piecewise expanding maps, it will take us some effort to verify the lower bound for the examples in Sections 6 and 7. The upper bound follows almost immediately from [17] and [8] (see the proof of Lemma A.1).

We turn to condition (IIb). Let a_1 and a_2 be two arbitrary parameter values in I such that $a_1 < a_2$ and fix an integer $j \ge 1$. We require that, for a.e. $y \in K(a_1)$ the following holds. To each point $x \in K(a_1)$ satisfying $T_{a_1}^j(x) = y$ there is an associated point $x' \in K(a_2)$ satisfying $T_{a_2}^j(x') = y$ and having the same combinatorics as x, i.e. $\mathrm{symb}_{a_2}(T_{a_2}^i(x')) = \mathrm{symb}_{a_1}(T_{a_1}^i(x))$, $0 \le i < j$. In other words, we require that the combinatorics of T_{a_1} should be 'contained' in the combinatorics of T_{a_2} and, furthermore, if $\omega \in \mathcal{P}_j(a_1)$ and $\omega' \in \mathcal{P}_j(a_2)$ have the same combinatorics, then the image by $T_{a_1}^j$ of ω should be contained in the image by $T_{a_2}^j$ of ω'. Condition (IIb) is rather restrictive, see Remark 4.1.

(IIb) For all $a_1, a_2 \in I$, $a_1 \le a_2$, and $j \ge 1$ there is a mapping

$$\mathcal{U}_{a_1,a_2,j} : \mathcal{P}_j(a_1) \to \mathcal{P}_j(a_2),$$

such that, for all $\omega \in \mathcal{P}_j(a_1)$,

$$\mathrm{symb}_{a_1}(T_{a_1}^i(\omega)) = \mathrm{symb}_{a_2}(T_{a_2}^i(\mathcal{U}_{a_1,a_2,j}(\omega))), \quad 0 \le i < j, \tag{17}$$

and, in particular,

$$T^j_{a_1}(\omega) \subset T^j_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega)). \tag{18}$$

Remark 4.1. A simple example of a one-parameter family not satisfying (IIb) are the maps $T_a(x) = \beta x + \alpha \bmod 1$, $\beta > 1$ and $\alpha \in (0,1)$, where the parameter a can be chosen to be either β or α. These maps are studied, e.g., in [7] and [6]. However, using the special property that for every fixed map T_a the derivative of T_a is constantly equal to β, it is possible to verify condition (II) directly (at least for large β). The main ingredient in verifying condition (II) for this one-parameter family is that one can show that the number of elements in the partitions $\mathcal{P}_j(a)$, $j \geq 1$, are bounded above by a constant C times β^j where the constant C is independent on the parameter value a. This property, that one, roughly speaking, can switch from considering the maps T_a to counting the number of partition elements, is also used in [15]. To keep this paper in a reasonable size we will not investigate this family T_a. For the parameter choice $a = \beta$, a.s. typicality in the case when $X(a) \equiv x \in [0,1]$ is shown in [6].

4.2. **Conditions (I), (IIa), and (IIb) imply condition (II).** We prove first a distortion lemma. Let T_a, $a \in I$, be a one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii), and let $X : I \to [0,1]$ be a to this family associated C^1 map. Let $J \subset I$ be an interval of length $1/n$. If condition (I) is satisfied, then, for large n, the length of J is huge compared to the length of an element $\omega \in \mathcal{P}_n|J$, which, by (I), can be estimated from above as $|\omega| \leq C_0/\lambda^n$. Nevertheless, as part b) of the following lemma asserts, the interval J is small enough to have good distortion estimates which will enable us to compare the map $x_j : J \to [0,1]$ with the map $T^j_{a_J} : [0,1] \to [0,1]$ where a_J is the right boundary point of J.

Lemma 4.2. *There exists a constant $C_3 \geq 1$ such that the following holds.*

a) *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies condition (I), then for $\omega \in \mathcal{P}_j|I$, $j \geq 1$,*
$$\frac{1}{C_3} \leq \left| \frac{x_j'(a)}{x_j'(a')} \right| \leq C_3,$$
for all $a, a' \in \omega$.

b) *If the one-parameter family T_a, $a \in I$, satisfies condition (IIb), then we have the following distortion estimate. Let $n \geq 1$ and $a_1, a_2 \in I$ such that $a_1 \leq a_2$ and $a_2 - a_1 \leq 1/n$. For $\omega \in \mathcal{P}_j(a_1)$, $1 \leq j \leq 2n$, we have*
$$\frac{1}{C_3} \leq \left| \frac{T^j_{a_1}{}'(x)}{T^j_{a_2}{}'(x')} \right| \leq C_3,$$
for all $x \in \omega$ and $x' \in \mathcal{U}_{a_1,a_2,j}(\omega)$.

Remark 4.3. If $a_1 = a_2$ in Lemma 4.2 b), then we get a well-known distortion estimate for piecewise expanding $C^{1,1}(L)$ maps.

Proof. We proof first part b), which is the more difficult part. Fix $\tau \geq 1$ such that
$$\max\{4L/\tau, L\lambda/(\lambda - 1)\tau\} \ll \delta_0.$$
The constant C_3 in Lemma 4.2 b) will be greater than $(\Lambda/\lambda)^\tau$. Hence, for $2n \leq \tau$, the distortion estimate in b) is trivially satisfied and we can assume that $\tau < j \leq 2n$. Observe that, by (IIb), the set $T^j_{a_1}(\omega)$ is contained in the set $T^j_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega))$. Fix a point y in $T^j_{a_1}(\omega)$ and let, for $1 \leq i \leq j$,
$$r_i \in T^{j-i}_{a_1}(\omega), \quad s_i \in T^{j-i}_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega)),$$

be the pre-images of y, i.e. $T^i_{a_1}(r_i) = T^i_{a_2}(s_i) = y$. Note that, by (IIb), we have $\mathrm{symb}_{a_1}(r_i) = \mathrm{symb}_{a_2}(s_i)$. Let $k_i = \mathrm{symb}_{a_1}(r_i)$.

Claim. *The distance between r_i and s_i, $1 \le i \le j$, satisfies*

$$|r_i - s_i| \le \frac{L\lambda}{\lambda - 1}\frac{1}{n}. \tag{19}$$

Proof. In order to show (19), we will show

$$|r_i - s_i| \le \frac{L}{n}\sum_{l=0}^{i-1}\frac{1}{\lambda^l}, \tag{20}$$

for $1 \le i \le j$. Since y has for both parameters a_1 and a_2 a pre-image $r_1 = T^{-1}_{a_1}(y)$ and $s_1 = T^{-1}_{a_2}(y)$, which lies in D_{k_1}, it follows, by (iii) in Subsection 2.1, that y has a pre-image in $D_{k_1}(a)$ for all parameter values a in the interval $[a_1, a_2]$ and, furthermore, the corresponding map $a \mapsto T^{-1}_a$ is $\mathrm{Lip}(L)$. Hence, we have

$$|r_1 - s_1| = |T^{-1}_{a_1}(y) - T^{-1}_{a_2}(y)| \le L(a_2 - a_1) \le \frac{L}{n}. \tag{21}$$

Assume now that we have shown (20) for some $1 \le i < j$. Since $a_2 - a_1 \le 1/n$ it follows, by (i), that the length of the intersection of $D_{k_{i+1}}(a_1)$ and $D_{k_{i+1}}(a_2)$ is at least $\delta_0 - 2L/n$. If z lies in this intersection, then, by (ii), the map $a \mapsto T_a(z)$ is $\mathrm{Lip}(L)$ on the interval $[a_1, a_2]$. Thus, the length of the intersection of $T_{a_1}(D_{k_{i+1}}(a_1))$ and $T_{a_2}(D_{k_{i+1}}(a_2))$ is at least $\delta_0 - 4L/n$. Since, by the assumption on τ, $\delta_0 - 4L/n \approx \delta_0$ and $|r_i - s_i| \le L\lambda/(\lambda - 1)n \ll \delta_0$, we deduce that at least one of the following two situations occurs:

- The branch of $T^{-1}_{a_1}$ which maps r_i to $D_{k_{i+1}}(a_1)$ is defined on the whole interval $[r_i, s_i]$.
- The branch of $T^{-1}_{a_2}$ which maps s_i to $D_{k_{i+1}}(a_2)$ is defined on the whole interval $[r_i, s_i]$.

Assuming the first situation occurs, we obtain, by (3),

$$|T^{-1}_{a_1}(r_i) - T^{-1}_{a_1}(s_i)| \le \frac{1}{\lambda}|r_i - s_i|,$$

and, as in (21), we derive that

$$|T^{-1}_{a_1}(s_i) - T^{-1}_{a_2}(s_i)| \le \frac{L}{n}.$$

It follows that

$$|r_{i+1} - s_{i+1}| = |T^{-1}_{a_1}(r_i) - T^{-1}_{a_2}(s_i)| \le \frac{1}{\lambda}|r_i - s_i| + \frac{L}{n} \le \frac{L}{n}\sum_{k=0}^{i}\frac{1}{\lambda^k}.$$

We can do a similar calculation when the second situation occurs, which concludes the proof. □

By a similar reasoning as in the proof of (19), we note that at least one of the following two situations occurs:

- $[r_i, s_i] \subset D_{k_i}(a_1)$.
- $[r_i, s_i] \subset D_{k_i}(a_2)$.

If the first situation occurs, we have, by (ii), that the map $a \mapsto T'_a(s_i)$ is Lip(L) on the interval $[a_1, a_2]$. Combined with (19) and since $x \mapsto T_{a_1}(x)$ is $C^{1,1}(L)$ on $[r_i, s_i]$, we obtain

$$\begin{aligned} |T'_{a_1}(r_i)| &\le |T'_{a_1}(s_i)| + L|r_i - s_i| \le |T'_{a_2}(s_i)| + L(a_2 - a_1) + L|r_i - s_i| \\ &\le |T'_{a_2}(s_i)| + \frac{2L\lambda}{\lambda - 1}\frac{1}{n}. \end{aligned}$$

If the second situation occurs, it follows in a similar way that

$$|T'_{a_2}(s_i)| \ge |T'_{a_1}(r_i)| - \frac{2L\lambda}{\lambda - 1}\frac{1}{n}.$$

For $\tau < i \le j$, let $t_i = r_i$ and $\alpha_i = a_1$ if the first situation occurs and $t_i = s_i$ and $\alpha_i = a_2$ otherwise. Altogether, we obtain

$$\begin{aligned} \left|\frac{T^j_{a_1}{}'(x)}{T^j_{a_2}{}'(x')}\right| &\le \left(\frac{\Lambda}{\lambda}\right)^\tau \prod_{i=\tau+1}^{j} \frac{|T'_{a_1}(T^{j-i}_{a_1}(x))|}{|T'_{a_2}(T^{j-i}_{a_2}(x'))|} \\ &\le \left(\frac{\Lambda}{\lambda}\right)^\tau \prod_{i=\tau+1}^{j} \frac{|T'_{a_1}(r_i)| + L|T^{j-i}_{a_1}(\omega)|}{|T'_{a_2}(s_i)| - L|T^{j-i}_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega))|} \\ &\le \left(\frac{\Lambda}{\lambda}\right)^\tau \prod_{i=\tau+1}^{j} \frac{|T'_{\alpha_i}(t_i)| + 2L\lambda(\lambda-1)^{-1}n^{-1} + L\lambda^{-i}}{|T'_{\alpha_i}(t_i)| - 2L\lambda(\lambda-1)^{-1}n^{-1} - L\lambda^{-i}}. \end{aligned} \tag{22}$$

(To ensure that the denominators are positive, we should also assume that τ was chosen so large that $2L\lambda(\lambda-1)^{-1}\tau^{-1} + L\lambda^{-\tau} < \lambda$.) Since $j \le 2n$, the product in the last term of inequality (22) is clearly bounded above by a constant independent on n. Hence, this shows the upper bound in the distortion estimate in part b). The lower bound is shown in the same way. This concludes the proof of part b).

The proof of part a) is similar but easier than the proof of part b). We will give only a sketch of the proof. Let $\omega \in \mathcal{P}_j|I$, $j \ge 1$, and $a, a' \in \omega$. By condition (I), we have

$$\left|\frac{x'_j(a)}{x'_j(a')}\right| \le C_0^2 \prod_{i=0}^{j-1} \left|\frac{T'_a(x_i(a))}{T'_{a'}(x_i(a'))}\right|.$$

The distance between $x_i(a)$ and $x_i(a')$, $1 \le i \le j-1$, satisfies, by (I),

$$|x_i(a) - x_i(a')| \le |x_i(\omega)| \le C_0^2 \lambda^{-(j-i)}.$$

This inequality is a counterpart to inequality (19), which is the part in b) where we used condition (IIb). Similarly as in the proof of part b) we derive

$$\left|\frac{x'_j(a)}{x'_j(a')}\right| \le C_0^2 \left(\frac{\Lambda}{\lambda}\right)^\tau \prod_{i=1}^{j-\tau} \frac{|T'_{\alpha_i}(t_i)| + 2LC_0^2\lambda^{-(j-i)}}{|T'_{\alpha_i}(t_i)| - 2LC_0^2\lambda^{-(j-i)}},$$

where either $\alpha_i = a$ and $t_i = x_i(a)$ or $\alpha_i = a'$ and $t_i = x_i(a')$, and τ is chosen so large that $2LC_0^2\lambda^{-\tau} < \lambda$. The product in this inequality is clearly bounded above by a constant independent on $j \ge 1$. This concludes the proof of Lemma 4.2. □

To prove Lemma 4.2 b), instead of property (18) in condition (IIb), it would be sufficient to assume that $\operatorname{dist}(T^j_{a_1}(\omega), T^j_{a_2}(\mathcal{U}_{a_1,a_2,j}(\omega))) \le 1/n$. However, to establish inequality (6) in condition (II), property (18) is essential.

Lemma 4.4. *Under the assumption that conditions (I) and (IIb) are satisfied, there is an integer $q \ge 1$ such that the following holds. Let $J \subset I$ be an open interval of length*

$1/n$ such that the right boundary point a_J of J is contained in I, and let $i \geq 0$. For each element $\tilde{\omega} \in \mathcal{P}_i|J$ and integer $1 \leq j \leq 2n$, there is an at most q-to-one map

$$\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J),$$

such that, for $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$, the image of ω is contained in the image of $\mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$, i.e.

$$x_{i+j}(\omega) \subset T^j_{a_J}(\mathcal{U}_{\tilde{\omega},a_J,j}(\omega)), \tag{23}$$

and we have the following distortion control:

$$\frac{1}{|T^j_a{}'(x_i(a))|} \leq C_3 \frac{1}{|T^j_{a_J}{}'(x)|}, \tag{24}$$

for all $a \in \omega$ and $x \in \mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$.

Proof. We define the map

$$\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$$

as follows. Let $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$ and $a \in \omega$. By the definition of the partitions associated to the parameter interval, $x_{i+l}(a) \notin \{c_0(a), ..., c_{p_1}(a)\}$, for all $0 \leq l < j$. Hence, there exists an element $\omega(x_i(a))$ in the partition $\mathcal{P}_j(a)$ containing the point $x_i(a)$. We set

$$\mathcal{U}_{\tilde{\omega},a_J,j}(\omega) = \mathcal{U}_{a,a_J,j}(\omega(x_i(a))),$$

where $\mathcal{U}_{a,a_J,j} : \mathcal{P}_j(a) \to \mathcal{P}_j(a_J)$ is the map given by (IIb). Note that the element $\omega' = \mathcal{U}_{a,a_J,j}(\omega(x_i(a)))$ has the same combinatorics as ω, i.e.

$$\text{symb}_{a_J}(T^l_{a_J}(\omega')) = \text{symb}(x_{i+l}(\omega)),$$

$0 \leq l < j$. Since there cannot be two elements in $\mathcal{P}_j(a_J)$ with the same combinatorics, the element ω' is independent on the choice of $a \in \omega$. It follows that the map $\mathcal{U}_{\tilde{\omega},a_J,j}$ is well-defined. By property (18) in condition (IIb), we have $T^j_a(\omega(x_i(a))) \subset T^j_{a_J}(\mathcal{U}_{\tilde{\omega},a_J,j}(\omega))$, for all $a \in \omega$. This implies (23). Since $j \leq 2n$ and $a_J - a \leq 1/n$, for $a \in \omega$, inequality (24) follows immediately from the distortion estimate in Lemma 4.2 b). In order to conclude the proof of Lemma 4.4, it is only left to show that the map $\mathcal{U}_{\tilde{\omega},a_J,j}$ is at most q-to-one for some integer $q \geq 1$. Let $i_0 = i_0(C_0, \lambda) \geq 0$ be so large that $|x_i'(a)| \geq L$ for all $i \geq i_0$ and parameter values $a \in I$ for which the derivative is defined (L is the Lipschitz constant introduced in Subsection 2.1). If $i \geq i_0$, using that the partition points $c_0(a), ..., c_{p_1}(a)$ are Lip(L), it is easy to show that the map $\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ is one-to-one. For $0 \leq i \leq i_0$, recall that, by condition (I), the partition $\mathcal{P}_i|I$ consists of only finitely many elements. Hence, setting $q = \#\{\omega \in \mathcal{P}_{i_0}|I\}$ we derive that the map $\mathcal{U}_{\tilde{\omega},a_J,j} : \mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ is at most q-to-one. □

Using Lemma 4.2 and Lemma 4.4, we can easily deduce the main statement of this section.

Proposition 4.5. *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies conditions (I), (IIa), and (IIb), then it satisfies condition (II).*

Proof. Let $J \subset I$ be an open interval of length $1/n$. We assume first that the right endpoint a_J of J lies in I. As in condition (II), let $\tilde{\omega} \in \mathcal{P}_i|J$, $i \geq 1$, and $1 \leq j \leq n$. Observe that, by condition (I), we have $|x_i'(a)|/|x_{i+j}'(a)| \leq C_0^2 |T^j_a{}'(x_i(a))|$. Let $\mathcal{U}_{\tilde{\omega},a_J,j}$: $\mathcal{P}_{i+j}|\tilde{\omega} \to \mathcal{P}_j(a_J)$ be the map provided by Lemma 4.4. By inequality (23), for each $\omega \in \mathcal{P}_{i+j}|\tilde{\omega}$, whenever there is a parameter value $a \in \omega$ such that $x_{i+j}(a) = y$, then there is also a point $x \in \mathcal{U}_{\tilde{\omega},a_J,j}(\omega)$ satisfying $T^j_{a_J}(x) = y$. Combined with the distortion estimate (24), we obtain

$$\sum_{\substack{a\in\tilde{\omega}\\ x_{i+j}(a)=y}} \frac{|x_i'(a)|}{|x_{i+j}'(a)|} \le C_0^2 \sum_{\substack{a\in\tilde{\omega}\\ x_{i+j}(a)=y}} \frac{1}{|T_a^j{}'(x_i(a))|} \le qC_0^2C_3 \sum_{\substack{x\in K(a_J)\\ T^j_{a_J}(x)=y}} \frac{1}{|T^j_{a_J}{}'(x)|},$$

for all but finitely many $y \in [0,1]$ (we exclude points y for which there exists a parameter value $a \in \tilde{\omega}$ which is not contained in any element of $\mathcal{P}_{i+j}|\tilde{\omega}$ and such that $x_{i+j}(a) = y$; by condition (I), the number of such points y is finite). Recall that condition (IIa) implies inequality (5) with the upper bound C_2^2. By applying (5) to the right hand side of the inequality above, we obtain condition (II) with the constant $C_1 = qC_0^2C_2^2C_3$. Since C_1 does not depend on a_J, we can drop the assumption, that the right boundary point of J is contained in I. This concludes the proof. □

5. Condition (III)

As already pointed out in the introduction, even if conditions (IIa) and (IIb) are in particular designed for verifying condition (II), it turns out that if conditions (I), (IIa), and (IIb) are satisfied, then also condition (III) is satisfied — at least on smaller intervals.

Proposition 5.1. *If the one-parameter family T_a, $a \in I$, with the associated map X satisfies conditions (I), (IIa), and (IIb), then, disregarding a finite number of parameter values in I, we can cover I by countably many intervals such that on each such interval condition (III) is satisfied.*

Proof. Fix an integer τ so large that $2^{1/\tau} \le \sqrt{\lambda}$. Observe that, if we set for a fixed $a \in I$,

$$\delta = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\},$$

then the following is trivially satisfied: For all $\omega \in \mathcal{P}_t(a)$, where $1 \le t \le \tau$, we have

$$|T_a^t(\omega)| \ge \delta.$$

We want that a similar property holds for the partition elements of the parameter space.

Claim. *Disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each interval $\tilde{I}$ there exists a constant $\delta_1 = \delta_1(\tilde{I}) > 0$ such that the following holds. Let $j \ge 1$ and $1 \le t \le \tau$. If $\omega \in \mathcal{P}_{j+t}|\omega'$ for some $\omega' \in \mathcal{P}_j|\tilde{I}$ and if ω is not adjacent to a boundary point of ω', then we have*

$$|x_{j+t}(\omega)| \ge \delta_1. \tag{25}$$

Proof. Even if the proof is rather straightforward, we have to be a bit careful. For $a \in I$, let

$$\kappa(a) = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\} > 0.$$

Since condition (IIb) holds, we can argue as in the proof of Lemma A.1 (see inequality (48)) and, disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each such interval $\tilde{I}$ there is a constant $\kappa_0 = \kappa_0(\tilde{I}) > 0$ such that

$$\kappa(a) \ge \kappa_0, \tag{26}$$

for all $a \in \tilde{I}$. Fix such a parameter interval $\tilde{I}$ and let $a \in \tilde{I}$. Recall that in Subsection 2.1 we defined $D_k(a)$, $1 \le k \le p_2$, to be the elements of $\mathcal{P}_1(a)$. Let $\mathcal{C}(\tilde{I})$ denote the following set of functions:

$$\mathcal{C}(\tilde{I}) = \{b : \tilde{I} \to [0,1] \ ; \ b(a) = \lim_{\substack{x \to c(a) \\ x \in D_k(a)}} T_a^t(x),\ 1 \le k \le p_2,\ 1 \le t \le \tau,$$
$$\text{and } c(a) \in \partial D_k(a) \text{ (s.t. } c \in C^0(\tilde{I}))\}.$$

By the first claim in the proof of Lemma A.1, the functions in $\mathcal{C}(\tilde{I})$ are continuous. Let $\omega \in \mathcal{P}_t(a)$, $1 \le t \le \tau$. Observe that the image of ω by T_a^t is of the form

$$T_a^t(\omega) = (b_1(a), b_2(a)),$$

for some functions $b_1, b_2 \in \mathcal{C}(\tilde{I})$ (if b_1 and b_2 are not uniquely defined, we choose them such that (27) below holds). By the continuity of the functions in $\mathcal{C}(\tilde{I})$ and by (26), we derive that for each parameter value $a_* \in \tilde{I}$ there exists a to ω associated element $\omega_* \in \mathcal{P}_t(a_*)$ such that

$$T_{a_*}^t(\omega_*) = (b_1(a_*), b_2(a_*)), \quad \text{and} \quad \text{symb}_{a_*}(T_{a_*}^i(\omega_*)) = \text{symb}_a(T_a^i(\omega)), \tag{27}$$

for $0 \le i < t$. By condition (IIb), it follows further that

$$(b_1(a_*), b_2(a_*)) \subset (b_1(a), b_2(a)), \tag{28}$$

for all $a_*, a \in \tilde{I}$ such that $a_* \le a$. Observe that from (26) it follows also that for each function $b \in \mathcal{C}(\tilde{I})$ there exists $1 \le k \le p_2$ such that

$$b(a) \in \text{closure}\{D_k(a)\}, \tag{29}$$

for all $a \in \tilde{I}$.

We turn to the partitions on the parameter space. Let $j_0 = j_0(C_0, \lambda) \ge 1$ be so large that $|x_j'(a)| \ge L$, for all $j \ge j_0$ and parameter values $a \in I$ for which the derivative is defined (L is the Lipschitz constant from Subsection 2.1). We consider first the case when $j \ge j_0$. Let $\omega' \in \mathcal{P}_j|\tilde{I}$, fix an integer $1 \le t \le \tau$, and denote by a_* the left boundary point of $\tilde{I}$. We will construct a map $\mathcal{U} : \mathcal{P}_{j+t}|\omega' \to \cup_{1 \le s \le t} \mathcal{P}_s(a_*)$ such that for each $\omega \in \mathcal{P}_{j+t}|\omega'$ not adjacent to a boundary point of ω' we have

$$T_{a_*}^s(\mathcal{U}(\omega)) \subset x_{j+t}(\omega), \quad \text{for some } 1 \le s \le t. \tag{30}$$

(Observe that for the construction of the map in Lemma 4.4 we consider instead of the left the right boundary point of the parameter interval and the inclusion is in the other direction. The construction regarding (30) is a bit more cumbersome.) Having constructed such a map $\mathcal{U}$, since the sizes of the elements in $\mathcal{P}_s(a_*)$, $1 \le s \le t$, are bounded from below by κ_0, by setting $\delta_1 = \kappa_0$, this immediately implies the assertion of the claim for the case $j \ge j_0$. Fix $\omega \in \mathcal{P}_{j+t}|\omega'$ not adjacent to a boundary point of ω'. Let

$$t_0 = \min\{s \ge 1 \ ; \ \exists\ \tilde{\omega} \in \mathcal{P}_{j+s}|\omega' \text{ s.t. } \omega \subset \tilde{\omega},\ \partial\tilde{\omega} \cap \partial\omega' = \emptyset \text{ and } \partial\tilde{\omega} \cap \partial\omega \ne \emptyset\},$$

and, for $i \ge 0$, if $\omega \notin \mathcal{P}_{j+t_i}|\omega'$, then let

$$t_{i+1} = \min\{s > t_i \ ; \ \exists\ \tilde{\omega} \in \mathcal{P}_{j+s}|\omega' \text{ s.t. } \omega \subset \tilde{\omega} \text{ and } \tilde{\omega} \notin \mathcal{P}_{j+t_i}|\omega'\};$$

and otherwise, we do not define t_{i+1}. Let $q \le t$ be the maximal integer for which t_q is defined and set $t_{q+1} = t + 1$. For $0 \le i \le q$, let ω_i be the element in $\mathcal{P}_{j+t_i}|\omega'$ containing ω. Observe that ω_i is also an element of $\mathcal{P}_{j+t_{i+1}-1}|\omega'$. In particular, we have $\omega_q = \omega$. We will show that for each ω_i there is an element $\omega_i^* \in \mathcal{P}_{t_i - t_0 + 1}(a_*)$ such that ω_i^* is also an element of $\mathcal{P}_{t_{i+1} - t_0}(a_*)$ and if $b_1, b_2 \in \mathcal{C}(\tilde{I})$ are the functions satisfying

$$T_{a_*}^{t_{i+1} - t_0}(\omega_i^*) = (b_1(a_*), b_2(a_*)), \tag{31}$$

then they satisfy also

(32) $\quad x_{j+t_{i+1}-1}(\omega_i)=(b_1(a_L),b_2(a_R))\quad\text{or}\quad x_{j+t_{i+1}-1}(\omega_i)=(b_1(a_R),b_2(a_L)),$

where a_L and a_R are the boundary points of ω_i. Combined with (28), this immediately implies (30) by setting $\mathcal{U}(\omega)=\omega_q^*$. Recall that by $c_0(a)<c_1(a)<...<c_{p_1}(a)$ we denote the boundary points of the $D_k(a)$, $1\le k\le p_2$, and these boundary points are Lipschitz in a with Lipschitz constant L. Since $j\ge j_0$ and since ω_0 is not adjacent to a boundary point of ω', we have that the image of ω_0 by x_{j+t_0-1} is of the form

$$x_{j+t_0-1}(\omega_0)=(c_k(a_L),c_{k+1}(a_R))\quad\text{or}\quad x_{j+t_0-1}(\omega_0)=(c_k(a_R),c_{k+1}(a_L))$$

where a_L and a_R are the boundary points of ω_0 and $0\le k<p_1$ (the assumption $j\ge j_0$ we used to avoid that the image is, e.g., of the form $(c_k(a_L),c_k(a_R))$). Now, let $\omega_0^*\in\mathcal{P}_1(a_*)$ be the element of the form $(c_k(a_*),c_{k+1}(a_*))$. By (26) and (29), we derive immediately that ω_0^* is also an element of $\mathcal{P}_{t_1-t_0}(a_*)$ and properties (31) and (32) are satisfied for w_0 and w_0^*. If $q=0$ then we are done. Otherwise, assume that we have shown (31) and (32) for $0\le i<q$. Let a_L and a_R denote the boundary points of ω_{i+1}. We have that one boundary point of $x_{j+t_{i+1}-1}(\omega_{i+1})$ is equal to $c_k(a_L)$ or $c_k(a_R)$ for some $0<k<p_1$, and the other coincides with a boundary point of $x_{j+t_{i+1}-1}(\omega_i)$ (this latter fact follows since, by the definition of t_0, ω_0 and ω have a common boundary point), i.e. $x_{j+t_{i+1}-1}(\omega_{i+1})$ is of the form $(b(a_L),c_k(a_R))$ or $(b(a_R),c_k(a_L))$ where $b\in\mathcal{C}(\tilde I)$ (if $c<b$ then we mean by (b,c) the interval (c,b)). Since we assumed that (31) and (32) are satisfied for the element ω_i with an associated element $\omega_i^*\in\mathcal{P}_{t_i-t_0+1}(a_*)$, we can apply (26) and (29) and we deduce that there is an element $\omega_{i+1}^*\in\mathcal{P}_{t_{i+1}-t_0+1}(a_*)$ such that $T_{a_*}^{t_{i+1}-t_0}(\omega_{i+1}^*)=(b(a_*),c_k(a_*))$. Applying (26) and (29) once more, we get that ω_{i+1}^* is also an element of $\mathcal{P}_{t_{i+2}-t_0}(a_*)$ and, furthermore, we deduce that properties (31) and (32) are satisfied for the elements ω_{i+1} and ω_{i+1}^*. This concludes the proof of the claim in the case when $j\ge j_0$. Observe that by condition (I) there are only finitely many elements in a partition $\mathcal{P}_j|\tilde I$. Hence, by setting the constant $\delta_1=\delta_1(\tilde I)$ equal to

$$\delta_1=\min\{\kappa_0,\min\{|\omega|\ ;\ \omega\in\mathcal{P}_j|\tilde I,\ 1\le j\le j_0\}\},$$

this concludes the proof of the claim. □

In the following, we restrict our considerations to an interval $\tilde I\subset I$ with an associated constant $\delta_1>0$, as described in the claim above, and verify condition (III) on this interval $\tilde I$. As in condition (III), let $J\subset\tilde I$ be an open interval of length $1/n$, where we assume $n\gg 1$, and fix an integer $1\le j\le 2n$. For each $\omega'\in\mathcal{P}_j|J$, we define the set

$$E_{\omega'}=\{\omega\in\mathcal{P}_{j+[\sqrt{n}]}|\omega'\ ;\ \nexists\,\tilde\omega\in\mathcal{P}_{j+k}|\omega',\quad 0\le k\le[\sqrt{n}],\ \text{s.t. }\tilde\omega\supset\omega\text{ and }|x_{j+k}(\tilde\omega)|\ge\delta_1\}.$$

If $1\le t\le\tau$, we derive from the claim above that

$$\#\{\omega\in\mathcal{P}_{j+t}|\omega'\ ;\ |x_{j+t}(\omega)|<\delta_1\}\le 2.$$

In other words only the element(s) in $\mathcal{P}_{j+t}|\omega'$ being adjacent to a boundary point of ω' can have a small image. By a repeated use of this fact we derive

$$\#\{\omega\in\mathcal{P}_{j+[\sqrt{n}]}|E_{\omega'}\}\le 2\cdot 2^{[\sqrt{n}]/\tau}\le 2\sqrt{\lambda}^{[\sqrt{n}]},$$

where in the last inequality we used the definition of τ. Applying condition (I), it follows that

$$|x_j(E_{\omega'})|\le C_0^2\frac{\#\{\omega\in\mathcal{P}_{j+[\sqrt{n}]}|E_{\omega'}\}}{\lambda^{[\sqrt{n}]}}\le\frac{2C_0^2}{\sqrt{\lambda}^{[\sqrt{n}]}}=:\gamma_n.$$

The exceptional set E in condition (III) is given by

$$E = \bigcup_{\omega' \in \mathcal{P}_j | J} E_{\omega'}.$$

Let a_J denote the right boundary point of J. Without loss of generality we can assume that $a_J \in I$. Set

$$\mathcal{C}_j := \{b \ ; \ b \in \partial T^i_{a_J}(\omega), \ 1 \leq i \leq j, \ \omega \in \mathcal{P}_j(a_J)\}.$$

By (I), we obtain

$$|E| = \sum_{\omega' \in \mathcal{P}_j | J} \int_{x_j(E_{\omega'})} \frac{1}{|x'_j(a_y)|} dy \leq \sum_{\omega' \in \mathcal{P}_j | J} C_0 \int_{x_j(E_{\omega'})} \frac{1}{|T^j_{a_y}{}'(X(a_y))|} dy,$$

where $a_y = (x_j|_{\omega'})^{-1}(y)$. Since conditions (I) and (IIb) are satisfied, we can apply Lemma 4.4 in the case where $i = 0$ and get

$$\begin{aligned} |E| &\leq \sum_{\omega \in \{\mathcal{U}_{J,a_J,j}(\omega') \ ; \ \omega' \in \mathcal{P}_j | J\}} C_0 C_3 \int_{\Gamma(\omega)} \frac{1}{|T^j_{a_J}{}'(x_y)|} dy \\ &\leq q \sum_{\omega \in \mathcal{P}_j(a_J)} C_0 C_3 \int_{\Gamma(\omega)} \frac{1}{|T^j_{a_J}{}'(x_y)|} dy, \end{aligned}$$

where $x_y = (T^j_{a_J}|_\omega)^{-1}(y)$, and

$$\Gamma(\omega) = [b_\omega, b_\omega + |x_j(E_{\omega'})|],$$

where $b_\omega \in \mathcal{C}_j$ denotes the left boundary point of $T^j_{a_J}(\omega)$. Recall that $|x_j(E_{\omega'})| \leq \gamma_n$. Finally, we can move the sum over the partition elements inside the integral and we derive that

$$|E| \leq q C_0 C_3 \sum_{b \in \mathcal{C}_j} \int_{[b, b+\gamma_n]} \sum_{\substack{x \in K(a_J) \\ T^j_{a_J}(x) = y}} \frac{1}{|T^j_{a_J}{}'(x)|} dy.$$

Since condition (IIa) is satisfied, we can apply inequality (5), and we get that the sum inside the integral above is bounded by the constant C_2^2. Recall that p_2 is the number of (maximal) smooth monotonicity domains for $T_{a_J}|_{K(a_J)}$, and observe that for each $b \in \mathcal{C}_j$ there is such a monotonicity domain $D \in \mathcal{P}_1(a_J)$ and a partition point $c \in \partial D$ such that

$$b = \lim_{\substack{x \to c \\ x \in D}} T^i_{a_J}(x),$$

for some $1 \leq i \leq j$. Thus, since $j \leq 2n$, we have

$$|\mathcal{C}_j| \leq |\mathcal{C}_{2n}| \leq 2n \cdot 2p_2.$$

Finally, for each $\varepsilon > 0$, we deduce that

$$|E| \leq 4 p_2 q C_0 C_2^2 C_3 n \gamma_n \leq \frac{\varepsilon}{n},$$

for $n \geq n_\varepsilon$, where n_ε can in fact be taken to grow less than polynomially in $1/\varepsilon$. This concludes the verification of condition (III) on the interval $\tilde{I}$. □

6. β-TRANSFORMATION

We apply Theorem 2.2 to a $C^{1,1}(L)$-version of β-transformations. Let the map $T : [0,\infty) \to [0,1]$ be piecewise $C^{1,1}(L)$ and $0 = b_0 < b_1 < ...$ be the associated partition, where $b_k \to \infty$ as $k \to \infty$. We assume that:

a) T is right continuous and $T(b_k) = 0$, for each $k \geq 0$.

b) For each $a > 1$,

$$1 < \inf_{x\in[0,1]} \partial_x T(ax) \quad \text{and} \quad \sup_{x\in[0,1]} \partial_x T(ax) < \infty.$$

See Figure 1. We define the one-parameter family $T_a : [0,1] \to [0,1]$, $a > 1$, by $T_a(x) = T(ax)$. There exists a unique a.c.i.p. μ_a for each T_a as the following lemma asserts.

Lemma 6.1. *For each $a > 1$ there exists a unique a.c.i.p. μ_a for T_a. The support $K(a)$ is an interval adjacent to 0 and its length $|K(a)|$ is piecewise constant in a where the number of discontinuities is countable. Furthermore, the following holds. Let $I \subsetneq (1,\infty)$ be a parameter interval on which $|K(a)|$ is constant and such that the left endpoint of I does not coincide with 1 or a point of discontinuity for $a \mapsto |K(a)|$. Then, there exists an integer $t \geq 1$ (independent on the parameter value $a \in I$), such that the support $K(a)$, $a \in I$, is obtained by iterating t times the interval of monotonicity adjacent to 0, i.e. $K(a) = \text{closure}\{T_a^t((0, b_1/a))\}$.*

The proof of Lemma 6.1 is not difficult but tedious. For completeness we add the proof in the end of this section. Henceforth in this section, $I \subsetneq (1,\infty)$ will always denote an interval as described in Lemma 6.1 and such that, for $a \in I$, the number of discontinuities of T_a inside $K(a)$ is constant, i.e. the number $\#\{k \geq 1 \; ; \; b_k/a \in \text{int}(K(a))\}$ is constant on I. For a fixed interval I it is now straightforward to check that the one-parameter family T_a, $a \in I$, fits into the model described in Subsection 2.1 fulfilling properties (i)-(iii). Now, we can state the main result of this section.

Theorem 6.2. *If for a C^1 map $X : I \to [0,1]$ condition (I) is satisfied, then $X(a)$ is typical for μ_a for a.e. $a \in I$.*

Remark 6.3. As the family T_a we could also consider other models as, e.g., $x \mapsto ag(x) \bmod 1$ where $g : [0,1] \to [0,1]$ is a $C^{1,1}(L)$ homeomorphism with a strict positive derivative. Even if this model is not included in the families described above, it would be easier to treat since, seen as a map from the circle into itself, it is non-continuous only in the point 0 which, in particular, implies that $K(a) = [0,1]$.

By Theorem 2.2 and Propositions 4.5 and 5.1, in order to proof Theorem 6.2, it is sufficient to check conditions (IIa) and (IIb). As we will show in the following subsection, there is a large class of maps X satisfying condition (I):

Corollary 6.4. *If $X : (1,\infty) \to (0,1]$ is C^1 such that $X'(a) \geq 0$, then $X(a)$ is typical for μ_a for a.e. $a > 1$.*

Remark 6.5. Observe that the map

$$X(a) \equiv \lim_{x\to b_k-} T(x),$$

$a > 1$, satisfies $X(a) > 0$ and $X'(a) \geq 0$, and, hence, Corollary 6.4 can be applied to these from a dynamical point of view important values.

Obviously, we can cover $(1,\infty)$ with a countable number of intervals $I \subset (1,\infty)$ as they are used in Theorem 6.2. Thus, in order to prove Corollary 6.4, it is sufficient to verify condition (I) for the family T_a restricted to such a parameter interval I. Henceforth, we will use the notation of Subsection 2.1 related to the family T_a, $a \in I$. In particular, recall that $\lambda > 1$ is defined to be a lower bound for the expansion in the family, and observe that here we have $c_k(a) = b_k/a$, for $0 < k < p_1$. First we will prove Corollary 6.4.

6.1. **Proof of Corollary 6.4.** Note that if the map X in Corollary 6.4 satisfies $X(a) \notin \text{int}(K(a))$ on I, then, by definition, the partition $\mathcal{P}_j|I$ would be empty for all $j \geq 1$ and, hence, condition (I) is not fulfilled. However, the following calculations in this subsection (see, in particular, (33)) show that, for $j \geq 1$, the derivative of x_j exists and is strictly positive for all but a finite number of points $a \in I$. Combined with property (4), we derive that, disregarding a countable number of points, we can cover I by a countable number of intervals $J \subset I$ such that for each such interval J there is an integer $j \geq 0$ such that $x_j|_J$ is C^1, $x_j'(a) \geq 0$, and $x_j(a) \in \text{int}(K(a))$ for all $a \in J$. Thus, by possibly redefining X as x_j and focusing on smaller parameter intervals, we can without loss of generality assume that $X(a) \in \text{int}(K(a))$ for all $a \in I$.

Let $j \geq 1$ and $\omega \in \mathcal{P}_j|I$. For $a \in \omega$ we have

$$\begin{aligned} x_j'(a) &= D_a T(a x_{j-1}(a)) = T'(a x_{j-1}(a))(x_{j-1}(a) + a x_{j-1}'(a)) \\ &= T_a'(x_{j-1}(a))(x_{j-1}(a)/a + x_{j-1}'(a)), \end{aligned}$$

and, hence, we derive

$$x_j'(a) = \sum_{i=0}^{j-1} T_a^{j-i}{}'(x_i(a)) \frac{x_i(a)}{a} + T_a^j{}'(X(a)) X'(a),$$

(recall that $x_0(a) = X(a)$). Furthermore, we obtain

$$\frac{x_j'(a)}{T_a^j{}'(X(a))} = \sum_{i=0}^{j-1} \frac{1}{T_a^i{}'(X(a))} \frac{x_i(a)}{a} + X'(a).$$

Let $\kappa = \inf_{a \in I} X(a)$ and $M = \sup_{a \in I} X'(a)$. By the assumptions on I and X, we have $\kappa > 0$ and $M < \infty$. Thus, for $a \in \omega$,

$$\frac{\kappa}{a_I} \leq \frac{x_j'(a)}{T_a^j{}'(X(a))} \leq \sum_{i=0}^{j-1} \frac{1}{\lambda^i} + M \leq \frac{\lambda}{\lambda - 1} + M, \tag{33}$$

where a_I denotes the right boundary point of I. This provides us with a lower and an upper bound in (I).

It is only left to show that the number of parameters $a \in I$ not contained in any element of the partition $\mathcal{P}_j|I$ is finite. We show this by induction. Note that the discontinuity points $c_k(a)$, $1 \leq k \leq p_1 - 1$, are equal to b_k/a (the partition points $c_0(a) \equiv 0$ and $c_{p_1}(a)$ are constant) and, thus, strictly decreasing in a. Since $X'(a) \geq 0$ and $X(a) \in \text{int}(K(a))$, for all $a \in I$, it follows that the number of parameters $a \in I$ such that $X(a) = c_k(a)$ for some $0 \leq k \leq p_1$ is finite. Hence, the number of parameters $a \in I$ not contained in any element of $\mathcal{P}_1|I$ is finite. Let $j \geq 1$ and assume that the number of a's not contained in any element of $\mathcal{P}_j|I$ is finite. Let $\omega \in \mathcal{P}_j|I$. By the first inequality in (33), it follows that $x_j'(a) > 0$, $a \in \omega$. Since the partition points $c_k(a)$, $0 \leq k \leq p_1$, of T_a are decreasing or constant in a it follows that the number of $a \in \omega$ such that $x_j(a) = c_k(a)$, $0 \leq k \leq p_1$, is finite. We derive that the number of parameters $a \in I$ not contained in any element of the partition $\mathcal{P}_{j+1}|I$ is finite. This concludes the proof of Corollary 6.4.

6.2. **Condition (IIa).** The verification of condition (IIb) in the next subsection does not make use of condition (IIa). Hence, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C(I) \geq 1$ such that for each $a \in I$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1} (otherwise, disregarding a finite number of points, by Lemma A.1, we can cover the interval I by a countable number of subintervals on each of which this is true and then proceed with these subintervals instead of I). To conclude the verification of condition (IIa) it is left to show that there exists a lower bound for φ_a on the whole of $K(a)$.

To make the definition of the intervals $J_i(a)$ below work, we assume that the interval $J(a)$ is closed to the left. Recall that, by property (i) in Subsection 2.1, we have $c_k(a) > c_{k-1}(a)+\delta_0$, $1 \leq k \leq p_1$, for some constant $\delta_0 = \delta_0(I) > 0$. Let $\varepsilon = \min\{(\lambda-1)/2C, \lambda\delta_0\}$ and take $l \geq 1$ so large that $\lambda^l/2C > 1$. We claim that $[0,\varepsilon) \subset T_a^l(J(a))$. Let $J_0(a) = J(a)$ and assume that we have defined the interval $J_{i-1}(a) \subset J(a)$, $i \geq 1$, where $J_{i-1}(a)$ is a (not necessarily maximal) interval of monotonicity for T_a^{i-1}. If $[0,\varepsilon) \subset T_a^i(J_{i-1}(a))$, we stop and do not define $J_i(a)$. If $[0,\varepsilon)$ is not contained in $T_a^i(J_{i-1}(a))$ then, since $J_{i-1}(a)$ is a monotonicity interval for T_a^{i-1} and by the definition of ε (combined with property (i) and property a) of T_a), it follows that there can lie at most one partition point $c_k(a)$ in the image $T_a^{i-1}(J_{i-1}(a))$. If there is no partition point in this image then we let $J_i(a) = J_{i-1}(a)$, which is in this case also a monotonicity interval for T_a^i. If there is a partition point $c_k(a) \in T_a^{i-1}(J_{i-1}(a))$, then we define $J_i(a) \subset J_{i-1}(a)$ to be the interval of monotonicity for T_a^i such that $T_a^{i-1}(J_i(a)) = T_a^{i-1}(J_{i-1}(a)) \cap [0, c_k(a))$. Note that $|T_a^{i-1}(J_{i-1}(a)) \cap [c_k(a),1]| < \varepsilon/\lambda$, since otherwise we would have $[0,\varepsilon) \subset T_a^i(J_{i-1}(a))$. Assuming that $J_l(a)$ is defined, we obtain

$$
\begin{aligned}
|T_a^l(J_l(a))| &\geq \lambda(|T_a^{l-1}(J_{l-1}(a))| - \varepsilon/\lambda) \geq \lambda^l|J_0(a)| - \varepsilon\frac{\lambda^l-1}{\lambda-1} \\
&\geq \lambda^l(1/C - 1/2C) \geq \lambda^l/2C > 1,
\end{aligned}
$$

where we used the definitions of ε and l. Since $J_l(a)$ is a monotonicity interval for T_a^l, this is a contradiction and it follows that the maximal integer $i \geq 0$ such that $J_i(a)$ is defined is strictly smaller than l. Hence, $T_a^l(J(a))$ contains $[0,\varepsilon)$ as claimed above. This immediately implies that there is an integer $l' \geq 1$ independent on the parameter $a \in I$ such that $[0, c_1(a)) \subset T_a^{l'}(J(a))$.

Combined with Lemma 6.1 we derive that there is an integer $j \geq 1$ independent on $a \in I$ such that, $K(a) = \text{closure}\{T_a^j(J(a))\}$. Now, by the Perron-Frobenius equality, it follows that, for $a \in I$,

$$
\varphi_a(y) \geq \sum_{\substack{x \in J(a) \\ T_a^j(x) = y}} \frac{\varphi_a(x)}{|T_a^j{}'(x)|} \geq \frac{1}{C\Lambda^j}, \tag{34}
$$

for a.e. $y \in K(a)$. This concludes the proof of a lower bound for φ_a on the whole of $K(a)$.

6.3. **Condition (IIb).** We can verify condition (IIb) by induction over $j \geq 1$. Let $a_1, a_2 \in I$ such that $a_1 \leq a_2$. Note that $\mathcal{P}_1(a) = \{(c_k(a), c_{k+1}(a))\ ;\ 0 \leq k < p_2\}$ where $c_k(a) = b_k/a$ for $0 < k < p_1$. Thus, if $1 \leq k < p_2$ then we clearly have $T_{a_1}((c_{k-1}(a_1), c_k(a_1))) = T_{a_2}((c_{k-1}(a_2), c_k(a_2)))$. The point $c_{p_1}(a) \in (b_{p_1-1}/a, b_{p_1}/a)$ is constant since the length $|K(a)|$ is constant. It follows that $T_a(c_{p_1}(a))$ is increasing in a, which implies that $T_{a_1}((c_{p_1-1}(a_1), c_{p_1}(a_1))) \subset T_{a_2}((c_{p_1-1}(a_2), c_{p_1}(a_2)))$. Hence, (IIb) holds for $j = 1$. Assume that (IIb) holds for $j \geq 1$. Let $\tilde{\omega} \in \mathcal{P}_j(a_1)$ and $\tilde{\omega}' = \mathcal{U}_{a_1,a_2,j}(\tilde{\omega})$ the corresponding element in $\mathcal{P}_j(a_2)$. Note that the image by T_a^i, $i \geq 1$, of an element in $\mathcal{P}_i(a)$ is always adjacent to 0. Since $T_{a_1}^j(\tilde{\omega}) \subset T_{a_2}^j(\tilde{\omega}')$ and the $c_k(a)$'s are decreasing (or constant in the case $k = 0$ and $k = p_1$), it follows immediately that for every element $\omega \in \mathcal{P}_{j+1}(a_1)|\tilde{\omega}$ there is a unique element $\omega' \in \mathcal{P}_{j+1}(a_2)|\tilde{\omega}'$ fulfilling

$\text{symb}_{a_1}(T^i_{a_1}(\omega)) = \text{symb}_{a_2}(T^i_{a_2}(\omega'))$, $0 \le i < j+1$, and $T^{j+1}_{a_1}(\omega) \subset T^{j+1}_{a_2}(\omega')$. Defining $\mathcal{U}_{a_1,a_2,j+1}(\omega) = \omega'$ shows that (IIb) holds also for $j+1$.

6.4. **Proof of Lemma 6.1.** For $a > 1$ let μ_a be an a.c.i.p. for T_a with support $K(a)$ and let $J \subset K(a)$ be an open interval. Since T_a is expanding there exists an integer $j \ge 1$ such that $T^j_a : J \to [0,1]$ is not any longer continuous. It follows that there exists an $\varepsilon > 0$ such that $T^j_a(J)$ contains $[0,\varepsilon)$. If T_a had more than one a.c.i.p. then, by [17], there would exists two a.c.i.p.'s with disjoint supports (disregarding a finite number of points). This shows that the a.c.i.p. μ_a is unique.

For each $a > 1$ we define a number $y(a) \in (0,1]$ and an integer $t(a) \ge 1$. The number $y(a)$ will be the right boundary point of the support $K(a)$ and $t(a)$ will be so large that the image by $T^{t(a)}_a$ of the interval $[0, b_1/a)$ will be equal to $[0, y(a))$. Let

$$y_1(a) = \lim_{x \to b_1-} T(x) \Big(= \lim_{x \to b_1/a-} T_a(x) \Big),$$

and $t_1(a) = 1$. Assume that, for $i \ge 1$, $y_i(a)$ and $t_i(a)$ are defined. Let

$$k_{i,1}(a) = \max\big\{k \ge 1 \ ; \ \frac{b_k}{a} \le y_i(a)\big\}$$

and set

$$y_{i,1}(a) = \max_{1 \le k \le k_{i,1}(a)} \lim_{x \to b_k-} T(x).$$

Assume that both $k_{i,j}(a)$ and $y_{i,j}(a)$ are defined for $j \ge 1$. Let

$$k_{i,j+1}(a) = \max\big\{k \ge 1 \ ; \ \frac{b_k}{a} \le y_{i,j}(a)\big\}.$$

If $k_{i,j+1}(a) = k_{i,j}(a)$ we do not define $y_{i,j+1}(a)$. Otherwise, let

$$y_{i,j+1}(a) = \max_{1 \le k \le k_{i,j+1}(a)} \lim_{x \to b_k-} T(x).$$

Since, for fixed $a > 1$, there are only finitely many $k \ge 1$ such that $b_k/a \in [0,1]$, $y_{i,j}(a)$ is only defined for finitely many $j \ge 1$. Let $j \ge 1$ be maximal such that $y_{i,j}(a)$ is defined.

1) If $y_{i,j}(a) = y_i(a)$ we do not define $y_{i+1}(a)$ and $t_{i+1}(a)$.
2) If $y_{i,j}(a) > y_i(a)$ and $\lim_{x \to y_{i,j}(a)-} T_a(x) \le y_{i,j}(a)$ we set $y_{i+1}(a) = y_{i,j}(a)$ and $t_{i+1}(a) = t_i(a) + j$.
3) If we are not in case 1) nor in case 2), it follows that

$$y_{i,j}(a) \in (b_{k_{i,j}(a)}/a, b_{k_{i,j}(a)+1}/a)$$

and $T_a(y_{i,j}(a)) > y_{i,j}(a)$. We set

$$y_{i+1}(a) = \lim_{x \to b_{k_{i,j}(a)+1}-} T(x).$$

Taking l minimal such that $T^l_a(y_{i,j}(a)) \ge b_{k_{i,j}(a)+1}/a$, we set $t_{i+1}(a) = t_i(a) + j + l < \infty$.

Observe that if the cases 1) or 2) occur, $y_{i+2}(a)$ will not be defined. Only when the case 3) occurs, $y_{i+2}(a)$ is possibly defined. Since, for fixed $a > 1$, there are only finitely many $k \ge 1$ such that $b_k/a \in [0,1]$, case 3) can occur only a finite number of times, which implies that $y_i(a)$ and $t_i(a)$ are defined only for a finite number of $i \ge 1$. We set $y(a) = y_i(a)$ and $t(a) = t_i(a)$ where i is the maximal number for which $y_i(a)$ and $t_i(a)$ are defined. (Note that $t(a)$ is finite.) By the constructions of $y(a)$ and $t(a)$ it follows that

$$[0, y(a)) = T^l_a([0, b_1/a)),$$

for all $l \geq t(a)$. Hence, the support $K(a)$ of $\mu(a)$ coincides with the interval $[0, y(a)]$. Furthermore, by the construction of $y(a)$, we deduce that $y(a)$ is not decreasing in a and for each $a > 1$ there exists a $k \geq 1$ such that $y(a) = \lim_{x \to b_k -} T(x)$. It follows that $y(a)$ is piecewise constant. It is straightforward to check that $y(a)$ is right continuous and $\tilde{a} > 1$ is a point of discontinuity for $a \mapsto y(a)$ if and only if $y(\tilde{a})$ is a fixed point for $T_{\tilde{a}}$ and different from 1. Furthermore, $\lim_{a \to \tilde{a}+} t(a) = \infty$ and possibly also $\lim_{a \to 1+} t(a) = \infty$. But on a parameter interval $I \subset (1, \infty)$ where $y(a)$ is constant, $t(a)$ is not increasing in a. Hence, if the left boundary point of I is not adjacent to 1 or a point of discontinuity for $a \mapsto y(a)$ then $t(a)$ is bounded from above on I. This concludes the proof of Lemma 6.1.

7. Tent maps

In this section we apply Theorem 2.2 to skew tent maps. Instead of considering skew tent maps defined on the unit interval as it is done in the introduction, we take the same representation as in [11], i.e. we define the skew tent map with slopes α and $-\beta$ where $\alpha, \beta > 1$, by the formula

$$T_{\alpha,\beta}(x) = \begin{cases} 1 + \alpha x & \text{if } x \leq 0, \\ 1 - \beta x & \text{otherwise,} \end{cases}$$

(see Remark 7.3). The turning point of $T_{\alpha,\beta}$ is 0, $T_{\alpha,\beta}(0) = 1$ and, by Lemma 3.1 in [11], if $\alpha^{-1} + \beta^{-1} \geq 1$ then the interval $[T_{\alpha,\beta}(1), 1] (= [1 - \beta, 1])$ is invariant under $T_{\alpha,\beta}$ (if $\alpha^{-1} + \beta^{-1} < 1$ then there exists no invariant interval of finite positive length). For two parameter couples (α, β) and (α', β') we take the same order relation as the one which appears in [11], i.e. we shall write $(\alpha', \beta') > (\alpha, \beta)$ if $\alpha' \geq \alpha$, $\beta' \geq \beta$, and at least one of these inequalities is sharp. Fix (α_0, β_0) and (α_1, β_1) in the set $\{(\alpha, \beta)\ ;\ \alpha, \beta > 1 \text{ and } \alpha^{-1} + \beta^{-1} \geq 1\}$ such that $(\alpha_1, \beta_1) > (\alpha_0, \beta_0)$. Let

$$\alpha : [0,1] \to [\alpha_0, \alpha_1] \quad \text{and} \quad \beta : [0,1] \to [\beta_0, \beta_1]$$

be functions in $C^1([0,1])$ such that $(\alpha(0), \beta(0)) = (\alpha_0, \beta_0)$, $(\alpha(1), \beta(1)) = (\alpha_1, \beta_1)$, and, for all $a \in [0,1]$, if $\alpha_0 \neq \alpha_1$ then $\alpha'(a) > 0$ and if $\beta_0 \neq \beta_1$ then $\beta'(a) > 0$. Observe that $\alpha(a), \beta(a) > 1$, and $\alpha(a)^{-1} + \beta(a)^{-1} \geq 1$, for all $a \in [0,1]$. We define the one-parameter family T_a as the family of skew tent maps given by

$$T_{\alpha(a),\beta(a)} : [T_{\alpha(a),\beta(a)}(1), 1] \to [T_{\alpha(a),\beta(a)}(1), 1], \qquad a \in [0,1].$$

By [9], since T_a has only two intervals of monotonicity, there exists a unique a.c.i.p. μ_a. Observe that even if T_a is now defined on a larger interval than the one-parameter families described in Subsection 2.1, the definitions of the partitions $\mathcal{P}_j(a)$ and $\mathcal{P}_j|[0,1]$ in Subsection 2.2 still apply. The main statement of this section is the following.

Theorem 7.1. *For a.e. parameter $a \in [0,1]$ the turning point 0 is typical for μ_a.*

In contrast to the β-transformation, it is more difficult to state typicality for other, not so specific points as, e.g., the turning point. However, we will show that conditions (IIa) and (IIb) are satisfied for the one-parameter family of skew tents maps T_a, $a \in [0,1]$. Hence, given a C^1 function $Y : [0,1] \to \mathbb{R}$ (such that $Y(a) \in [T_a(1), 1]$), it is sufficient to check condition (I) in order to obtain a.s. typicality for Y.

Corollary 7.2. *If the one-parameter family T_a, $a \in [0,1]$, with the associated map $a \mapsto Y(a)$ satisfies condition (I), then $Y(a)$ is typical for μ_a, for a.e. $a \in [0,1]$.*

The calculations in Subsection 7.1 below show that if the function $y_j(a) = T_a^j(Y(a))$ has, for some $j \geq 1$, a high enough initial expansion in a, it will imply condition (I) for T_a with the associated map $a \mapsto \tilde{Y}(a) = y_j(a)$. This makes it easy to check condition (I) in Corollary 7.2 numerically.

If $\alpha \le \beta/(\beta^2-1)$ then $T_{\alpha,\beta}$ is renormalizable, see, e.g., [11]. More precisely, $T^2_{\alpha,\beta}(1)$ is greater or equal than the unique fixed point in $(0,1)$ and $T^2_{\alpha,\beta}$ restricted either to the interval $[T_{\alpha,\beta}(1), T^3_{\alpha,\beta}(1)]$ or to the interval $[T^2_{\alpha,\beta}(1), 1]$ is affinely conjugated to $T_{\beta^2,\alpha\beta}$ restricted to the interval $[T_{\beta^2,\alpha\beta}(1), 1]$. Observe that the new slopes $\alpha' = \beta^2$ and $-\beta' = -\alpha\beta$ still satisfy $\alpha', \beta' > 1$ and $(\alpha')^{-1} + (\beta')^{-1} \ge 1$ (the latter inequality follows from the assumption $\alpha \le \beta/(\beta^2-1)$). Since the function $\beta \mapsto \beta/(\beta^2-1)$ is decreasing for $\beta > 1$, we have that if T_0 is not renormalizable then not either T_a, $a \in [0,1]$, is renormalizable. Now, assume for the moment that T_a is renormalizable for each $a \in [0,1]$ and consider the one-parameter family defined by $\tilde{T}_a = T_{\beta(a)^2,\alpha(a)\beta(a)}$. Note that if we show typicality of the turning point for the family $\tilde{T}_a$, for a.e. $a \in [0,1]$, this implies a.s. typicality of the turning point for the original family T_a. Furthermore, if we verify conditions (IIa) and (IIb) for the family $\tilde{T}_a$, this implies that conditions (IIa) and (IIb) also hold for the family T_a. Since the a-derivative of $\alpha(a)\beta(a)$ is positive and the a-derivative of $\beta(a)^2$ is non-negative, the new one-parameter family $\tilde{T}_a$ fits into the family of skew tent maps described in the beginning of this section. Furthermore, it is known that for each $a \in [0,1]$, T_a is at most a finite number of times renormalizable where this number is bounded above by a constant only dependent on (α_0, β_0) and not on the parameter a (this can easily be derived by looking, e.g., at the topological entropy of T_a, see [11] page 137). Altogether, we derive that in order to prove Theorem 7.1 (and therewith also Corollary 7.2) we can without loss of generality restrict ourself to the case when T_a, $a \in [0,1]$, is not renormalizable, i.e. we assume that

$$\alpha_0 > \beta_0/(\beta_0^2 - 1). \tag{35}$$

Observe that it is only possible for the parameter $a = 1$ to satisfy the equality $\alpha(a)^{-1} + \beta(a)^{-1} = 1$. Thus, since we are only interested in Lebesgue almost every parameter we can neglect skew tent maps whose slopes satisfy $\alpha^{-1} + \beta^{-1} = 1$, i.e. we assume that

$$\alpha_1^{-1} + \beta_1^{-1} > 1. \tag{36}$$

For non-renormalizable T_a it will follow from Subsection 7.2 that the support $K(a)$ of the a.c.i.p. μ_a is the whole invariant interval $[T_a(1), 1]$. Hence, if ψ_a is the affine map from $[0,1]$ onto $[T_a(1), 1]$ with, say, positive derivative it is straightforward to check that the one-parameter family $\psi_a \circ T_a \circ \psi_a^{-1} : [0,1] \to [0,1]$, $a \in [0,1]$, fits into the model described in Subsection 2.1, satisfying properties (i)-(iii).

Remark 7.3. Observe that the length of the invariant interval $K(a)$ for T_a is bounded from below by 1 and from above by β_1. Hence, the estimates to be established in conditions (I), (IIa), and (IIb) for the family T_a and the to it affinely conjugated family on the unit interval will differ only by constants, which are uniformly in a bounded above, and below away from zero. Therefore we will continue with the representation T_a and do not switch to skew tent maps defined on the unit interval. The partitions defined in Subsection 2.2 are defined in an analog way for the family T_a.

Let $X(a) = T_a^3(0)$ (if we started with an iteration of 0 lower than the third, then by our definition of the $\mathcal{P}_j|[0,1]$'s, all of these partitions would be empty). To prove Theorem 7.1 it is sufficient to verify conditions (I), (IIa), and (IIb) for the family T_a, $a \in [0,1]$, with the associated map X. Henceforth, we will use the notations of Subsection 2.1 related to the family T_a, $a \in [0,1]$. The constants λ and Λ can be chosen as

$$\lambda = \min\{\alpha_0, \beta_0\} \qquad \text{and} \qquad \Lambda = \max\{\alpha_1, \beta_1\}.$$

The main computation needed for the verifications of (I) and (IIb) is already done in a paper by Misiurewicz and Visinescu [11] (see Lemma 3.3 and 3.4 therein), where they show monotonicity of the kneading sequence for skew tent maps. (A reader not familiar with the basic notions and facts of kneading theory can find them in [5].) For the later use we state here the main result in [11].

Theorem 7.4. *Let* (α,β) *and* (α',β') *be in the set* $\{(\alpha,\beta)\ ;\ \alpha,\beta>1$ *and* $\alpha^{-1}+\beta^{-1}\geq 1\}$*. If* $(\alpha',\beta')>(\alpha,\beta)$ *then the kneading sequence of* $T_{\alpha',\beta'}$ *is strictly greater than the kneading sequence of* $T_{\alpha,\beta}$*.*

Proof. See Theorem A in [11]. □

Since the derivatives of $\alpha(a)$ and $\beta(a)$ are non-negative and at least one of them is positive, we obtain strict monotonicity of the kneading sequence for our family T_a.

Remark 7.5. We could also formulate and prove certain $C^{1,1}(L)$-versions of Theorem 7.1 and Corollary 7.2. But it is difficult for us to formulate very general statements as it is, e.g., done in Section 6 for the β-transformation. To mention nevertheless an example, one could show, by the methods in this section, almost sure typicality for the turning point of one-parameter families $\tilde{T}_a$ of $C^{1,1}(L)$ unimodal maps, which are of the form $\tilde{T}_a=T_a\circ g$ where T_a is a fixed family of skew tent maps as described above but with a representation such that the T_a's map the unit interval into itself, and $g:[0,1]\to[0,1]$ is a $C^{1,1}(L)$ homeomorphism satisfying $g'(x)\approx 1$. However, to keep this paper in a reasonable size we will here not investigate such possible $C^{1,1}(L)$-versions of skew tent maps.

7.1. **Condition (I).** We note first that, for $j\geq 1$, the number of parameter values $a\in[0,1]$ not contained in any partition element of $\mathcal{P}_j|[0,1]$ is finite. In fact, if $a\in[0,1]$ is not contained in any element of $\mathcal{P}_j|[0,1]$, then $x_i(a)=T_a^{i+3}(0)=0$ for some $0\leq i<j$. Hence, the kneading sequence of T_a ends with C and has length smaller than $j+3$. But, by the strict monotonicity of the kneading sequence, T_a can have such a kneading sequence only for finitely many parameters $a\in[0,1]$.

Let $j\geq 1$, $0\leq\tau<j$, and $\omega\in\mathcal{P}_j|[0,1]$. For $a\in\omega$, we have

$$\begin{aligned}
x_j'(a)=D_a(T_a(x_{j-1}(a)))&=\begin{cases}\alpha(a)x_{j-1}'(a)+\alpha'(a)x_{j-1}(a) & \text{if } x_{j-1}(a)\leq 0,\\ -\beta(a)x_{j-1}'(a)-\beta'(a)x_{j-1}(a) & \text{otherwise,}\end{cases}\\
&=T_a'(x_{j-1}(a))\cdot\begin{cases}x_{j-1}'(a)+\frac{1}{T_a'(x_{j-1}(a))}\alpha'(a)x_{j-1}(a) & \text{if } x_{j-1}(a)\leq 0,\\ x_{j-1}'(a)-\frac{1}{T_a'(x_{j-1}(a))}\beta'(a)x_{j-1}(a) & \text{otherwise,}\end{cases}\\
&=T_a^{j-\tau\,\prime}(x_\tau(a))\cdot\\
&\quad\Bigg(x_\tau'(a)+\underbrace{\sum_{i=1}^{j-\tau}\frac{1}{T_a^{i\,\prime}(x_\tau(a))}\cdot\begin{cases}\alpha'(a)x_{\tau+i-1}(a) & \text{if } x_{\tau+i-1}(a)\leq 0,\\ -\beta'(a)x_{\tau+i-1}(a) & \text{otherwise.}\end{cases}}_{(*)}\Bigg).
\end{aligned}$$

Let $M=\max_{a\in[0,1]}\{\alpha'(a)|1-\beta_1|,\beta'(a)\}<\infty$ and $M'=\max_{a\in[0,1]}|X'(a)|$. We have

$$|(*)|\leq\frac{M}{\lambda-1},$$

and, by setting $\tau=0$, we obtain in condition (I) the upper bound

$$\left|\frac{x_j'(a)}{T_a^{j\,\prime}(X(a))}\right|\leq M'+\frac{M}{\lambda-1}.$$

To establish a lower bound in condition (I) is more delicate. We will use some derivative estimates given in [11]. To this end we will look at the kneading sequences of the T_a's. Every kneading sequence of a map T_a in our family starts with RL and is smaller or equal than the sequence RL^∞. In fact, by (36) and by the monotonicity of the kneading sequence (Theorem 7.4), the kneading sequence of $T_1 (= T_{\alpha_1,\beta_1})$ is strictly smaller than RL^∞. Let $1 \le m_1 < \infty$ be the integer such that the kneading sequence of T_1 starts with $RL^{m_1}R$ or is equal to $RL^{m_1}C$. From [11] we derive the following result.

Proposition 7.6. *There exists a constant $\kappa > 0$ such that for $\omega \in \mathcal{P}_j|[0,1]$, $j \ge 0$, and $a \in \omega$, we have*

$$\left|\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0)\right|, \ \left|\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0)\right| \ge \kappa \beta_0^{\left[\frac{j}{m_1}\right]}, \tag{37}$$

and, furthermore,

$$\operatorname{sign}(\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0)) = \operatorname{sign}(\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0)) = \operatorname{sign}(T_a^{j+2\,\prime}(1)). \tag{38}$$

Proof. The proof of Proposition 7.6 follows from Lemma 3.3 and 3.4 in [11]. For this note that for each $a \in [0,1]$, the integer $m \ge 1$ such that the kneading sequence of T_a starts with RL^mR or is equal to RL^mC is smaller or equal than m_1. Observe also that x_j in [11] corresponds to x_{j-2} in our setting.

Actually, Lemma 3.4 in [11] is only formulated for the case when $j \ge m$. But considering Lemma 3.4 i) in [11] it is easy to deduce that Proposition 7.6 also holds when $0 \le j < m$. □

Property (38) will be essential to verify condition (IIb).

For $j \ge 0$, we have

$$x_j'(a) = \alpha'(a)\partial_\alpha T^{j+3}_{\alpha(a),\beta(a)}(0) + \beta'(a)\partial_\beta T^{j+3}_{\alpha(a),\beta(a)}(0), \tag{39}$$

for all a contained in an element of $\mathcal{P}_j|[0,1]$. Since at least one of the derivatives $\alpha'(a)$ and $\beta'(a)$ is uniformly bounded away from 0, by (37) and (38), $|x_j'|$ is uniformly growing and we can fix an integer $j_0 \ge 1$ such that

$$|x_{j_0}'(a)| \ge \frac{M}{\lambda - 1} + 1.$$

Thus, by setting $\tau = j_0$ in the formula for the derivative of x_j in the beginning of this subsection, we obtain that for all $\omega \in \mathcal{P}_j|[0,1]$, $j \ge j_0$, and $a \in \omega$,

$$|x_j'(a)| \ge |T_a^{j-j_0\,\prime}(x_{j_0}(a))|,$$

which implies

$$\left|\frac{x_j'(a)}{T_a^{j\,\prime}(X(a))}\right| \ge \frac{1}{\Lambda^{j_0}}.$$

Furthermore, by (37), (38), and (39), there is a constant $\kappa' > 0$ such that $|x_j'(a)| \ge \kappa'$, for all $1 \le j < j_0$. This concludes the proof of a lower bound in condition (I).

7.2. **Condition (IIa).** The verification of condition (IIb) in the next subsection does not make use of condition (IIa). Hence, as in the first paragraph in Subsection 6.2, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C([0,1]) \ge 1$ such that for each $a \in [0,1]$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. In the remaining part of this section we will establish a lower bound for φ_a on the whole of $K(a)$.

If $\alpha = \beta/(\beta^2 - 1)$ then the kneading sequence of $T_{\alpha,\beta}$ is RLR^∞ (see Lemma 3.2 and its proof in [11]). By (35) and by the monotonicity of the kneading sequence (Theorem 7.4), there exists a non-negative integer m_0, which is either equal to 0 or even, such that the kneading sequence of $T_0 = T_{\alpha_0,\beta_0}$ starts with $RLR^{m_0}L$ or is equal to $RLR^{m_0}C$. This in turn implies that for each $a \in [0,1]$ there exists an integer $0 \le m \le m_0$, which is either equal to 0 or even, such that the kneading sequence of T_a starts with RLR^mL or is equal to RLR^mC. Fix $a \in [0,1]$ and let $0 \le m \le m_0$ be the corresponding integer.

Lemma 7.7. *Let $J \subset [T_a(1), 1]$ be an interval adjacent to 0 and let $j \ge 1$ be the first time such that $\#\{\omega \in \mathcal{P}_j(a)|J\} = 2$. If $j \le m+2$ then $T_a^{m+4}(J) \supset (T_a(1), 1)$.*

Proof. Since $T_a((0,1)) = (T_a(1), 1)$ it is enough to show that $T_a^{m+3}(J) \supset (0,1)$. Observe that, by the definition of j, 0 is contained in $T_a^j(J)$. If $j = 1$ then, since 0 is a boundary point of J, it follows that $T_a(J)$ contains the interval $(0,1)(= (0, T_a(0)))$.

If $j = 2$, then $T_a^2(J)$ contains the interval $(T_a^2(0), 0)$. If $m = 0$ then $T_a^2(0) \le 0$ and it follows that $T_a^3(J) \supset (0,1)$. If $m \ge 2$ (recall that m is even), then we derive inductively that, for $2 < i \le m+2$, $T_a^i(J) \supset (T_a^i(0), 1)$ if i is odd and $T_a^i(J) \supset (0, T_a^i(0))$ if i is even. Hence, $T_a^{m+2}(J) \supset (0, T_a^{m+2}(0))$ and, by the fact that $T_a^{m+3}(0) \le 0$, it follows that $T_a^{m+3}(J) \supset (0,1)$.

If $2 < j \le m+2$ then $T_a^j(0) > 0$. Hence, $T_a^j(J)$ contains the interval $(0, T_a^j(0))$. Observing that this implies that j is even, we can argue as in the case when $j = 2$ and deduce that $T_a^{m+3}(J) \supset (0,1)$. □

Let $J(a)$ be the interval of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. Let $j_0 \ge 1$ be the first time such that $\#\{\omega \in \mathcal{P}_{j_0}(a)|J(a)\} = 2$. We define $J_0(a)$ to be the interval in $\mathcal{P}_{j_0}(a)|J(a)$ satisfying

$$|T_a^{j_0-1}(J_0(a))| \ge |T_a^{j_0-1}(J(a))|/2 \tag{40}$$

(if both intervals in $\mathcal{P}_{j_0}(a)|J(a)$ satisfy this then we choose one arbitrarily). Let $j_1 > j_0$ be the first time such that $\#\{\omega \in \mathcal{P}_{j_1}(a)|J_0(a)\} = 2$. Assume now that $J_{i-1}(a)$ and j_i are defined for some $i \ge 1$. If $j_i - j_{i-1} \le m+2$ we stop and do not define $J_i(a)$. Otherwise, let $J_i(a)$ be the interval in $\mathcal{P}_{j_i}(a)|J_{i-1}(a)$ satisfying

$$|T_a^{j_i-1}(J_i(a))| \ge |T_a^{j_i-1}(J_{i-1}(a))|/2 \tag{41}$$

(if both intervals in $\mathcal{P}_{j_i}(a)|J_{i-1}(a)$ satisfy this then we choose one arbitrarily). We define $j_{i+1} > j_i$ to be the first time such that $\#\{\omega \in \mathcal{P}_{j_{i+1}}(a)|J_i(a)\} = 2$. The size of the images of the $J_i(a)$'s is growing in i:

Lemma 7.8. *There exists a constant $\tilde{\lambda} > 2$ independent on the parameter a such that if $J_k(a)$ and j_{k+1} are defined for some $k \ge 1$, then, for all $0 \le i < k$, we have*

$$|T_a^{j_{i+1}-1}(J_i(a))| \ge \tilde{\lambda}|T_a^{j_i-1}(J_i(a))|.$$

Proof. Since there are at least $m+3$ iterations between j_i and j_{i+1} and since the right boundary point of $T_a^{j_i}(J_i(a))$ is 1, we have

$$\begin{aligned}|T_a^{j_{i+1}-1}(J_i(a))| &= |T_a^{j_{i+1}-j_i-1}{}'(1)| \cdot |T_a'|_{T_a^{j_i-1}(J_i(a))}| \cdot |T_a^{j_i-1}(J_i(a))| \\ &= \underbrace{|T_a^{j_{i+1}-j_i-3}{}'(T_a^2(1))|\alpha(a)\beta(a)\min\{\alpha(a),\beta(a)\}}_{(*)}|T_a^{j_i-1}(J_i(a))|.\end{aligned}$$

If $m \ge 2$ (recall that m is even), then $j_{i+1} - j_i - 3 \ge 2$ and the kneading sequence of T_a starts with RLR^2. Therefore $(*) \ge \alpha(a)\beta(a)^3\min\{\alpha(a),\beta(a)\}$. By (35), $\alpha(a) > \max\{\beta(a)/(\beta(a)^2-1), 1\}$. Hence,

$$(*) \ge \inf_{\beta > 1}\left\{\beta^3\max\{\beta/(\beta^2-1),1\}\min\{\max\{\beta/(\beta^2-1),1\},\beta\}\right\} \ge 4,$$

where it is straightforward to verify the last inequality.

If $m = 0$ then $(*) = \beta(a)\alpha(a)\min\{\alpha(a),\beta(a)\}$. In the case when $1 < \beta(a) < 2$ we have a better lower bound for $\alpha(a)$ than the one above. Given $1 < \beta < 2$, note that the kneading sequence of $T_{(\beta-1)^{-1},\beta}$ is equal to RLC. Since $m = 0$ it follows that $\alpha(a) \geq \max\{(\beta(a)-1)^{-1}, 1\}$ and we obtain

$$(*) \geq \beta(a)\max\{(\beta(a)-1)^{-1},1\}\min\{\max\{(\beta(a)-1)^{-1},1\},\beta(a)\}.$$

This lower bound is not good enough since the minimum of the function

$$\beta\max\{(\beta-1)^{-1},1\}\min\{\max\{(\beta-1)^{-1},1\},\beta\},$$

which is attained in the point $\beta = 2$, is equal to 2. However, if $\beta(a) \approx 2$ then, since $\alpha(a) \geq \alpha_0 > 1$, we derive that $(*) \geq 2+\varepsilon$ for some $\varepsilon > 0$. Setting $\tilde{\lambda} = \min\{4, 2+\varepsilon\}$, this concludes the proof. □

Assuming that $J_k(a)$ and j_{k+1} are defined for some $k \geq 1$, by (40), (41), and Lemma 7.8, it follows that

$$|T_a^{j_k-1}(J_{k-1}(a))| \geq \Big(\frac{\tilde{\lambda}}{2}\Big)^k \frac{|J(a)|}{2} \geq \Big(\frac{\tilde{\lambda}}{2}\Big)^k \frac{1}{2C},$$

where $(\tilde{\lambda}/2)^k$ is growing in k. The length of the interval $T_a^{j_k-1}(J_{k-1}(a))$ is bounded above by the length of the invariant interval $[T_a(1),1]$, which in turn is bounded by β_1. This implies that the number $k \geq 0$ for which $J_k(a)$ can be defined, is bounded above by a number independent on $a \in [0,1]$. Let k be maximal such that $J_k(a)$ and j_{k+1} are defined. It follows that $j_{k+1} - j_k \leq m+2$. Since the interval $T_a^{j_k-1}(J_k(a))$ is adjacent to 0, we can apply Lemma 7.7 and we obtain

$$T_a^{j_k+m+3}(J_k(a)) \supset (T_a(1),1).$$

Clearly, the number of iterations between successive j_i's is bounded above by a number independent on a, and the integer m is bounded above by m_0. Altogether, we derive that there is an iteration $j \geq 1$ (independent on the parameter a), such that for each $a \in [0,1]$, $T_a^j(J(a)) = [T_a(1),1]$. Finally, we can apply inequality (34) and we obtain a lower bound for φ_a on the whole of $[T_a(1),1]$. Observe that $T_a^j(J(a)) = [T_a(1),1]$ implies that $K(a) = [T_a(1),1]$.

7.3. **Condition (IIb).** The main ingredient in verifying condition (IIb) is property (38) stated in Proposition 7.6. Observe that, by (38), (39), and the definition of x_j, we have

$$\operatorname{sign}(D_a T_a^m(0)) = \operatorname{sign}(T_a^{m-1\,\prime}(1)), \tag{42}$$

for all $m \geq 3$ and parameter values a contained in an element of $\mathcal{P}_{m-3}|[0,1]$.

We verify condition (IIb) by induction over $j \geq 1$. In fact, we will show the following statement. For each $j \geq 1$ there exists a map as described in condition (IIb) and further, if $a_1, a_2 \in [0,1]$ such that $a_1 < a_2$ and $\omega \in \mathcal{P}_{a_1}$, then the boundary points of $T_a^j(\mathcal{U}_{a_1,a,j}(\omega))$ are continuous in $a \in [a_1,a_2]$. For $j = 1$ this statement can easily be verified observing that, by the properties of the maps α and β, $T_a(1)$ is constant or continuously decreasing and $T_a^2(1)$ is continuously increasing in $a \in [0,1]$. Assume now that the statement holds for some $j \geq 1$. Take $\tilde{\omega} \in \mathcal{P}_j(a_1)$ and, for $a \in [a_1,a_2]$, let $\tilde{\omega}(a) = \mathcal{U}_{a_1,a,j}(\tilde{\omega})$ be the to it associated element in $\mathcal{P}_j(a)$. Since $T_{a_1}^j(\tilde{\omega}) \subset T_a^j(\tilde{\omega}(a))$ and the turning point 0 is constant in a, it follows that for each (of the maximal two) element $\omega \in \mathcal{P}_{j+1}(a_1)|\tilde{\omega}$ there is a unique element $\omega(a) \in \mathcal{P}_{j+1}(a)|\tilde{\omega}(a)$ fulfilling

$$\operatorname{symb}_a(T_a^i(\omega(a))) = \operatorname{symb}_{a_1}(T_{a_1}^i(\omega)), \tag{43}$$

for $0 \leq i < j+1$. In particular this is true for $a = a_2$. Setting $\mathcal{U}_{a_1,a_2,j+1}(\omega) = \omega(a_2)$, this verifies property (17) in condition (IIb). Using the induction assumption it is straightforward to derive the continuity of the boundary points of $T_a^{j+1}(\omega(a))$ for $a \in [a_1, a_2]$. So, it is only left to show that property (18) is satisfied, i.e.

$$T_{a_1}^{j+1}(\omega) \subset T_{a_2}^{j+1}(\omega(a_2)). \tag{44}$$

For $a \in [a_1, a_2]$, let x_a be, say, the left boundary point of $\omega(a)$. Observe that, by (43), we have $\operatorname{sign}(T_a^i{}'|_{\omega(a)}) \equiv \operatorname{sign}(T_{a_1}^i{}'|_\omega)$ for all $1 \le i < j+1$. Let $\sigma = \operatorname{sign}(T_{a_1}^{j+1}{}'|_\omega)$. We have to show that

$$\sigma T_{a_2}^{j+1}(x_{a_2}) \le \sigma T_{a_1}^{j+1}(x_{a_1}). \tag{45}$$

To this end, we make use of the following general fact for skew tent maps. For $a \in [0, 1]$, the image by T_a^i, $i \ge 1$, of a boundary point x of an element in $\mathcal{P}_i(a)$ is of the form

$$T_a^i(x) = T_a^m(0),$$

for some integer $1 \le m \le i+2$. Applied to the boundary point x_a, $a \in [a_1, a_2]$, we denote by $1 \le m(a) \le j+3$ the minimal integer such that $T_a^{j+1}(x_a) = T_a^{m(a)}(0)$. By the strict monotonicity of the kneading sequence there can only be finitely many $a_0 \in [a_1, a_2]$ such that $T_{a_0}^k(0) = T_{a_0}^l(0)$, for $1 \le k \ne l \le j+3$. Hence, by the continuity of $T_a^{j+1}(x_a)$, it follows that, disregarding an at most finite number of points, we can cover $[a_1, a_2]$ by open intervals $J \in [a_1, a_2]$ on which the integer $m(a)$ is constant. Fix such an interval J and let m denote the to it associated integer. In order to show (45), we prove that

$$\sigma T_{a'}^m(0) \le \sigma T_a^m(0), \tag{46}$$

for all $a, a' \in J$ such that $a < a'$. Since $a \mapsto T_a^{j+1}(x_a)$ is continuous on $[a_1, a_2]$, we can extend (46) to parameters a and a' lying in the closure of J, from which we deduce (45). In order to establish (46), it is sufficient to show that, for all $a \in J$, the derivative $D_a T_a^m(0)$ exists and

$$\operatorname{sign}(D_a T_a^m(0)) = -\sigma \qquad \text{or} \qquad \operatorname{sign}(D_a T_a^m(0)) = 0. \tag{47}$$

The cases when $m = 1, 2$ or 3 are a bit special, so we treat them one by one. If $m = 1$ then $T_a^{j+1}(x_a) = 1$ for all $a \in J$, and (47) is satisfied. If $m = 2$ then, for $a \in J$, we have $T_a^{j+1}(x_a) = T_a(1) = 1 - \alpha(a)$ which is the left boundary point of $K(a)$. It follows that $\sigma = +1$. The derivative of α is non-negative in a and, hence, (47) is satisfied. If $m = 3$ then $T_a^j(x_a) = T_a(1)$ for all $a \in J$. It follows that $\operatorname{sign}(T_a^j{}'|_{\omega(a)})$ and $\operatorname{sign}(T_a'|_{T_a^j(\omega(a))})$ are both equal to $+1$ and, hence, we have $\sigma = +1$. By (42), $\operatorname{sign}(D_a T^3(0)) = -1$ which implies (47).

Finally, we turn to the case when $m > 3$. Observe that since m was chosen minimal, we have that $T_a^i(0) \ne 0$ for all $1 \le i < m$ and all $a \in J$. It follows that $J \subset \omega'$ for some element $\omega' \in \mathcal{P}_{m-3}|[0, 1]$. By (42), we deduce that the derivatives $D_a T^m(0)$ and $D_a T^{m-1}(0)$ exist and are non-zero on J. Note that since $T_a^j(x_a) \ne 0$ this implies that x_a is in fact also the left boundary point of $\tilde{\omega}(a)$. Thus, by the induction assumption, we obtain

$$\operatorname{sign}(T_{a_1}^j{}'|_\omega) T_{a'}^j(x_{a'}) \le \operatorname{sign}(T_{a_1}^j{}'|_\omega) T_a^j(x_a),$$

for all $a, a' \in [a_1, a_2]$ such that $a < a'$, which implies that $\operatorname{sign}(D_a T_a^{m-1}(0))$ is equal to $-\operatorname{sign}(T_{a_1}^j{}'|_\omega)$. On the other hand, by (42), we derive that

$$\operatorname{sign}(D_a T_a^m(0)) \operatorname{sign}(D_a T_a^{m-1}(0)) = \operatorname{sign}(T_a'(T_a^{m-1}(0))).$$

Since $\operatorname{sign}(T_a'(T_a^{m-1}(0))) = \operatorname{sign}(T_{a_1}'|_{T_{a_1}^j(\omega)})$, it follows that $\operatorname{sign}(D_a T_a^m(0)) = -\sigma$, which concludes the proof of (47) in the case when $m > 3$.

If x_a, $a \in [a_1, a_2]$, denotes the right boundary point of $\omega(a)$, we can do an analog argument to show that

$$\sigma T_{a_2}^{j+1}(x_{a_2}) \geq \sigma T_{a_1}^{j+1}(x_{a_1}).$$

Combined with (45) this implies inequality (44) and, thus, this concludes the verification of condition (IIb).

8. Markov partition preserving one-parameter families

Assume that we have a one-parameter family $T_a : [0,1] \to [0,1]$, $a \in I$, as described in Subsection 2.1, satisfying properties (i)-(iii). We require additionally that for each $a \in I$ the intervals $D_1(a), ..., D_{p_2}(a)$ have the following Markov property.

(M) For each $1 \leq k \leq p_2$ there exists $0 \leq i_k^L < i_k^R \leq p_1$ (independent on a), such that, for all $a \in I$,

$$T_a(D_k(a)) = (c_{i_k^L}(a), c_{i_k^R}(a)),$$

and, furthermore, these images are constant, i.e.

$$c_{i_k^L}(a) \equiv c_{i_k^L} \quad \text{and} \quad c_{i_k^R}(a) \equiv c_{i_k^R}.$$

Theorem 8.1. *If the one-parameter family T_a, $a \in I$, satisfies the Markov property (M) and if for a C^1 map $X : I \to [0,1]$ condition (I) is fulfilled, then $X(a)$ is typical for μ_a, for a.e. $a \in I$.*

Example 8.2. Let

$$\tilde{T}_a(x) = \begin{cases} \frac{x}{a} & \text{if } x < a, \\ \frac{x-a}{1-a} & \text{otherwise,} \end{cases}$$

and $g : [0,1] \to [0,1]$ a $C^{1,1}(L)$ homeomorphism such that $\inf_x g'(x) > 0$ and such that the set

$$I = \{a \in (0,1) \ ; \ \inf_x \tilde{T}_a'(g(x))g'(x) > 1\}$$

is non-empty. Clearly, I is an (open) interval. We define the one-parameter family $T_a : [0,1] \to [0,1]$ as

$$T_a(x) = \tilde{T}_a(g(x)), \qquad a \in I.$$

By [17], since T_a has only one point of discontinuity, there exists a unique a.c.i.p. μ_a. From the verification of condition (IIa) in the proof of Theorem 8.1, it will follow that $\mathrm{supp}(\mu_a) = [0,1]$.

Proposition 8.3. *If $X : I \to (0,1)$ is a C^1 map such that $X'(a) \leq 0$, then $X(a)$ is typical for μ_a, for a.e. parameter $a \in I$.*

Proof. To fit the one-parameter family T_a into the model described in Subsection 2.1, we restrict the family to a smaller parameter interval $\tilde{I} \subsetneq I$ such that $\tilde{I}$ does not have a boundary point in common with I. Since I can be covered by a countable number of such intervals $\tilde{I}$, in order to prove Proposition 8.3, it is sufficient to consider the family T_a, $a \in \tilde{I}$. By the choice of $\tilde{I}$, it follows that there exist constants $1 < \lambda \leq \Lambda < \infty$ such that for every $a \in \tilde{I}$,

$$\lambda \leq \inf_{x \in [0,1]} T_a'(x) \qquad \text{and} \qquad \sup_{x \in [0,1]} T_a'(x) \leq \Lambda.$$

Furthermore, for $a \in \tilde{I}$, T_a is piecewise $C^{1,1}(\tilde{L})$ where

$$\tilde{L} = L \cdot \sup_{a \in \tilde{I}} \{a^{-1}, (1-a)^{-1}\}.$$

Now, one checks easily that the one-parameter family T_a, $a \in \tilde{I}$, fits into the model described in Subsection 2.1 satisfying properties (i)-(iii). Hence, we can apply Theorem 8.1 to this family. Clearly, T_a satisfies the Markov property (M). In order to proof a.s. typicality, it is only left to verify condition (I). By a similar calculation as it is done in Subsections 6.1 and 7.1, we derive, for $\omega \in \mathcal{P}_j|\tilde{I}$, $j \geq 1$, the following formula for the derivative $x_j'(a)$, $a \in \omega$:

$$x_j'(a) = T_a^j{}'(X(a))X'(a) - \sum_{i=0}^{j-1} T_a^{j-i}{}'(x_i(a)) \cdot \begin{cases} \frac{g(x_i(a))}{ag'(x_i(a))} & \text{if } g(x_i(a)) < a, \\ \frac{1-g(x_i(a))}{(1-a)g'(x_i(a))} & \text{otherwise.} \end{cases}$$

Note that this derivative is strictly negative. We obtain

$$\frac{x_j'(a)}{T_a^j{}'(X(a))} = X'(a) - \sum_{i=0}^{j-1} \frac{1}{T_a^i{}'(X(a))} \cdot \begin{cases} \frac{g(x_i(a))}{ag'(x_i(a))} & \text{if } g(x_i(a)) < a, \\ \frac{1-g(x_i(a))}{(1-a)g'(x_i(a))} & \text{otherwise.} \end{cases}$$

Set

$$s = \inf_{a \in \tilde{I}} X'(a) \qquad \text{and} \qquad \kappa = \inf_{a \in \tilde{I}} \{g(X(a)), 1 - g(X(a))\}.$$

By the choice of $\tilde{I}$, the constant s is bounded from below and κ is strictly positive. Thus, for $a \in \tilde{I}$ and $j \geq 1$, we deduce that

$$\frac{\kappa}{\sup_x g'(x)} \leq \left| \frac{x_j'(a)}{T_a^j{}'(X(a))} \right| \leq s + \sum_{i=0}^{j-1} \frac{1}{\lambda^i} \cdot \frac{1}{\inf_x g'(x)},$$

where the first term is positive and the last one bounded from above. Hence, to conclude the verification of condition (I), it is only left to show that, for $j \geq 1$, the number of $a \in \tilde{I}$, which are not contained in any element of $\mathcal{P}_j|\tilde{I}$ is finite. By the choice of X and since the point of discontinuity of T_a is strictly increasing in a, there can only be one point in the inner of the interval $\tilde{I}$ not belonging to an element of the partition $\mathcal{P}_1|\tilde{I}$. Assume that for some $j \geq 1$ there are only finitely many points in $\tilde{I}$, which are not contained in any element of the partition $\mathcal{P}_j|\tilde{I}$. For $\omega \in \mathcal{P}_j|\tilde{I}$, we have that $x_j(\omega) \subset (0,1)$ and the derivative of x_j is negative. Hence, there is at most one point $a \in \omega$ satisfying $x_j(a) = a$ and which has to be excluded in the partition $\mathcal{P}_{j+1}|\omega$. It follows that there are only finitely many points not belonging to the partition $\mathcal{P}_{j+1}|\tilde{I}$, which concludes the verification of condition (I) and, hence, the proof of the Proposition 8.3. □

We turn to the proof of Theorem 8.1.

Proof. In order to proof Theorem 8.1, it is sufficient to verify conditions (IIa) and (IIb). We first verify condition (IIa). To verify (IIb) we observe that, since T_a is preserving a Markov structure, there exists even a bijection

$$\mathcal{U}_{a_1,a_2,j} : \mathcal{P}_j(a_1) \to \mathcal{P}_j(a_2),$$

for all $a_1, a_2 \in I$ and $j \geq 1$, satisfying (17). Since, by (M), the images $T_a(D_k(a))$ are constant, we have that, for all $\omega \in \mathcal{P}_j(a_1)$,

$$T_{a_1}^j(\omega) = T_{a_2}^j(\mathcal{U}_{a_1,a_2,j}(\omega)).$$

As in the first paragraph in Subsection 6.2, by Lemma A.1, we can without loss of generality assume that there is a constant $C = C(I) \geq 1$ such that for each $a \in I$ the density φ_a is bounded from above by C and, further, there exists an interval $J(a)$ of length C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}. Since for each $a \in I$ the expansion of T_a is at least λ, we derive that there is an integer $i \geq 1$ independent on a such that the number of elements in $\mathcal{P}_i|J(a)$ is greater or equal than 3. An element $\omega \in \mathcal{P}_i|J(a)$, which is not adjacent to a boundary point of $J(a)$, has, by

(M), image $T_a^i(\omega) = T_a(D_k(a))$ for some $1 \le k \le p_2$. By our assumption on the one-parameter family T_a, the measure μ_a is ergodic. It follows that there is an integer $j \ge i$ such that $K(a) = \text{closure}\{T_a^{j-i}(D_k(a))\}$. Furthermore, we can take $j \ge i$ not depending on $1 \le k \le p_2$. Thus, for almost every $y \in K(a)$, there exists a point $x \in J(a)$ such that x is mapped to y after j iterations, i.e. $T_a^j(x) = y$. Now, inequality (34) provides us with a lower bound for the density. Note that from this argument follows that $\text{supp}(\mu_a) = [0,1]$ in Example 8.2. □

Appendix A

Lemma A.1. *Let $T_a : [0,1] \to [0,1]$, $a \in I$, be a one-parameter family as described in Subsection 2.1, satisfying properties (i)-(iii) and condition (IIb). Disregarding a finite number of parameters in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that on each interval $\tilde{I}$ the following holds. There exists a constant $C = C(\tilde{I}) \ge 1$ such that for each $a \in \tilde{I}$ the density φ_a of μ_a is bounded above by C and, further, there exists an interval $J(a) \subset [0,1]$ of size C^{-1} such that φ_a restricted to $J(a)$ is bounded from below by C^{-1}.*

Proof. For each $a \in I$ it follows from [17] p.496 line 5 and [8] p.484 line 6, that the variation over the unit interval of the density φ_a is bounded above by a constant

$$C_v(a) = \frac{3}{\kappa(a)(\lambda^\tau - 3)},$$

where the integer $\tau \ge 1$ is chosen so large that $3/\lambda^\tau < 1$ and the number $\kappa(a)$ is given by

$$\kappa(a) = \min\{|\omega| \ ; \ \omega \in \mathcal{P}_\tau(a)\} > 0.$$

Claim. *For $j \ge 1$, let $(s_0, ..., s_{j-1})$ be a sequence of symbols $s_i \in \{1, ..., p_2\}$, $0 \le i < j$. If $a_0 \in I$ is a parameter value such that there exists an element $\omega(a_0) \in \mathcal{P}_j(a_0)$ satisfying*

$$\text{symb}_{a_0}(T_{a_0}^i(\omega(a_0))) = s_i, \qquad 0 \le i < j,$$

then there is a neighborhood U of a_0 in I such that for all $a \in U$ there is an element $\omega(a) \in \mathcal{P}_j(a)$ having the same combinatorics as $\omega(a_0)$, i.e. $\text{symb}_a(T_a^i(\omega(a))) = s_i$, $0 \le i < j$. Furthermore, the boundary points of $\omega(a)$ and $T_a^j(\omega(a))$ depend continuously on $a \in U$.

Proof. We prove the claim by induction over $j \ge 1$. The proof is easy but a bit cumbersome, so we will give only a sketch of it. We do not make use of condition (IIb). For $j = 1$ the elements in $\mathcal{P}_1(a)$ corresponding to the symbols $s_0 \in \{1, ..., p_2\}$ are the intervals $D_k(a)$, $1 \le k \le p_2$. The boundary points of these intervals are the partition points $c_k(a)$, $0 \le k \le p_1$, which are, by property (i), continuous functions on I. Using property (ii), one can show by an easy calculation that the boundary points of $T_a(D_k(a))$ are continuous on I. Now, assume that the statement holds for some $j \ge 1$. Fix a sequence $(s_0, ..., s_j)$ of symbols in $\{1, ..., p_2\}$. Let $a_0 \in I$ be a parameter such that there exists an element $\omega(a_0) \in \mathcal{P}_{j+1}(a_0)$ satisfying $\text{symb}_{a_0}(T_{a_0}^i(\omega(a_0))) = s_i$, for all $0 \le i < j+1$ (if there is no such a parameter a_0 for which the element $\omega(a_0)$ exists then there is nothing to show). Let $\tilde{\omega}(a_0) \in \mathcal{P}_j(a_0)$ be the element containing $\omega(a_0)$. By the induction assumption there exists a neighborhood V of a_0 in I such that for all $a \in V$ there is an element $\tilde{\omega}(a) \in \mathcal{P}_j(a)$ having the same combinatorics as $\tilde{\omega}(a_0)$ and the boundary points of $\tilde{\omega}(a)$ and $T_a^j(\tilde{\omega}(a))$ depend continuously on $a \in V$. Note that if $y(a_0)$ is a boundary point of $T_a^j(\omega(a_0))$ then it is equal to a partition point $c_k(a_0)$, $0 \le k \le p_1$, or it is a boundary point of $T_{a_0}^j(\tilde{\omega}(a_0))$. By the continuity of the boundary points of $T_a^j(\omega(a))$ on V and the continuity of $a \mapsto c_k(a)$, we deduce that there exists a neighborhood $U \subset V$

of a_0 in I such that for each $a \in U$ there exists an element $\omega(a) \in \mathcal{P}_{j+1}(a)$ having the same combinatorics as $\omega(a_0)$. Since the boundary points of $T_a^j(\omega(a))$ are continuous on U, we can once more apply property (ii) to deduce that also the boundary points of $T_a^{j+1}(\omega(a))$ are continuous on U. The continuity of the boundary points of $\omega(a)$ follows by a repeated use of property (iii). □

Let $(s_0, ..., s_{j-1})$ be a sequence of symbols $s_i \in \{1, ..., p_2\}$, and for each $a \in I$ let $\omega(a) \in \mathcal{P}_j(a)$ be — if it exists — the to it associated element as in the claim above. Writing $|\omega(a)| = 0$ if such an element does not exists it follows immediately from the claim that the map $a \mapsto |\omega(a)|$ is continuous on I. Furthermore, by condition (IIb), if $|\omega(a_0)| > 0$ for some $a_0 \in I$, then $|\omega(a)| > 0$ for all $a \geq a_0$. This implies that the map $a \mapsto \kappa(a)$ is piecewise continuous on I with only a finite number of discontinuities. Hence, disregarding a finite number of parameter values in I, we can cover I by a countable number of intervals $\tilde{I} \subset I$ such that for each such interval $\tilde{I}$ there is a constant $\kappa_0 = \kappa_0(\tilde{I}) > 0$ such that

$$\kappa(a) \geq \kappa_0, \tag{48}$$

for all $a \in \tilde{I}$. It follows that that there is a constant $C_v = C_v(\tilde{I}) \geq 1$ such that the variation of φ_a is bounded from above by C_v for all $a \in \tilde{I}$. Since $\int_0^1 \varphi_a(x)dx = 1$, this immediately implies that φ_a is bounded from above by $C_v + 1$. To establish a lower bound on a subinterval of $K(a)$, we observe the following.

Claim. *If the variation over $[0,1]$ of a function $\varphi : [0,1] \to \mathbb{R}_+$ is bounded from above by a constant $C_v \geq 1$, and if $\int_0^1 \varphi(x)dx = 1$, then there exists an interval J of length $1/2C_v$ such that $\varphi(x) \geq 1/3C_v$ for all $x \in J$.*

Proof. Let $N = [2C_v]$, divide the unit interval into N disjoint intervals $J_1, ..., J_N$ of length $1/N$, and, for $1 \leq l \leq N$, set $m_l = \inf\{\varphi(x) \; ; \; x \in J_l\}$ and $M_l = \sup\{\varphi(x) \; ; \; x \in J_l\}$. Since $1 = \int_0^1 \varphi(x)dx \leq \sum_{l=1}^N M_l/N$, it follows that $N \leq \sum_{l=1}^N M_l$. If $m_l < 1/3C_v$, for all $1 \leq l \leq N$, it would follow that the variation of φ is strictly greater than $\sum_{l=1}^N (M_l - 1/3C_v) \geq N(1 - 1/3C_v) \geq C_v$, where the last inequality follows since $C_v \geq 1$. Hence, at least for one $1 \leq l \leq N$, $m_l \geq 1/3C_v$. □

Setting $C = 3C_v$ this concludes the proof of Lemma A.1. □

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References

[1] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. of Math. (2) 122 (1985), no.1, 1–25.

[2] M. Björklund and D. Schnellmann, *Almost sure equidistribution in expansive families*, Indag. Math. 20 (2) (2009), 167–177.

[3] K. Brucks and M. Misiurewicz, *The trajectory of the turning point is dense for almost all tent maps*, Ergodic Theory Dynam. Systems 16 (1996), no. 6, 1173–1183.

[4] H. Bruin, *For almost every tent-map, the turning point is typical*, Fund. Math. 155 (1998), no.3, 215–235.

[5] P. Collet and J.-P. Eckmann, "Iterated Maps on the Interval as Dynamical Systems", Birkhäuser, Boston, 1980.

[6] B. Faller and C.-E. Pfister, *A point is normal for almost all maps* $\beta x + \alpha \bmod 1$ *or generalized* β*-maps*, Ergodic Theory Dynam. Systems 29 (2009), 1529–1547.

[7] P. Gora, *Invariant densities for generalized* β*-maps*, Ergodic Theory Dynam. Systems 27 (2007), no. 5, 1583–1598.

[8] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), 481–488.
[9] T.-Y. Li and J.A. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. 235 (1978), 183–192.
[10] P. Mattila, "Geometry of Sets and Measures in Euclidean Spaces (Cambridge studies in advanced mathematics)", Cambridge University Press, Cambridge, 1995.
[11] M. Misiurewicz and E. Visinescu, *Kneading sequences of skew tent maps*, Ann. Inst. H. Poincar Probab. Statist. 27 (1991), no. 1, 125–140.
[12] M. Pollicott, *Normal points for generic hyperbolic maps*, to appear in Fund. Math.
[13] V.A. Rohlin, *Exact endomorphisms of a Lebesgue space*, (Russian) Akad. Nauk SSSR Ser. Mat. 25 (1961), 499–530. (English) Amer. Math. Soc. Transl. Ser. 2, volume 39 (1964), 1–36.
[14] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergodic Theory Dynam. Systems 17 (1997), no. 3, 675-694.
[15] D. Schnellmann, *Non-continuous weakly expanding skew-products of quadratic maps with two positive Lyapunov exponents,* Ergodic Theory Dynam. Systems 28 (2008), no. 1, 245–266.
[16] G. Wagner, *The ergodic behaviour of piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete 46 (1979), no. 3, 317–324.
[17] S. Wong, *Some metric properties of piecewise monotonic mappings of the unit interval*, Trans. Amer. Math. Soc. 246 (1978), 493–500.

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