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Dimension and measure of baker-like skew-products of β-transformations

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Abstract

We consider a generalisation of the baker's transformation, consisting of a skew-product of contractions and a β-transformation. The Hausdorff dimension and Lebesgue measure of the attractor is calculated for a set of parameters with positive measure. The proofs use a new transverality lemma similar to Solomyak's [5]. This transversality, which is applicable to the considered class of maps holds for a larger set of parameters than Solomyak's transversality.

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1 Introduction

In [1], Alexander and Yorke considered fat baker's transformations. These are maps on the square $[0, 1) \times [0, 1)$, defined by

$$(x, y) \mapsto \begin{cases} (\lambda x, 2y) & \text{if } y < 1/2 \\ (\lambda x + 1 - \lambda, 2y - 1) & \text{if } y \geq 1/2 \end{cases},$$

where $\frac{1}{2} < \lambda < 1$ is a parameter. They showed that the SRB-measure of this map is a product of Lebesgue-measure and the distribution of the

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corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm\lambda^k$. Together with Erdös' result [2], this implies that if λ is an inverse of a Pisot-number, then the SRB-measure is singular with respect to the Lebesgue measure on $[0,1) \times [0,1)$.

In [2], Solomyak proved that for almost all $\lambda \in (\frac{1}{2}, 1)$, the distribution of the corresponding Bernoulli convolution $\sum_{k=1}^{\infty} \pm\lambda^k$ is absolutely continuous with respect to Lebesgue measure. Hence this implies that the SRB-measure of the fat baker's transformation is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$. Solomyak's proof used a transversality property of power series of the form $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$, where $a_k \in \{-1, 0, 1\}$. More precisely, Solomyak proved that there exists a $\delta > 0$ such that if $x \in (0, 0.64)$ then

$$|g(x)| < \delta \quad \Longrightarrow \quad g'(x) < -\delta. \tag{1}$$

The constant 0.64 is an approximation of a root to a power series and cannot be improved to something larger than this root.

In this paper we consider maps of the form

$$(x,y) \mapsto \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases},$$

where $0 < \lambda < 1$ and $1 < \beta < 2$. Using the above mentioned transversality of Solomyak one can prove that for almost all $\lambda \in (0, 0.64)$ and $\beta \in (1, 2)$ the SRB-measure is absolutely continuous with respect to Lebesgue measure provided $\lambda\beta > 1$, and the Hausdorff dimension of the SRB-measure is $1 + \frac{\log \beta}{\log 1/\lambda}$ provided $\lambda\beta < 1$.

A problem with this approach is that the condition $\lambda < 0.64$ is very restrictive when β is close to 1. Then the above method yields no λ for which the SRB-measure is absolutely continuous, and it does not give the dimension of the SRB-measure for any $\lambda \in (0.64, 1/\beta)$.

We prove that these results about absolute continuity and dimension of the SRB-measure hold for sets of (β, λ) of positive Lebesgue measure, even when $\lambda > 0.64$. This is done by extending the interval on which the transversality property (1) holds. This can be done in our setting, since in our class of maps, not every sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in \{-1, 0, 1\}$ occurs in the power series $g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$ that we need to consider in the proof. To control which sequences that occur, we will use some results of Brown and Yin [6] and Kwon [7] on natural extensions of β-shifts.

The paper is organised as follows. In Section 2 we recall some facts about β-transformations and β-shifts. We then present the results of Brown and Yin, and Kwon in Sections 3 and 4. In Section 5 we state our results, and give the proofs in Section 7. The transversality property is stated and proved in Section 6.

2 β-shifts

Let $\beta > 1$ and define $f_\beta\colon [0,1] \to [0,1)$ by $f_\beta(x) = \beta x \mod 1$. For $x \in [0,1]$ we associate a sequence $d(x,\beta) = (d_k(x,\beta))_{k=1}^\infty$ defined by $d_k(x,\beta) = [\beta f_\beta^{k-1}(x)]$ where $[x]$ denotes the integer part of x. If $x \in [0,1]$, then $x = \phi_\beta(d(x,\beta))$, where

$$\phi_\beta(i_1, i_2, \ldots) = \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}$$

This representation, among others, of real numbers was studied by Rényi [4]. He proved that there is a unique probability measure μ_β on $[0,1]$ invariant under f_β and equivalent to Lebesgue measure. We will use this measure in Section 7.

We let S_β^+ denote the closure in the product topology of the set $\{\, d(x,\beta) : x \in [0,1) \,\}$. The compact symbolic space S_β^+ together with the left shift σ is called a β-shift.

If we define $d_-(1,\beta)$ to be the limit in the product topology of $d(x,\beta)$ as x approaches 1, we have the equality

$$S_\beta^+ = \{\, (a_1, a_2, \ldots) \in \{0, 1, \ldots, [\beta]\}^{\mathbb{N}} : \sigma^k(a_1, a_2, \ldots) \leq d_-(1,\beta) \ \forall k \geq 0 \,\},$$

where σ is the left-shift. This was proved by Parry in [3], where he studied the β-shifts and their invariant measures. Note that $d_-(1,\beta) = d(1,\beta)$ if and only if $d(1,\beta)$ contains infinitely many non-zero digits.

The map $\phi_\beta\colon S_\beta^+ \to [0,1]$ is not necessarily injective, but we have $d(\cdot,\beta) \circ f_\beta = \sigma \circ d(\cdot,\beta)$.

3 Symmetric β-shifts

Let $\beta > 1$ and consider S_β^+. The natural extension of (S_β^+, σ) can be realised as (S_β, σ), with

$$S_\beta = \{\, (\ldots, a_{-1}, a_0, a_1, \ldots) : (a_n, a_{n+1}, \ldots) \in S_\beta^+ \ \forall n \in \mathbb{Z} \,\},$$

where σ is the left shift on bi-infinite sequences.

We will use the concept of cylinder sets only in S_β. A cylinder set is a subset of S_β of the form

$$[a_{-n}, a_{-n+1}, \ldots, a_0] = \{\, (\ldots, b_{-1}, b_0, b_1, \ldots) \in S_\beta : a_k = b_k \ \forall k = -n, \ldots, 0 \,\}.$$

We define S_β^- to be the set

$$\begin{aligned} S_\beta^- &= \{\, (b_1, b_2, \ldots) : \exists (a_1, a_2, \ldots) \in S_\beta^+ \text{ s.t. } (\ldots, b_2, b_1, a_1, a_2, \ldots) \in S_\beta \,\} \\ &= \{\, (b_1, b_2, \ldots) : (\ldots, b_2, b_1, 0, 0, \ldots) \in S_\beta \,\}. \end{aligned}$$

We will be interested in the set S of β for which $S_\beta^+ = S_\beta^-$. This set was considered by Brown and Yin in [6]. We now describe the properties of S that we will use later on.

Consider a sequence of the digits a and b. Any such sequence can be written in the form

$$(a^{n_1}, b, a^{n_2}, b, \ldots),$$

where each n_k is a non-negative integer or ∞. We say that such a sequence is allowable if $a \in \mathbb{N}$, $b = a - 1$, and $n_1 \geq 1$. If the sequence $(n_1, n_2, \ldots)$ is also allowable, we say that $(a^{n_1}, b, a^{n_2}, b, \ldots)$ is derivable, and we call $(n_1, n_2, \ldots)$ the derived sequence of $(a^{n_1}, b, a^{n_2}, b, \ldots)$. For some sequences, this operation can be carried out over and over again, generating derived sequences out of derived sequences. We have the following theorem.

Theorem 1 (Brown–Yin [6], Kwon [7])**.** *$\beta \in S$ if and only if $d(1, \beta)$ is derivable infinitely many times.*

The "only if"-part was proved by Brown and Yin in [6] and the "if"-part was proved by Kwon in [7]. Using this characterisation of S, Brown and Yin proved that S has the cardinality of the continuum, but its Hausdorff dimension is zero.

4 Examples

Let us find some numbers in S to see better what the theorems in Section 5 say. We do this by first finding some sequences that are infinitely derivable, and then we find the corresponding β by solving the equation $1 = \phi_\beta(d(1, \beta))$. Let us first remark that the sequence $(1, 0, 0, \ldots)$ is its own derived sequence.

The sequence $d(1, \beta) = (1, 1, 0, (1, 0)^\infty)$ is clearly derivable infinitely many times. It's derived sequence is $(2, 1, 1, \ldots)$, and the derived sequence of this sequence is $(1, 0, 0, \ldots)$. One finds numerically that the corresponding β is given by $\beta = 1.801938\ldots$ and that $1/\beta = 0.554958\ldots$

There are however smaller numbers in the set S. Consider the sequence $d(1, \beta) = (1, 0, (1, 0, 0)^\infty)$. It's derived sequence is $(1, 1, 0, (1, 0)^\infty))$, which derives to $(2, 1, 1, \ldots)$, and so on. Solving for β we find that $\beta = 1.558980\ldots$ and $1/\beta = 0.641445\ldots$

Let us now show that $\inf S = 1$. For all natural n, let β_n be such that

$$d(1, \beta_n) = (1, 0^n, (1, 0^{n+1})^\infty).$$

Then, for $n \geq 2$, the derived sequence of $d(1, \beta_n)$ is the sequence $d(1, \beta_{n-1})$. Hence all sequences $d(1, \beta_n)$ are infinitely derivable, and so $\beta_n \in S$. Moreover it is clear that $\beta_n \to 1$ as $n \to \infty$. See Table 1.

n	β_n	$1/\beta_n$
1	1.558980...	0.641445...
2	1.438417...	0.695209...
3	1.365039...	0.732580...
4	1.315114...	0.760390...
5	1.278665...	0.782066...

Table 1: Some numerical values of β_n.

5 Results

Let $0 < \lambda < 1$ and $1 < \beta < 2$. Put $Q = [0, 1) \times [0, 1)$ and define $T \colon Q \to Q$ by

$$T_{\beta,\lambda}(x, y) = \begin{cases} (\lambda x, \beta y) & \text{if } y < 1/\beta \\ (\lambda x + 1 - \lambda, \beta y - 1) & \text{if } y \geq 1/\beta \end{cases}.$$

Denote by ν the 2-dimensional Lebesgue measure on Q. For any $n \in \mathbb{N}$ we define the measure

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu \circ T_{\beta,\lambda}^{-n}.$$

The unique SRB-measure of $T_{\beta,\lambda}$ is the weak limit of ν_n as $n \to \infty$.

The SRB-measures are characterised by the property that their conditional measures along unstable manifolds are equivalent to Lebesgue measure. The existence of such measures was established for invertible maps by Pesin [8] and extended to non-invertible maps by Schmeling and Troubetzkoy [9]. We denote the SRB-measure of $T_{\beta,\lambda}$ by μ_{SRB}. It is clearly unique by a simple Hopf-argument. The support of μ_{SRB} is the set

$$\Lambda = \text{closure} \bigcap_{n=0}^{\infty} T_{\beta,\lambda}^n(Q).$$

If $\lambda\beta < 1$, it is easy to see that the Hausdorff dimension of Λ is at most $1 + \frac{\log \beta}{\log 1/\lambda}$. The following theorem states that for a set of parameters of positive Lebesgue measure, this bound is optimal.

Theorem 2. *Let $1 < \beta < 2$ and $\gamma = \inf\{\, \beta' \in S : \beta' \geq \beta \,\}$. Then for Lebesgue almost every $\lambda \in (0, 1/\gamma)$ the Hausdorff dimension of the SRB-measure of $T_{\beta,\lambda}$ is $1 + \frac{\log \beta}{\log 1/\lambda}$.*

Recall from Section 4 that $\inf S = 1$. This implies that when β gets close to 1, Theorem 2 gives the dimension of the SRB-measure for a large set of $\lambda > 0.64$, which is not obtainable using Solomyaks transversality from [5], described in the introduction.

In the area-expanding case, when $\lambda\beta > 1$, we have the following theorem.

Theorem 3. *For any $\gamma \in S$, there is an $\varepsilon > 0$ such that for all β with $1/\beta \in (1/\gamma, 1/\gamma+\varepsilon)$, and Lebesgue almost every $\lambda \in (1/\beta, 1/\gamma+\varepsilon)$ the* SRB*-measure of $T_{\beta,\lambda}$ is absolutely continuous with respect to Lebesgue measure.*

Since $\inf S = 1$ there are β arbitrarily close to 1 for which we have a set of λ of positive Lebesgue measure, where the SRB-measure is absolutely continuous. In particular, this means that for these parameters, the set Λ has positive 2-dimensional Lebesgue measure.

6 Transversality

Fix $\beta > 1$. Consider the set of power series of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k, \tag{2}$$

where $\boldsymbol{a} = (a_1, a_2, \dots)$ and $\boldsymbol{b} = (b_1, b_2, \dots)$ are sequences in S_β^+.

Lemma 1. *There exist $\varepsilon > 0$ and $\delta > 0$ such that for any g of the form (2), $x \in [0, 1/\beta + \varepsilon]$ and $|g(x)| < \delta$ implies that $g'(x) < -\delta$.*

Proof. Let us first prove the transversality property for $x \in [0, 1/\beta]$. Assume that no such δ exists. Then there is a sequence g_n of power series and a sequence of numbers $x_n \in [0, 1/\beta]$, such that $\lim_{n\to\infty} g_n(x_n) = 0$ and $\liminf_{n\to\infty} g_n'(x_n) \geq 0$.

We can take a subsequence such that g_n converges term-wise to a series $g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k$, with $\boldsymbol{a}, \boldsymbol{b} \in S_\beta^+$, and x_n converges to some number x_0. Clearly, $g(x_0) = 0$ and $g'(x_0) \geq 0$, so $x_0 \neq 0$.

Let $\beta_0 = 1/x_0 \geq \beta$. Then $\boldsymbol{a}, \boldsymbol{b} \in S_{\beta_0}^+$ and $g(x_0) = 0$ implies that

$$\phi_{\beta_0}(a_1, a_2, \dots) - \phi_{\beta_0}(b_1, b_2, \dots) = \sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} - \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = -1.$$

Since both sums are in $[0, 1]$, we conclude that

$$\sum_{k=1}^{\infty} \frac{a_k}{\beta_0^k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{b_k}{\beta_0^k} = 1.$$

We must therefore have $(a_1, a_2, \ldots) = (0, 0, \ldots)$. This implies that $g'(x) < 0$ for all $x \in (0, 1/\beta]$, contradicting the fact that $g'(x_0) \geq 0$.

We have now proved the transversality for $x \in [0, 1/\beta]$. Let us now prove transversality for $x \in [0, 1/\beta + \varepsilon]$, where ε is to be determined. Arguing exactly as above, we get a series $g(x)$ and an x_0 such that $g(x_0) = 0$ and $g'(x_0) \geq 0$. However, this time we have $x_0 \in (0, 1/\beta + \varepsilon]$.

Let

$$h_1(x) = \sum_{k=1}^{\infty} a_k x^k \quad \text{and} \quad h_2(x) = \sum_{k=1}^{\infty} b_k x^k.$$

Then $h_2(1/\beta) \le 1$. Moreover we have $0 \le h_2'(x) \le \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$. Therefore we have

$$h_2(x_0) \le 1 + \int_{1/\beta}^{1/\beta+\varepsilon} \frac{\mathrm{d}x}{(1-x)^2} = 1 + \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}.$$

Since $h_1(x_0) - h_2(x_0) = -1$, we must have $0 \le h_1(x_0) \le \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$.

Since all terms in $h_1(x)$ are non-negative we must have $a_i = 0$ for $i \le k$, where k is the largest number such that $(1/\beta+\varepsilon)^k \ge \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$. This implies that $h_1'(x) \le \sum_{i=k+1}^{\infty} ix^{i-1} = \frac{(k+1)x^k + kx^{k+1}}{(1-x)^2}$. Moreover we have $x_0^{k+1} \le (1/\beta+\varepsilon)^{k+1} < \frac{\varepsilon}{(1-1/\beta-\varepsilon)(1-1/\beta)}$, so that

$$h_1'(x_0) \le \frac{\varepsilon}{(1-1/\beta+\varepsilon)(1-1/\beta)} \frac{k+1+kx_0}{x_0(1-x_0)^2} \le \frac{\varepsilon(2k+1)}{(1-1/\beta+\varepsilon)^4 x_0}.$$

To estimate $h_2'(x_0)$ from below we argue as follows. It is clear that h_2 is strictly convex and $h_2(0) = 0$. This implies that $h_2'(x_0) > \frac{h_2(x_0)}{x_0} \ge \frac{1}{x_0}$.

Now we want to choose ε so that $h_1'(x_0) - h_2'(x_0) < 0$, to get a contradiction. Using the estimates from above, we see that it is enough to choose ε so small that

$$\frac{\varepsilon(2k+1)}{(1-1/\beta+\varepsilon)^4 x_0} - \frac{1}{x_0} < 0 \quad \Longleftrightarrow \quad \varepsilon < \frac{(1-1/\beta+\varepsilon)^4}{2k+1}.$$

Hence, it is sufficient to choose $\varepsilon < \frac{(1-1/\beta)^4}{2k+1}$. Assume that $\varepsilon \le \frac{1-1/\beta}{2}$. For k we have the estimate $k \le \frac{\log \varepsilon}{\log \frac{1+1/\beta}{2}}$, so we get the sufficient condition

$$\varepsilon < \frac{\left(1-1/\beta\right)^4}{\frac{2\log\varepsilon}{\log\frac{1+1/\beta}{2}} + 1},$$

or equivalently,

$$\frac{2}{\log \frac{1+1/\beta}{2}} \varepsilon \log \varepsilon + \varepsilon < \left(1-1/\beta\right)^4.$$

Using that $\varepsilon \le \frac{-\varepsilon \log \varepsilon}{\log \frac{2}{1-1/\beta}}$, we see that this inequality is satisfied if

$$-\varepsilon \log \varepsilon \Big(\frac{2}{\log \frac{2}{1+1/\beta}} + \frac{1}{\log \frac{2}{1-1/\beta}} \Big) < \left(1-1/\beta\right)^4.$$

Finally we use that $-\varepsilon \log \varepsilon < \frac{3}{4}\sqrt{\varepsilon}$, and conclude that any

$$\varepsilon \leq \frac{16}{9} \frac{\left(1 - 1/\beta\right)^8}{\left(\frac{2}{\log \frac{2}{1+1/\beta}} + \frac{1}{\log \frac{2}{1-1/\beta}}\right)^2}$$

works. ☐

7 Proofs

It is not hard to see that Λ satisfies

$$\Lambda = \{ (x, y) : \exists \boldsymbol{a} \in S_\beta \text{ such that } x = \pi_1(\boldsymbol{a}, \lambda),\ y = \pi_2(\boldsymbol{a}, \beta) \},$$

where

$$\pi_1(\boldsymbol{a}, \lambda) = (1 - \lambda) \sum_{k=0}^{\infty} a_{-k} \lambda^k,$$
$$\pi_2(\boldsymbol{a}, \beta) = \sum_{k=1}^{\infty} a_k \beta^{-k}.$$

Let $\pi(\boldsymbol{a}, \beta, \lambda) = (\pi_1(\boldsymbol{a}, \lambda), \pi_2(\boldsymbol{a}, \beta))$ and transfer the measure μ_{SRB} to a measure η on S_β by $\eta = \mu_{\text{SRB}} \circ \pi(\cdot, \beta, \lambda)$. We take a closer look at this measure before we start the proofs.

Recall, from Section 2, the probability measure μ_β on $[0, 1]$ that is invariant under f_β and equivalent to Lebesgue measure. We get a shift-invariant measure on S_β^+ by taking $\mu_\beta \circ \phi_\beta$ and it can be extended in the natural way to a shift-invariant measure η_β on S_β.

It is easy to see that the conditional measures of $\eta_\beta \circ \pi(\cdot, \beta, \lambda)^{-1}$ along the unstable manifolds are equivalent to Lebesgue measure, so by the uniqueness of μ_{SRB}, η_β must coincide with η. Thus, η is independent of λ and satisfies

$$\begin{aligned} \eta([a_{-n} \ldots a_0]) &= \mu_\beta\Big(\phi_\beta\big(\{ (x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0 \}\big)\Big) \\ &\leq K \text{ diamater } \Big(\phi_\beta\big(\{ (x_i)_{i=1}^\infty \in S_\beta^+ : x_1 \ldots x_{n+1} = a_{-n} \ldots a_0 \}\big)\Big) \\ &\leq K\beta^{-n+1}, \end{aligned} \tag{3}$$

where $K < \infty$ is a constant.

Proof of Theorem 2. Let $\beta > 1$ and pick any $\beta' > \beta$ such that $\beta' \in S$.

Consider a sequence $\boldsymbol{a} \in S_\beta$ and the corresponding point $p = \pi(\boldsymbol{a}, \beta, \lambda)$. The local unstable manifold of $\boldsymbol{a}$ is the set of sequences $\boldsymbol{b}$ such that $a_k = b_k$ for $k \leq 0$. For $\lambda < 1/\beta$, π is injective on S_β so the local unstable manifold of p is unique. If $\lambda \geq 1/\beta$, then π is not injective on S_β, so the local unstable manifold of p is not unique. In both cases the local unstable manifold of p corresponding to $\boldsymbol{a}$ is the set of points $q = \pi(\boldsymbol{b}, \beta, \lambda)$ such that $\boldsymbol{b}$ is in the unstable manifold of $\boldsymbol{a}$. It consists of vertical line segments and in some cases single points.

For η-almost every sequence $\boldsymbol{a}$ the local unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$ corresponding to $\boldsymbol{a}$, contains a line segment of positive length. Note that the length does not depend on λ.

Let ω_δ be the set of sequences $\boldsymbol{a}$, such that the corresponding local unstable manifold of $\pi(\boldsymbol{a}, \beta, \lambda)$ has a length of at least $\delta > 0$. Take $\delta > 0$ so that ω_δ has positive η-measure. Then the set $\Omega_\delta = \pi(\omega_\delta, \beta, \lambda)$ has the same positive μ_{SRB}-measure.

Consider the restriction of μ_{SRB} to Ω_δ and project this measure to $[0,1) \times \{0\}$. Let $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ denote this projection.

Take an interval $I = (c, d)$ with $0 < c < d < 1/\beta'$. Let t be a number in $(0, 1)$. We estimate the quantity

$$J(t) = \int_I \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{1}{|x_1 - x_2|^t} \, \mathrm{d}\mu_{\mathrm{SRB}}^{\mathrm{s}}(x_1) \mathrm{d}\mu_{\mathrm{SRB}}^{\mathrm{s}}(x_2) \mathrm{d}\lambda,$$

If this integral converges, then for Lebesgue almost every $\lambda \in I$, the dimension of $\mu_{\mathrm{SRB}}^{\mathrm{s}}$ is at least t, and so the dimension of μ_{SRB} is at least $1 + t$. Writing $J(t)$ as an integral over the symbolic space we have that

$$J(t) = \int_I \int_{\omega_\delta} \int_{\omega_\delta} \frac{1}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\eta(\boldsymbol{a}) \mathrm{d}\eta(\boldsymbol{b}) \mathrm{d}\lambda.$$

Since η does not depend on λ we can change order of integration and write

$$J(t) = \int_{\omega_\delta} \int_{\omega_\delta} \int_I \frac{1}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\lambda \mathrm{d}\eta(\boldsymbol{a}) \mathrm{d}\eta(\boldsymbol{b}).$$

By making use of the transversality from Lemma 1 it is apparent that for $\boldsymbol{a}$ and $\boldsymbol{b}$ with $a_j = b_j$ for $j = -k+1, \ldots, 0$ and $a_{-k} \neq b_{-k}$, we have

$$\begin{aligned} &\int_I \frac{1}{|\pi_1(\boldsymbol{a}, \lambda) - \pi_1(\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\lambda \\ &\qquad = \int_I \lambda^{-kt} \frac{1}{|\pi_1(\sigma^{-k}\boldsymbol{a}, \lambda) - \pi_1(\sigma^{-k}\boldsymbol{b}, \lambda)|^t} \, \mathrm{d}\lambda \leq C c^{-kt}, \end{aligned}$$

where C only depends on t and β. We can write $S_\beta \times S_\beta = A \cup B$, where

$$\begin{aligned} A = &\bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \ldots, a_0]} [0, a_{-k+1}, \ldots, a_0] \times [1, a_{-k+1}, \ldots, a_0] \\ &\cup \bigcup_{k=1}^{\infty} \bigcup_{[a_{-k+1}, \ldots, a_0]} [1, a_{-k+1}, \ldots, a_0] \times [0, a_{-k+1}, \ldots, a_0], \end{aligned}$$

and

$$B = \bigcup_{\boldsymbol{a} \in S_\beta} \{\boldsymbol{a}\} \times \{\boldsymbol{a}\}.$$

Since $\eta(\boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in S_\beta$, we can replace $\omega_\delta \times \omega_\delta$ by A in the estimates, and we get

$$\begin{aligned} J(t) &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1},\ldots,a_0]} 2Cc^{-kt} \int_{[0,a_{-k+1},\ldots,a_0]} \int_{[1,a_{-k+1},\ldots,a_0]} \mathrm{d}\eta \mathrm{d}\eta \\ &\leq \sum_{k=1}^{\infty} \sum_{[a_{-k+1},\ldots,a_0]} 2CKc^{-kt}\beta^{-k} \int_{[1,a_{-k+1},\ldots,a_0]} \mathrm{d}\eta \\ &\leq 2CK \sum_{k=0}^{\infty} c^{-kt}\beta^{-k}, \end{aligned}$$

by (3) and the fact that η is a probability measure. This series converges provided that $t < \frac{\log \beta}{\log 1/c}$.

We have now proved that for a.e. λ in $I = (c, d)$, the dimension of the SRB-measure is at least $1 + \frac{\log \beta}{\log 1/c}$. To get the result of the theorem, we let $\varepsilon > 0$ and write $I = (0, 1/\beta')$ as a union of intervals $I_n = (c_n, d_n)$ such that $\frac{\log \beta}{\log 1/c_n} > \frac{\log \beta}{\log 1/d_n} - \varepsilon$. Then the dimension is at least $1 + \frac{\log \beta}{\log c_n} \geq 1 + \frac{\log \beta}{\log 1/\lambda} - \varepsilon$ for a.e. $\lambda \in I$. Since ε and β' was arbitrary this proves the theorem. □

Sketch of the proof of Theorem 3. The proof is analogous to the proof of Theorem 2, using the technique of Peres and Solomyak [10]. The only difference from that paper is that we have β-shifts instead of full shifts and we handle this as in the proof of Theorem 2. In particular Lemma 1 gives an interval $I = (1/\gamma, 1/\gamma + \varepsilon)$ for which the set

$$\{\lambda \in I : |g(\lambda)| \leq \rho\}$$

has Lebesgue measure at most $2\delta^{-1}\rho$, for any $\rho > 0$ and any g of the form

$$g(x) = 1 + \sum_{k=1}^{\infty} (a_k - b_k) x^k,$$

where $\boldsymbol{a} = (a_1, a_2, \ldots)$ and $\boldsymbol{b} = (b_1, b_2, \ldots)$ are sequences in $S_\gamma^+ = S_\gamma^-$. □

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