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The Sharkovsky Theorem: A natural direct proof

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1. INTRODUCTION

In this note f is a continuous function from an interval into itself. The interval need not be closed or bounded, although this is usually assumed in the literature. The point of view of dynamical systems is to study iterations of f: for a given point x_0 one investigates the sequence defined by setting $x_1 := f(x_0)$, $x_2 := f(x_1)$, $x_3 := f(x_2)$, and so on. This sequence is called the f-*orbit* of x_0, or just the *orbit* of x_0 for short.

It is particularly interesting when this sequence repeats. In this case we say that x_0 is a *periodic point*, and we refer to the number of distinct points in the orbit or *cycle* $\mathcal{O} := \{x_0, x_1, x_2, \dots\}$ as the *period* of x_0.[1] Equivalently, the period of x_0 is the smallest positive integer m such that $x_m = x_0$. A *fixed point* is a periodic point of period 1, that is, a point x_0 such that $f(x_0) = x_0$. If f has a periodic point of period m, then m is called a *period for* (or *of*) f.

Given a continuous map of an interval one may ask what periods it can have. The genius of Alexander Sharkovsky lay in realizing that there is a structure to the set of periods.

1.1. The Sharkovsky Theorem. The Sharkovsky Theorem involves the following ordering of the set $\mathbb{N}$ of positive integers, which is now known as the Sharkovsky ordering:

$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2{\cdot}3 \rhd 2{\cdot}5 \rhd 2{\cdot}7 \rhd \cdots \rhd 2^2{\cdot}3 \rhd 2^2{\cdot}5 \rhd 2^2{\cdot}7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

This is a total ordering; we write $m \rhd l$ or $l \lhd m$ whenever m is to the left of l.

It is crucial that the Sharkovsky ordering has the following *doubling property*:

$$m \rhd l \text{ if and only if } 2m \rhd 2l. \tag{1}$$

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[1] Dynamicists usually refer to m as the *least* period.

This is because the odd numbers greater than 1 appear at the left end of the list, the number 1 appears at the right end, and the rest of $\mathbb{N}$ is included by successively doubling these end pieces, and inserting these doubled strings inward:

$$\text{odds}, 2\cdot\text{odds}, 2^2\cdot\text{odds}, 2^3\cdot\text{odds}, \dots, 2^3\cdot 1, 2^2\cdot 1, 2\cdot 1, 1.$$

Sharkovsky showed that this ordering describes which numbers can be periods for a continuous map of an interval.

Theorem 1.1 (Sharkovsky Forcing Theorem [13, 14])**.** *If m is a period for f and $m \rhd l$, then l is also a period for f.*

This shows that the set of periods of a continuous interval map is a *tail* of the Sharkovsky order. A tail is a set $\mathscr{T} \subset \mathbb{N}$ such that $s \rhd t$ for all $s \notin \mathscr{T}$ and all $t \in \mathscr{T}$. There are three types of tails: $\{m\} \cup \{l \in \mathbb{N} \mid l \lhd m\}$ for some $m \in \mathbb{N}$, the set $\{\dots, 16, 8, 4, 2, 1\}$ of all powers of 2, and $\varnothing$.

The following complementary result is sometimes called the converse to the Sharkovsky Theorem, but is proved in Sharkovsky's original papers.

Theorem 1.2 (Sharkovsky Realization Theorem [13, 14])**.** *Every tail of the Sharkovsky order is the set of periods for some continuous map of an interval into itself.*

The Sharkovsky Theorem is the union of Theorem 1.1 and Theorem 1.2: A subset of $\mathbb{N}$ is the set of periods for a continuous map of an interval to itself if and only if the set is a tail of the Sharkovsky order. We reproduce a proof of the Realization Theorem in Section 7 at the end of this note.

All proofs of the Sharkovsky Theorem that we know are elementary, no matter how ingenious; the Intermediate-Value Theorem is the deepest ingredient. There is variation in the clarity of the proof strategy and its implementation. Our aim is to present, with all details, a direct proof of the Forcing Theorem that is conceptually simple and involves no artificial case distinctions. Indeed, its directness provides additional information (Section 8).

The standard proof of the Sharkovsky Forcing Theorem discards the m-cycle in the hypothesis and instead studies orbits of odd period with the property that their period comes earlier in the Sharkovsky sequence than any other period for that map. It shows that such an orbit is of a special type, known as a Štefan cycle, and then that such a cycle forces the presence of periodic orbits with Sharkovsky-lesser periods. The second stage of the proof considers various cases in which the period that comes earliest in the Sharkovsky order is even. Finally, this approach requires special treatment of the case in which the set of periods consists of all powers of 2.

We extract the essence of the first stage of the standard proof to produce an argument that does not need Štefan cycles, and we replace the second stage of the standard proof by a simple and natural induction. Our main idea is to select a salient sequence of orbit points and to prove that this sequence "spirals out" in essentially the same way as the Štefan cycles considered in the standard proof.

1.2. **History.** A capsule history of the Sharkovsky Theorem is in [11], and [1] provides much context. The first result in this direction was obtained by Coppel [5] in the 1950s: every point converges to a fixed point under iteration of a continuous map of a closed interval if the map has no periodic points of period 2; it is an easy corollary that a continuous map must have 2 as a period if it has any periodic points that are not fixed. This amounts to 2 being the penultimate number in the Sharkovsky ordering.

Sharkovsky obtained the results described above and reproved Coppel's theorem in a series of papers published in the 1960s [13, 14]. He also worked on other aspects of one-dimensional dynamics (see, for instance, [15, 16, 17]). Sharkovsky appears to have been unaware of Coppel's paper. His work did not become known outside eastern Europe until the second half of the 1970s. In 1975 this Monthly published a famous paper, "Period three implies chaos" [10] by Li and Yorke, which included the result that the presence of a periodic point of period 3 implies the presence of periodic points of all other periods.[2] This amounts to 3 being the initial number in the Sharkovsky order. Some time later Yorke attended a conference in East Berlin, and during a river cruise a Ukrainian participant approached him. Although they had no language in common, Sharkovsky (for it was he) managed to convey, with translation by Lasota and Mira, that unbeknownst to Li and Yorke (and perhaps all of western mathematics) he had proved his results about periodic points of interval mappings well before [10], even though he did not at the time say what that result was.

Besides introducing the idea of chaos to a wide audience, Li and Yorke's paper was to lead to global recognition of Sharkovsky's work. Within a few years of [10] new proofs of the Sharkovsky Forcing Theorem appeared, one due to Štefan [18], and a later one, which is now viewed as the "standard" proof, due to Block, Guckenheimer, Misiurewicz, and

[2] It should not be forgotten that Li and Yorke's work contains more than a special case of Sharkovsky's: "chaos" is not just "periods of all orders".

Young [3], Burkart [4], Ho and Morris [9], and Straffin [19]. Nitecki's paper [12] provides a lovely survey from that time. Alsedà, Llibre, and Misiurewicz improved this standard proof [1] and also gave a beautiful proof of the realization theorem, which we reproduce in Section 7.

The result has also been popular with contributors to the Monthly. We mention here a short proof of one step in the standard proof [2] and several papers by Du [6, 8, 7]. Reading the papers by Du inspired the work that resulted in this article.

1.3. **Related work.** There is a wealth of literature related to periodic points for one-dimensional dynamical systems. [1] is a good source of pertinent information. There is a characterization of the exact structure of a periodic orbit whose period comes earliest in the Sharkovsky order for a specific map. There is also work on generalizations to other permutation patterns (how particular types of periodic points force the presence of others, and how intertwined periodic orbits do so), to different one-dimensional spaces (that look like the letter "Y", the letter "X," or a star "∗"), and to multivalued maps.

2. Intervals, covering relations, and cycles

In the remainder of this note, f^n will denote the n-fold composition of f with itself. We can then write the orbit of a point x as $\{f^n(x) \mid n = 0,\dots\}$, and a periodic point with period m becomes a fixed point of f^m (and of f^{2m}, $f^{3m}\dots$). Thus, if $f^n(x) = x$, then the period of x is a factor of n.

Definition 2.1. We say that an interval I *covers* an interval J and write $I \xrightarrow{f} J$ if $J \subset f(I)$. We usually omit f and simply write $I \to J$ instead.

The Intermediate-Value Theorem allows us to translate knowledge of how intervals are moved around into information about the presence of periodic points. This is the content of the next three lemmas.

Lemma 2.2. *If $[a_1,a_2] \xrightarrow{f} [a_1,a_2]$, then f has a fixed point in $[a_1,a_2]$.*

Proof. If $b_1, b_2 \in [a_1,a_2]$ with $f(b_i) = a_i$, then $f(b_1) - b_1 \le 0 \le f(b_2) - b_2$. By the Intermediate-Value Theorem, $f(x) - x = 0$ for some x between b_1 and b_2. □

Lemma 2.3 (Itinerary Lemma)**.** *If $J_0 \dots, J_{n-1}$ are closed bounded intervals and $J_0 \to \cdots \to J_{n-1} \to J_0$ (this is called a* loop *or n*-loop of intervals*) then there is a point x that* follows the loop*, that is, $f^i(x) \in J_i$ for $0 \le i < n$ and $f^n(x) = x$.*

Proof. We write $I \rightarrowtail J$ if $f(I) = J$. If $I \to J$, there is an interval $K \subset I$ such that $K \rightarrowtail J$ because the intersection of the graph of f with the rectangle $I \times J$ contains a minimal arc that joins the top and bottom sides of the rectangle. We can choose K to be the projection to I of such an arc.

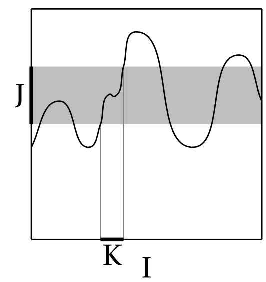


FIGURE 1. Finding $K \rightarrowtail J$

Thus there is a closed bounded interval $K_{n-1} \subset J_{n-1}$ such that $K_{n-1} \rightarrowtail J_0$. Then $J_{n-2} \to K_{n-1}$, and so there is $K_{n-2} \subset J_{n-2}$ such that $K_{n-2} \rightarrowtail K_{n-1}$. Inductively, there are closed bounded intervals $K_i \subset J_i$, $0 \le i < n$, such that

$$K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_{n-1} \rightarrowtail J_0.$$

Any $x \in K_0$ satisfies $f^i(x) \in K_i \subset J_i$ for $0 \le i < n$ and $f^n(x) \in J_0$. Since $K_0 \subset J_0 = f^n(K_0)$, Lemma 2.2 implies that f^n has a fixed point in K_0. □

We wish to ensure that the period of the point x found in Lemma 2.3 is n and not a proper divisor of n, such as for the 2-loop $[-1,0] \rightleftarrows [0,1]$ of $f(x) = -2x$, which is followed only by the fixed point 0.

Definition 2.4. We say that a loop $J_0 \to \cdots \to J_{n-1} \to J_0$ of intervals is *elementary* if every point that follows it has period n.[3]

With this notion, the conclusion of Lemma 2.3 gives us:

Proposition 2.5. *For an elementary loop $J_0 \to \cdots \to J_{n-1} \to J_0$ there is a periodic point with period n that follows the loop.*

This makes it interesting to give convenient criteria for being elementary. The simplest is that any loop of length 1 is elementary (since the period of a point that follows such a loop must be a factor of 1). A criterion with wider utility is:

Lemma 2.6. *A loop $J_0 \to \cdots \to J_{n-1} \to J_0$ of intervals is elementary if it is not followed by either endpoint of J_0 and the interior* $\mathrm{Int}(J_0)$ *of J_0 is disjoint from each of $J_1, \ldots, J_{n-1}$, i.e.,* $\mathrm{Int}(J_0) \cap \bigcup_{i=1}^{n-1} J_i = \varnothing$.

[3]This is a different use of the word “elementary” from the one in [1].

Proof. If x follows the loop, then $x \in \operatorname{Int}(J_0)$ because $x \in J_0$ and it is not an endpoint. If $0 < i < n$ then $f^i(x) \notin \operatorname{Int}(J_0)$ because it is in J_i, and so $x \neq f^i(x)$. Thus x has period n. □

2.1. **$\mathcal{O}$-intervals and $\mathcal{O}$-forced covering relations.** A closed bounded interval whose endpoints belong to a cycle $\mathcal{O}$ of f is called an $\mathcal{O}$*-interval.*

In the rest of the paper the above ideas will be applied to $\mathcal{O}$-intervals. We will use only information that can be obtained from the action of f on $\mathcal{O}$ and therefore applies to all continuous maps f for which $\mathcal{O}$ is a cycle.

In particular, all of the covering relations $I \to J$ of $\mathcal{O}$-intervals considered in the rest of the paper are $\mathcal{O}$*-forced.* By this we mean that J lies in the $\mathcal{O}$-interval whose endpoints are the leftmost and rightmost points of $f(I \cap \mathcal{O})$. By our standing assumption that f is continuous and the Intermediate-Value Theorem, this implies $I \to J$. We say that a loop of $\mathcal{O}$-intervals is $\mathcal{O}$*-forced* if every arrow in it arises from an $\mathcal{O}$-forced covering relation.

Because in the remainder of the paper these are the only covering relations we will use, the symbol "$\to$" will henceforth denote $\mathcal{O}$-forced covering relations.

3. Examples

The first example is the most celebrated special case of the Sharkovsky Theorem: that period 3 implies all periods. The second and third examples apply the same method to longer cycles and illustrate how our choice of $\mathcal{O}$-intervals differs from that made in the standard proof. The last example illustrates our induction argument, which is built on the doubling structure of the Sharkovsky order.

3.1. **Period 3 implies all periods.** A 3-cycle comes in two versions that are mirror-images of one another. In Figure 2, the dashed arrows indi-

FIGURE 2. 3-cycles

cate that $x_1 = f(x_0)$, $x_2 = f(x_1)$ and $x_0 = f(x_2)$. In both pictures, I_1 is the $\mathcal{O}$-interval with endpoints x_0 and x_1, and I_0 is the $\mathcal{O}$-interval with endpoints x_0 and x_2. The endpoints of I_1 are mapped to the very left and right points of the cycle, so we have the $\mathcal{O}$-forced covering relations $I_1 \to I_1$ and $I_1 \to I_0$. The endpoints of I_0 are mapped to those of I_1, and so $I_0 \to I_1$ is $\mathcal{O}$-forced. We summarize these covering relations by writing $\circlearrowleft I_1 \rightleftarrows I_0$.

Since $I_1 \to I_1$, Lemma 2.2 implies that I_1 contains a fixed point of f.

The endpoints of I_1 cannot follow the cycle $I_1 \to I_0 \to I_1$ because they are periodic points with period 3, whereas a point that follows this cycle must have period 1 or 2. By Lemma 2.6, f has a point with period 2.

No point of $\mathcal{O}$, and hence no endpoint of I_0, has three consecutive iterates in the interval I_1. Hence by Lemma 2.6 the loop

$$I_0 \to \overbrace{I_1 \to I_1 \to \cdots \to I_1}^{l-1 \text{ copies of } I_1} \to I_0$$

is elementary if $l > 3$. Thus, f has a periodic point of period l for each $l > 3$.

This shows a special case of the Sharkovsky Theorem: the presence of a period-3 point causes every positive integer to be a period.

3.2. **A 7-cycle.** Consider a 7-cycle $\mathcal{O}$ and $\mathcal{O}$-intervals as in Figure 3.

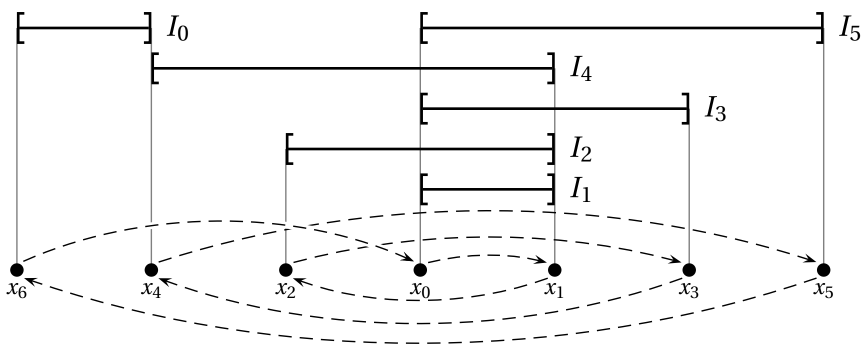


FIGURE 3. A 7-cycle

Again, we write $x_i = f^i(x_0)$ and $I_1 = [x_0, x_1]$ and so on, as indicated. With this choice of intervals we get the following $\mathcal{O}$-forced covering relations:

(1) $I_1 \to I_1$ and $I_0 \to I_1$,
(2) $I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_0$, and
(3) $I_0 \to I_5, I_3, I_1$.

This information can be summarized in a graph as follows:

$$\begin{tikzcd}
& I_1 \arrow[loop above] \arrow[r] & I_2 \arrow[dr] & \\
I_0 \arrow[ur] \arrow[rrr] \arrow[dr, shift left] & & & I_3 \arrow[dl] \\
& I_5 \arrow[ul, shift left] & I_4 \arrow[l] &
\end{tikzcd} \tag{2}$$

From this graph we read off the following loops.

(4) $I_1 \to I_1$,
(5) $I_0 \to I_5 \to I_0$,
(6) $I_0 \to I_3 \to I_4 \to I_5 \to I_0$,
(7) $I_0 \to I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_0$,
(8) $I_0 \to I_1 \to I_1 \to \cdots \to I_1 \to I_2 \to I_3 \to I_4 \to I_5 \to I_0$ with 3 or more copies of I_1.

$I_1 \to I_1$ is elementary because it has length 1, and the remaining loops are elementary by Lemma 2.6 because $\mathrm{Int}(I_0) \cap I_j = \varnothing$ if $1 \le j \le 5$ and the loops cannot be followed by an endpoint of I_0 for reasons familiar from the previous example. The lengths of these loops are 1, 2, 4, 6 and anything larger than 7, which proves that this cycle forces every period $l \lhd 7$.

The standard proof uses a different choice of $\mathcal{O}$-intervals to study this example: the interval I_i for each i with $2 \le i \le 5$ is replaced by the interval between x_i and x_{i-2}. With this alternative choice one still obtains the covering relations (1)–(3), but our choice of $\mathcal{O}$-intervals adapts better to other situations such as that in the next example.

3.3. **A 9-cycle.** Figure 4 shows a 9-cycle $\mathcal{O}$ for which we chose $\mathcal{O}$-intervals $I_0, \dots, I_5$ such that $\mathrm{Int}(I_0) \cap I_j = \varnothing$ if $1 \le j \le 5$ and the covering relations in the graph (2) above are satisfied. The arguments in Subsection 3.2 apply word-for-word to show that there are elementary loops, and hence periodic orbits, of length 1, 2, 4, 6 and anything larger than 7.

The endpoints $x_0, \dots, x_6$ of the intervals in Figure 4 spiral outwards from the "center" $c := (x_0 + x_1)/2$ like the corresponding points in Figure 3, but now they do not constitute the entire cycle $\mathcal{O}$ and we do not have $f(x_i) = x_{i+1}$ for every i.

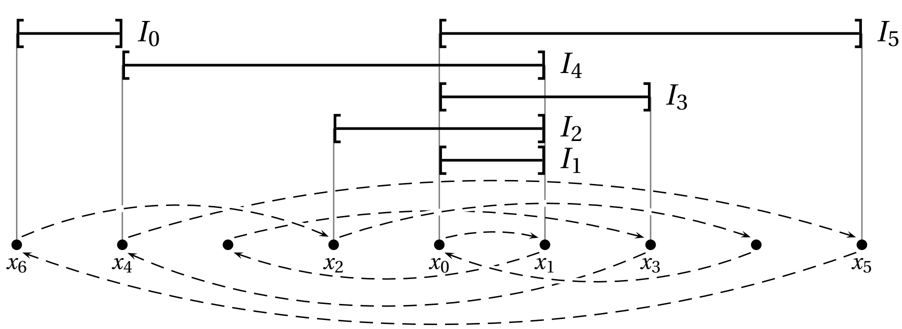


FIGURE 4. A 9-cycle

The sequence $x_0, \dots, x_6$ is chosen using the algorithm explained in Section 5. The main idea in this algorithm is that one does not always choose $x_{i+1} = f(x_i)$, but moves inwards towards the center c if this will make $f(x_{i+1})$ lie further from c. Figure 5 illustrates this with the graph of a simple function f that exhibits the cycle $\mathcal{O}$.

Starting from a point $(x_i, f(x_i))$ on the graph of $\mathcal{O}$ one moves horizontally to the diagonal, then vertically to the point $(f(x_i), f^2(x_i))$ on the graph. Then, if possible, one skips to a point on the graph of $\mathcal{O}$ that is closer to c in the horizontal direction and further from c in the vertical direction; this point will be $(x_{i+1}, f(x_{i+1}))$. Such skips happen in step 2 and 3 of this example.

The process terminates when the sequence has spiralled out past a point (x_6 here) whose image under f is on the same side of c as the point itself.

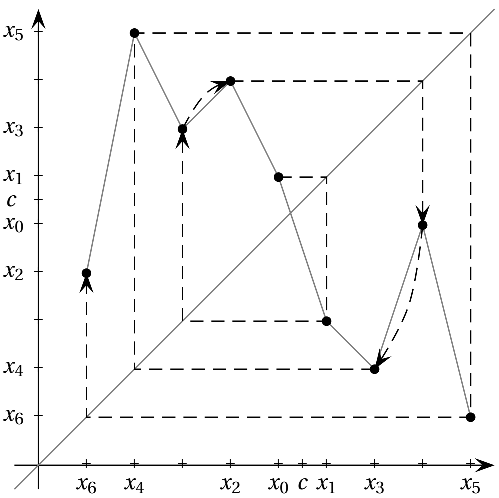


FIGURE 5. The spiral of the points x_i

In the next section we abstract the properties of the endpoints of the intervals $I_0, I_1, \dots$ that are essential to the above argument.

3.4. **A 6-cycle.** Consider the 6-cycle in Figure 6.

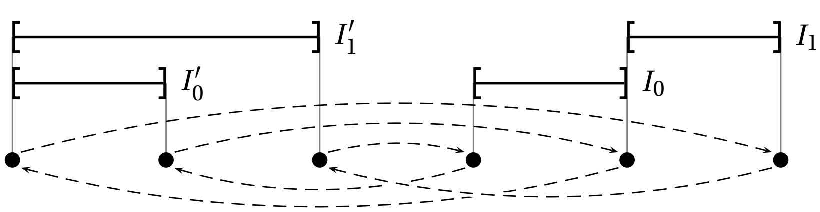


FIGURE 6. A 6-cycle

The salient feature here is that the 3 points in the left half are mapped to the 3 points in the right half and vice versa. Therefore, the 3 points in the right half form a cycle for the second iterate f^2. As in Subsection 3.1 we have the covering relations $I_1 \xrightarrow{f^2} I_1$, $I_1 \xrightarrow{f^2} I_0$, and $I_0 \xrightarrow{f^2} I_1$ for the intervals I_0 and I_1 shown in Figure 6. We can conclude as before that f^2 has elementary loops of all lengths.

For f itself we choose two additional intervals I'_0 and I'_1 by taking I'_j to be the shortest $\mathcal{O}$-interval that contains $f(I_j \cap \mathcal{O})$.

We now illustrate a recursive method we will use later: we show how to associate with an elementary k-loop for f^2 an elementary $2k$-loop for f itself. In the present example this then tells us that every even number is a period.

Consider an elementary k-loop for f^2 made using the covering relations $I_1 \xrightarrow{f^2} I_1$, $I_1 \xrightarrow{f^2} I_0$, and $I_0 \xrightarrow{f^2} I_1$. Replace each occurrence of "$I_1 \xrightarrow{f^2}$" by "$I_1 \xrightarrow{f} I'_1 \xrightarrow{f}$" and each occurrence of "$I_0 \xrightarrow{f^2}$" by "$I_0 \xrightarrow{f} I'_0 \xrightarrow{f}$" and note that this produces a $2k$-loop for f that is not a repeated shorter loop. We show that it is elementary using the *definition* of elementary. Suppose a point p follows the $2k$-loop under f. We need to show that it has period $2k$ for f. Observe that p follows the original elementary k-loop under f^2 and hence has period k for f^2. On the other hand, the iterates of p under f are alternately to the left and the right of the middle interval (x_0, x_1) since the $2k$-loop for f alternates between primed and unprimed intervals. Therefore, the orbit of p consists of $2k$ distinct points; there are k even iterates on the right and k odd iterates on the left. This means that the period of p for f is $2k$. Since k was arbitrary, this shows that this 6-cycle forces all even periods (as well as period 1 due to the interval $[x_0, x_1]$ in the center, which covers itself under f).

In the next 3 sections we prove the Sharkovsky Forcing Theorem 1.1. We first show that the existence of a special sequence in an m-cycle $\mathcal{O}$ produces all desired cycles. Next we construct such a sequence under a mild assumption on $\mathcal{O}$. Finally we reduce the general case to this latter one.

4. Štefan sequences produce cycles

Let $\mathcal{O}$ be an m-cycle with $m \geq 2$ of a continuous map f on an interval.

Definition 4.1. Let p be the rightmost of those points in $\mathcal{O}$ for which $f(p) > p$, and q the point of $\mathcal{O}$ to the immediate right of p.

We define the *center* of $\mathcal{O}$ by $c := (p+q)/2$. For $x \in \mathcal{O}$ we denote by $\mathcal{O}_x \subset \mathcal{O}$ the set of points of $\mathcal{O}$ in the closed interval bounded by x and c. That is, $\mathcal{O}_x = \mathcal{O} \cap [x, p]$ when $x \leq p$, and $\mathcal{O}_x = \mathcal{O} \cap [q, x]$ when $x \geq q$.

We say that a point $x \in \mathcal{O}$ *switches sides* if c is between x and $f(x)$.

From the examples in Section 3 we extract the following desirable properties of a sequence of points of $\mathcal{O}$.

Definition 4.2. A sequence $x_0, \dots, x_n$ of points in $\mathcal{O}$ is called a *Štefan sequence* if

(Š1) $\{x_0, x_1\} = \{p, q\}$.
(Š2) $x_1, \dots, x_n$ are on alternating sides of the center c and the sequences (x_{2j}) and (x_{2j+1}) are both strictly monotone (necessarily moving away from c).
(Š3) If $1 \le j \le n-1$, then x_j switches sides and $x_{j+1} \in \mathcal{O}_{f(x_j)}$.
(Š4) x_n does not switch sides.

Remark 4.3. The condition $x_{j+1} \in \mathcal{O}_{f(x_j)}$ in (Š3) means that $c < x_{j+1} \le f(x_j)$ or $c > x_{j+1} \ge f(x_j)$.

(Š2) implies that $x_0, \dots, x_n$ are pairwise distinct. Hence $n+1 \le m$ and so $n < m$. Figure 2 and Figure 3 show Štefan sequences that happen to consist of the entire cycle; we have $n+1 = m$ in these cases. Figure 4 provides an illustration in which a Štefan cycle is a proper subset of the orbit and $n+1 < m$.

(Š1) and (Š4) together imply that $n \ge 2$ and hence $m \ge 3$. Note that for $m = 1$ the Sharkovsky Forcing Theorem is vacuously true and for $m = 2$ it is an application of Lemma 2.2.

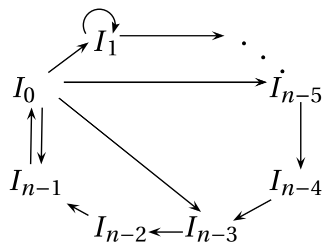
Proposition 4.4. *Suppose that the m-cycle $\mathcal{O}$ has a Štefan sequence. If $l \lhd m$, then f has an $\mathcal{O}$-forced elementary l-loop of $\mathcal{O}$-intervals and hence a periodic point with least period l.*

Given a Štefan sequence $x_0, \dots, x_n$ we define the desired $\mathcal{O}$-intervals $I_0, \dots, I_{n-1}$ as follows. For $1 \le j < n$, we take I_j to be the shortest interval that contains x_0, x_1 and x_j, while I_0 is defined to be the $\mathcal{O}$-interval with endpoints x_n and x_{n-2}. It follows from (Š2) that $\mathrm{Int}(I_0) \cap I_j = \varnothing$ if $1 \le j < n$.

Proposition 4.5. *With I_j chosen as above, we have the following $\mathcal{O}$-forced covering relations.*

(1) $I_1 \to I_1$ *and* $I_0 \to I_1$.
(2) $I_1 \to I_2 \to \cdots \to I_{n-1} \to I_0$.
(3) $I_0 \to I_{n-1}, I_{n-3}, \dots$.

They can be summarized in a graph as follows:



Proof. (1) We will, in fact, prove that $I_j \to I_1$ for $j = 0, \dots, n-1$. This amounts to showing that $f(I_j)$ contains x_0 and x_1.

By (Š2) and (Š3) the endpoints of I_j for $j = 1, \dots, n-1$ are on opposite sides of c and both switch sides. The endpoints of I_0 are on the same side of c, but one switches sides and the other does not, by (Š4). In either case $f(I_j)$ contains points of $\mathcal{O}$ on both sides of c and must therefore contain x_0 and x_1 by (Š1) and the definition of c.

(2) It suffices to show for $1 \le j \le n-1$ that $f(I_j)$ contains x_0, x_1 and x_{j+1}. We have already seen that x_0 and x_1 are in $f(I_j)$. Since $f(x_j)$ is also in the interval $f(I_j)$, this implies that $\mathcal{O}_{f(x_j)} \subset f(I_j)$. It follows from this and (Š3) that $x_{j+1} \in f(I_j)$ as well.

(3) It suffices to show that $f(I_0)$ contains x_0, x_1 and all of the points $x_{n-1}, x_{n-3}, \dots$ of $\mathcal{O}$ that are on the opposite side of c from x_n. We have already seen that $f(I_0)$ contains x_0 and x_1. But x_{n-2} is in I_0 and it follows from (Š3) that $f(x_{n-2})$ is at least as far from c as x_{n-1}, which is further from c than $x_{n-3}, x_{n-5}, \dots$, by (Š2). Consequently the points $x_{n-1}, x_{n-3}, \dots$ lie in $f(I_0)$. □

From the graph in Proposition 4.5 we read off the following loops:

(L1) $I_1 \to I_1$;
(L2) $I_0 \to I_{n-(l-1)} \to I_{n-(l-2)} \to \cdots \to I_{n-2} \to I_{n-1} \to I_0$ for even $l \le n$;
(L3) $I_0 \to I_1 \to I_1 \to \cdots \to I_1 \to I_2 \to \cdots \to I_{n-1} \to I_0$ with $r \ge 1$ repetitions of I_1.

Proof of Proposition 4.4. If $l \lhd m$ then there are 3 cases.

If $l = 1$ we use that the loop (L1) has length 1 and is hence elementary.

If $l \le n$ is even, (L2) provides a loop of length l.

If $l > m$, then (L3) with $r = l - n - 1 > m - n \ge 2$ provides a loop of this length.

The fact that $\mathrm{Int}(I_0) \cap I_j = \varnothing$ if $1 \le j < n$ combined with Lemma 2.6 will tell us that these loops are elementary once we show that they cannot be followed by a point of $\mathcal{O}$. This holds for the loops in (L2) because they have length $l \le n < m$ (Remark 4.3) and for the loops in (L3) when they have $r > 2$ repetitions of I_1. □

5. Constructing a Štefan sequence

The Sharkovsky Forcing Theorem would be immediate from Proposition 4.4 if every cycle had a Štefan sequence. However, the cycle in Figure 6 has no Štefan sequence because every point switches sides. We now show that this is the only obstacle to finding a Štefan sequence.

Proposition 5.1. *A cycle with more than one point contains a Štefan sequence unless every point switches sides.*

Proof. Let $\mathcal{O}$ be a cycle with $m \geq 2$ points.

First we identify a set $\mathcal{S} \subset \mathcal{O}$, which contains the points of $\mathcal{O}$ that are candidates to be non-final terms in a Štefan cycle. Let M be the maximal $\mathcal{O}$-interval containing $[p,q]$ such that all points of $\mathcal{O}$ that are in M switch sides; $\mathcal{O} \cap M$ is thus the set of all $x \in \mathcal{O}$ such that every point of $\mathcal{O}_x$ switches sides. The set $\mathcal{S}$ consists of those $x \in \mathcal{O} \cap M$ such that f maps x further from c than any other point in $\mathcal{O}_x$. Equivalently, $x \in \mathcal{O} \cap M$ is in $\mathcal{S}$ if $\mathcal{O}_{f(w)} \subset \mathcal{O}_{f(x)}$ for all $w \in \mathcal{O}_x$. Note that $p, q \in \mathcal{S}$.

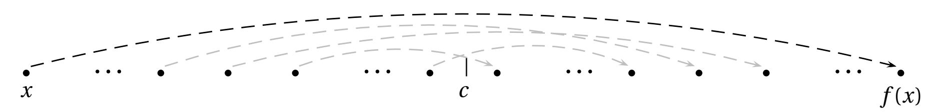


FIGURE 7. $x \in \mathcal{S}$

We now define a map $\sigma\colon \mathcal{S} \to \mathcal{O}$, which will take an element of a Štefan sequence to its successor in the sequence. We always choose $\sigma(x) \in \mathcal{O}_{f(x)}$; since $x \in \mathcal{S}$ this ensures that x and $\sigma(x)$ are on opposite sides of c.

(i) If $f(x) \notin M$, we can take $\sigma(x)$ to be any point of $\mathcal{O}_{f(x)}$ that does not switch sides. In this case $\sigma(x) \notin \mathcal{S}$.

(ii) If $f(x) \in M$, then $\sigma(x)$ is the point of $\mathcal{O}_{f(x)}$ that maps furthest from c, i.e.,

$$f(\mathcal{O}_{f(x)}) \subset \mathcal{O}_{f(\sigma(x))}.$$

By construction (see Figure 8, for example) we have $\sigma(x) \in \mathcal{S}$ in this case.

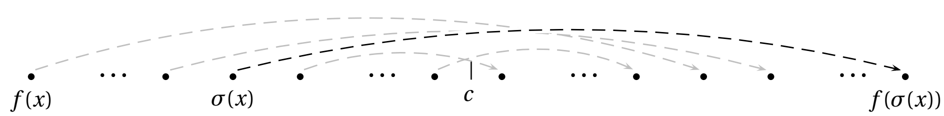


FIGURE 8. The successor map σ in case (ii)

We noted that x and $\sigma(x)$ are on opposite sides of c, so $\sigma^2(x)$, if defined, is again on the same side as x. It is crucial for obtaining the outward spiraling in (Š2) that $\sigma^2(x)$ is further from c than x, i.e., that $\sigma^2(x) \notin \mathcal{O}_x$.

Lemma 5.2. *If there is an $x \in \mathcal{S}$ such that $\sigma^2(x) \in \mathcal{O}_x$, then all points of $\mathcal{O}$ switch sides.*

Proof. In order to have $\sigma^2(x)$ defined and in $\mathcal{O}_x$, we must have x, $y := \sigma(x)$ and $z := \sigma(y) = \sigma^2(x)$ all in $\mathcal{S}$. Moreover $\sigma(x)$ and $\sigma(y)$ are both obtained using case (ii) in the definition of σ. Hence

$$f(\mathcal{O}_{f(x)}) \subset \mathcal{O}_{f(\sigma(x))} = \mathcal{O}_{f(y)}$$

and

$$f(\mathcal{O}_{f(y)}) \subset \mathcal{O}_{f(\sigma(y))} = \mathcal{O}_{f(z)}.$$

Since $z = \sigma^2(x) \in \mathcal{O}_x$ and $x \in \mathcal{S}$, we have

$$\mathcal{O}_{f(z)} \subset \mathcal{O}_{f(x)}.$$

Combining the above inclusions shows that $\mathcal{O}_{f(x)} \cup \mathcal{O}_{f(y)}$ is mapped into itself by f. Since f is a *cyclic* permutation of $\mathcal{O}$, the only nonempty f-invariant subset of $\mathcal{O}$ is $\mathcal{O}$ itself. Thus $\mathcal{O} = \mathcal{O}_{f(x)} \cup \mathcal{O}_{f(y)}$. But all points of this set switch sides because x and y are in $\mathcal{S}$. □

To conclude the proof of Proposition 5.1 we now suppose that there is a point of $\mathcal{O}$ that does not switch sides and show that this implies the existence of a Štefan cycle.

The contrapositive of Lemma 5.2 implies that we cannot have both $\sigma(p) = q$ and $\sigma(q) = p$. Therefore we can choose $\{x_0, x_1\} = \{p, q\}$ in such a way that $x_2 := \sigma(x_1) \neq x_0$ and then continue to choose $x_{i+1} = \sigma(x_i)$ while $x_i \in \mathcal{S}$.

We now verify that this produces a Štefan sequence.

Our choice of $\{x_0, x_1\} = \{p, q\}$ gives (Š1).

To check (Š2) note that successive terms lie on alternating sides of c because x and $\sigma(x)$ are on opposite sides of c. To check that the sequence spirals outward note first that our choice of x_0 and x_1 ensures that $x_2 \notin \mathcal{O}_{x_0}$. Thereafter, Lemma 5.2 shows that $x_{i+2} = \sigma^2(x_i) \notin \mathcal{O}_{x_i}$, i.e., x_{i+2} lies further from c than x_i.

This implies in particular that the terms of the sequence are pairwise distinct. Since they lie in the finite set $\mathcal{O}$, the sequence terminates. We label the last term x_n and note that it necessarily arises from (i) in the definition of σ. Hence x_n does not switch sides, which implies (Š4).

To check (Š3) we note first that for $j < n$ we have $x_j \in \mathcal{S} \subset M$, and x_j therefore switches sides. Finally, $x_{j+1} = \sigma(x_j) \in \mathcal{O}_{f(x_j)}$ by definition of σ. □

Proposition 5.1 and Proposition 4.4 give the following main case of the Sharkovsky Theorem.

Proposition 5.3. *If an m-cycle $\mathcal{O}$ with $m \ge 2$ contains a point that does not switch sides, then for each $l \lhd m$ there is an elementary, $\mathcal{O}$-forced l-loop of $\mathcal{O}$-intervals, and hence an l-cycle.*

6. Proof of the Sharkovsky Forcing Theorem

To prove the Sharkovsky Forcing Theorem it remains to reduce the case of a cycle in which all points switch sides to the main case of Proposition 5.3. We use that the left and right halves of such a cycle are cycles for f^2 of half the length.

Proposition 6.1. *An m-cycle $\mathcal{O}$ has an $\mathcal{O}$-forced elementary l-loop of $\mathcal{O}$-intervals for each $l \lhd m$.*

By Proposition 2.5, this implies the Sharkovsky Forcing Theorem 1.1.

Proof. We proceed by induction on m. Proposition 6.1 is vacuously true for $m = 1$ since there is no $l \lhd 1$.

Suppose now that Proposition 6.1 is known for all cycles of length less than m. Let $\mathcal{O}$ be an m-cycle. If there is a point that does not switch sides, then the conclusion of Proposition 6.1 follows by Proposition 5.3.

Otherwise, all points switch sides. Write $L := \min \mathcal{O}$ and $R := \max \mathcal{O}$. Then $\mathcal{O}_L$ (see Definition 4.1) contains the points in $\mathcal{O}$ to the left of c, $\mathcal{O}_R$ contains those to the right of c, and f swaps these sets: $f \restriction \mathcal{O}_L$ is a bijection from $\mathcal{O}_L$ to $\mathcal{O}_R$ and $f \restriction \mathcal{O}_R$ is a bijection from $\mathcal{O}_R$ to $\mathcal{O}_L$, so $\mathcal{O}_L$ and $\mathcal{O}_R$ have the same number of points, and m is even.

Since m is even, it follows from the doubling property (1) that $l \lhd m$ if and only if $l = 1$ or $l = 2k$ with $k \lhd m/2$. Therefore we need to show that f has an elementary 1-loop as well as an elementary $\mathcal{O}$-forced $2k$-loop of $\mathcal{O}$-intervals for each $k \lhd m/2$.

As the elementary 1-loop we can take the middle $\mathcal{O}$-interval $[p, q]$, since $p = \max \mathcal{O}_L$ and $q = \min \mathcal{O}_R$.

To find the required $2k$-loops, we use the inductive assumption and the fact that $\mathcal{O}_L$ and $\mathcal{O}_R$ are cycles of length $m/2$ for the second iterate f^2. Proposition 6.1 can be applied to either of these cycles. Using $\mathcal{O}_R$, we find that f^2 has an elementary $\mathcal{O}_R$-forced k-loop of $\mathcal{O}_R$-intervals for each $k \lhd m/2$. The induction will be complete once we show that these give rise to elementary $2k$-loops for f itself.

To that end, consider an elementary k-loop

$$I_0 \xrightarrow{f^2} I_1 \xrightarrow{f^2} I_2 \xrightarrow{f^2} \cdots \xrightarrow{f^2} I_{k-1} \xrightarrow{f^2} I_0 \tag{3}$$

of $\mathcal{O}_R$-intervals for f^2. For later convenience we set $I_k := I_0$. Let I_i' be the shortest closed interval that contains $f(I_i \cap \mathcal{O}) \subset \mathcal{O}_L$. These are $\mathcal{O}$-intervals and by construction we have the $\mathcal{O}$-forced covering relation

$I_i \xrightarrow{f} I'_i$ for each i, $0 \le i < k$. The remainder of the proof consists of showing that this produces an $\mathcal{O}$-forced $2k$-loop

$$I_0 \xrightarrow{f} I'_0 \xrightarrow{f} I_1 \xrightarrow{f} I'_1 \xrightarrow{f} \cdots \xrightarrow{f} I_{k-1} \xrightarrow{f} I'_{k-1} \xrightarrow{f} I_0 \tag{4}$$

for f that is elementary.

To see that this is an $\mathcal{O}$-forced loop we show that we also have the covering relations $I'_i \xrightarrow{f} I_{i+1}$ and that they are $\mathcal{O}$-forced. Since $I_i \xrightarrow{f^2} I_{i+1}$ and this covering is $\mathcal{O}_R$-forced, there are points $a_i, b_i \in I_i \cap \mathcal{O}_R$ such that the closed interval between $f^2(a_i)$ and $f^2(b_i)$ contains I_{i+1}. But then $a'_i := f(a_i)$ and $b'_i := f(b_i)$ are in $I'_i \cap \mathcal{O}$ and the closed interval between $f(a'_i) = f^2(a_i)$ and $f(b'_i) = f^2(b_i)$ contains I_{i+1}, as required.

It remains to show that the loop in (4) is elementary. Consider a periodic point x for f that follows the loop (4). It is a periodic point for f^2 that follows the elementary loop (3) and hence has period k with respect to f^2. Therefore k points of its f-orbit (the even iterates) lie in $\mathcal{O}_R$. Since the intervals in the loop (4) are alternately to the right and the left of the center, so are the iterates of x under f. Therefore another k points (the odd iterates) lie in $\mathcal{O}_L$, and the orbit has length $2k$. Hence x has period $2k$ with respect to f, and (4) is elementary. □

7. THE SHARKOVSKY REALIZATION THEOREM

An elegant proof of the Sharkovsky Realization Theorem 1.2 is given in [1]. It reveals one Sharkovsky tail at a time as one increases h in the family of truncated tent maps

$$T_h\colon [0,1] \to [0,1], \quad x \mapsto \min(h, 1-2|x-1/2|) \quad \text{for} \quad 0 \le h \le 1.$$

This family has several key properties.

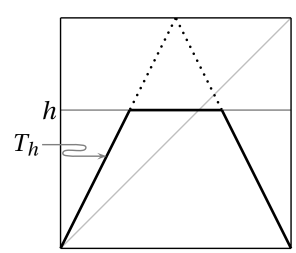


FIGURE 9. Truncated tent maps

(a) T_1 has a 3-cycle $\{2/7, 4/7, 6/7\}$ and hence has all natural numbers as periods by the Sharkovsky Forcing Theorem 1.1.

(b) Any cycle $\mathcal{O} \subset [0,h)$ of T_h is a cycle for T_1, and any cycle $\mathcal{O} \subset [0,h]$ of T_1 is a cycle for T_h.

What makes the proof so elegant is that h plays three roles: as a parameter, as the maximum value of T_h, and as a point of an orbit. The key idea is to let $h(m) := \min\{\max \mathcal{O} \mid \mathcal{O}$ is an m-cycle of $T_1\}$ for $m \in \mathbb{N}$. (We can write "min" instead of "inf" because T_1 has a finite number of periodic points for each period.[4]) From this and (b) we obtain:

(c) T_h has an l-cycle $\mathcal{O} \subset [0,h)$ if and only if $h(l) < h$.

(d) The orbit of $h(m)$ is an m-cycle for $T_{h(m)}$, and all other cycles for $T_{h(m)}$ lie in $[0,h(m))$.

From (d) and the Sharkovsky Forcing Theorem 1.1 we see that if $l \lhd m$, then $T_{h(m)}$ has an l-cycle that lies in $[0,h(m))$; it follows from (c) that $h(l) < h(m)$. By symmetry,

(e) $h(l) < h(m)$ if and only if $l \lhd m$.

We see from (c), (d), and (e) that for any $m \in \mathbb{N}$ the set of periods of $T_{h(m)}$ is the tail of the Sharkovsky order consisting of m and all $l \lhd m$.

The set of all powers of 2 is the only other tail of the Sharkovsky order (besides $\varnothing$, which is the set of periods of the translation $x \mapsto x+1$ on $\mathbb{R}$). We have $h(2^\infty) := \sup_k h(2^k) > h(2^k)$ by (e) for all $k \in \mathbb{N}$, so $T_{h(2^\infty)}$ has 2^k-cycles for all k by (c). Suppose $T_{h(2^\infty)}$ has an m-cycle with m not a power of 2. By Theorem 1.1 $T_{h(2^\infty)}$ also has a $2m$-cycle. Since the m-cycle and the $2m$-cycle are disjoint, at least one of them is contained in $[0,h(2^\infty))$, contrary to (c) and (e).

8. Conclusion

It may be of interest to note that the proof given here provides more information than the statement of the Sharkovsky Forcing Theorem 1.1. Indeed, the loops in (L3) on page 12 are also not followed by a point of $\mathcal{O}$ when

- $r = 1$: since this loop has length $n < m$.
- $r = 2$ and $n < m-1$: since this loop has length $n+1 < m$.

Therefore Proposition 4.4 can be amplified to the following:

Proposition 8.1. *If an m-cycle $\mathcal{O}$ contains a Štefan sequence of length $n < m-1$, then $\mathcal{O}$ forces periods $l = 1$ (from (L1)), $l \geq n$ (from (L3)), and even $l \leq n$ (from (L2)).*

This includes periods that precede m in the Sharkovsky order.

An extreme instance is given by a cycle in which the point q chosen at the beginning of Proposition 5.1 is $\max \mathcal{O}$ and $f(q) = \min \mathcal{O}$, i.e., a

[4] Inspection of the graph of T_1^n shows that it has exactly 2^n fixed points.

cycle of the form $\bullet \cdots \bullet \; \bullet$. The 3 points shown here constitute a Štefan sequence with $n = 2$, which forces period 3 and hence all periods.

Another way in which additional information can be extracted by keeping track of patterns arises in connection with cycles whose length is 2^k for some k. If such a cycle $\mathcal{O}$ contains a point that does not switch sides, then there is a Štefan sequence of length $n < m-1$, and Proposition 8.1 shows that $\mathcal{O}$ forces all periods $l \geq n$, in particular for some odd such l, and hence there are periods that are not powers of 2. Morover, if all points of $\mathcal{O}$ do switch sides, the reduction in the proof of Proposition 6.1 yields a cycle of length 2^{k-1} for f^2 to which one can apply the previous reasoning: It either forces a period that is not a power of 2 or all its points switch sides. In the latter case one can again reduce a step. If this keeps happening until one has reduced to period 2 for $f^{2^{k-1}}$, then we say that $\mathcal{O}$ is *simple*, and we have observed that if a continuous map has only powers of 2 as periods, then all cycles must be simple.

Conversely, if there is a cycle of length 2^k for any $k > 1$ that is not simple, then it forces a period that is not a power of 2.

These observations illustrate that our method can make use of more information than just the period of the cycle from which one starts; this differs from the standard proof, which begins by discarding the initial orbit. Like our proof, refinements of Sharkovsky's Theorem systematically take into account "patterns" instead of just periods.

The definition of a Štefan sequence implies that if $n = m-1$, there will be only one point of $\mathcal{O}$, namely x_{m-1}, that does not switch sides. The point x_{m-1} must be either the leftmost or rightmost point of $\mathcal{O}$ and the sequence $x_0, x_1, \dots$ must spiral outwards clockwise or counterclockwise as shown:

$$x_{m-1} \ \dots \ x_4 \ x_2 \ x_0 \ x_1 \ x_3 \ \dots \ x_{m-2} \quad \text{or} \quad x_{m-2} \ \dots \ x_3 \ x_1 \ x_0 \ x_2 \ x_4 \ \dots \ x_{m-1}\,.$$

Furthermore we must have $f(x_i) = x_{i+1}$ for $0 \leq i < m-1$. These orbits are called *Štefan cycles*. They are central to the standard proof of the Sharkovsky Theorem. Our proof is more direct because we do not need these cycles, but they inspired our definition of Štefan sequences.

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