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THE LÖWNER-KUFAREV REPRESENTATIONS FOR DOMAINS WITH ANALYTIC BOUNDARIES

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Abstract. We consider the Löwner-Kufarev differential equations generating univalent maps of the unit disk onto domains bounded by analytic Jordan curves. A solution to the problem of the maximal lifetime shows how long a representation of such functions admits using infinitesimal generators analytically extendable outside of the unit disk. We construct a Löwner-Kufarev chain consisting of univalent quadratic polynomials and compare the the Löwner-Kufarev representations of bounded and arbitrary univalent functions.

1. Introduction

Löwner introduced [2] his equation to represent a dense subclass of the class S of the univalent conformal maps $f(z) = z + a_2 z^2 + \dots$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by the limit

$$f(z) = \lim_{t\to\infty} e^t w(z,t), \quad z \in \mathbb{D}, \tag{1}$$

where $w(z,t) = e^{-t}z + a_2(t)z^2 + \dots$ is a solution to the equation

$$\frac{dw}{dt} = -w\frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(z,0) \equiv z. \tag{2}$$

Here the driving term $u(t)$ is a continuous function on $t \in [0,\infty)$. Functions $w(z,t)$ map $\mathbb{D}$ onto $\Omega(t) \subset \mathbb{D}$. Later on Pommerenke [4, 5] described governing evolution equations in partial and ordinary derivatives, known now as the Löwner-Kufarev equations due to Kufarev's work [1]

$$\frac{dw}{dt} = -wp(w,t), \quad w(z,0) \equiv z, \tag{3}$$

$$\frac{\partial F(z,t)}{\partial t} = z\frac{\partial F(z,t)}{\partial z}p(z,t), \quad F(z,0) = f(z), \tag{4}$$

for $z \in \mathbb{D}$ and for almost all $t \geq 0$. Here the function p belongs to the Carathéodory class C which means that $p(z,t)$, Re $p(z,t) > 0$, is analytic for $z \in \mathbb{D}$ and measurable for $t \geq 0$, $p(z,t) = 1 + p_1(t)z + p_2(t)z^2 \dots$. We will denote the class of these functions $p(z,t)$ with fixed $t \geq 0$ by the same symbol C if it will not lead to contradiction.

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Pommerenke proved that given a subordination chain of domains $D(t)$, $t \in [0, T]$, there exists $p \in C$ such that the conformal mapping $F : \mathbb{D} \to D(t)$ solves equation (4). Conversely, given an initial univalent function $f(z)$ and $p \in C$, let us ask a question whether the solution $F(z,t)$ to (4) generates a subordination chain of simply connected domains $F(\mathbb{D}, t)$. The univalence condition can be obtained by combination of known results of [5], see also [3].

Theorem A. [3] *Given a function $p \in C$, the solution to equation (4) is unique, analytic and univalent with respect to $z \in \mathbb{D}$ for almost $t \geq 0$ if and only if the initial condition $f(z)$ is taken in the form (1) where the function $w(z,t)$ is the solution to equation (3) with the same driving function p.*

The connection between solutions $F(z,t)$ to (4) and $w(z,t)$ to (3) is given by $w(z,t) = F^{-1}(f(z), t)$ or $F(z,t) = f(w^{-1}(z,t))$. This approach requires the extension of $f(w^{-1}(z,t))$ into $\mathbb{D}$ because $w(z,t)$ range within $\mathbb{D}$ but do not fill it. This is the reason why $F(z,t)$ may be non-univalent if the criterion of Theorem A fails.

According to Pommerenke [5], each function $p(z,t) \in C$ generates by (1), (3) a unique function $f \in S$. The reciprocal statement is not true. In general, a function $f \in S$ can be determined by different functions $p \in C$. Essentially this relates to functions $f \in S$ which map $\mathbb{D}$ onto domains bounded by Jordan analytic curves.

The Löwner equation (2) was an excellent tool to solve numerous extremal problems in the class S, the Bieberbach conjecture among them. The great advantage is that extremal functions of regular problems solve (1)-(2). This gives a chance to apply the classical calculus of variations, optimization methods and other powerful approaches. Every time extremal configurations are one-slit or finitely many-slit domains with boundaries along trajectories of quadratic differentials.

Recently the new trends in geometric function theory called attention to evolution processes for domains with smooth boundaries, C^∞ smooth in particular. We refer to the survey [3] by Markina and Vasil'ev who showed the structural role of the Witt algebra as a background of the Löwner-Kufarev contour evolution. Besides, the conformal anomaly and the Virasoro algebra appear in [3] as a quantum or stochastic effect in the stochastic version of the Löwner equation. Surely, Löwner chains for domains with smooth boundaries are not compact, i.e., in general, the function f in (1) is not in the same class with $w(z,t)$.

The present article deals with solutions to the Löwner-Kufarev equations (3)-(4) which map $\mathbb{D}$ onto domains with analytic boundaries. This class is not compact as well, and the representation of $f \in S$ by (1), (3) is not unique. However we consider a problem of the maximal lifetime for this process.

In Section 2 we prove Theorem 1 which gives the criterion for using $p(\cdot, t) = 1$, $0 \leq t \leq t_0$, in a representation (1) of $f \in S$. Theorem 2 shows how long a representation (1), (3) for functions f analytically extendable to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ admits using $p(\cdot, t)$ which is also analytically extendable to $\overline{\mathbb{D}}$.

In Section 3 we construct a Löwner-Kufarev chain consisting of univalent quadratic polynomials. Surely, the construction ideas work for univalent polynomials of arbitrary powers.

In Section 4 we compare the Löwner-Kufarev representations of bounded and arbitrary univalent functions and give a criterion for representations of bounded functions.

2. The Löwner-Kufarev evolution of domains with analytic boundaries

The function $p(\cdot, t) = 1$ in (3), (4) plays an evident extremal role, and the question is whether it can be used in the Löwner-Kufarev evolution process.

Theorem 1. *Suppose $f(z) = \lim_{t\to\infty} e^t w(z,t)$ where $w(z,t)$ is a solution to the Löwner-Kufarev equation (3) and f maps $\mathbb{D}$ onto a domain $D = f(\mathbb{D})$. Then it is possible to choose $p(w,t) = 1$, $t \in [0, t_0]$, for a certain $t_0 > 0$ if and only if D is bounded by an analytic Jordan curve.*

Proof. The function $f(z)$ serves the initial data $f(z) = F(z, 0)$ in the Löwner-Kufarev evolution $F(z,t)$ solving equation (4). Hence, $w(z,t) = F^{-1}(f(z), t)$ or

$$f(z) = F(w(z,t), t) \tag{5}$$

with solutions $w(z,t)$ to the Löwner-Kufarev equation (3). The choice $p(w,t) = 1$, $0 \le t \le t_0$, in (3) implies that $w(z,t) = e^{-t}z$, $0 \le t \le t_0$. Thus $f(z) = F(e^{-t}z, t)$. However, both f and $F(\cdot, t)$ are defined analytically in $\mathbb{D}$. Therefore $F(e^{-t_0}z, t_0)$ admits an analytic continuation onto $\mathbb{D}(t_0) = \{z : |z| < e^{t_0}\}$. Similarly, $f(z)$ admits an analytic continuation onto $\mathbb{D}(t_0)$. This is possible if and only if $f(\mathbb{D})$ is bounded by an analytic Jordan curve.

To end the proof we should show that there is $p(w,t)$, $0 \le t < \infty$, such that $f(z) = \lim_{t\to\infty} e^t w(z,t)$. Indeed, the function $e^{-t_0}F(z, t_0)$ can be obtained as $e^{-t_0}F(z, t_0) = \lim_{\tau\to\infty} e^{\tau} w(z,\tau)$ where $w(z,\tau)$ is a solution to (3) with a certain function $\tilde{p}(w,\tau)$. Therefore there exists a Löwner-Kufarev evolution $G(z,\tau) = e^{\tau}z + \dots$ solving (4) with the initial data $G(z, 0) = e^{-t_0}F(z, t_0)$. The function $e^{t_0}G(z,\tau) = e^{\tau+t_0}z + \dots$ also forms the subordination chain which satisfies the same equation (4). It remains to denote $t = \tau + t_0$ and $F(z,t) = e^{t_0}G(z, t - t_0)$, $t \ge t_0$. Now $F(z,t)$, $0 \le t < \infty$, $F(z, 0) = f(z)$, forms the subordination chain satisfying (4) with $p(z,t) = 1$ for $0 \le t \le t_0$, and $p(z,t) = \tilde{p}(z, t - t_0)$ for $t > t_0$. The same function p generates $f(z)$ by (3). This completes the proof. □

Theorem 1 is true for functions f extendable from $\mathbb{D}$ on $\mathbb{D}(t_0)$. Solutions $F(z,t)$, $0 \le t \le t_0$, to (4) with $p(z,t) = 1$ map $\mathbb{D}$ onto domains with analytic boundaries. We will try to preserve the latter property as far as possible with suitable $p(z,t)$.

Let $f(z) = z + a_2 z^2 + \dots$ be analytically extendable from $\mathbb{D}$ on a simply connected domain B containing the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$ and map B one-to-one onto a domain Ω_1. Suppose that the conformal radius of Ω_1 with respect to 0 equals e^{t_1}.

Denote $\Omega := f(\mathbb{D})$. There exists $F(z, t_1) = e^{t_1}z + b_2 z^2 + \dots$, $F(\mathbb{D}, t_1) = \Omega_1$, and $w(z, t_1) := F^{-1}(f(z), t_1)$, $w(\mathbb{D}, t_1) := E$. Then $\mathbb{D} \setminus E$ is the doubly-connected domain which can be mapped by $\zeta = h(w)$ onto the annulus $\{\zeta : \rho < |\zeta| < 1\}$ so that h is analytically extended on the boundary, $h(\partial\mathbb{D}) = \{\zeta : |\zeta| = 1\}$ and $h(\partial E) = \{\zeta : |\zeta| = \rho\}$.

Denote $h^{-1}(\{\zeta : |\zeta| = r\}) := L_r$, $\rho \le r \le 1$. The analytic curve L_r bounds the simply connected domain E_r, $E = E_\rho$. Then $F(z, t_1)$ maps E_r onto $F(E_r, t_1) := \Omega_r$, $\Omega = \Omega_\rho$. The family $\{E_r\}$, $\rho \le r \le 1$, forms the subordination chain of domains with analytic boundaries. The corresponding Löwner chain is formed by the family $\{F(w_r(z), t_1)\}$, $\rho \le r \le 1$, where w_r maps $\mathbb{D}$ onto E_r. The conformal radius $c(r)$ of E_r with respect to 0 increases from e^{-t_1} to 1 as r varies from ρ to 1. The equality $c(r) = e^{t-t_1}$, $0 \le t \le t_1$, determines an increasing function $r = r(t) = c^{-1}(e^{t-t_1})$, $r(0) = \rho$, $r(t_1) = 1$. So $w_{r(t)}(z) = e^{t-t_1} z + c_2 z^2 + \dots$.

Denote $w_{r(t)}(z) := w(z, t)$. The Löwner chain $\{G(z, t)\} := \{F(w(z, t), t_1)\}$, $0 \le t \le t_1$, satisfies the Löwner-Kufarev differential equation

$$(6) \qquad \frac{\partial G(z,t)}{\partial t} = z\frac{\partial G(z,t)}{\partial z} p(z,t), \quad G(z,0) = f(z), \quad G(z,t_1) = F(z,t_1),$$

$0 \le t \le t_1$, $G(z, t) = e^t z + d_2 z^2 + \dots$, with $p(z, t) \in C$.

Finally, as in the proof of Theorem 1, it remains to continue $p(z, t)$ for $t > t_1$. Similarly, the function $e^{-t_1} G(z, t_1)$ can be obtained as $e^{-t_1} G(z, t_1) = \lim_{\tau\to\infty} e^\tau w(z, \tau)$ for a solution $w(z, \tau)$ to (3) with a certain function $\tilde{p}(w, \tau)$. Hence, there exists an evolution $H(z, \tau) = e^\tau z + \dots$ solving (4) such that $H(z, 0) = e^{-t_1} G(z, t_1)$. The function $e^{t_1} H(z, \tau) = e^{\tau+t_1} z + \dots$ forms the subordination chain which satisfies (4). Denote $t = \tau + t_1$ and $G(z, t) = e^{t_1} H(z, t - t_1)$, $t \ge t_1$. Now $G(z, t)$, $0 \le t < \infty$, $G(z, 0) = f(z)$, forms the subordination chain satisfying (4) with $p(z, t)$ from (6) for $0 \le t \le t_1$, and $p(z, t) = \tilde{p}(z, t - t_1)$ for $t > t_1$. The same function p generates $f(z)$ by (3).

The above reasonings proved the following theorem.

Theorem 2. *Suppose $f(z) = \lim_{t\to\infty} e^t w(z, t)$ where $w(z, t)$ is a solution to the Löwner-Kufarev equation (3), f is analytically extendable from $\mathbb{D}$ on a simply connected domain B containing $\overline{\mathbb{D}}$ and maps B one-to-one onto a domain Ω_1 having the conformal radius e^{t_1} with respect to 0. Then it is possible to choose $p(\cdot, t)$ in (3) such that $p(z, t)$ satisfies (6) for $0 \le t \le t_1$ and $p(z, t) = \tilde{p}(z, t)$ for $t > t_1$. In this case all the domains $w(\mathbb{D}, t)$ and $F(\mathbb{D}, t)$, $0 \le t \le t_1$, where $F(z, t)$ satisfies (4) with the same $p(z, t)$ and the initial data $F(z, 0) = f(z)$, are bounded by analytic Jordan curves.*

Remark that Roth and Schippers [6] considered a "C^m injective homotopy of closed curves". In this sense the family of curves $\{L_r\}$, $\rho \le r \le 1$, in the proof of Theorem 2 forms the "analytic injective homotopy" under assumption that the conformal radius $c(r)$ of E_r is a real analytic function of $r \in (\rho, 1)$. It is interesting to compare Theorems 1-2 with the results of Roth and Schippers [6] who established the existence of solutions to the Löwner-Kufarev equation (4) with sufficiently smooth initial infinitesimal generators $p(z, 0) \in C$. Namely, they proved the following theorem.

Theorem B. [6] *Let $f(z) : \mathbb{D} \to D_0$ be a one-to-one and onto holomorphic mapping such that $f(0) = 0 \in D_0$. Assume that $f \in C^3(\overline{\mathbb{D}})$, and that the boundary of D_0 is a simple curve. For any $p(z) \in C \cap C^2(\overline{\mathbb{D}})$, there exists a Löwner-Kufarev* *chain $F(z,t)$ defined on an interval $[0,T]$, $F(z,0) = f(z)$, satisfying the Löwner-Kufarev partial differential equation (4) such that $p(z,0) = p(z)$.*

It follows from Theorem 2 that if f is analytically extendable to a neighborhood of $\mathbb{D}$, then there exists a Löwner-Kufarev chain defined on an interval $[0,T]$ satisfying the Löwner-Kufarev partial differential equation (4) with $p(z,t)$ analytically extendable on $\overline{\mathbb{D}}$, $p(\mathbb{D},t)$ is a subset of the right half-plane, $0 \le t \le T$. Theorem 2 gives the maximum of T.

3. Quadratic polynomial evolution

In Section 3 we call attention to univalent polynomials. They map $\mathbb{D}$ onto domains with analytic boundaries if the critical points of a polynomial lie outside $\overline{\mathbb{D}}$. We restrict the consideration to quadratic univalent polynomials to clarify the features which can be generalized for arbitrary non-linear univalent polynomials.

A quadratic polynomial

$$f(z) = z + a_2 z^2 \tag{7}$$

is univalent in $\mathbb{D}$ if and only if $|a_2| \le 1/2$. We ask the question whether it can be represented by (1) where solutions $w(z,t)$, $0 < t < \infty$, to (3) are quadratic polynomials as well.

Let $\alpha(t)$, $0 < t < T < \infty$, be a complex-valued non-vanishing continuously differentiable function such that $4e^t|\alpha(t)| < 1$ and Re $p(w,t) > 0$, where

$$p(w,t) = \frac{2 + (1 - \frac{\alpha'(t)}{\alpha(t)})(\sqrt{1 + 4e^t\alpha(t)w} - 1)}{\sqrt{1 + 4e^t\alpha(t)w} + 1}, \quad w \in \mathbb{D}, \quad 0 < t \le T, \tag{8}$$

Denote the class of these functions $\alpha(t)$ with $\alpha(0) = 0$ by $A(T)$.

Theorem 3. *Let $\alpha \in A(T)$. Then*

$$f(z) = z + \alpha(T)z^2 = \lim_{t\to\infty} e^t w(z,t), \quad z \in \mathbb{D},$$

where $w(z,t)$ is a solution to the Löwner-Kufarev equation (3) with $p(w,t)$ given by (8) for $0 \le t \le T$, and $p(w,t) = 1$ for $t > T$. For every $t > 0$, $w(z,t)$ is a quadratic univalent polynomial.

Proof. Denote

$$w(z,t) := f(z,t) = e^{-t}(z + \alpha(t)z^2), \quad z \in \mathbb{D}, \quad 0 \le t \le T.$$

Then

$$z = f^{-1}(w,t) = \frac{2e^t w}{1 + \sqrt{1 + 4e^t\alpha(t)w}}, \quad w \in f(\mathbb{D},t),$$

the continuous branch of the square root is determined by

$$\frac{f^{-1}(w,t)}{w}\Big|_{w=0} = e^t.$$

We find that

$$-\frac{1}{f(z,t)}\frac{\partial f(z,t)}{\partial t} = \frac{1 + (\alpha(t) - \alpha'(t))z}{1 + \alpha(t)z} = \frac{1 + (\alpha(t) - \alpha'(t))f^{-1}(w,t)}{1 + \alpha(t)f^{-1}(w,t)} := p(w,t).$$

The function $p(w,t)$ in the right-hand side of this formula is defined for $w \in f(\mathbb{D},t)$, $0 \le t \le T$. Being extended to $\mathbb{D}$, $p(w,t)$ corresponds to (8). Therefore the quadratic polynomials $f(z,t)$, $0 \le t \le T$, are univalent in $\mathbb{D}$. It remains to put $p(w,t) := 1$ for $t > T$ which implies that the solution $w(z,t)$ to (3) is given by

$$w = f(z,t) = e^{T-t} f(z,T), \quad t > T,$$

and completes the proof. □

Remark 1. *Though the function family $\{w(z,t)\}_{t>0}$ in Theorem 3 consists of quadratic polynomials, in the case when $\alpha'(t) \neq 0$ and $\alpha(t) \neq \alpha(T)$ for $0 < t < T$ neither the function $p(w,t)$ in (8) nor solutions $F(z,t)$ to (4) are polynomials.*

Indeed, $p(w,t)$ is not a polynomial according to (8). The conditions of Remark 1 imply that $w(z,t)$ and $w(z,T)$ have different critical points for $0 < t < T$, and $F(w,t) = f(f^{-1}(w,t))$ is not analytic at the critical point of $f(z,t)$. Therefore $F(w,t)$ is not a polynomial.

Remark 2. *It is impossible to put $T = \infty$ in Theorem 3 and obtain a non-degenerate quadratic polynomial $f(z) = z + \alpha z^2$.*

Indeed, if $\alpha(t)$ tends to $\alpha \neq 0$ as $t \to \infty$, then the condition $4e^t|\alpha(t)| < 1$ breaks for t large enough.

Along with Theorem 3, we can construct qualitatively a family of quadratic polynomials solving equation (3). Let

$$p(w,t) = 1 + \sum_{n=1}^{\infty} p_n(t) w^n, \quad w(z,t) = e^{-t}(z + \sum_{n=2}^{\infty} a_n(t) z^n).$$

Quadratic polynomials $w(z,t)$ have vanishing coefficients $a_3(t) = \dots = a_n(t) = \dots = 0$. Expand the both sides in (3) in powers of z, equate coefficients at the same powers of z and obtain the differential equations for coefficients

$$\frac{da_2}{dt} = -p_1(t)e^{-t}, \quad a_2(0) = 0, \tag{9}$$

$$\frac{da_3}{dt} = -2p_1(t)a_2(t)e^{-t} - p_2(t)e^{-2t}, \quad a_3(0) = 0, \tag{10}$$

$$\frac{da_4}{dt} = -p_1(t)a_2^2(t)e^{-t} - 3p_2(t)a_2(t)e^{-2t} - p_3(t)e^{-3t}, \quad a_4(0) = 0, \tag{11}$$

and so on. Consider the coefficient $p_1(t)$ as the driving function for $a_2(t)$ according to (9) which gives

$$a_2(t) = -\int_0^t p_1(\tau)e^{-\tau} d\tau.$$

To force the next coefficients $a_3(t), a_4(t), \dots$ vanish we require according to (10)-(11) that

$$p_2(t) = -2e^t p_1(t) a_2(t),$$

$$p_3(t) = -e^{2t}p_1(t)a_2^2(t) - 3e^t p_2(t)a_2(t),$$

and further. So all the coefficients $p_2(t), p_3(t), \dots$ are expressed in terms of the only driving function $p_1(t)$. It remains to verify that $\operatorname{Re} p(w,t) > 0$ for $w \in \mathbb{D}$ to be sure that the family $\{w(z,t)\}$ form the Löwner subordination chain. However the requirement $\operatorname{Re} p(w,t) > 0$ is not necessary for univalent quadratic polynomials $w(z,t)$. They can preserve univalence though do not form the univalent subordination chain.

4. The Löwner-Kufarev embedding of the class of bounded functions

It is known that every function $f \in S$ is represented by (1), (3) with a certain function $p(w,t) \in C$. From the other side, every bounded function $f \in S$, $|f(z)| < M$ for $z \in \mathbb{D}$, is represented as $f(z) = Mw(z, \log M)$ where $w(z,t)$ again is a solution to (3) with the corresponding function $p(w,t) \in C$. Denote by $S(M)$ the class of functions $f \in S$ satisfying $|f(z)| < M$ in $\mathbb{D}$. Put the question how $S(M)$ is embedded in S in the Löwner-Kufarev sense. In other words, we should represent

$$f(z) = Mw(z, logM) \in S(M) \;\text{ as }\; f(z) = \lim_{t\to\infty} e^t w(z,t) \in S$$

where $w(z,t)$ is a solution to (3).

One of the ways to embed $S(M)$ in S is proposed in the following theorem.

Theorem 4. *Let $f \in S(M)$ be represented by $f(z) = Mw(z, \log M)$ where $w(z,t)$ is a solution on $t \in [0, \log M]$ to (3) with a function $p(w,t) \in C$, $0 \le t \le \log M$, in its right-hand side. Then $f(z) = \lim_{t\to\infty} e^t w(z,t)$ where $w(z,t)$ solves (3) with the function $\tilde p(w,t) \in C$ such that $\tilde p(w,t) = p(w,t)$ for $0 \le t \le \log M$ and $\tilde p(w,t) = 1$ for $t > \log M$.*

Proof. The solution $w(z,t)$ to (3) with the function $\tilde p \in C$ satisfies the relation $w(z,t) = e^{-t} Mw(z, \log M)$ for $t > \log M$ which completes the proof. □

Remark 3. *The corresponding function $F(z,t)$ solving (4) with the initial data $F(z,0) = f(z)$ and the function $\tilde p \in C$ as in Theorem 4 satisfies the relation $F(z,t) = e^t z$ for $t > \log M$.*

In connection with Theorem 4 we suggest a criterion for bounded Löwner-Kufarev domain evolutions.

Proposition 1. *Let a function $p(z,t) = 1 + \sum_{n=1}^{\infty} p_n(t) z^n$ be analytic for $z \in \mathbb{D}$ and measurable for $t \ge 0$, and $\operatorname{Re} p(z,t) > \beta > 0$ in $\mathbb{D} \times [0,\infty)$. Then the function $f(z)$ given by (1) is bounded where $w(z,t)$ is the solution to the Cauchy problem (3).*

Proof. For $0 < \beta < 1$, the function

$$\zeta = h(z) = \frac{1 + (1-2\beta)z}{1-z}$$

maps $\mathbb{D}$ onto the half-plane $\{\zeta : \text{Re}\,\zeta > \beta\}$. Let $p(z,t)$ satisfy the conditions of Proposition 1. Then the Schwarz lemma implies that

$$\text{Re}\ p(z,t) \geq \frac{1-(1-2\beta)|z|}{1+|z|}, \quad z \in \mathbb{D}.$$

Apply this inequality to the real part of $d\log w$ in the Löwner-Kufarev equation (3) and obtain the differential inequality

$$\frac{1}{|w|}\frac{d|w|}{dt} \leq -\frac{1-(1-2\beta)|w|}{1+|w|}. \tag{12}$$

Separate the variables and integrate inequality (12) on $[0,t]$ to get

$$\int_{|z|}^{|w(z,t)|} \frac{(1+|w|)d|w|}{|w|(1-(1-2\beta)|w|)} \leq -t. \tag{13}$$

Calculations give

$$e^t|w|(1-(1-2\beta)|w|)^{2(1-\beta)/(2\beta-1)} \leq |z|(1-(1-2\beta)|z|)^{2(1-\beta)/(2\beta-1)} \tag{14}$$

for $\beta \neq 1/2$, and

$$e^t|w|e^{|w|} \leq |z|e^{|z|} \tag{15}$$

for $\beta = 1/2$. Going to the limit as $t \to \infty$ in (14)-(15) we find that

$$|f(z)| \leq (2\beta)^{2(1-\beta)/(2\beta-1)}$$

for $\beta \neq 1/2$, and

$$|f(z)| \leq e$$

for $\beta = 1/2$ which completes the proof. □

The conditions of Proposition 1 can be weakened in the way that $p(w,t) \in C$ is an arbitrary function for $0 \leq t \leq T = \log M$ and satisfies $\text{Re}\ p(w,t) > \beta > 0$ for $t > T$. In this case we separate the variables and integrate inequality (12) on $[T,t]$ to get

$$\int_{|w(z,T)|}^{|w(z,t)|} \frac{(1+|w|)d|w|}{|w|(1-(1-2\beta)|w|)} \leq T-t.$$

Now calculations give

$$e^t|w|(1-(1-2\beta)|w|)^{2(1-\beta)/(2\beta-1)} \leq M|w(z,T)|(1-(1-2\beta)|w(z,T)|)^{2(1-\beta)/(2\beta-1)}$$

for $\beta \neq 1/2$, and

$$e^t|w|e^{|w|} \leq M|w(z,T)|e^{|w(z,T)|}$$

for $\beta = 1/2$. Going to the limit as $t \to \infty$ in the last inequalities we find that

$$|f(z)| \leq M(2\beta)^{2(1-\beta)/(2\beta-1)}$$

for $\beta \neq 1/2$, and

$$|f(z)| \leq Me$$

for $\beta = 1/2$.

However, neither Proposition 1 nor its weakened version are necessary for boundedness of $f(z)$. For example, let a function $p(w,t) = p(w) \in C$ have the value set in the right half-plane which touches the imaginary axis and omits a neighborhood of the origin. Then the function $1/p(w) \in C$ has the bounded value set in the right half-plane which touches the imaginary axis. Equation (3) generates by (1) the starlike function $f(z)$ satisfying

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \frac{1}{p(z)}.$$

The function $f(z) \in S$ is bounded together with $1/p(z) \in C$.

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